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Note on Unitary Symmetry in Strong Interactions*)

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Assuming invariance of theory under three-dimensional unitary group, various consequences have been investigated. Both Sakata's and Gell-Mann's scheme can be treated in the same fashion and in a simpler way. Mass formula for particles belonging to the same irreducible representation has been derived and compared with experiments.

§ 1. Introduction

The purpose of this note is to investigate consequences of the three-dimensional unitary group (denoted as U_8 hereafter), which is a certain generalization of the usual isotopic space group. Though many authors^{1),2),8} have examined this problem, our procedure is simpler and some new results have been obtained. Also, we can treat different schemes of U_8 such as Sakata's^{1),2} or Gell-Mann's⁸ on the same footing by our method.

First of all, we shall give some motivations for introducing U_3 . All known interactions obey certain symmetries, i.e. they are subject to the corresponding transformation groups. We can classify all known groups appearing in the studies of elementary particles into the following three categories.

(I) Space-group

- (i) Lorentz group (ii) Charge conjugation
- (II) Isotopic-groups
 - (i) Isotopic spin rotation $R_3^{(I)}$
 - (ii) Baryon gauge transformation $R_2^{(B)}$
 - (iii) Charge gauge transformation $R_2^{(Q)}$
 - (iv) Strangeness gauge transformation $R_2^{(S)}$
 - (v) Leptonic gauge transformation $R_2^{(L)}$
- (III) Gauge-transformation of the 2nd kind
 - (i) Electro-magnetic field
 - (ii) Yang-Mills field

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In this list, we have included the charge conjugation into the space-group, because of the TCP theorem. These three groups of transformations are correlated with each other in some degree, but here we do not go into details. Furthermore, we restrict ourselves only in the study of the iso-space groups (II), in this paper. Moreover, we do not take account of leptons also, though they might be treated on the same footing.⁴⁾ Then, the groups (II) consist of 4 groups. However, by virtue of the Nakano-Nishijima-Gell-Mann formula, we have one following relation:

$$Q = I_3 + 1/2 \cdot (N+S). \tag{1}$$

Thus, only 3 out of the 4 groups are independent. So, the known strong interactions have to be invariant under the following group G:

$$G = R_{3}^{(I)} \times R_{2}^{(B)} \times R_{2}^{(Q)}.$$

Now, for the moment, let us suppose that the nature obeys some higher symmetry than this. Then, the invariant group U of this higher symmetry must include G as a sub-group. One of them including G is U_3 , which is relatively uncomplicated. This is one motivation for adopting U_3 . Besides, we may note that the 3-dimension is the minimum dimension for non-trivial representation of the group G. This may be taken as another motivation for U_3 .⁵⁾

In the next section, we shall give the classification of particles belonging to a given irreducible representation by means of restricting U_3 into U_2 (twodimensional unitary group). In § 3 we shall give applications of U_3 . Furthermore, the following mass formula will be proved:

$$M = a + b \cdot S + c \cdot [I(I+1) - 1/4 \cdot S^{2}].$$
(2)

This relation holds for particles belonging to a given irreducible representation of U_3 , and S and I stand for the strangeness and isospin of particles contained in the representation, respectively. This formula has been proved in the lowest order perturbation violating U_3 -symmetry of the type $\overline{A}A$, but in any orders for the strong U_3 -invariant interactions. The proof of Eq. (2) will be given in the Appendix. As an application of Eq. (2), we note that if N, A, Σ and Ξ belong to an irreducible representation as in the Gell-Mann scheme, we have

$$1/2 \cdot [M_N + M_E] = 3/4 \cdot M_A + 1/4 \cdot M_2,$$

which is satisfied in good accuracy. Another application of our formula Eq. (2) is that the mass of a neutral-isoscalar meson π_0' would be given by

$$M(\pi_0') = 4/3 \cdot M(K) - 1/3 \cdot M(\pi) \simeq 600 \text{ Mev},$$

where π_0' is the meson belonging to the same representation as π , K and \overline{K} mesons. Similarly, we should have

$$M(K^*) = 3/4 \cdot M(\omega) + 1/4 \cdot M(\rho)$$

where ρ , ω and K^* are bosons representing resonant states of $(\pi \cdot \pi)$, $(\pi \cdot \pi \cdot \pi)$ and $(\pi \cdot K)$ system, respectively. We note that this relation is satisfied within an error of 12%.

§ 2. Classification of particles in U_3

The three-dimensional unitary group U_3 is defined by the following transformation on a vector ϕ_{μ} ($\mu = 1, 2, 3$):

$$\phi_{\mu} \rightarrow \sum_{\lambda=1, 2, 3} a_{\mu}^{\lambda} \phi_{\lambda} \quad (\mu = 1, 2, 3)$$
 (3)

where a_{μ}^{λ} satisfies

$$\sum_{\mu=1, 2, 3} (a_{\mu}^{\lambda})^{*} a_{\mu}^{\nu} = \delta_{\lambda}^{\nu} \quad (\nu, \lambda = 1, 2, 3).$$
(4)

In the Sakata model,⁶ we identify ϕ_1 , ϕ_2 and ϕ_3 with the proton, the neutron and the Λ , respectively. However, this is not the only way. We shall assume that ϕ_1 , and ϕ_2 form an isotopic doublet and ϕ_3 an isotopic singlet. As for other quantum numbers, we can assign according to the following cases:

- (a) ϕ_1 , ϕ_2 and ϕ_3 have the baryon number N=1. ϕ_1 and ϕ_2 have the strangeness quantum number S=0, ϕ_3 has the strangeness S=-1.
- (b) We do not assign any baryon numbers to ϕ_1 , ϕ_2 and ϕ_3 , but assign Y=0 for ϕ_1 and ϕ_2 , and Y=-1 for ϕ_3 where Y stands for the hypercharge Y=N+S.
- (c) We do not assign any baryon numbers to ϕ_1 , ϕ_2 and ϕ_3 , but assign a new quantum number $Z=N+3\cdot S$ as Z=1 for ϕ_1 and ϕ_2 , and Z=-2 for ϕ_3 .

The first assignment (a) corresponds to the usual Sakata model, and the second one (b) is practically the same as the Gell-Mann scheme,³⁾ and so we refer to it as "Gell-Mann scheme" for simplicity,^{*)} though not exactly. The third scheme is actually convenient if we consider the unitary-unimodular group of 3 dimensions instead of U_3 , and so refer to it as "the unitary-unimodular scheme". We may give possible schemes other than (a), (b) and (c), but it will not be so fruitful.

First, let us consider the case (a) (referred to as "Sakata scheme" hereafter). In this scheme, consider a special transformation:

$$\phi_1 \rightarrow \mathcal{E}_1 \phi_1, \qquad \phi_2 \rightarrow \mathcal{E}_2 \phi_2, \qquad \phi_3 \rightarrow \mathcal{E}_3 \phi_3$$
$$|\mathcal{E}_{\mu}| = 1 \quad (\mu = 1, 2, 3).$$
 (5)

This is a special transformation of Eqs. (3) and (4). Then, a component of every tensor $T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}$ would transform as

^{*)} Note added in proof: Exactly the same scheme has been proposed by Y. Yamaguchi in 1960, so that we should call it as Yamaguchi-Gell-Mann scheme hereafter. Y. Yamaguchi: private communication.

$$T \longrightarrow \mathcal{E}_1^{\ \alpha} \mathcal{E}_2^{\ \beta} \mathcal{E}_3^{\ \gamma} T.$$

In our case, the baryon number N and the strangeness S is obviously given by

$$N = \alpha + \beta + \gamma$$

$$S = -\gamma.$$
(6)

Now, all irreducible tensor representation of U_3 are characterized by three integers f_1 , f_2 and f_3 satisfying a condition $f_1 \ge f_2 \ge f_3$. We shall denote it as $U_3(f_1, f_2, f_3)$, hereafter. The dimension of the representation is given⁷ by

$$D = 1/2 \cdot (f_1 - f_2 + 1) (f_1 - f_3 + 2) (f_2 - f_3 + 1).$$
(7)

Also, comparing the character of $U_3(f_1, f_2, f_3)$ with Eq. (6), we find that the baryon number N of this representation is

$$N = f_1 + f_2 + f_3. \tag{8}$$

Now, to specify sub-quantum numbers S and the isospin I in $U_3(f_1, f_2, f_3)$, we fix the direction of the 3rd component ϕ_3 . So, we restrict ourselves within the two-dimensional unitary group U_2 , whose irreducible representations are specified by two integers f_1' , and f_2' satisfying $f_1' \ge f_2'$ and will be referred to as $U_2(f_1', f_2')$. Then, the branching rule⁸⁾ for this decomposition tells us that U_3 can be decomposed according as

$$U_{3}(f_{1}, f_{2}, f_{3}) \xrightarrow{}_{(f_{1}', f_{2}')} U_{2}(f_{1}', f_{2}'), \qquad (9)$$

where we sum over all possible integer pairs (f_1', f_2') satisfying the following conditions:

$$f_1 \ge f_1' \ge f_2 \ge f_2' \ge f_3.$$
 (10)

The decomposition Eq. (9) is an analogue of the well-known decomposition of R_4 into R_3 (R_n being the *n*-dimensional rotation group).

$$R_4(l, l') \to \sum_{L=|l-l'|}^{(l+l')} R_3(L).$$

Now, two-dimensional unitary group is a product of two-dimensional unitaryunimodular group (which we can identify as the usual isotopic rotation group) and a gauge group, which defines the nucleon charge. Then, the isospin I is immediately given by

$$I = 1/2 \cdot (f_1' - f_2') \tag{11}$$

and also, comparing the character of $U_2(f_1', f_2')$ with Eq. (6), we get

$$S = (f_1' + f_2') - (f_1 + f_2 + f_3).$$
(12)

In this way, we could specify sub-quantum numbers S and I. Furthermore, we note⁹⁾ that two representations $U_3(f_1, f_2, f_3)$ and $U_3(-f_3, -f_2, -f_1)$ are contragradient to each other, i.e. they are charge-conjugate of each other in our case.

This remark does not apply to the cases (b) and (c), since the nucleon number is not defined in these cases.

In order to explain our procedure, consider various cases :

(i) $(f_1, f_2, f_3) = (1, 0, 0)$

This is a 3-dimensional representation by Eq. (7) and the decomposition Eqs. (9) and (10) tells us two choices $(f_1', f_2') = (1, 0)$ or (0, 0). By Eqs. (8), (11) and (12), N=1 and the former belongs to (I=1/2, S=0), and the latter to (I=0, S=-1). So the natural identification would be the triplet (p, n, Λ) . (ii) $(f_1, f_2, f_3) = (1, 0, -1)$

By Eqs. (7) and (8), this is a boson representation with 8 components. Also, by the remark given after Eq. (12), it must be self-conjugate, i.e. it must contain a particle and its anti-particle together. Now, the decomposition Eqs. (9) and (10) gives us the choice $(f'_1, f'_2) = (1, 0)$, (0, -1), (1, -1) and (0, 0), and by Eqs. (11) and (12) they have (I=1/2, S=1), (I=1/2, S=-1), (I=1, S=0), and (I=0, S=0), respectively. By the remark given in the beginning, the first two must be charge conjugate of each other and the last two must be selfconjugate under charge conjugation operation. Natural identification would be (K_+, K_0) , $(\overline{K}_+, \overline{K}_0)$, (π_+, π_0, π_-) and π_0' , where the last one is a new pseudoscalar boson. We may identify the newly found states K^* , \overline{K}^* , ρ and ω mesons under the same category.

(iii) $(f_1, f_2, f_3) = (2, 0, -1)$

This is a fermion state with 15 components by Eqs. (7) and (8), and they contain the following particles by Eqs. (10), (11) and (12).

$$(I=1/2, S=-2), (I=1, S=-1), (I=0, S=-1), (I=1/2, S=0), (I=1, S=+1), (I=3/2, S=0).$$

We might identify the first four as Σ , Σ , Λ and N, respectively, but then we have two other unwanted particles. This interpretation is originally due to Yamaguchi²⁾ but as we will see in a later section this identification seems to give small masses for (I=1, S=1) and (I=3/2, S=0) particles so as to make them stable, and so it would be more natural to adopt the case (i) as representing Λ and N. Furthermore, if we take the viewpoint (ii) for bosons, then (I=1/2, S=-2) has to be identified still as Σ particles. This is because the transition $\Xi \rightarrow \Lambda + \overline{K}$ must be possible and therefore Ξ (and also Σ since $\Sigma \rightarrow \Lambda + \pi$) has to be in a product representation $U_3(1, 0, 0) \times U_3(1, 0, -1)$. However,¹⁰) we have

$$U_3(1, 0, 0) \times U_3(1, 0, -1) = U_3(2, 0, -1) + U_3(1, 1, -1)$$

+ $U_3(1, 0, 0)$

but $U_3(1, 1, -1)$ and $U_3(1, 0, 0)$ do not contain a particle with (I=1/2, S=-2). As for Σ , the same argument shows that it must belong either to $U_3(2, 0, -1)$ or to $U_3(1, 1, -1)$. Ikeda et al.¹⁾ identify (I=3/2, S=0) in $U_3(2, 0, -1)$ as N* (the first π -N scattering resonance), then the spin of Σ has to be 3/2, since N* has the space-spin 3/2. Similarly, (I=1, S=-1) and (I=0, S=-1) states in $U_3(2, 0, -1)$ may be interpreted as $Y_1^*(\pi \cdot \Lambda \text{ scattering resonance})$ and Y_0^* ($\pi \cdot \Sigma$ scattering resonance), respectively. Then, they must have spin 3/2 also. In this case, we have to assign $U_3(1, 1, -1)$ for Σ .

(iv) $(f_1, f_2, f_3) = (1, 1, -1)$

This is a fermion state with six components. We have (I=1/2, S=0), (I=0, S=+1) and (I=1, S=-1), and the last one may be interpreted as Σ . However, we have a new state with (I=0, S=+1), so, we should observe a resonance for the reaction K_++n scattering, which has not so far been found experimentally.

Up to now, we have investigated the case (a), i.e. the Sakata-scheme. Now, let us consider the case (b). In this case, we cannot assign any baryon numbers to ϕ_{μ} , so that Eq. (8) has no meaning as to indicate the baryon number. Eq. (11) is unchanged as before, but in Eq. (12), S has to be replaced by Y, so that in our scheme (b), we have

$$I = 1/2 \cdot (f_1' - f_2')$$

$$Y = (f_1' + f_2') - (f_1 + f_2 + f_3).$$
(13)

In this case, the representation (1, 0, -1) gives four states; (I=1/2, Y=1), (I=1/2, Y=-1), (I=1, Y=0) and (I=0, Y=0). As for bosons, our assignment is unchanged, since S and Y are the same for bosons. So, we can assign $(\pi, K, \overline{K}, \pi_0')$ and $(\rho, K^*, \overline{K^*}, \omega)$ to $U_3(1, 0, -1)$. A new phenomenon is that we can also assign $(N, \Xi, \Sigma, \Lambda)$ to $U_3(1, 0, -1)$ since the nucleon number is no longer defined and the corresponding quantum numbers Y and I can be given correctly. This is exactly the same as in Gell-Mann's scheme, though the starting points are quite different. As we shall see in the next section, our scheme is essentially the same as Gell-Mann's. We may note the following decomposition:¹⁰

$$egin{aligned} U_{\mathfrak{s}}(1,\,0,\,-1) imes U_{\mathfrak{s}}(1,\,0,\,-1) &= 2U_{\mathfrak{s}}(1,\,0,\,-1) + U_{\mathfrak{s}}(0,\,0,\,0) + U_{\mathfrak{s}}(2,\,0,\,-2) \ &+ U_{\mathfrak{s}}(2,\,-1,\,-1) + U_{\mathfrak{s}}(1,\,1,\,-2) \end{aligned}$$

so that Y_1^* , Y_0^* and N^* in the Gell-Mann scheme have to be included in one of the right-hand side, since they decay into one-boson and one-fermion state. This will be treated in a forthcoming paper.

Finally, we may study the consequence of our scheme (c). This was given, since it is more natural when we think of the unitary-unimodular group of 3dimension (we refer to it as SL(3)) rather than U_3 . In SL(3), there is no distinction between covariant and contravariant tensors. This is because a constant totally anti-symmetric tensor $\epsilon^{\lambda\mu\nu}$ is invariant under SL(3), so that ϕ^{λ} behaves like $\epsilon^{\lambda\mu\nu}T_{\mu\nu}$ where $T_{\mu\nu}$ is a tensor. More generally, we have that the representation (f_1, f_2, f_3) , which we have written¹¹) as $U_3(f_1, f_2, f_3)$ up to now, is the same representation as (f_1+e, f_2+e, f_3+e) where e is an arbitrary integer. Then, obviously Eqs. (12) or (13) is not invariant under SL(3), since it is not invariant under $f_{\mu} \rightarrow f_{\mu} + e(\mu = 1, 2, 3)$ and $f_{\mu}' \rightarrow f_{\mu}' + e(\mu = 1, 2)$. Invariant quantum numbers under SL(3) under our decomposition Eq. (9) are given by

$$Z = 3(f_1' + f_2') - 2(f_1 + f_2 + f_3),$$

$$I = 1/2(f_1' - f_2')$$
(14)

where $Z=N+3\cdot S$. We omit the details for these derivations. In this case, we can repeat the same procedures as before, but it gives almost the same results as in the case (a), so we will not go too far. Here we may note also that if we give up additivity of quantum numbers, we may assign $Z=3\cdot Y+N(N-1)$ for Eq. (14). In this case, we can assign (1, 0, -1) both for bosons and fermion, and we have the same result as Gell-Mann's again. We shall not consider our case (c) any longer in this paper, and restrict ourselves only in discussions of the cases (a) and (b).

§ 3. Tensor representation and applications

First, let us consider the Sakata scheme (a), and we take the representations $U_3(1, 0, 0)$ and $U_3(1, 0, -1)$ for (Λ, n, p) and $(\pi, \pi_0', K, \overline{K})$ systems, respectively. Then, p, n and Λ can be represented by a vector ψ_{μ} .

$$\psi_1 = p, \ \psi_2 = n, \ \psi_3 = \Lambda \tag{15}$$

and (π, π_0', K, K) can be represented by a traceless tensor f_{μ}^{μ} , so that $f_{\mu}^{\mu} = 0$. The identification is

$$\pi_{+} = f_{1}^{2}, \ \pi_{-} = f_{2}^{1}, \ \pi_{0} = \frac{1}{\sqrt{2}} (f_{1}^{1} - f_{2}^{2}), \ \pi_{0}' = -\frac{3}{\sqrt{6}} f_{3}^{3},$$

$$K_{+} = f_{1}^{3}, \ K_{0} = f_{2}^{3}, \ \overline{K_{+}} = f_{3}^{1}, \ \overline{K_{0}} = f_{3}^{2}$$
(16)

and also $(\rho, \omega, K^*, \overline{K^*})$ can be represented by a traceless tensor F_{ν}^{μ} exactly in the same fashion as Eq. (16) by replacing $\pi \rightarrow \rho$, $\pi_0' \rightarrow \omega$, $K \rightarrow K^*$, $\overline{K} \rightarrow \overline{K^*}$. Actually, F_{ν}^{μ} has a vector suffix due to space-spin, but we omit it for simplicity.

The invariant interactions among baryon-boson and among boson-boson would be given by

$$H_1 = ig \,\overline{\psi}_{\mu} \,\gamma_5 \,\psi_{\nu} \,f_{\mu}{}^{\nu}, \tag{17}$$

$$H_2 = ig F_{\nu}^{\mu} \cdot (f_{\lambda}^{\nu} \cdot \partial f_{\mu}^{\lambda} - \partial f_{\lambda}^{\nu} \cdot f_{\mu}^{\lambda})$$
(18)

where the repeated indices mean summations over 1, 2 and 3. In Eq. (17), we note that $\vec{\psi}_{\mu}$ behaves as a contra-variant vector ϕ^{μ} . Using the representations Eqs. (15) and (16), these Hamiltonians can be written as

$$H_1 = ig \frac{1}{\sqrt{2}} \overline{N} \gamma_5 (\boldsymbol{\tau} \cdot \boldsymbol{\pi}) N + ig \overline{N} \gamma_5 \Lambda K + ig \overline{\Lambda} \gamma_5 N \overline{K}$$

$$+ig\frac{1}{\sqrt{6}}(\overline{N}\gamma_{5}N-2\overline{A}\gamma_{5}\Lambda)\pi_{0}', \qquad (17)'$$

$$H_{2}=\frac{ig}{\sqrt{2}}\rho(\overline{K}\tau\partial K-\partial \overline{K}\tau K)+\sqrt{2}\cdot g\cdot\rho(\pi\times\partial\pi)$$

$$+\frac{ig}{\sqrt{2}}\overline{K}^{*}\tau[K\partial\pi-(\partial K)\pi]+\frac{3}{\sqrt{6}}ig\overline{K}^{*}[K\partial\pi_{0}'-\partial K\pi_{0}']$$

$$+\frac{ig}{\sqrt{2}}[\pi(\partial\overline{K})-\partial\pi\overline{K}]\cdot\tau K^{*}+\frac{3}{\sqrt{6}}ig[\pi_{0}'\partial\overline{K}-\partial\pi_{0}'\overline{K}]K^{*}$$

$$+\frac{3}{\sqrt{6}}ig\omega[\overline{K}\partial K-\partial\overline{K}K]. \qquad (18)'$$

We note that Eq. (18)' agrees with that given by Gell-Mann.³⁾

Now, let us consider the Gell-Mann scheme (b). Here, as for bosons, Eq. (16) is unchanged. For baryons, we introduce two traceless tensors N_{ν}^{μ} and M_{ν}^{μ} (so that $M_{\mu}^{\mu} = N_{\mu}^{\mu} = 0$) as representing

$$\Sigma_{+} = N_{1}^{2}, \ \Sigma_{-} = N_{2}^{1}, \ \Sigma_{0} = \frac{1}{\sqrt{2}} (N_{1}^{1} - N_{2}^{2}), \ \Lambda = -\frac{3}{\sqrt{6}} N_{3}^{3},$$
 (19a)

$$p = N_{1}^{3}, \ n = N_{2}^{3}, \ \overline{\Xi}_{-} = N_{3}^{1}, \ \overline{\Xi}_{0} = N_{3}^{2},$$

$$\overline{\Sigma}_{-} = M_{1}^{2}, \ \overline{\Sigma}_{+} = M_{2}^{1}, \ \overline{\Sigma}_{0} = \frac{1}{\sqrt{2}} (M_{1}^{1} - M_{2}^{2}), \ \overline{\Lambda} = -\frac{3}{\sqrt{6}} M_{3}^{3}, \qquad (19b)$$

$$\overline{\Xi}_{-} = M_{1}^{3}, \ \overline{\Xi}_{0} = M_{2}^{3}, \ \overline{p} = M_{3}^{1}, \ \overline{n} = M_{3}^{2}.$$

$$H_3 = ig M_{\nu}^{\mu} \gamma_5 N_{\lambda}^{\nu} f_{\mu}^{\lambda}, \qquad (20a)$$

$$H_4 = ig M_{\nu}^{\mu} \gamma_5 f_{\lambda}^{\nu} N_{\mu}^{\lambda}. \tag{20b}$$

Explicit calculation gives

$$H_{3} = \frac{ig}{\sqrt{2}} \overline{N} \gamma_{5}(\boldsymbol{\tau} \cdot \boldsymbol{\pi}) N + \frac{g}{\sqrt{2}} (\overline{\boldsymbol{\Sigma}} \gamma_{5} \times \boldsymbol{\Sigma}) \boldsymbol{\pi} + \frac{g}{\sqrt{6}} [i \overline{\boldsymbol{\Sigma}} \boldsymbol{\pi} \gamma_{5} \boldsymbol{\Lambda} + \text{c.c.}]$$

$$+ \frac{g}{\sqrt{6}} [\bar{\boldsymbol{\Lambda}} \gamma_{5} \boldsymbol{\Sigma} \tau_{2} \boldsymbol{K} + \text{c.c.}] - \frac{\sqrt{6}}{3} g [i \overline{N} \gamma_{5} \boldsymbol{K} \boldsymbol{\Lambda} + \text{c.c.}]$$

$$- \frac{g}{\sqrt{2}} [\overline{\boldsymbol{K}} \boldsymbol{\tau} \tau_{2} \overline{\boldsymbol{\Sigma}} \gamma_{5} \boldsymbol{\Sigma} + \text{c.c.}]$$

$$- \frac{ig}{\sqrt{6}} \pi_{0}' [2 (\overline{\boldsymbol{\Sigma}} \gamma_{5} \boldsymbol{\Sigma}) + \bar{\boldsymbol{\Lambda}} \gamma_{5} \boldsymbol{\Lambda} - \overline{\boldsymbol{\Sigma}} \gamma_{5} \boldsymbol{\Sigma} - \overline{N} \gamma_{5} \boldsymbol{N}], \qquad (21a)$$

$$H_{4} = \frac{g}{\sqrt{6}} [i \bar{\boldsymbol{\Lambda}} \gamma_{5} \boldsymbol{\Sigma} \boldsymbol{\pi} + \text{c.c.}] - \frac{g}{\sqrt{2}} (\overline{\boldsymbol{\Sigma}} \gamma_{5} \times \boldsymbol{\Sigma}) \boldsymbol{\pi}$$

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$$-\frac{ig}{\sqrt{2}}\overline{\mathcal{B}}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\gamma_{5}\overline{\mathcal{B}}$$

$$+\frac{g}{\sqrt{6}}[i\overline{N}\gamma_{5}KA + \text{c.c.}]$$

$$-\frac{2g}{\sqrt{6}}[\bar{A}\gamma_{5}\overline{\mathcal{B}}\tau_{2}K + \text{c.c.}]$$

$$+\frac{g}{\sqrt{2}}[i\overline{N}\gamma_{5}\boldsymbol{\tau}K\boldsymbol{\Sigma} + \text{c.c.}]$$

$$+\frac{ig}{\sqrt{6}}\pi_{0}'[\overline{\boldsymbol{\Sigma}}\gamma_{5}\boldsymbol{\Sigma} + \overline{\mathcal{B}}\gamma_{5}\overline{\mathcal{B}} - \bar{A}\gamma_{5}A - 2\overline{N}\gamma_{5}N] \qquad (21b)$$

where we have put

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix}, \quad \boldsymbol{\Xi} = \begin{pmatrix} -\boldsymbol{\Xi}_0 \\ \boldsymbol{\Xi}_- \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} K_+ \\ K_0 \end{pmatrix}, \quad \boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix},$$

and Eqs. (21a) and (21b) are connected with L_D and L_F of Gell-Mann³⁾ by

$$H_{3} = \frac{1}{2\sqrt{2}} [L_{D} + L_{F}],$$
$$H_{4} = \frac{1}{2\sqrt{2}} [L_{D} - L_{F}],$$

when we take the same coupling constants.

As applications of our formalism, we may think of the boson-baryon scattering in the case of the Sakata scheme. In this case, we can form the following invariants of which the S-matrix element is a linear combination:

$$T_{\nu}{}^{\mu}f_{\lambda}{}^{\nu}\tilde{f}_{\mu}{}^{\lambda}, \ T_{\nu}{}^{\mu}\tilde{f}_{\lambda}{}^{\nu}f_{\mu}{}^{\lambda}, \ T_{\mu}{}^{\mu}f_{\beta}{}^{a}\tilde{f}_{a}{}^{\beta}$$

where we have put $T_{\nu}^{\mu} = \vec{\psi}_{\mu} \phi_{\nu}$, and f, and \tilde{f} represent for incoming and outgoing bosons. From this, we can prove the following identities among total cross-sections.

$$\sigma(\pi_{+}+p) = \sigma(K_{+}+p), \quad \sigma(K_{-}+n) = \sigma(\pi_{+}+A),$$

$$\sigma(\pi_{-}+p) = \sigma(K_{-}+p) = \sigma(K_{+}+A), \text{ etc.}$$

$$\sigma(\pi_{0}'+p) = 1/3 \cdot \sigma(\pi_{0}+p) + 2/3 \cdot \sigma(K_{0}+p).$$

These have been derived also by Hara and Singh.¹²⁾ They are also investigating similar identities in the case of Gell-Mann scheme. We can get similar identities among magnetic moments of baryons. In the case of Sakata-scheme, let us assume that the electromagnetic current j_{μ} has a transformation property as T_1^1 component of a tensor T_{ν}^{μ} . This can be taken, since the usual current $ie\bar{p}\gamma_{\mu}p$ has such form. Then, the method mentioned in the above immediately gives

 $\mu(\Lambda) = \mu(n)$ and also we can prove that K_0 and $\overline{K_0}$ have no electromagnetic structures. This is because we can prove $\langle K_0 | j_{\mu} | K_0 \rangle = \langle \overline{K_0} | j_{\mu} | \overline{K_0} \rangle$ similarly, but j_{μ} changes its sign under charge conjugation, and therefore $\langle K_0 | j_{\mu} | K_0 \rangle$ has to be identically zero.

In the case of Gell-Mann scheme (b), we can give some relations among magnetic moments of baryons. By the same reason as in the above, let us assume that the electromagnetic current j_{μ} behaves as T_1^1 of a tensor T_{ν}^{μ} , with respect to U_3 . We have to take the expectation value of j_{μ} , i.e. T_1^1 . From invariance, we have

$$\langle T_{\nu}^{\mu} \rangle = a M_{\lambda}^{\mu} N_{\nu}^{\lambda} + b M_{\nu}^{\lambda} N_{\lambda}^{\mu} + c \cdot \delta_{\nu}^{\mu} \cdot (M_{\beta}^{\alpha} N_{\alpha}^{\beta})$$

where M and N represent baryons as in Eq. (19) and we have omitted spinor indices. By putting $\mu = \nu = 1$, and comparing with Eq. (19), we have $\mu(p) = a + c$, $\mu(n) = c$, etc. Then, we have the following relations:

$$\mu(p) = \mu(\Sigma_{+}),$$

$$\mu(\Xi_{0}) = \mu(n),$$

$$\mu(\Xi_{-}) = \mu(\Sigma_{-}),$$

$$\mu(\Lambda) = 1/6 \cdot [\mu(p) + \mu(\Sigma_{-}) + 4\mu(n)],$$

$$\mu(\Sigma_{0}) = 1/2 \cdot [\mu(\Sigma_{+}) + \mu(\Sigma_{-})].$$
(22)

Furthermore, if we demand that T_{ν}^{μ} is traceless, i.e. $T_{\mu}^{\mu}=0$, then we should have a+b+3c=0 and then this condition gives one more relation:

$$\mu(\Lambda) = (1/2)\,\mu(n). \tag{23}$$

Relations Eqs. (22) and (23) have been given also by Coleman and Glaschow¹³⁾ by somewhat more direct method. We note that they used $T_{\beta}{}^{\alpha} = M_{\lambda}{}^{\alpha}N_{\beta}{}^{\lambda}$ $-M_{\beta}{}^{\lambda}N_{\lambda}{}^{\alpha}$, so that obviously $T_{\alpha}{}^{\alpha} = 0$ is satisfied. From our derivation, however, it is clear that the explicit form for $T_{1}{}^{1}$ is unnecessary.

We can give other applications of our method for the weak leptonic decays of bosons and fermions. In case of the strangeness-violating leptonic decays, the interaction Hamiltonian would be given by

$$H_1 = G \ \mathfrak{Y}_{\mu} [\overline{\nu} \gamma_{\mu} (1 + \gamma_5) e + \overline{\nu} \gamma_{\mu} (1 + \gamma_5) \mu] + \mathrm{c.c.}$$
(24)

where \mathfrak{F}_{μ} is the strangeness-violating current. Let us consider the case of Gell-Mann scheme, and assume that \mathfrak{F}_{μ} has the transformation property as $T_1^{\mathfrak{F}}$ component of a tensor T_{ν}^{μ} , so that it has the same character as K_+ . Then, we may construct two tensors $M_{\lambda}^{\mathfrak{F}}N_1^{\lambda}$ and $M_1^{\lambda}N_{\lambda}^{\mathfrak{F}}$ out of M and N, and it would be natural to take

$$\mathfrak{F}_{\mu} = aM_{\lambda}^{3}N_{1}^{\lambda} + bM_{1}^{\lambda}N_{\lambda}^{3}$$

$$= a \left[\frac{1}{\sqrt{6}} (\overline{\mathcal{F}}_{-} \cdot \Lambda) + (\overline{\mathcal{F}}_{0} \cdot \mathcal{F}_{+} + \frac{1}{\sqrt{2}}\overline{\mathcal{F}}_{-} \cdot \mathcal{F}_{0}) - \frac{\sqrt{6}}{3}(\overline{\Lambda} \cdot p) \right]$$
(25)

$$+b\left[\frac{1}{\sqrt{6}}(\bar{A}\cdot p)+(\bar{\Sigma}_{-}\cdot n+\frac{1}{\sqrt{2}}\bar{\Sigma}_{0}\cdot p)-\frac{\sqrt{6}}{3}(\bar{\Xi}_{-}\cdot A)\right]$$

where we omitted γ -matrices. Of course, this behaves as a component of an isotopic spinor¹⁴ in the usual isospin assignment.

§4. Applications of mass formula

If there are no interactions violating U_3 symmetry, all particles belonging to the same irreducible representation have to have the same mass, the same spin and parity. So we should have the same mass for pion and kaon, which is not true. We must therefore have some interactions violating U_3 . According to Yamaguchi,²⁾ we may suppose that such interactions may be moderately strong, as compared with the very strong U_3 -conserving interactions. Our purpose in this note is to investigate the result of mass-splitting among particles in a given irreducible representation due to this moderately strong U_3 -violating interaction. In the Appendix, we shall prove that the mass splitting is given by the following formula.^{*)}

$$M = a + b \cdot S + c \cdot [1/4 \cdot S^2 - I(I+1)].$$
(26)

Eq. (26) has been proved in the lowest order perturbation for such U_s -violating interaction with the transformation property T_s^3 of a tensor T_{ν}^{μ} but in any orders for U_s -conserving very strong interactions. In Eq. (26), a, b and c are constants which do not depend upon such sub-quantum numbers as the strangeness S and isospin I, but may depend upon the nature of the interaction and upon the irreducible representation to be considered. Eq. (26) may be rewritten as

$$M = a' + b'Y + c'[1/4 \cdot Y^2 - I(I+1)]$$
(27)

if we use the hypercharge Y=N+S instead of S. Formula Eqs. (26) or (27) holds for both the Sakata and the Gell-Mann scheme. For the details, the reader may consult the Appendix.

Now, in this section, we shall investigate the result of Eqs. (26) or (27). First, let us consider boson system $(\pi, \pi_0, K \text{ and } \overline{K})$. An application of (26) or (27) immediately gives that we have a relation

$$M(K) = 1/2 \cdot [M(K) + M(K)] = 3/4 \cdot M(\pi_0') + 1/4 \cdot M(\pi).$$
(28)

From this, we can calculate the mass of π_0' with $M(\pi_0') \simeq 600$ Mev. It is interesting to note that a similar value has been predicted by other methods.¹⁶ The same formula as Eqs. (28) holds for the $(\omega, \rho, K^*, \overline{K^*})$ system.

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$$M(K^*) = 1/2 \cdot [M(K^*) + M(K^*)] = 3/4 \cdot M(\omega) + 1/4 \cdot M(\rho).$$
(29)

^{*)} A similar formula has already been suggested by R. P. Feynman at Gatlingburg Conference held in 1958.

The calculated value for $M(K^*)$ by using $M(\omega)$ and $M(\rho)$ is 780 Mev, compared to the experimental value 885 Mev. This relation Eq. (29) holds as long as $(\rho, \omega, K^*, \overline{K}^*)$ belongs to the same irreducible representation. Previously we have assigned (1, 0, -1) for these, but another possibility is that these may belong to 27-dimensional representation (2, 0, -2) instead of the 8-dimensional $U_3(1, 0, -1)$ representation. Then, the method of § 2 tells us that we have 5 more states $(I=1, S=\pm 2), (I=3/2, S=\pm 1)$ and (I=2, S=0) in addition to $(\rho, \omega, K^*, \overline{K}^*)$. Then, we can use our formula Eq. (26) and we can calculate the mass of these states in terms of $M(\rho)$ and $M(\omega)$, to get

$$M(I=1, S=\pm 2) \simeq 770$$
 Mev,
 $M(I=3/2, S=\pm 1) \simeq 720$ Mev, $M(I=2, S=0) \simeq 700$ Mev.

However, we do not observe I=3/2 resonance for the K- π system, and so this value for $M(I=3/2, S=\pm 1)$ contradicts the experiment. Accordingly, it seems that our assignment of (1, 0, -1) for $(\rho, \omega, K^*, \overline{K^*})$ is more reasonable than that of (2, 0, -2). The above argument equally applies both to the Sakata and the Gell-Mann schemes.

As for baryons, let us first consider the Gell-Mann scheme; then $(\Lambda, \Sigma, N, \Xi)$ belongs to $U_3(1, 0, -1)$ representation. Then, by using Eq. (27), we have a relation

$$1/2[M(N) + M(\mathcal{Z})] = 3/4 \cdot M(\Lambda) + 1/4 \cdot M(\mathcal{Z})$$
(30)

which is satisfied with good accuracy.

In the case of Sakata scheme, we do not have such relation unless we include $(N, \Xi, \Lambda, \Sigma)$ in $U_{\mathfrak{s}}(2, 0, -1)$ representation as we mentioned in § 2. Then, we have Eq. (30) still. However, $U_{\mathfrak{s}}(2, 0, -1)$ representation contains two other states with (I=3/2, S=0) and (I=0, S=+1). We can calculate the masses of these particles by Eq. (26) and by using the experimental masses of N, Λ , and Σ . Then, we get

$$M(I=3/2, S=0) \simeq 1050 \text{ Mev} (< M(N) + M(\pi)),$$

 $M(I=0, S=+1) \simeq 770 \text{ Mev} (< M(N))$

which seems to have too small masses not to be detected experimentally. Thus, this assignment originally due to Yamaguchi would not be so good. Therefore, we take the view that $U_3(2, 0, -1)$ represents Ξ , N^* , Y_0^* , Y_1^* , etc., as has been mentioned in § 2. In this case, we have the following relations:

$$M(Y_{1}^{*}) = 1/2 \cdot [M(\Xi) + M(N^{*})],$$

$$M(I=1/2, S=0) = 1/2 \cdot [M(Y_{0}^{*}) + M(I=1, S=+1)],$$

$$M(I=1, S=+1) = M(Y_{1}^{*}) + 2[M(Y_{0}^{*}) - M(\Xi)].$$
(31)

The first relation gives us $M(Y_1^*) \simeq 1280$ Mev by using the experimental values

for $M(\Xi)$ and $M(N^*)$ and it should be compared to the experimental value of $M(Y_1^*) \simeq 1385$ Mev. Similarly, the last two equations give us

$$M(I=1, S=+1) \simeq 1560$$
 Mev,
 $M(I=1/2, S=0) \simeq 1480$ Mev

where we have used the experimental masses for Y_0^* and Y_1^* . Consequently, we may identify the (I=1/2, S=0) state as the 2nd pion-nucleon resonance, if it corresponds to the $p_{3/2}$ resonance instead of the usual $d_{3/2}$ resonance. As for (I=1, S=+1), resonance for K_++n or K_++p scattering has not been discovered yet, and this gives a trouble to this scheme.

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Appendix

Derivation of Mass Formula

Here, we shall prove the mass formula Eq. (26).

Let us consider infinitesimal U_3 transformation. Then, the infinitesimal generator A_{ν}^{μ} of U_3 satisfies the Lie equation:

$$[A_{\beta}{}^{\alpha}, A_{\nu}{}^{\mu}] = \delta_{\beta}{}^{\mu} \cdot A_{\nu}{}^{\alpha} - \delta_{\nu}{}^{\alpha} \cdot A_{\beta}{}^{\mu}.$$
(A·1)

This relation holds actually for general linear transformation of arbitrary dimension. The unitary restriction gives

$$(A_{\beta}{}^{\alpha})^{\dagger} = A_{\alpha}{}^{\beta} \tag{A.2}$$

where Q^{\dagger} means the hermitian conjugate of Q. For comparison's sake, our A_{μ}^{μ} is related to Ikeda et al.¹⁾'s $X_{\mu\nu}$ by

$$A_{\nu}^{\mu} = -1/2 \cdot [(1+i) X_{\nu\mu} + (1-i) X_{\mu\nu}],$$

$$X_{\mu\nu} = -1/2 \cdot [(1+i) A_{\nu}^{\mu} + (1-i) A_{\mu}^{\nu}].$$
 (A·3)

However, their notation $X_{\mu\nu}$ makes the mixed tensor character of A_{ν}^{μ} obscure.

For an arbitrary mixed tensor T_{ν}^{μ} , the commutation relation is given by

$$[A_{\beta}{}^{\alpha}, T_{\nu}{}^{\mu}] = \delta_{\beta}{}^{\mu} \cdot T_{\nu}{}^{\alpha} - \delta_{\nu}{}^{\alpha} \cdot T_{\beta}{}^{\mu}.$$
(A·4)

Comparing this with Eq. (A·1), we see that A_{ν}^{μ} has the property of a mixed tensor.

Generalized Casimir operators of our Lie algebra can be given by

$$M_{1} = A_{\mu}^{\mu} = \langle A \rangle,$$

$$M_{2} = A_{\nu}^{\mu} \cdot A_{\mu}^{\nu} = \langle A \cdot A \rangle,$$

$$M_{3} = A_{\nu}^{\mu} \cdot A_{\lambda}^{\nu} \cdot A_{\mu}^{\lambda} = \langle A \cdot A \cdot A \rangle$$
(A.5)

where the repeated indices mean summation over 1, 2 and 3, and we used the notations $\langle Q \rangle$ and defined product tensor $Q \cdot R$ of two tensor Q_{μ}^{μ} and R_{μ}^{μ} by

$$\langle Q \rangle = Q_{\mu}^{\mu},$$

$$(Q \cdot R)_{\nu}^{\mu} = Q_{\lambda}^{\mu} \cdot R_{\nu}^{\lambda}.$$
(A·6)

It is easy to see that M_1 , M_2 and M_3 commute with all A_{ν}^{μ} and therefore they commute with each other. Thus, they are constants in a given irreducible representation. Again, we will give a relation between our M_i and N, M, M' of Ikeda et al.¹⁾

$$N = -M_1,$$

 $M = 1/2 \cdot M_2,$
 $M' = -1/2 \cdot M_3 + 3/4 \cdot M_2 - 1/4 \cdot (M_1)^2,$

and so the relation between eigenvalues of M_i and f_1 , f_2 , f_3 of $U_3(f_1, f_2, f_3)$ is given¹⁾ by

$$\begin{split} M_{1} &= -(f_{1} + f_{2} + f_{3}), \\ M_{2} &= (f_{1}^{2} + f_{2}^{2} + f_{3}^{2}) + 2(f_{1} - f_{3}), \\ M_{3} &= -(f_{1}^{3} + f_{2}^{3} + f_{3}^{3}) + [-3/2 \cdot f_{1}^{2} + 3/2 \cdot f_{2}^{2} + 9/2 \cdot f_{3}^{2}] \qquad (A \cdot 7) \\ &- 1/2 \cdot (f_{1} + f_{2} + f_{3})^{2} + (2f_{1} + 2f_{2} - 4f_{3}). \end{split}$$

Note that $M_4 = \langle A \cdot A \cdot A \rangle$, etc., are unnecessary. They are given as functions of M_1 , M_2 and M_3 as will be seen shortly.

Now, we will prove the following theorem. [Theorem I]

In any irreducible representations of U_3 , any mixed tensors T_{ν}^{μ} can be regarded as a linear combination:

$$T_{\nu}^{\mu} = a \cdot \delta_{\nu}^{\mu} + b \cdot A_{\nu}^{\mu} + c (A \cdot A)_{\nu}^{\mu}.$$
 (A.8)

Eq. (A.8) means that it holds good when we take matrix elements of both sides in a given irreducible representation. Constants a, b and c are independent of tensor suffices μ and ν and of sub-quantum numbers S and I of the representation, but may depend upon f_1 , f_2 and f_3 and upon the nature of the tensor T_{ν}^{μ} . Eq. (A.8) is an analogue of the so-called vector algebra in R_3 , i.e.

$$\langle J, m | V_{\mu} | J, m' \rangle = \langle J | | V | | J \rangle \langle J, m | J_{\mu} | J, m' \rangle$$

where $V_{\mu}(\mu=1, 2, 3)$ is a vector in R_3 , and J_{μ} means the angular momentum operator in R_3 .

Before proving our theorem, we will show that this equation will give the desired mass formula Eq. (26).

First, let us consider the case of Sakata scheme. In that case, the nucleon number N, the strangeness quantum number S, and the isotopic spin operator I are defined¹⁾ by

$$N = -\langle A \rangle,$$

$$S = A_3^3,$$

$$I_+ = (I_1 + iI_2) = -A_1^2, \ I_- = (I_1 - iI_2) = -A_2^1,$$

$$I_3 = 1/2 (A_2^2 - A_1^1).$$

(A·9)

Now, let us suppose that the mass-splitting interaction is given by T_s^3 which has the same property as $\overline{\Lambda} \cdot \Lambda$ in the case of Sakata model. Then, the mass splitting is given by diagonal matrix element of T_s^3 .

$$\Delta M = \langle i | T_3^3 | i \rangle.$$

Then, noting (A9) and

 $(A \cdot A)_{\mathfrak{s}}^{\mathfrak{s}} = 1/2 \cdot \langle A \cdot A \rangle + 1/2 \cdot S^2 + 1/2 (\mathfrak{s} \cdot S - \langle A \rangle) - (\mathbf{I})^2 - 1/4 \cdot (S - \langle A \rangle)^2,$

we find that our theorem I (Eq. $(A \cdot 8)$) gives the desired mass formula Eq. (26).

In the case of Gell-Mann scheme, we have only to replace S by Y, hence we get Eq. (27). In this case, N is simply a parameter to distinguish representations.

Now, let us prove our theorem Eq. $(A \cdot 8)$. First, we will show the following lemma.

[Lemma I]

In the three-dimensional space, suppose that a tensor $S^{\alpha\beta}_{\mu\nu}$ is anti-symmetric with respect to exchanges of α and β and of μ and ν and furthermore $S^{\mu\beta}_{\mu\nu}=0$, i.e. traceless; then $S^{\alpha\beta}_{\mu\nu}$ is identically zero. Schematically, this means that $S^{\alpha\beta}_{\mu\nu} = -S^{\beta\alpha}_{\mu\nu} = -S^{\alpha\beta}_{\nu\mu}$ and $S^{\mu\beta}_{\mu\nu} = 0 \rightarrow S^{\alpha\beta}_{\mu\nu} \equiv 0$.

Let us consider a tensor

$$T^{\alpha\beta\gamma}_{\mu\nu\lambda} = S^{\alpha\beta}_{\mu\nu} \cdot \delta^{\gamma}_{\lambda} - S^{\alpha\gamma}_{\mu\nu} \cdot \delta^{\beta}_{\lambda} - S^{\gamma\beta}_{\mu\nu} \cdot \delta^{\alpha}_{\lambda} + S^{\gamma\beta}_{\lambda\nu} \cdot \delta^{\alpha}_{\mu} - S^{\gamma\alpha}_{\lambda\nu} \cdot \delta^{\beta}_{\mu} - S^{\alpha\beta}_{\lambda\nu} \cdot \delta^{\gamma}_{\mu}.$$

Then, $T^{\alpha\beta\gamma}_{\mu\nu\lambda}$ is totally anti-symmetric for any two exchanges of α , β and γ and satisfies traceless condition $T^{\alpha\beta\gamma}_{\alpha\nu\lambda}=0$. However, such tensor must be identically zero in the three-dimensional space, since only non-zero independent component must be $T^{123}_{\mu\nu\lambda}$ and by traceless-condition, this has to be identically zero, (for example, consider the case $\mu=1$). Thus we have $T^{\alpha\beta\gamma}_{\mu\nu\lambda}\equiv 0$. Then, by putting $\gamma=\nu$ and summing over ν , we find

$$T^{\alpha\beta\nu}_{\mu\nu\lambda} = S^{\alpha\beta}_{\mu\lambda} - S^{\alpha\beta}_{\lambda\mu} = 2S^{\alpha\beta}_{\mu\lambda} \equiv 0. \quad (Q.E.D.)$$

Our lemma I is not surprising at all, since such tensor $S^{\alpha\beta}_{\mu\nu}$ must be an irreducible representation in U_n but such type of irreducible representation is not possible in U_3 . (However, it is possible in $U_n(n \ge 4)$ and has signature (1, 1, -1, -1) in U_4 .) [Lemma II]

In U_3 , for any two arbitrary tensors M_{ν}^{μ} and N_{ν}^{μ} , we have the following identities:

$$\begin{split} & (M_{\nu}{}^{\mu}\cdot N_{\beta}{}^{\alpha}+M_{\beta}{}^{\alpha}\cdot N_{\nu}{}^{\mu})-(M_{\nu}{}^{\alpha}\cdot N_{\beta}{}^{\mu}+M_{\beta}{}^{\mu}\cdot N_{\nu}{}^{\alpha}) \\ &=\delta_{\nu}{}^{\mu}[\langle M\rangle N_{\beta}{}^{\alpha}+M_{\beta}{}^{\alpha}\langle N\rangle-(M\cdot N){}_{\beta}{}^{\alpha}-M_{\beta}{}^{\lambda}\cdot N_{\lambda}{}^{\alpha}] \\ & -\delta_{\nu}{}^{\alpha}[\langle M\rangle N_{\beta}{}^{\mu}+M_{\beta}{}^{\mu}\langle N\rangle-(M\cdot N){}_{\beta}{}^{\mu}-M_{\beta}{}^{\lambda}\cdot N_{\lambda}{}^{\mu}] \\ & -\delta_{\beta}{}^{\mu}[\langle M\rangle\cdot N_{\nu}{}^{\alpha}+M_{\nu}{}^{\alpha}\cdot\langle N\rangle-(M\cdot N){}_{\nu}{}^{\alpha}-M_{\nu}{}^{\lambda}\cdot N_{\lambda}{}^{\alpha}] \\ & +\delta_{\beta}{}^{\alpha}[\langle M\rangle\cdot N_{\nu}{}^{\mu}+M_{\nu}{}^{\mu}\cdot\langle N\rangle-(M\cdot N){}_{\nu}{}^{\mu}-M_{\nu}{}^{\lambda}\cdot N_{\lambda}{}^{\mu}] \\ & -(\delta_{\nu}{}^{\mu}\cdot\delta_{\beta}{}^{\alpha}-\delta_{\nu}{}^{\alpha}\cdot\delta_{\beta}{}^{\mu})\cdot[\langle M\rangle\cdot\langle N\rangle-\langle M\cdot N\rangle]. \end{split}$$

[Proof]

Define a tensor $Q_{\mu\nu}^{\alpha\beta}$ by

$$Q^{\mathfrak{a}\mathfrak{g}}_{_{\mu\nu}} = (M_{_{\mu}}{}^{^{\alpha}} \cdot N_{_{\nu}}{}^{^{\beta}} - M_{_{\mu}}{}^{^{\beta}} \cdot N_{_{\nu}}{}^{^{\alpha}}) - (M_{_{\nu}}{}^{^{\alpha}} \cdot N_{_{\mu}}{}^{^{\beta}} - M_{_{\nu}}{}^{^{\beta}} \cdot N_{_{\mu}}{}^{^{\alpha}}) \cdot$$

Then, $Q^{\alpha\beta}_{\mu\nu}$ is anti-symmetric for exchanges of α and β and of μ and ν . Furtheremore, construct a new tensor $S^{\alpha\beta}_{\mu\nu}$ by

$$S^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta}_{\mu\nu} - (\delta^{\alpha}_{\mu} \cdot Q^{\lambda\beta}_{\lambda\nu} + \delta^{\beta}_{\mu} \cdot Q^{\alpha\lambda}_{\lambda\nu} + \delta^{\alpha}_{\nu} \cdot Q^{\lambda\beta}_{\mu\lambda} + \delta^{\beta}_{\nu} \cdot Q^{\alpha\lambda}_{\mu\lambda}) + 1/2 \cdot (\delta^{\alpha}_{\mu} \cdot \delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu} \cdot \delta^{\beta}_{\mu}) Q^{\lambda\beta}_{\lambda\theta}.$$

We can see that $S^{\alpha\beta}_{\mu\nu}$ satisfies the conditions of lemma I, and must be identically zero. This gives the desired identity. (Q.E.D.) [Theorem II]

In U_3 , for any tensor T_{ν}^{μ} and for infinitesimal operator A_{ν}^{μ} , which satisfy the commutation relations Eqs. (A·1) and (A·4), we have the following identity.

$$\begin{split} 2 \cdot \left[\left(A \cdot T \cdot A \right)_{\nu}^{\mu} + \left(T \cdot A \cdot A \right)_{\nu}^{\mu} + \left(A \cdot A \cdot T \right)_{\nu}^{\mu} \right] - \left(2 \langle A \rangle + 9 \right) \cdot \left[\left(A \cdot T \right)_{\nu}^{\mu} + \left(T \cdot A \right)_{\nu}^{\mu} \right] \\ - 2 \cdot \langle T \rangle \langle A \cdot A \rangle_{\nu}^{\mu} + \left[6 \langle A \rangle + 12 + \left(\langle A \rangle \right)^{2} \right] T_{\nu}^{\mu} \\ - 1/2 \cdot \left[\langle A \cdot A \rangle T_{\nu}^{\mu} + T_{\nu}^{\mu} \cdot \langle A \cdot A \rangle \right] + \left[6 \langle T \rangle + 2 \langle A \rangle \langle T \rangle - 2 \langle A \cdot T \rangle \right] A_{\nu}^{\mu} \\ + \delta_{\nu}^{\mu} \cdot \left(- \langle T \rangle \cdot \left[\left(\langle A \rangle \right)^{2} - \langle A \cdot A \rangle + 4 \langle A \rangle + 4 \right] + \left(2 \langle A \rangle + 6 \right) \langle A \cdot T \rangle \\ - 2 \langle A \cdot A \cdot T \rangle \right) \equiv 0. \end{split}$$

Note that $[\langle A \rangle, T_{\nu}^{\mu}] = 0$, $[\langle T \rangle, A_{\nu}^{\mu}] = 0$ but $[\langle A \cdot A \rangle, T_{\nu}^{\mu}] \neq 0$. [Theorem III]

$$6(A \cdot A \cdot A)_{\mu} - [6\langle A \rangle + 18] \cdot (A \cdot A)_{\mu} + [3 \cdot (\langle A \rangle)^2 - 3 \cdot \langle A \cdot A \rangle +$$

$$\begin{array}{l} +12\cdot\langle A\rangle +12]\cdot A_{\nu}^{\mu} \\ -\left[(\langle A\rangle)^{3}+4(\langle A\rangle)^{2}+4\langle A\rangle -3\langle A\rangle\cdot\langle A\cdot A\rangle +2\langle A\cdot A\cdot A\rangle \\ -6\langle A\cdot A\rangle\right]\delta_{\nu}^{\mu} \equiv 0. \end{array}$$

Theorem III can be obtained from theorem II by putting T=A. From this, we see that $(A \cdot A \cdot A)_{\nu}$ can be expressed as a linear combination of δ_{ν}^{μ} , A_{ν}^{μ} , $(A \cdot A)_{\nu}^{\mu}$ and $(A \cdot A \cdot A)_{\nu}^{\mu}$, and so $\langle A \cdot A \cdot A \cdot A \rangle$ is a function of $\langle A \rangle$, $\langle A \cdot A \rangle$ and $\langle A \cdot A \cdot A \rangle$. So are $\langle A^n \rangle$ $(n \geq 4)$, as has already been mentioned.

To prove theorem II, we put $M_{\nu}^{\mu} = N_{\nu}^{\mu} = A_{\nu}^{\mu}$ in lemma II, and multiply T_{α}^{ν} from the left, and using commutation relations Eqs. (A·1) and (A·4), we find our theorem II, when we change the indices suitably. We may give another direct proof of theorem II as follows. Any tensor $Q_{\star\lambda\mu\nu}^{\alpha\beta\gamma\theta}$ which is anti-symmetric with respect to any exchanges of two variables among α , β , γ and θ must be identically zero in U_3 . Therefore, we have

$$\sum_{P} (-1)^{P} T_{\pi}^{\alpha} \cdot A_{\lambda}^{\beta} \cdot A_{\nu}^{\gamma} \cdot \delta_{\mu}^{\theta} = 0$$

where P means permutations among α , β , γ and θ . Then putting $\pi = \beta$, $\gamma = \lambda$, $\theta = \nu$ and taking traces, we find our theorem II again after somewhat long calculations.

Now, we shall prove our theorem I, Eq. $(A \cdot 8)$. Using the commutation relations

$$\begin{bmatrix} M_3, T_{\nu}^{\mu} \end{bmatrix} = 3(A \cdot A \cdot T)_{\nu}^{\mu} - 3(T \cdot A \cdot A)_{\nu}^{\mu} - 3\begin{bmatrix} M_2, T_{\nu}^{\mu} \end{bmatrix},$$
$$\begin{bmatrix} M_2, S_{\nu}^{\mu} \end{bmatrix} = 2(A \cdot S)_{\nu}^{\mu} - 2(S \cdot A)_{\nu}^{\mu},$$

we can rewrite theorem II as follows.

$$3(T \cdot A \cdot A)_{\nu}^{\mu} - (T \cdot A)_{\nu}^{\mu} \cdot (2\langle A \rangle + 9) + T_{\nu}^{\mu} \cdot \lfloor 1/2 \cdot (\langle A \rangle)^{2} - 1/2 \cdot \langle A \cdot A \rangle + 3\langle A \rangle + 6 \rfloor$$

$$= -1/2 \cdot \llbracket M_{2}, \ (TA)_{\nu}^{\mu} - (\langle A \rangle + 3) T_{\nu}^{\mu} \rrbracket - 1/3 \llbracket M_{3}, \ T_{\nu}^{\mu} \rrbracket$$

$$+ (A \cdot A)_{\nu}^{\mu} \cdot \langle T \rangle - A_{\nu}^{\mu} \cdot \llbracket (\langle A \rangle + 3) \cdot \langle T \rangle - \langle T \cdot A \rangle \rrbracket$$

$$- \delta_{\nu}^{\mu} (\llbracket \langle A \rangle + 3 \rrbracket \langle T \cdot A \rangle - \langle T \cdot A \cdot A \rangle - 1/2 \cdot \langle T \rangle \cdot \llbracket (\langle A \rangle)^{2} - \langle A \cdot A \rangle + 4 \langle A \rangle + 4 \rrbracket).$$

$$(A \cdot 9)$$

Now, in a given irreducible representation, M_2 and M_3 are constants, so that matrix elements $\langle \alpha | [M_2, Q] | \beta \rangle = 0$ and $\langle \alpha | [M_3, Q] | \beta \rangle = 0$, hence we can omit the first and second terms in the right-hand side of Eq. (A·9) in our case. Thus, we have

$$\begin{split} 3(T \cdot A \cdot A)_{\nu}{}^{\mu} - (T \cdot A)_{\nu}{}^{\mu}(2\langle A \rangle + 9) + T_{\nu}{}^{\mu} \cdot \left[1/2 \cdot (\langle A \rangle)^{2} - 1/2 \cdot \langle A \cdot A \rangle + 3\langle A \rangle + 6\right] \\ &= (A \cdot A)_{\nu}{}^{\mu} \cdot \langle T \rangle - A_{\nu}{}^{\mu} \cdot \left[(\langle A \rangle + 3)\langle T \rangle - \langle T \cdot A \rangle\right] \\ &- \partial_{\nu}{}^{\mu} (\left[\langle A \rangle + 3\right] \langle T \cdot A \rangle - \langle T \cdot A \wedge A \rangle - \langle T \cdot A \rangle - \langle T \cdot A \rangle - \langle T \cdot A \rangle$$

 $-1/2 \cdot \langle T \rangle \cdot [\langle A \rangle]^2 - \langle A \cdot A \rangle + 4 \langle A \rangle + 4]). \tag{A.10}$

Eq. (A·10) is true when we take any matrix elements in a given irreducible representation. Now, T_{*}^{*} is arbitrary, as long as it satisfies the commutation relation Eq. (A·4), and so we can replace T by $T \cdot A$ and $T \cdot A \cdot A$ in Eq. (A·10). For quantities like $T \cdot A \cdot A \cdot A$ or $T \cdot (A \cdot A \cdot A \cdot A)$, we use our theorem III and we can reduce them to a linear combination of T, $T \cdot A$ and $T \cdot A \cdot A$. Then, Eq. (A·10) gives three equations of the form

$$a_{1i}(T \cdot A \cdot A)_{\nu}^{\mu} + a_{2i}(T \cdot A)_{\nu}^{\mu} + a_{3i}(T)_{\nu}^{\mu}$$

= $b_{1i}(A \cdot A)_{\nu}^{\mu} + b_{2i}(A)_{\nu}^{\mu} + b_{3i} \cdot \delta_{\nu}^{\mu}$. (*i* = 1, 2, 3) (A \cdot 10)

We can give an explicit form for a_{ij} and b_{ij} , but as it is a little complicated, here we simply remark that a_{ij} are functions of only $\langle A \rangle$, $\langle A \cdot A \rangle$ and $\langle A \cdot A \cdot A \rangle$, i.e. a_{ij} depend only upon f_1 , f_2 and f_3 by Eq. (A·7), b_{ij} depend upon f_1 , f_2 and f_3 , and also upon $\langle T \rangle$, $\langle T \cdot A \rangle$ and $\langle T \cdot A \cdot A \rangle$, which are constants in the irreducible representation which we are considering. We can solve Eq. (A·10), since the determinant det (a_{ij}) is, in general, not identically zero; thus we get

$$T_{\nu}^{\mu} = a \cdot \delta_{\nu}^{\mu} + b \cdot A_{\nu}^{\mu} + c \left(A \cdot A \right)_{\nu}^{\mu}$$

and two other equations for $(T \cdot A)_{\mu}^{\mu}$ and $(T \cdot A \cdot A)_{\mu}^{\mu}$. This is the desired formula theorem I.

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