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PREDICTION INTERVALS FOR WARRANTY RESERVES AND CASH FLOWS†

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This paper presents estimates of cash flows associated with a given warranty policy. Included in the results are prediction intervals for warranty reserves and cash flows. A numerical example is given to illustrate these results.
(MARKETING—WARRANTY POLICY; ACCOUNTING; MARKETING—PRICING)

1. Introduction

Often the only means a consumer has for differentiating between two products of equal quality and price is the products' warranties. In fact, the product warranty is becoming an increasingly important dimension of competitive strategy. Warranties are important to buyers and sellers alike. Buyers need warranties to satisfy their need for assurance that the product will perform satisfactorily. Sellers use warranties mainly for promotional and protectional purposes. Warranties that involve a refund of the purchase price, or a replacement, often have strong promotional characteristics. Protection is achieved by the design of the warranty policy to protect the producer from exceptional claims of customers.

A seller creates a contingent liability when he sells a product and offers some type of warranty to a customer. Previous researchers have concentrated on estimating expected total warranty reserves to meet warranty claims when customers exercise their warranties. The knowledge of total expected warranty liability will help manufacturers plan operations more effectively, since an accurate knowledge of warranty costs allows more accurate profit expectations which may, in turn, lead to unanticipated marketing advantages.

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The present value of the rebates paid in the future are less than the face value of those rebates shown by Amato [2], [3]. However, for accounting purposes, the Accounting Principles Board does not recommend the application of the present value measurement (valuation) technique to estimates of contractual or other obligations assumed in connection with sale of property, goods, or service, for example, a warranty for product performance [1]. Hence, on the balance sheet, the total warranty liability estimate will be an estimate without considering the time value of money. For additional comments on accounting for warranty expenses see [5], [6].

Estimates of the total warranty reserves and the expected present value of the reserves have been presented in [2], [3], and [4]. The knowledge of the total expected warranty liability and the present value of the same are indeed useful for decision making. However, because of the random nature of failure of each item, it is very unlikely that all items will fail at any one particular time. Therefore, the cash flows associated with warranty liability should also be studied. A cash flow for the purpose of this paper may be defined as the expected total rebates to be paid during a given year when the customers execute their warranties.

The upper bounds on total warranty reserves liability should be investigated as well as upper bounds on cash flows in order to analyze the uncertainty or risk involved. These bounds are essentially the prediction[1] interval for the value of total warranty reserves and the present value of warranty reserves and cash flows for a given level of risk.

This paper is devoted to the study of the protectional aspect of warranties. Presented in the next section are methods of estimating expected cash flow and prediction intervals for total warranty reserves and the present value of warranty reserves and the cash flows. The notation and terminology used in [4] will be followed.

2. Prediction Intervals and Cash Flows

Assume:

c = constant unit product price, including warranty cost,

t = time of product failure, distributed exponentially,

W = duration of warranty period,

m = product mean time to failure (MTTF),

N = product lot size for warranty reserve determination typically one year's production for planning purposes,

R = warranty reserve cost for lot size N,

$PV(R)$[2] = present value of warranty reserve cost for lot size N,

r = expected warranty reserve cost per unit of product = $E(R)/N$,

θ = the firm's discount rate[3],

ϕ = expected change per period in the general price level.

It is assumed that c, W, m, N, θ and ϕ are known fixed parameters. The distribution of R must be known in order to set prediction intervals for actual warranty rebates. The total rebate, R, is the sum of N individual rebates, $r_i(t)$, where $r_i(t)$ is the rebate paid on the ith item if failure occurs at time t. According to the Central Limit

[1] Prediction Interval is used to denote the smallest interval for which there is a probability of $1-\alpha$ that the random variable will fall within that interval.

[2] $PV(R)$ is the same as R^* in Amato [2].

[3] Seldom if ever are warranty reserves ever funded, therefore θ is defined differently than in Amato [2].

Theorem, the sum of N independent identically distributed random variables approaches a normal distribution for large N. The amounts paid on individual items, $r_i(t)$, are independent and identically distributed since t is distributed exponentially and failures are assumed independent. Thus the total rebate paid, R, is normally distributed with mean $N \cdot E(r_i(t))$ and Variance $N \cdot V(r_i(t))$.

By similar reasoning, the present value of warranty reserves $PV(R)$ is normally distributed with mean $N \cdot E(PV(r_i(t))$ and variance $N.V(PV(r_i(t)))$. Since expected reserve is a special case of a present value analysis, when $\theta = \phi = 0$, only the present value of warranty reserves will be considered. The expected present value of warranty reserves is found in [2], [3] using a discrete discounting procedure. However, a continuous discounting procedure is used in this paper to estimate the mean and variance. $r_i(t) = c(1 - t/w)$ for $t < w$ and zero otherwise for the case of linear prorated rebates. Since $PV(r_i(t))$ is a random variable, where t is exponentially distributed, the expected present value of the rebate paid per unit is given by

$$PV(r) = E(PV(r_i(t))) = \int_0^w c(1 - t/w)\exp(-(\theta + \phi)t)(1/m)\exp(-t/m)\,dt$$

or

$$PV(r) = cs/m\left[1 - s/w(1 - \exp(-w/s))\right], \tag{1}$$

where $s = (\theta + \phi + 1/m)$. The variance of $PV(r_i(t))$ is given by

$$\begin{aligned} V(PV(r)) &= E(PV(r_i(t))) - PV(r)^2 \\ &= \int_0^w (c(1 - t/w)\exp(-(\theta + \phi)t))^2(1/m)\exp(-t/m)\,dt - PV(r)^2 \\ &= c^2x/m\left[1 - 2x/w\{1 - x/w(1 - \exp(-w/x))\}\right] - PV(r)^2 \end{aligned} \tag{2}$$

where

$$x = 1/(2\theta + \phi + 1/m).$$

Since the present value of warranty reserves is normally distributed with mean and variance given by $N \cdot PV(r)$ and $N \cdot V(PV(r))$ a prediction interval for the present value of warranty reserves can be constructed as

$$\begin{aligned} PV(R) \in NPV(r) \pm Z_{\alpha/2}N^{1/2}\{c^2x/m[1 - 2x/w \\ \times \{1 - x/w(1 - \exp(-w/x))\}] - PV(r)^2\}^{1/2} \end{aligned} \tag{3}$$

where $PV(r)$ is defined in 1, and x is defined in 2.

The procedure for price determination outlined in Menke [4] and Amato [2] is followed: w/m is used to determine the value of r/c, and the product price is calculated as

$$c = c'/(1 - r/c), \tag{4}$$

where c' is the unit price before the warranty cost is added. Once c has been determined, the prediction interval for the present value of total warranty reserves can be calculated from equation 3.

It is necessary to determine the amount of warranty cost for time period s to $s + 1$ for the analysis of the cash flow resulting from a warranty policy. This expected cost is given by

$$r_s = \int_s^{s+1} r_i(t) f(t)\,dt.$$

Thus, the expected total warranty cost or cash flow, CF_s, for N items for time period s to $s + 1$ is

$$E(CF_s) = Nr_s = N\int_s^{s+1} r_i(t) f(t)\,dt.$$

$$N\int_s^{s+1} c(1 - t/w)1/m\exp(-t/m)\,dt \tag{5}$$

$$= Ncm/w\left[1 - w/m + y)(\exp(-z) - \exp(-y) + 1/m\exp(-z)\right]$$

where $z = (s + 1)/m$ and $y = s/m$.

Since the concern is with the time period s to $s + 1$, the variance of $r_i(t)$ is given by

$$V(r_i(t)) = E\left(r_i(t)^2\right) - r_s^2.$$

Thus, a prediction interval for total warranty cost for time period s to $s + 1$ is given by

$$E(CF_s) \pm Z_{\alpha/2}N^{1/2}\Big[c^2/w^2\exp(-s/m)\big\{(-q^2 + 2mq - 2m^2)\exp(-1/m) + (q + 1)^2 - 2m(q + 1) + 2m^2\big\} - r_s^2\Big]^{1/2} \tag{6}$$

where $q = w - s - 1$.

Consider the numerical example in which

$w = 5$ years $=$ duration of warranty,
$m = 10$ years $=$ mean product life,
$c' = \$10.00 =$ unit price before warrant cost,
$N = 5000$,
$\theta + \phi = 0.1$

to illustrate the use of the above results.

From equation (1), $r/c = 0.184$. This means that in this case the present value of the reserves is 18.4% of the price of the product. Hence, the price by equation (4) should be

$$c = 10/(1 - 0.184) = 12.25$$

The expected present value of the reserves for 5000 units is then (5000)(12.25)(0.184) = \$11,270. The 95% prediction limit from equation 3 is (\$10,770, \$11,761). The expected actual amount to be paid out on warranty claims is obtained from equation 1 with $\theta + \phi = 0$ and is of the amount of 0.213 of the sales price per unit. The total expected amount paid out is 13,050.

The cash flows from equations 5 and 6 for this example are presented in Table 1. For example, in the first year of the warranty rebates totalling between \$4,805 and \$5,706 will be paid.

TABLE 1

Time				Control Limits 95%	
From	To	Expected	Variance	Lower	Upper
0	1	5,255	43,072	4,805	5,706
1	2	3,700	23,905	3,365	4,035
2	3	2,394	11,233	2,164	2,624
3	4	1,302	3,796	1,168	1,435
4	5	397	508	348	445

3. Summary

Previous authors have estimated the expected value and the expected present value of warranty reserves. This paper has extended their work to incorporate estimates for prediction intervals and cash flows. These estimates are needed both as a measure of risk and for planning. The promotional aspects of the warranty offered must be incorporated in the decision of the warranty length along with these measures of the protection afforded.

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