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NOTE ON THE TIGHTNESS OF THE METRIC ON THE SET OF COMPLETE SUB σ-ALGEBRAS OF A PROBABILITY SPACE

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The purpose of this note is to show that the usual metric on the set of complete sub σ-algebras of a probability space is very tight; a recent result from E. B. Boylan on equi-convergence of martingales follows and is thereby, we believe, better understood.

Given a probability space $(\Omega, \mathscr{A}, P)$, a metric can be introduced on the set S of all complete sub σ-algebras $\mathscr{B}$ of $\mathscr{A}$ by letting (see [1])

$$d(\mathscr{B}_1, \mathscr{B}_2) = \sup_{B_1 \in \mathscr{B}_1} \inf_{B_2 \in \mathscr{B}_2} P(B_1 \triangle B_2) + \sup_{B_2 \in \mathscr{B}_2} \inf_{B_1 \in \mathscr{B}_1} P(B_1 \triangle B_2)$$

where $\triangle$ denotes symmetric difference of sets as usual; here a sub σ-algebra $\mathscr{B}$ of $\mathscr{A}$ is said to be complete (relatively to $\mathscr{A}$) if every set $A \in \mathscr{A}$ of probability zero belongs to $\mathscr{B}$. The following proposition is then the main result of this note.

PROPOSITION. *Let $\mathscr{B}$, $\mathscr{B}'$ be two complete sub σ-algebras of $\mathscr{A}$ in $(\Omega, \mathscr{A}, P)$ such that $\mathscr{B} \subset \mathscr{B}'$. Then there exists a set $A \in \mathscr{B}$ such that*

$$P(A^c) \leqq 4d(\mathscr{B}, \mathscr{B}') \qquad \textit{and} \qquad A \cap \mathscr{B}' = A \cap \mathscr{B}.$$

Conversely if $A \in \mathscr{B}'$ is such that $A \cap \mathscr{B}' = A \cap \mathscr{B}$, then $d(\mathscr{B}, \mathscr{B}') \leqq P(A^c)$.

When $A \in \mathscr{A}$ and $\mathscr{B} \in S$, we denote by $A \cap \mathscr{B}$ the σ-algebra of subsets of A which are of the from $A \cap B$ for a $B \in \mathscr{B}$; when $A \in \mathscr{B}$, then this class $A \cap \mathscr{B}$ is also the σ-algebra of subsets of A belonging to $\mathscr{B}$.

PROOF.

(a) When $\mathscr{B} \subset \mathscr{B}'$, the distance $d(\mathscr{B}, \mathscr{B}')$ can be characterized as the smallest positive real number d for which the following implication holds

$$B' \in \mathscr{B}', \; P^{\mathscr{B}}(B') \leqq \tfrac{1}{2} \text{ a.s.} \Rightarrow P(B') \leqq d,$$

where $P^{\mathscr{B}}(B')$ denotes the conditional expectation of the indicator function $1_{B'}$ of B' with respect to $\mathscr{B}$. This is easily proved as follows.

Because $P(B \triangle B') = E[|P^{\mathscr{B}}(B') - 1_B|]$ when $B \in \mathscr{B}$ and $B' \in \mathscr{B}'$, we have for every $B' \in \mathscr{B}'$

$$\begin{aligned} \inf_{B \in \mathscr{B}} P(B \triangle B') &= \inf_{B \in \mathscr{B}} E[|P^{\mathscr{B}}(B') - 1_B|] \\ &= E\{\min[P^{\mathscr{B}}(B'), 1 - P^{\mathscr{B}}(B')]\} \end{aligned}$$

the infimum being for instance achieved by the $\mathscr{B}$-set $\{P^{\mathscr{B}}(B') > \frac{1}{2}\}$. Hence

$$d(\mathscr{B}, \mathscr{B}') = \sup_{B' \in \mathscr{B}'} E\{\min[P^{\mathscr{B}}(B'), 1 - P^{\mathscr{B}}(B')]\}.$$

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Now we use the hypothesis $\mathscr{B} \subset \mathscr{B}'$ to assert that for every $B' \in \mathscr{B}'$, the set $B^* = B' \triangle \{P^{\mathscr{B}}(B') > \frac{1}{2}\}$ is also in $\mathscr{B}'$; since

$$P^{\mathscr{B}}(B^*) = \min[P^{\mathscr{B}}(B'), 1 - P^{\mathscr{B}}(B')] \leqq \tfrac{1}{2}$$

it is not hard to see that

$$d(\mathscr{B}, \mathscr{B}') = \sup[P(B''); B'' \in \mathscr{B}', P^{\mathscr{B}}(B'') \leqq \tfrac{1}{2}]$$

as announced.

(b) If $\mathscr{B}, \mathscr{B}' \in S$ still verify $\mathscr{B} \subset \mathscr{B}'$, let A be a $\mathscr{B}$-atom of $\mathscr{B}'$ [i.e. a set $A \in \mathscr{B}'$ such that $A \cap \mathscr{B}' = A \cap \mathscr{B}$] with largest possible conditional expectation $P^{\mathscr{B}}(A)$; the existence of such a set has been proved in [2], Theorem 1, page 258. Let us show now that $P(A^c) \leqq 4d(\mathscr{B}, \mathscr{B}')$.

By Theorem 2 of [2], we may find a set $C \in \mathscr{B}'$ such that

$$P^{\mathscr{B}}(C) \leqq \tfrac{1}{2} \leqq P^{\mathscr{B}}(C) + P^{\mathscr{B}}(A) .$$

Then we let

$$D = C\{P^{\mathscr{B}}(A) \leqq \tfrac{1}{4}\} + A\{\tfrac{1}{4} < P^{\mathscr{B}}(A) \leqq \tfrac{1}{2}\} + A^c\{P^{\mathscr{B}}(A) > \tfrac{1}{2}\} .$$

This set D belongs to $\mathscr{B}'$ and is such that

$$\tfrac{1}{4}[1 - P^{\mathscr{B}}(A)] \leqq P^{\mathscr{B}}(D) \leqq \tfrac{1}{2} \quad \text{a.s.}$$

as is easily checked on each of the three sets $\{P^{\mathscr{B}}A \leqq \frac{1}{4}\}$, $\{\frac{1}{4} < P^{\mathscr{B}}A \leqq \frac{1}{2}\}$ and $\{\frac{1}{2} < P^{\mathscr{B}}(A)\}$ on which respectively $P^{\mathscr{B}}(D) = P^{\mathscr{B}}(C)$, $P^{\mathscr{B}}(A)$, or $1 - P^{\mathscr{B}}(A)$. The properties of this set D imply with the aid of (a) that

$$d(\mathscr{B}, \mathscr{B}') \geqq P(D) \geqq \tfrac{1}{4}[1 - P(A)] .$$

The direct part of the proposition is thus proved. The converse is immediate; indeed if $A \in \mathscr{B}'$ is such that $A \cap \mathscr{B}' = A \cap \mathscr{B}$, then for every $B' \in \mathscr{B}'$ there exists a $B \in \mathscr{B}$ for which $AB' = AB$ and then $P(B \triangle B') \leqq P(A^c)$; hence $d(\mathscr{B}, \mathscr{B}') \leqq P(A^c)$. □

The following easy corollary has an interest only for equi-integrable sets of functions (for which $\delta_H(a) \downarrow 0$ as $a \nearrow \infty$).

COROLLARY. *Let H be a set of real integrable functions defined on a probability space $(\Omega, \mathscr{A}, P)$ and let $\mathscr{B}, \mathscr{B}'$ be two sub σ-algebras of $\mathscr{A}$ such that $\mathscr{B} \subset \mathscr{B}'$. Then the following inequality holds*

$$\sup_{f \in H} ||E^{\mathscr{B}}(f) - E^{\mathscr{B}'}(f)||_1 \leqq 16ad(\mathscr{B}, \mathscr{B}') + 4\delta_H(a)$$

for every real $a > 0$, provided one lets

$$\delta_H(a) = \sup_{f \in H} \textstyle\int_{\{|f|>a\}} |f| \, dP .$$

PROOF. Let A be a set with the properties stated in the preceding proposition and let f be a $\mathscr{B}'$-integrable function. Then it is easily checked (see Lemma 2, page 257 of [2]) that $E^{\mathscr{B}}(f1_A) = fP^{\mathscr{B}}(A)$ a.s. on A; hence

$$\begin{aligned}||E^{\mathscr{B}}(f1_A) - f1_A||_1 &= \int_{A^c}|E^{\mathscr{B}}(f1_A)|\,dP + \int_A|f|[1 - P^{\mathscr{B}}(A)]\,dP \\ &\leqq \int_\Omega E^{\mathscr{B}}(|f|)[1_{A^c} + P^{\mathscr{B}}(A^c)]\,dP \\ &= 2\int_\Omega |f|\,P^{\mathscr{B}}(A^c)\,dP\,.\end{aligned}$$

On the other hand

$$||E^{\mathscr{B}}(f1_{A^c}) - f1_{A^c}||_1 \leqq 2||f1_{A^c}||_1 = 2\int_\Omega |f|\,1_{A^c}\,dP\,.$$

The addition of the two inequalities gives that

$$||E^{\mathscr{B}}(f) - f||_1 \leqq 2\int |f|[P^{\mathscr{B}}(A^c) + 1_{A^c}]\,dP$$

for every $\mathscr{B}'$-integrable function f; the inequality remains valid for every integrable f, if $E^{\mathscr{B}'}(f)$ is substituted for f in the first member and then

$$\begin{aligned}||E^{\mathscr{B}}f - E^{\mathscr{B}'}(f)||_1 &\leqq 2\int_\Omega |f|[P^{\mathscr{B}}(A^c) + 1_{A^c}]\,dP \\ &\leqq 4aP(A^c) + 4\int_{\{|f|>a\}} |f|\,dP \\ &\leqq 16ad(\mathscr{B}, \mathscr{B}') + 4\int_{\{|f|>a\}} |f|\,dP\,.\end{aligned}$$

By taking the supremum of the extreme members over H, we obtain the formula of the corollary. □

Boylan has recently proved [1] that for any equi-integrable subset H of $L^1(\Omega, \mathscr{A}, P)$ and for any monotone sequence $(\mathscr{B}_n, n \in N)$ of sub σ-algebras of $\mathscr{A}$ the L^1-convergence $\lim_{n\to\infty} E^{\mathscr{B}_n}(f) = E^{\mathscr{B}_\infty}(f)$ holds *uniformly* on H, provided $\mathscr{B}_\infty$ denotes the limiting σ-algebra of the increasing or decreasing sequence $(\mathscr{B}_n, n \in N)$ and provided $d(\mathscr{B}_n, \mathscr{B}_\infty) \to 0$ as $n \uparrow \infty$. This theorem can be readily obtained from the preceding corollary, since by this result

$$\begin{aligned}\sup_H ||E^{\mathscr{B}_n}(f) - E^{\mathscr{B}_\infty}(f)|| &\leqq 16ad(\mathscr{B}_n, \mathscr{B}_\infty) + 4\delta_H(a) \\ &\to 0\end{aligned}$$

when $n \nearrow \infty$ and then $a \nearrow \infty$.

REFERENCES

[1] Boylan, E. S. (1971). Equi-convergence of martingales. *Ann. Math. Statist.* **42** 552–559.
[2] Neveu, J. (1967). Atomes conditionnels d'espace de probabilité. *Symposium on Probability Methods in Analysis; Lecture Notes in Mathematics* **31** 256–271, Springer Verlag, Berlin.