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Notes on a Double Inequality for Ratios of any Two Neighbouring Non-zero Bernoulli Numbers

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Abstract In the paper, the author notes on a double inequality published in “Feng Qi, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, Journal of Computational and Applied Mathematics 351 (2019), 1-5; Available online at https://doi.org/10.1016/j.cam.2018.10.049.”

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1. Introduction

We recall from [[1], p. 804, 23.1.1] and [[2], p. 3, (1.1)] that the Bernoulli numbers B_n can be generated by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}=1-\frac{z}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!}$$

for $|z|<2\pi$. It is easy to verify that the function

$$\frac{x}{e^x-1}-1+\frac{x}{2}$$

is even in $x\in\mathbb{R}$. Consequently, all the Bernoulli numbers B_{2k+1} for $k\in\mathbb{N}$ equal 0.

To discover explicit formulas, recurrent formulas, closed expressions, and integral representations of the Bernoulli numbers B_{2k} for $k\in\mathbb{N}$ is a classical topic. For recently published results, please refer to the papers [3-12] and closely related references therein.

To bound the Bernoulli numbers B_{2k} for $k\in\mathbb{N}$ by inequalities is an alternative topic. In [[1], p. 805, 23.1.15], [[13], Theorem 1.1], [[2], p. 14, (1.23) and p. 23, Exercise 1.2], and the papers [14,15,16], some inequalities for bounding the Bernoulli numbers B_{2k} were established and collected. Most of these inequalities have been refined or sharpened in [17] by the double inequality

$$\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\alpha-2k}}\le|B_{2k}|\le\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\beta-2k}} \tag{1}$$

for $k\in\mathbb{N}$, where $\alpha=0$ and

$$\beta=2+\frac{\ln(1-6/\pi^2)}{\ln 2}=0.649\dots$$

are the best possible in the sense that they can not be replaced respectively by any bigger and smaller constants in the double inequality (1).

To study the differences $|B_{2k+2}|-|B_{2k}|$ and the ratios $\frac{|B_{2k+2}|}{|B_{2k}|}$ for $k\in\mathbb{N}$ is also an interesting topic. In the newly published paper [18], the ratios $\frac{|B_{2k+2}|}{|B_{2k}|}$ for $k\in\mathbb{N}$, which is equivalent to the differences $\ln|B_{2k+2}|-\ln|B_{2k}|$, were bounded by the double inequality

$$\begin{aligned}\frac{2^{2k-1}-1}{2^{2k+1}-1}\frac{(2k+1)(2k+2)}{\pi^2}&<\frac{|B_{2k+2}|}{|B_{2k}|}\\&<\frac{2^{2k}-1}{2^{2k+2}-1}\frac{(2k+1)(2k+2)}{\pi^2}.\end{aligned} \tag{2}$$

Motivated by the double inequality (2) and by the fact that the function $\frac{2^{2k+x}-1}{2^{2k+2+x}-1}$ is strictly increasing in $x\ne-2(k+1)$ for all $k\in\mathbb{N}$, we naturally pose a problem: what are the best constants α and β such that the double inequality

$$\begin{aligned}\frac{2^{2k+\beta}-1}{2^{2k+2+\beta}-1}\frac{(2k+1)(2k+2)}{\pi^2}&<\frac{|B_{2k+2}|}{|B_{2k}|}\\&<\frac{2^{2k+\alpha}-1}{2^{2k+2+\alpha}-1}\frac{(2k+1)(2k+2)}{\pi^2}.\end{aligned} \tag{3}$$

is valid for all $k \in \mathbb{N}$?

In [[2], p. 5, (1.14)], it was listed that

$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k), \quad k \in \mathbb{N},$$

where the Riemann zeta function ζ can be defined [19,20,21] by the series $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ under the condition $\Re(z) > 1$ and by analytic continuation elsewhere.

$$\frac{|B_{2k+2}|}{|B_{2k}|} = \frac{(2k+1)(2k+2)}{\pi^2} \frac{1}{4} \frac{\zeta(2k+2)}{\zeta(2k)} \tag{4}$$

for $k \in \mathbb{N}$. By virtue of (4), the double inequality (3) can be rewritten as

$$\frac{2^{2k+\beta}-1}{2^{2k+2+\beta}-1} < \frac{1}{4} \frac{\zeta(2k+2)}{\zeta(2k)} < \frac{2^{2k+\alpha}-1}{2^{2k+2+\alpha}-1} \tag{5}$$

which can be further reformulated as

$$\Bigl(1 - \frac{1}{2^{2k+\beta}}\Bigr) \zeta(2k) < \Bigl(1 - \frac{1}{2^{2k+2+\beta}}\Bigr) \zeta(2k+2)$$

and

$$\Bigl(1 - \frac{1}{2^{2k+2+\alpha}}\Bigr) \zeta(2k+2) < \Bigl(1 - \frac{1}{2^{2k+\alpha}}\Bigr) \zeta(2k).$$

Let

$$S_\theta(x) \triangleq \Bigl(1 - \frac{1}{2^{2x+\theta}}\Bigr) \zeta(2x), \quad x \in [1, \infty), \quad \theta \in \mathbb{R}.$$

Then

$$S_{\theta}'(x) = \frac{1}{2^{2x+\theta-1}} \Bigl[\bigl(2^{2x+\theta} - 1\bigr) \zeta'(2x) + (\ln 2) \zeta(2x)\Bigr].$$

In order that the function $S_\theta(x)$ is strictly increasing (or strictly decreasing, respectively) on $[1, \infty)$, it is necessary and sufficient that

$$\bigl(2^{2x+\theta} - 1\bigr) \zeta'(2x) + (\ln 2) \zeta(2x) \gtreqless 0$$

on $[1, \infty)$, which can be rearranged as

$$2^\theta \lesseqgtr \Bigl[1 - \frac{(\ln 2) \zeta(2x)}{\zeta'(2x)}\Bigr] \frac{1}{2^{2x}} \to \begin{cases} 1, & x \to \infty; \\ \frac{1}{4} - \frac{\pi^2 \ln 2}{24 \zeta'(2)} = 0.55\ldots, & x \to 1^+. \end{cases}$$

Consequently, in order that the function $S_\theta(x)$ for $x \in [1, \infty)$ and the sequence $S_\theta(k)$ with $k \in \mathbb{N}$ are strictly increasing (or strictly decreasing, respectively), it is necessary that $\theta \ge 0$ (or

$$\theta \le \frac{\ln\Bigl[\frac{1}{4} - \frac{\pi^2 \ln 2}{24 \zeta'(2)}\Bigr]}{\ln 2} = -0.85\ldots,$$

respectively). The double inequality (5) can also be reformulated as

$$2^{2+\beta} < \frac{1}{2^{2k}} \frac{4 - \zeta(2k+2)/\zeta(2k)}{1 - \zeta(2k+2)/\zeta(2k)}$$

and

$$2^{2+\alpha} > \frac{1}{2^{2k}} \frac{4 - \zeta(2k+2)/\zeta(2k)}{1 - \zeta(2k+2)/\zeta(2k)}.$$

Since

$$\frac{1}{2^{2k}} \frac{4 - \zeta(2k+2)/\zeta(2k)}{1 - \zeta(2k+2)/\zeta(2k)} \to \begin{cases} \frac{\pi^2 - 60}{4(\pi^2 - 15)} = 2.44\ldots, & k \to 1, \\ 4, & k \to \infty, \end{cases}$$

It follows that the necessary conditions are $\alpha \ge 0$ and

$$\beta \le \frac{\ln \frac{\pi^2 - 60}{\pi^2 - 15}}{\ln 2} - 4 = -0.711\ldots.$$

This implies that the right-hand side inequality in (2) is sharp, but the left-hand side inequality in (2) perhaps can be improved. In conclusion, we guess that the double inequality (3) is valid if and only if $\alpha \ge 0$ and

$$\beta \le \frac{\ln\Bigl[\frac{1}{4} - \frac{\pi^2 \ln 2}{24 \zeta'(2)}\Bigr]}{\ln 2} = -0.85\ldots.$$

Since

$$\lim_{x \to 1^+} \Bigl\{\Bigl[1 - (\ln 2) \frac{\zeta(x)}{\zeta'(x)}\Bigr] \frac{1}{2^x}\Bigr\} = \frac{1}{2}$$

and

$$\lim_{x \to \infty} \Bigl\{\Bigl[1 - (\ln 2) \frac{\zeta(x)}{\zeta'(x)}\Bigr] \frac{1}{2^x}\Bigr\} = 1,$$

we guess that the function

$$\Bigl(1 - \frac{1}{2^{x+\theta}}\Bigr) \zeta(x), \quad x \in (1, \infty)$$

is strictly increasing (or strictly decreasing, respectively) if and only if $\theta \le -1$ (or $\theta \ge 0$, respectively).

The double inequality (2) has been cited and applied in the papers [22-29].

Can one generalize the inequality (2) to the case for the Bernoulli polynomials?

This paper and [18] are respectively extracted from the preprints [30,31,32,33].

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