PHYSICAL REVIEW D VOLUME 14, NUMBER 4 15 AUGUST 1976

Notes on black-hole evaporation

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This paper examines various aspects of black-hole evaporation. A two-dimensional model is investigated where it is shown that using fermion-boson cancellation on the stress-energy tensor reduces the energy outflow to zero, while other noncovariant techniques give the Hawking result. A technique for replacing the collapse by boundary conditions on the past horizon is developed which retains the essential features of the collapse while eliminating some of the difficulties. This set of boundary conditions is also suggested as the most natural set for a preexistent black hole. The behavior of particle detectors under acceleration is investigated where it is shown that an accelerated detector even in flat spacetime will detect particles in the vacuum. The similarity of this case with the behavior of a detector near the black hole is brought out, and it is shown that a geodesic detector near the horizon will not see the Hawking flux of particles. Finally, the work of Berger, Chitre, Nutku, and Moncrief on scalar geons is corrected, and the spherically symmetric coupled scalar-gravitation Hamiltonian is presented in the hope that someone can apply it to the problem of black-hole evaporation.

INTRODUCTION

Hawking[1,2] has recently advanced arguments to suggest that black holes evaporate, due to quantum particle creation, and behave as though they had an effective temperature of $(8\pi M)^{-1}$, with M the mass of the black hole.

This paper is an attempt to investigate the assumptions which lead to this result, and so discover whether such an evaporation is physically reasonable or is the result of difficulty in defining physical particles on a non-flat-background spacetime.

Section I will examine the collapse of a spherical shell of matter to a black hole in a two-dimensional model. Here the field equations for massless fields are exactly solvable. Following the procedure used by Hawking gives a constant particle production rate long after collapse in this two-dimensional example, just as in the four-dimensional problem.

This particle production is associated with an energy flow out to infinity only if the energy-momentum tensor is regularized in a particular, noncovariant fashion at infinity.

A variety of other renormalization methods are suggested. The only one which is actually investigated gives no energy flow out of the black hole, no black-hole evaporation.

This technique used for regularization in two dimensions is to cancel the infinities in the boson energy-momentum tensor (taken as a massless scalar field) with those in the fermion energy-momentum tensor (taken as that of a massless spin-$\frac{1}{2}$ field). Such a cancellation could be extended to four dimensions where the worst divergences in the energy-momentum tensor would be canceled (i.e., the ω^4 divergences). The two-dimensional result suggests that black-hole evaporation may be linked to these divergent parts of the energy-momentum tensor.

Section II of the paper discusses an attempt to replace the difficult collapse problem with an equivalent formulation in full empty Schwarzschild spacetime. Here the collapsing body is replaced by boundary conditions on the past horizon of the black hole. In particular, the definition of positive frequency for normal modes which come out of the past horizon of the black hole is changed from the naive formulation. Instead of defining positive frequency for these modes via the timelike Killing vector, it is defined for the initial data on the past horizon via a null vector field which is of Killing type on the horizon (only).

The Feynman propagator is briefly investigated in the two methods of defining positive frequencies. It is suggested that the usual definition of positive frequencies leads to a propagator divergent on the future horizon, which the alternative method does not.

Replacing the collapsing star with these boundary conditions on the past horizon gets rid of the difficulty of solving the scalar, neutrino, etc. field equations in the time-dependent metric of a collapsing star. Analyzing this problem by the same method used by Hawking leads to exactly the result obtained by him a long time after the collapse of the star, i.e., a constant outward flux of particles as if the black hole had a temperature $(8\pi M)^{-1}$.

Section III discusses the problem of particle detection by model particle detectors. It is shown that an accelerated detector in flat spacetime will detect quanta even in the vacuum. A discussion of the applicability to the black-hole evaporation process is given.

Section IV is a digression to a completely different possible attack on the black-hole evaporation problem. Essentially it is a revision of the work of Berger, Chitre, Moncrief, and Nutku[3] (BCMN). The Hamiltonian for a spherically symmetric scalar field coupled via Einstein's equations to a spherically symmetric gravitational field is derived which corrects the results of BCMN. Unfortunately, the Hamiltonian is highly nonlinear and nonpolynomial as well as having a nonlocal Hamiltonian density. As a result, no definite results have been obtained, nor has any real line of attack been found as yet. It is presented only in the hope that a solution of this problem would clarify the black-hole evaporation problem.

Units are chosen throughout such that $G=\hbar=c=k$ (Boltzmann's const) $=1$. All quantities are therefore dimensionless and are measured in terms of Planck units ($L_P \sim 10^{-33}$ cm, $t_P \sim 10^{-44}$ sec, $M_P \sim 10^{-5}$ g, $T_P \sim 10^{+31}$ °K). In order to minimize use of strange letters, note that the coordinates U, V, u, v are defined independently and differently in Secs. I and II. Note further that in Sec. I the use of a caret designates the symbol as a specified function rather than a variable.

I. TWO-DIMENSIONAL COLLAPSE

The two-dimensional metric which I will be investigating in this section is the restriction to two dimensions of a collapsing-shell metric:

$$ds^2 = \begin{cases} d\tau^2 - dr^2, & r < \hat{R}(\tau) \\ \left(1 - 2\dfrac{M}{r}\right)dt^2 - \dfrac{dr^2}{1-2M/r}, & r > \hat{R}(\tau). \end{cases} \tag{1.1}$$

The shell radius is given by the equation $r = \hat{R}(\tau)$, with $\hat{R}(\tau)$ given, for example, by the equation

$$\hat{R}(\tau) = \begin{cases} R_0, & \tau < 0 \\ R_0 - \nu\tau, & \tau > 0. \end{cases} \tag{1.2}$$

I assume $\nu \simeq 1$ and $R_0 > 2M$. I also assume the shell is rapidly collapsing with $\nu > 1 - [4M/(R_0 + 2M)]$. This ensures that an inward-going light ray emitted just as collapse begins cannot bounce off $r=0$ and escape before the shell has passed through its Schwarzschild radius. These restrictions on the speed of infall of the shell are made for convenience of later mathematics only, and do not affect the conclusions.

The time coordinates τ and t inside and outside the shell, respectively, are related by demanding that the path length along the shell be the same in both coordinate systems. This gives the relation

$$\frac{dt}{d\tau} = \left\{ \frac{\hat{R}(\tau)}{[\hat{R}(\tau) - 2M]^2} \left[\hat{R} - 2M + \left(\frac{d\hat{R}}{d\tau}\right)^2 2M \right] \right\}^{1/2}$$
$$= \begin{cases} (1-2M/R_0)^{-1/2}, & \tau < 0 \\ \left[\dfrac{R_0 - \nu\tau}{(R_0 - 2M - \nu\tau)^2} (R_0 - 2M - \nu\tau + 2M\nu^2) \right]^{1/2}, & \tau > 0. \end{cases} \tag{1.3}$$

Also define the advanced and retarded null coordinates

$$V = \tau + r - R_0, \quad U = \tau - r + R_0, \tag{1.4a}$$

$$u = t - r^* + R_0^*, \quad v = t + r^* - R_0^*. \tag{1.4b}$$

Here the asterisk indicates a function of the form

$$r^* = r + 2M\ln(r - 2M). \tag{1.4c}$$

The coordinates u, U and v, V have been chosen so that the shell begins to collapse at retarded time $u = U = 0$ and at advanced time $v = V = 0$.

In these coordinates the metric (1.1) is given by

$$ds^2 = \begin{cases} dU\,dV & \text{inside the shell}, \\ (1-2M/r)du\,dv & \text{outside the shell}, \end{cases} \tag{1.5}$$

where r is interpreted as an implicit function of u, v via (1.4b) and (1.4c).

These coordinates are related by the equations

$$\frac{du}{dU} = \begin{cases} (1-2M/R_0)^{-1/2}, & U, u < 0 \\ \dfrac{R}{(1+\nu)(R-2M)} \left[\nu + \left(1 - \dfrac{2M(1-\nu^2)}{R}\right)^{1/2} \right], & U, u > 0 \end{cases} \tag{1.6}$$

$$R = \hat{R}(U/(1+\nu)),$$

$$\frac{dv}{dV} = \begin{cases} (1-2M/R_0)^{-1/2}, & V, v < 0 \\ \dfrac{R}{(1-\nu)(R-2M)} \{[1 - 2M(1-\nu^2)/R]^{1/2} - \nu\}, & V, v > 0 \end{cases}$$

$$R = \hat{R}(V/(1-\nu)). \tag{1.7}$$

To differentiate between the coordinates v, u and the functions of V, U defined by these equations, the latter are designated by $\hat{v}, \hat{u}$ and the (functional, not algebraic) inverses of the functions by

$$\hat{U} = (\hat{u}^{-1}), \quad \hat{V} = (\hat{v}^{-1}).$$

From equations (1.6) and (1.7) one obtains

$$\hat{u}(U) = \begin{cases} (1-2M/R_0)^{-1/2}U, & U < 0 \\ -4M\ln\{1 - \nu U/[(1+\nu)(R_0 - 2M)]\} + U + O(1-\nu), & U > 0 \end{cases} \tag{1.8}$$

$$\hat{v}(V)=\begin{cases}(1-2M/R_0)^{-1/2}\,V\,, & V<0\\ V+O(1-\nu)\,, & V>0\end{cases} \tag{1.9}$$

Now consider the massless scalar and Dirac equations restricted to two dimensions:

$$\phi^{,\mu}{}_{;\mu}=(-g)^{-1/2}\frac{\partial}{\partial x^\mu}\left[g^{\mu\nu}(-g)^{1/2}\frac{\partial}{\partial x^\nu}\right]\phi=0\,, \tag{1.10}$$

$$\gamma^\mu\nabla_\mu\psi=0\,,\quad (1+i\gamma^5)\psi=0 \tag{1.11}$$

The scalar field equation is straightforward. In (1.11), ψ is assumed to be a 4-spinor, and the γ^μ are the restriction to two dimensions of the four-dimensional Dirac matrices. The operators $\nabla_\mu=\partial/\partial x^\mu-\Gamma_\mu$ are defined so that the Dirac matrices have zero derivatives, as in four dimensions.

Both (1.10) and (1.11) are conformally invariant with $\phi\to\phi$ and $\psi\to\Omega^{1/2}\psi$ when we conformally transform the metric $g^{\mu\nu}\to\Omega^2g^{\mu\nu}$. As all two-dimensional metrics are conformally flat, this greatly simplifies the solution of Eqs. (1.10) and (1.11). Using U,V or u,v coordinates, the scalar equation simply becomes

$$\frac{\partial}{\partial U}\frac{\partial}{\partial V}\phi=\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi=0\,, \tag{1.12}$$

with solutions

$$\begin{aligned}\phi&=\hat{F}(V)+\hat{G}(U)\\&=\hat{f}(v)+\hat{g}(u)\,,\end{aligned} \tag{1.13a}$$

where

$$\hat{f}(v)=\hat{F}(\hat{V}(v))\,,\quad \hat{g}(u)=\hat{G}(\hat{U}(u))\,. \tag{1.13b}$$

In two dimensions, the centrifugal barrier which prevents any particle flow through $r=0$ is absent. In order to mimic the effect of such a barrier, I demand that there be no net radial flux at $r=0$. As the flux is proportional to ($\phi^*\partial\phi/\partial r$ − complex conjugate), this may be accomplished by demanding that some linear combination $\alpha\phi+\beta\partial\phi/\partial r$ be zero at $r=0$. (α,β must be fixed real constants so that this property will be valid for all linear combinations of solutions.) I have chosen to take $\phi=0$ (i.e., $\beta=0$) as the appropriate condition at $r=0$. Any other choice would lead to the same conclusions as does this particular choice. At most they would introduce a phase shift in the outgoing wave when compared to this case.

The line $r=0$ corresponds to $U=V+2R_0$. The requirement that the modes vanish on this line leads to the relation

$$\hat{G}(U)=-\hat{F}(U-2R_0)\,. \tag{1.14}$$

The full solution outside the shell will then be

$$\phi(u,v)=\hat{f}(v)+\hat{f}(\hat{v}[\hat{U}(u)-2R_0])\,. \tag{1.15}$$

The energy-momentum tensor for a solution ϕ is given by

$$T_{\mu\nu}=\phi^*_{,(\mu}\phi_{,\nu)}-\tfrac{1}{2}g_{\mu\nu}\,\phi^*_{,\alpha}\phi^{,\alpha}\,, \tag{1.16}$$

where the asterisk indicates complex conjugation, and the parentheses around indices indicate symmetrization. For the general solution, the various components of $T_{\mu\nu}$ are

$$\begin{aligned}T_{uu}&=\left(\frac{\partial\hat{v}}{\partial U}\frac{\partial\hat{U}}{\partial u}\right)^2|\hat{f}'(\hat{v}[\hat{U}(u)-2R_0])|^2\,,\\ T_{vv}&=|\hat{f}'(v)|^2\,,\quad T_{uv}=T_{vu}=0\,.\end{aligned} \tag{1.17}$$

(The prime denotes the derivative of the function with respect to its argument.)

In particular, if one selects $f(v)$ to be a normal mode,

$$f(v)=e^{-i\omega v}/(2\pi\omega)^{1/2}\,,$$

one obtains

$$\begin{aligned}T_{vv}&=|\omega|/2\pi\,,\\ T_{uu}&=|\omega|[\hat{v}'(\hat{U}(u)-2R_0)\hat{U}'(u)]^2/2\pi\\ &=\begin{cases}|\omega|/2\pi\,, & u<0\\ |\omega|\,|\hat{U}'(u)|^2/[2\pi(1-2M/R_0)]\,, & u>0\,.\end{cases}\end{aligned} \tag{1.18}$$

The metric (1.1) has a timelike Killing vector outside of the shell where the Killing vector η is given by $\partial/\partial t$. The energy current[4] can be defined as

$$J_E^\nu=\eta^\mu T_\mu{}^\nu\,. \tag{1.19}$$

Expressing η^μ in u,v coordinates we find that $T_v{}^u$ represents the inward flowing energy current across a u = constant surface, whereas $T_u{}^v$ represents the outward-flowing energy current across a v = constant surface. The normal modes therefore have a constant energy flow inward with an equal outward flow before the onset of collapse ($u<0$). After the onset of collapse ($u>0$), the inward flow is still constant but the outward flow of energy decays rapidly, $\hat{U}'(u)$ going to zero exponentially as $u\to\infty$. [This can be seen from Eq. (1.8) for $U\to(R_0-2M)(1+\nu)/\nu$.]

The solution of the neutrino equations is more complicated. One must first choose a representation for the Dirac matrices. We define the matrices $\gamma_u,\gamma_v,\tilde{\gamma}_U,\tilde{\gamma}_V$ as follows:

$$\begin{aligned}\gamma_u&=\begin{cases}(1-2M/r)^{1/2}\gamma_b\,, & r>\hat{R}\\ [\hat{U}'(u)\hat{V}'(v)]^{1/2}\gamma_b\,, & r<\hat{R}\end{cases}\\ \gamma_v&=\begin{cases}(1-2M/r)^{1/2}\gamma_a\,, & r>\hat{R}\\ [\hat{U}'(u)\hat{V}'(v)]^{1/2}\gamma_a\,, & r<\hat{R}\end{cases}\end{aligned} \tag{1.20}$$

$$\tilde{\gamma}_U=[\hat{u}'(U)\hat{v}'(V)]^{1/2}\gamma_u\,,\quad \tilde{\gamma}_V=[\hat{u}'(U)\hat{v}'(V)]^{1/2}\gamma_v\,, \tag{1.21}$$

with

$$\gamma_a = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix},$$

$$\gamma_b = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{1.22}$$

The matrix γ^5 is given by

$$\gamma^5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

The γ and $\tilde{\gamma}$ are two different representations in the two coordinate systems and are related by

$$\gamma_v = \hat{V}'(v) S \tilde{\gamma}_v S^{-1}, \quad \gamma_u = \hat{U}'(u) S \tilde{\gamma}_u S^{-1}, \tag{1.23}$$

where

$$S = \cosh\theta + \sinh\theta(\gamma_a\gamma_b - \gamma_b\gamma_a), \quad \theta = \ln\{[\hat{V}'(v)\hat{u}'(U)]^{-1/4}\}. \tag{1.24}$$

The solution to the neutrino equations

$$\gamma^\mu \nabla_\mu \psi = 0, \quad (1 + i\gamma^5)\psi = 0 \tag{1.25}$$

in the representation of Eq. (1.20) is

$$\psi = \mathfrak{Q} \times \begin{cases} (1 - 2M/r)^{-1/4}, & r > R \\ [\hat{U}'(u)\hat{V}'(v)]^{-1/4}, & r < R \end{cases} \tag{1.26a}$$

$$\mathfrak{Q} = \begin{bmatrix} f(u) \\ g(v) \\ f(u) \\ g(v) \end{bmatrix}.$$

In the representation of Eq. (1.21) the solutions are

$$\tilde{\psi} = \tilde{\mathfrak{Q}} \times \begin{cases} [(1 - 2M/r)\hat{u}'(U)\hat{v}'(V)]^{-1/4}, & r > R \\ 1, & r < R \end{cases} \tag{1.26b}$$

with $\tilde{f}(U)$, $\tilde{g}(V)$ the appropriate components of $\tilde{\mathfrak{Q}}$.

From the relation $\tilde{\psi} = S^{-1}\psi$ we obtain

$$\tilde{f}(U) = f(\hat{u}(U))[\hat{u}'(U)]^{1/2}, \quad \tilde{g}(V) = g(\hat{v}(V))[\hat{v}'(V)]^{1/2}. \tag{1.27}$$

The reflection condition at $r = 0$ is satisfied by demanding

$$\tilde{g}(U - 2R_0) = \tilde{f}(U)$$

or

$$f(u) = [\hat{U}'(u)\hat{v}'(\hat{U}(u) - 2R_0)]^{1/2} g(\hat{v}(\hat{U}(u) - 2R_0)). \tag{1.28}$$

The energy-momentum tensor for a solution ψ is given by

$$T_{\mu\nu}(\psi) = \frac{i}{2}\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi + \text{complex conjugate}, \tag{1.29}$$

where $\bar{\psi}$ is the Dirac adjoint of ψ.

As for the scalar field, $T_{\mu\nu}$ is conformally invariant, trace free, and divergence free, with components

$$\begin{aligned} T_{vv} &= 2\,\mathrm{Re}[ig^*(v)g'(v)], \\ T_{uu} &= 2\,\mathrm{Re}[i\,|\hat{U}'(u)\hat{v}'(\hat{U}(u) - 2R_0)|^2 \\ &\quad \times g^*(\hat{v}(\hat{U}(u) - 2R_0))g'(\hat{v}(\hat{U}(u) - 2R_0))], \\ T_{uv} &= T_{vu} = 0. \end{aligned} \tag{1.30}$$

If we define the behavior of $g(v)$ on a $u = \text{const}$ (e.g., $u = -\infty$) surface as a normal-mode solution $g(v) = (4\pi)^{-1/2}e^{-i\omega v}$ we find

$$\begin{aligned} T_{vv} &= \omega/2\pi, \\ T_{uu} &= \omega\,|\hat{U}'(u)\hat{v}'(\hat{U}(u) - 2R_0)|^2/2\pi, \\ T_{uv} &= T_{vu} = 0. \end{aligned} \tag{1.31}$$

For $\omega > 0$ this is identical to the previous result for scalar fields [e.g., (1.18)] and is of opposite sign for $\omega < 0$.

Having solved the classical wave equations, one can quantize these fields in a straightforward manner. Φ and Ψ are now regarded as operator fields which we can expand in the previously defined normal modes [i.e., modes which go as $e^{-i\omega v}/(2\pi\omega)^{1/2}$ near $u = -\infty$]:

$$\begin{aligned} \Phi &= \sum_{\omega > 0} (a_\omega\phi_\omega + b_\omega^\dagger\phi_\omega^*), \\ \Psi &= \sum_{\omega > 0} (c_\omega\psi_\omega + d_\omega^\dagger\psi_{-\omega}). \end{aligned} \tag{1.32}$$

The operators a, b and their Hermitian conjugates $a^\dagger, b^\dagger$ obey the usual commutation relations while c, d and their Hermitian conjugates obey the usual anticommutation relations. The operators a_ω, b_ω, c_ω, d_ω are interpreted as the annihilation operators for the modes of energy $|\omega|$ whereas their Hermitian conjugates are the creation operators.

This particular choice of creation and annihilation operators corresponds to the physical demand that no particles be impinging on the star from infinity.[2] The incoming particle states are, in other words, identified with positive-frequency states near infinity. This identification is strengthened by the analysis in Sec. III of the response of a model particle detector to the field.

The vacuum state $|0\rangle$ is now defined by requiring that it be annihilated by all of the annihilation operators:

$$0=a_\omega|0\rangle=b_\omega|0\rangle=c_\omega|0\rangle=d_\omega|0\rangle. \tag{1.33}$$

Note that we have not defined the vacuum by minimizing some positive-definite-operator expectation value (e.g., the Hamiltonian), but have defined the vacuum as the state with no incoming particles.

To continue, the energy-momentum operator for scalar fields is given by

$${}_s\mathfrak{T}_{\mu\nu}=\{\Phi^\dagger_{,(\mu},\Phi_{,\nu)}\}-\tfrac{1}{2}g_{\mu\nu}\{\Phi^\dagger_{,\alpha},\Phi^{,\alpha}\} \tag{1.34}$$

and for neutrino fields by

$${}_n\mathfrak{T}_{\mu\nu}=\frac{i}{2}[\bar{\Psi},\gamma_{(\mu}\nabla_{\nu)}\Psi]+\text{H.c.} \tag{1.35}$$

$\{,\}$ denotes the anticommutator while $[,]$ denotes the commutator. Using the expansion of Φ and Ψ in normal modes (1.32) and the definition of the vacuum $|0\rangle$, the vacuum expectation value of ${}_s\mathfrak{T}_{\mu\nu}$ for scalars becomes

$$\begin{aligned}\langle 0|{}_s\mathfrak{T}_{\mu\nu}|0\rangle=\tfrac{1}{2}\sum_{\omega,\omega'}\big[&\langle 0|a^\dagger_\omega a_{\omega'}+a_{\omega'}a^\dagger_\omega|0\rangle T_{\mu\nu}(\phi_\omega,\phi_{\omega'})+\langle 0|b_\omega b^\dagger_{\omega'}+b^\dagger_{\omega'}b_\omega|0\rangle T_{\mu\nu}(\phi^*_\omega,\phi^*_{\omega'})\\&+\langle 0|b_\omega a_{\omega'}+a_{\omega'}b_\omega|0\rangle T_{\mu\nu}(\phi^*_\omega,\phi_{\omega'})+\langle 0|a^\dagger_\omega b^\dagger_{\omega'}+b^\dagger_{\omega'}a^\dagger_\omega|0\rangle T_{\mu\nu}(\phi_\omega,\phi^*_{\omega'})\big]\\=\tfrac{1}{2}\sum_\omega &T_{\mu\nu}(\phi_\omega),\end{aligned}$$

where

$$T_{\mu\nu}(\phi_1,\phi_2)=(\phi^*_{1,(\mu}\phi_{2,\nu)}+\tfrac{1}{2}g_{\mu\nu}\phi^*_{1,\alpha}\phi_2{}^{,\alpha}).$$

Similarly, the neutrino vacuum expectation value is

$$\langle 0|{}_n\mathfrak{T}_{\mu\nu}|0\rangle=-\tfrac{1}{2}\sum_{\omega>0}T_{\mu\nu}(\psi_\omega)+\tfrac{1}{2}\sum_{\omega<0}T_{\mu\nu}(\psi_\omega).$$

From Eqs. (1.18) and (1.31), the neutrino and scalar vacuum expectation values are equal but of opposite sign term by term. Furthermore, $\langle 0|\mathfrak{T}_{\mu\nu}|0\rangle$ is infinite for both neutrinos and scalar particles, the divergence going as ω^2. One possible method of regularizing the vacuum expectation values is to sum the contributions due to the scalar field and due to the neutrinos. This gives a vacuum expectation for the sum of scalar and neutrino energy-momentum tensors of exactly zero everywhere. The zero-point oscillation energy for the normal modes of the boson field exactly cancel the energy of the filled negative-energy sea of the fermions.

In the four-dimensional black-hole formation problem, this procedure would be expected to cancel only the worst divergences of $T_{\mu\nu}$. However, the Hawking energy flow would be largely eliminated. This suggests that the Hawking energy flow may be a result of the worst divergences in $T_{\mu\nu}$ which may disappear when the proper renormalization procedure is found.

The above renormalization procedure is certainly not the only possibility. One could, for example, normal-order $\mathfrak{T}_{\mu\nu}$ according to the "in" operators giving an expectation value of exactly zero everywhere. Such a procedure, however, amounts to little more than assuming the answer one is trying to calculate. A gravitational field could surely lead to vacuum polarization effects in $\mathfrak{T}_{\mu\nu}$.

One could also attempt to apply methods such as dimensional regularization,[5] which, however, seems to me to be applicable only to the case of almost flat spacetimes or highly symmetrical spacetimes. Black-hole evaporation exists only because of the formation of a horizon, a decidedly nonflat spacetime phenomenon.

Another renormalization technique is the subtraction of geometrical counterterms $\langle 0|\mathfrak{T}_{\mu\nu}|0\rangle$.[6] Such subtraction terms occur in the Lagrangian density of the form

$$\sqrt{-g}\,(\lambda_1+\lambda_2R+\lambda_3R_{\mu\nu}R^{\mu\nu}+\lambda_4R^2). \tag{1.36}$$

In an empty background spacetime, however, only the first of these leads to a nonzero subtraction term (i.e., $\lambda_1 g_{\mu\nu}$). It is, furthermore, capable of renormalizing only the trace $T^\alpha{}_\alpha$ of the stress-energy tensor. That this is insufficient to renormalize $T_{\mu\nu}$ can be seen by noticing that for a conformally invariant field (e.g., electromagnetic neutrino) the trace is already zero, whereas the divergences in T^{00} for example, are as bad as those for massive fields. The attempt by Parker and Fulling[7] to use these subtraction terms in cosmological spacetimes led to manifestly noncovariant expressions for their renormalized $T_{\mu\nu}$. They achieved covariance by using the subtraction technique to renormalize only T^{00}, and then using $T^{\mu\nu}{}_{;\nu}=0$ to find the spatial terms.

DeWitt[6] has suggested a formal procedure for regularizing $\mathfrak{T}^{\mu\nu}$ by associating its divergences with the divergences in the Feynman propagator $G(x,x')$, or rather in the difference between the Feynman propagator and the time-symmetric ($\frac{1}{2}$ advanced plus retarded) Green's function. By a technique suggested by Schwinger[8] for background electromagnetic fields, DeWitt obtains the Feynman propagator as a formal integral solution to the equation

$$(\nabla_\mu \nabla^\mu + m^2 + i\epsilon) G(x,x') = -\delta(x,x') \tag{1.37}$$

in the limit as $\epsilon \to 0$. He obtains

$$G(x,x') = i\int_0^\infty ds[g^{-1/4}(x)\langle x,s|x',0\rangle g^{-1/4}(x')], \tag{1.38a}$$

where

$$\langle x,s|x',0\rangle = \frac{-i}{(4\pi)^2}\frac{D^{1/2}(x,x')}{s^2} \times \exp\left[i\frac{\sigma(x,x')}{2s} - im^2 s + \Omega(x,x',s)\right]. \tag{1.38b}$$

Here $\sigma(x,x')$ is $\frac{1}{2}$ the square of the geodesic distance separating x, x',

$$D(x,x') = -\det\left[-\frac{\partial^2\sigma}{\partial x^\mu \partial x'^\nu}\right], \tag{1.39}$$

and Ω is a function which satisfies

$$i\frac{\partial\Omega}{\partial s} + D^{-1}(D\Omega^{;\mu})_{;\mu} + \Omega_{;\mu}\Omega^{;\mu} + is^{-1}\sigma^{;\mu}\Omega_{;\mu} = -D^{1/2}(D^{1/2}{}_{;\mu}{}^{;\mu}). \tag{1.40}$$

The solution obtained for Ω is, however, purely local. For example, if x, x' are both within a flat region of spacetime (e.g., within a spherical dust shell) a valid solution for Ω is $\Omega = 0$. This would then imply that the Feynman propagator is defined purely locally by Eq. (1.38), no matter what has occurred in the spacetime outside of the dust shell, and independent of the motion of the dust shell. The Feynman propagator is, however, a nonlocal Green's function which cannot be defined by a purely local technique, i.e., by examining the behavior only for x and x' near each other.

It seems that Eq. (1.37) must be augmented by global boundary conditions in order to obtain the Feynman propagator as a solution. These global boundary conditions will also imply global conditions on Ω which would also be expected to alter the behavior of Ω for small values of s. Further research on this problem is being done.[29]

DeWitt has also suggested an alternative to the above method, which is not *a priori* equivalent, due to the interchange of integration and limit procedures on a divergent expression. This technique is to evaluate $\mathfrak{T}_{\mu\nu}$ by separating the points at which the two field operators in $\mathfrak{T}_{\mu\nu}$ are evaluated, and then discarding any terms which depend on the magnitude of the separation for small separations. He has related this also to the adiabatic regularization scheme of Parker and Fulling.[9] Application to the full four-dimensional black-hole evaporation process has not yet been made. (See, however, Davies, Fulling, and Unruh[28] for an application to the two-dimensional collapse.)

A more naive prescription for regularizing the expression for the energy flow out through infinity is given by the following procedure.

We will be concerned here with the flow of energy through a surface near infinity. A conserved-energy-flow vector operator may be defined by

$$\mathfrak{I}^\mu = \mathfrak{T}_0{}^\mu. \tag{1.41}$$

The field operator Φ can be split near infinity into an ingoing and an outgoing part respectively,

$$\Phi = {}_I\Phi(v) + {}_O\Phi(u). \tag{1.42}$$

This split is exact in two dimensions, and applies only near infinity in four dimensions. Even in two dimensions, however, one would expect the following results to apply only near infinity.

We expand ${}_I\Phi(v)$ near infinity in terms of the ingoing modes ${}_I\phi_\omega = e^{-i\omega v}/(2\pi|\omega|)^{1/2}$, with $\omega > 0$ so that

$$_I\Phi = \sum_{\omega>0}\left[\frac{{}_Ia_\omega e^{-i\omega v}}{(2\pi|\omega|)^{1/2}} + \frac{{}_Ib_\omega^\dagger e^{i\omega v}}{(2\pi|\omega|)^{1/2}}\right]. \tag{1.43}$$

Similarly, ${}_O\Phi$ is expanded in terms of the modes which behave near $\mathscr{I}^+$ as

$${}_O\phi_\omega = \frac{e^{-i\omega u}}{(2\pi|\omega|)^{1/2}}$$

with operators ${}_Oa_\omega$ and ${}_Ob_\omega^\dagger$. As will be shown [Eq. (1.50)] the wave equation implies that these operators are linear combinations of the "in" operators ${}_Ia, {}_Ib^\dagger$,

$$\begin{aligned} {}_Oa_\omega &= \sum_{\omega'>0}[\alpha(\omega,\omega')_I a_{\omega'} + \beta(\omega,\omega')_I b_{\omega'}^\dagger], \\ {}_Ob_\omega^\dagger &= \sum_{\omega'>0}[\gamma(\omega,\omega')_I b_{\omega'}^\dagger + \sigma(\omega,\omega')_I a_{\omega'}]. \end{aligned} \tag{1.44}$$

We assume that the Φ field is in the "in" vacuum state $|0\rangle$ defined by ${}_Ia_\omega|0\rangle = {}_Ib_\omega|0\rangle = 0$ for all ω.

In defining the operator $\mathfrak{I}^\mu$, we expand Φ into ${}_I\Phi$ and ${}_O\Phi$ and then expand each of these with respect to their associated operators. Furthermore, we normal-order each term of Φ such that each ${}_Ia_\omega$ stands to the right of an operator ${}_Ia_\omega^\dagger$, and we proceed similarly for $({}_Ib_\omega, {}_Ib_\omega^\dagger)$, $({}_Oa_\omega, {}_Oa_\omega^\dagger)$, and $({}_Ob_\omega, {}_Ob_\omega^\dagger)$. A term such as ${}_Oa_\omega\,{}_Oa_\omega^\dagger$ therefore becomes ${}_Oa_\omega^\dagger\,{}_Oa_\omega$. In any expression which is ambiguous (e.g., ${}_{I}a_{\omega}\,{}_{O}a_{\omega}^{\dagger}$) we order the operators symmetrically. Then in the vacuum expectation value, $\langle 0|\mathfrak{J}^{\mu}|0\rangle$, all terms quadratic in ${}_{I}\Phi$ will be zero, leaving only terms of the form ${}_{I}\Phi^{\dagger}\,{}_{O}\Phi$ or of the form ${}_{O}\Phi^{\dagger}\,{}_{O}\Phi$.

Let us examine the energy flow out through $\mathscr{I}^{+}$.[10] The total energy flow between two retarded times u_1, u_2 will be given by

$$\begin{aligned}E(u_1,u_2)&=\int_{u_1}^{u_2}\sqrt{-g}\langle 0|\mathfrak{J}^{v}|0\rangle du\\&=\int_{u_1}^{u_2}\sqrt{-g}\,g^{uv}\langle 0|\mathfrak{J}_{u}|0\rangle du\\&=\int_{u_1}^{u_2}\sqrt{-g}\,g^{uv}\langle 0|\mathfrak{T}_{uu}+\mathfrak{T}_{vu}|0\rangle du\,.\end{aligned}\tag{1.45}$$

In two dimensions, the second term is zero by the trace-free property of $\mathfrak{T}_{\mu\nu}$, and near infinity $g^{uv}=2$ and $\sqrt{-g}=\frac{1}{2}$, leaving

$$\begin{aligned}E(u_1,u_2)&=\int_{u_1}^{u_2}\langle 0|\mathfrak{T}_{uu}|0\rangle du\\&=\int_{u_1}^{u_2}\langle 0|:{}_{O}\Phi^{\dagger}_{,u}\,{}_{O}\Phi_{,u}:|0\rangle du\,,\end{aligned}\tag{1.46}$$

where : : indicates normal-ordering with respect to the "out" operators. Expanding ${}_{O}\Phi$ into "out" modes gives

$$\begin{aligned}E(u_1,u_2)=\int_{u_1}^{u_2}du\sum_{\omega,\tilde\omega}\big[&\langle 0|{}_{O}a_{\omega}^{\dagger}\,{}_{O}a_{\tilde\omega}|0\rangle T_{uu}({}_{O}\phi_{\omega},{}_{O}\phi_{\tilde\omega})+\langle 0|{}_{O}b_{\omega}^{\dagger}\,{}_{O}b_{\tilde\omega}|0\rangle T_{uu}({}_{O}\phi_{\tilde\omega}^{*},{}_{O}\phi_{\tilde\omega}^{*})\\&+\langle 0|\{{}_{O}a_{\omega},{}_{O}b_{\tilde\omega}\}|0\rangle T_{uu}({}_{O}\phi_{\omega},{}_{O}\phi_{\tilde\omega})+\langle 0|\{{}_{O}a_{\omega}^{\dagger},{}_{O}b_{\tilde\omega}^{\dagger}\}|0\rangle T_{uu}({}_{O}\phi_{\omega},{}_{O}\phi_{\tilde\omega}^{*})\big]\,.\end{aligned}\tag{1.47}$$

Using the definition of $|0\rangle$ and the expansion of the "out" operators in terms of "in" operators, one obtains

$$\begin{aligned}\langle 0|{}_{O}a_{\omega}^{\dagger}\,{}_{O}a_{\tilde\omega}|0\rangle&=\sum_{\omega'}\beta(\omega,\omega')^{*}\beta(\tilde\omega,\omega')\,,\\\langle 0|{}_{O}b_{\omega}^{\dagger}\,{}_{O}b_{\tilde\omega}|0\rangle&=\sum_{\omega'}\sigma(\omega,\omega')\sigma(\tilde\omega,\omega')^{*}\,,\\\langle 0|\{{}_{O}a_{\omega},{}_{O}b_{\tilde\omega}\}|0\rangle&=\langle 0|\{{}_{O}a_{\omega}^{\dagger},{}_{O}b_{\tilde\omega}^{\dagger}\}|0\rangle^{*}\\&=\sum_{\omega'}\left[\beta(\omega,\omega')\gamma(\tilde\omega,\omega')^{*}+\sigma(\omega,\omega')^{*}\alpha(\tilde\omega,\omega')\right].\end{aligned}\tag{1.48}$$

Also note that

$$T_{uu}({}_{O}\phi_{\omega},{}_{O}\phi_{\tilde\omega})=\frac{(\omega\tilde\omega)^{1/2}e^{-i(\omega-\tilde\omega)u}}{2\pi}\,.\tag{1.49}$$

For $u_1,u_2\gg 0$, the last two terms of equation 1.47 will be proportional to $e^{\pm i(\omega+\tilde\omega)(u_1+u_2)/2}$, which is rapidly oscillating as a function of $\omega,\tilde\omega$ and will not contribute to $E(u_1,u_2)$. The first two terms, however, will contribute, and for $u_1,u_2>0$, their contribution per unit time will be constant. To show this we need an estimate for $\beta(\omega,\omega')$ and $\sigma(\omega,\omega')$.[11]

By the reality of the scalar equation one finds that $\sigma(\omega,\omega')=\beta(\omega,\omega')^{*}$. From Eq. (1.15) we obtain

$${}_{O}\Phi(u)={}_{I}\Phi(\hat v(\hat U(u)-2R_0))\,.\tag{1.50}$$

Therefore one readily finds that

$$\begin{aligned}\beta(\omega,\omega')&=\frac{i}{2}\int_{-\infty}^{\infty}du\,\frac{e^{i\omega u}}{(2\pi|\omega|)^{1/2}}\,\overleftrightarrow{\frac{\partial}{\partial u}}\,\frac{e^{i\omega'\hat v(\hat U(u)-2R_0)}}{(2\pi|\omega'|)^{1/2}}\\&=\frac{i}{4\pi}\int_{-\infty}^{v_0}dv\,\frac{e^{i\omega\hat u(\hat V(v)+2R_0)}}{(|\omega|)^{1/2}}\,\overleftrightarrow{\frac{\partial}{\partial v}}\,\frac{e^{i\omega' v}}{(|\omega'|)^{1/2}}\,.\end{aligned}\tag{1.51}$$

Substituting from (1.8) and (1.9) gives

$$\begin{aligned}\beta(\omega,\omega')=\frac{i}{4\pi}\Bigg[&\int_{-\infty}^{-R_0/(1-2M/R_0)^{1/2}}dv\,\frac{\exp[i\omega(v+2R_0)/(1-2M/R_0)^{1/2}]}{|\omega\omega'|^{1/2}}\,\overleftrightarrow{\frac{\partial}{\partial v}}e^{i\omega' v}\\&+\int_{-R_0/(1-2M/R_0)^{1/2}}^{v_0}dv\Bigg(\frac{A(\omega)}{|\omega\omega'|^{1/2}}\exp\{i\omega[-4M\ln(v_0-v)+(1-2M/R_0)^{1/2}(v-v_0)+O(1-\nu)]\}\\&\times\overleftrightarrow{\frac{\partial}{\partial v}}e^{i\omega' v}\Bigg)\Bigg]\,,\end{aligned}\tag{1.52}$$

$$v_0=\frac{(1-\nu)R_0-(1+\nu)2M}{(1-2M/R_0)^{1/2}\nu},$$

$$A(\omega)=\left(\frac{\nu}{(1+\nu)[R_0(R_0-2M)]^{1/2}}\right)^{-i4M\omega}e^{i\omega[2R_0+(1-2M/R_0)^{1/2}v_0]}$$

(note that $v_0<0$ because of the conditions on ν). For the purposes of calculating $\langle 0|\mathfrak{T}_{\mu\nu}|0\rangle$ a long time after the onset of collapse (i.e., large u) it is sufficient to approximate $\beta(\omega,\omega')$ by (see the Appendix)

$$\beta(\omega,\omega')\simeq-\frac{A(\omega)}{2\pi}\int_{-\infty}^{v_0}dv\left|\frac{\omega'}{\omega}\right|^{1/2}\exp[-i\omega 4M\ln(v_0-v)]e^{i\omega' v}$$
$$\simeq-\left|\frac{\omega'}{\omega}\right|^{1/2}\frac{A(\omega)e^{i\omega' v_0}}{2\pi}\Gamma(1-i4M\omega)(i\omega')^{+4i\omega M-1}. \tag{1.53}$$

Then we have

$$\int_{\omega'>0}d\omega'\beta(\omega,\omega')^*\beta(\tilde\omega,\omega')=(-1)^{+4i\omega M}i^{+4iM(\omega-\tilde\omega)}\frac{\Gamma(1+4M\omega i)\Gamma(1-4M\tilde\omega i)A(\omega)^*A(\tilde\omega)}{4\pi^2(\omega\tilde\omega)^{1/2}}\int_0^\infty d\omega'(\omega')^{-1-4i\omega M+4i\tilde\omega M}$$
$$\simeq\frac{e^{-4\pi\omega M}|\Gamma(1+i4M\omega)|^2}{8\pi M\omega}\delta(\omega-\tilde\omega)$$
$$\simeq\frac{e^{-4\pi\omega M}}{2\sinh(4M\pi\omega)}\delta(\omega-\tilde\omega). \tag{1.54}$$

Therefore one obtains

$$E(u_1,u_2)=(u_2-u_1)\int_0^\infty\frac{\omega e^{-4\pi\omega M}}{2\sinh(4M\pi\omega)}d\omega. \tag{1.55}$$

This is exactly the Hawking result if one remembers that in two dimensions the absorption amplitude of the black hole is unity. We have neglected terms in our approximation for $\beta(\omega,\omega')$ which will produce terms in $E(u_1,u_2)$ which die off at large u_1,u_2.

This normal-ordering prescription for determining the energy flow out through $\mathscr{I}^+$ also has some justification in that the "out" vacuum state $|0\rangle_O$ defined by ${}_Oa_\omega|0\rangle_O={}_Ob_\omega|0\rangle_O$ is a state of minimum-energy flow out through infinity—i.e., any n-particle "out" state has a higher energy flow through $\mathscr{I}^+$. The normal-ordering procedure therefore just subtracts off the expectation value of energy outflow in this state.

Furthermore, if one is interested in the energy flow out through an $r=\text{const}$ surface near infinity rather than through $\mathscr{I}^+$, one does not need to normal-order $\mathfrak{T}^{\mu\nu}$. The net radial flow[12] of energy through a surface of constant radius is given by $\mathfrak{F}^r$ which may be rewritten as

$$\mathfrak{F}^r=g^{rr}(-\mathfrak{T}_{uu}+\mathfrak{T}_{vv})$$
$$\sim-\mathfrak{T}_{uu}+\mathfrak{T}_{vv}\quad\text{as } r\to\infty. \tag{1.56}$$

As ${}_O\Phi$ depends on u and ${}_I\Phi$ only on v, this may be written as

$$\tfrac{1}{2}(-\{{}_O\Phi^\dagger_{,u},{}_O\Phi_{,u}\}+\{{}_I\Phi^\dagger_{,v},{}_I\Phi_{,v}\}). \tag{1.57}$$

Expanding ${}_O\Phi$ in the out modes ${}_O\Phi_\omega$ and ${}_I\Phi$ in the in modes ${}_I\Phi_\omega$, we find that this operator differs from the operator in which we normal-order the various terms of $\mathfrak{T}_{\mu\nu}$ with respect to the appropriate operators by the c-number field:

$$\tfrac{1}{2}\left\{\int_{\omega>0}d\omega[-T_{uu}({}_O\phi_\omega,{}_O\phi_\omega)]+\int_{\omega'>0}d\omega' T_{vv}({}_I\phi_{\omega'},{}_I\phi_{\omega'})\right\}. \tag{1.58}$$

Regrouping terms we obtain

$$\tfrac{1}{2}\int_{\omega>0}d\omega[-T_{uu}({}_O\phi_\omega,{}_O\phi_\omega)+T_{vv}({}_I\phi_\omega,{}_I\phi_\omega)]. \tag{1.59}$$

These two terms exactly cancel and one is left with the same expression for the energy flow to infinity as when one normal-ordered the various components of $\mathfrak{T}_{\mu\nu}$ with respect to their respective creation and annihilation operators.

As the above process applies to both scalar and neutrino fields we appear to have a contradiction. When the neutrino and scalar energy-momentum tensors were summed mode by mode, one obtained zero energy flow in two dimensions into or out of the black hole. Now, however, when one sums neutrino and scalar contributions by this technique, one obtains a net energy flow out through infinity. This contradiction comes about because the expressions with which we are working are not absolutely convergent expressions. By an appropriate grouping of terms in a conditionally convergent series, any result desired may be obtained. In the first case, the neutrino and scalar terms were summed first mode by mode, and then the total expectation value of the energy-momentum tensor was calculated. That order of summation guarantees that the energy-momentum tensor will be covariant.

In the latter method the procedure is inherently noncovariant. The time development of the mode ${}_I\phi_\omega$ is a linear combination of out modes ${}_O\phi_\omega$. We are therefore performing the mode summation for $\langle 0|\mathfrak{T}_{\mu\nu}|0\rangle$ in two different ways for the ingoing and outgoing portions of $\mathfrak{T}_{\mu\nu}$. Although both orders seem natural for their respective parts of $\mathfrak{T}_{\mu\nu}$, I would suggest that such a switch is invalid. Once an order of mode summation in calculating $\mathfrak{T}_{\mu\nu}$ is used in one part of spacetime, I would suggest it must be used everywhere. Otherwise, by an appropriate choice of order of summation of the modes, one could arrive at any answer one wanted.[13]

If this is correct, then the procedure of Fermi-Bose cancellation, in which the mode sum is defined so as to result in no inflow of energy into the black hole, suggests that there is also no outflow of energy from the black hole and that the quantum evaporation does not take place.

II. HAWKING PROCESS FOR ETERNAL BLACK HOLES

The procedure in this section was inspired by a study begun by Fulling[14,15] on alternative quantizations of flat spacetime. In particular, the natural quantization in Rindler coordinates leads to a quantization inequivalent to the normal Minkowski quantization. The resolution of the problem there leads directly to a quantization of the full Schwarzschild spacetime which has all of the features that Hawking found for a black hole formed by collapse of a star. As the collapse is replaced by certain natural boundary conditions on the past horizon of the black hole, this problem is mathematically much simpler than the collapse problem.

I will first discuss the Rindler-Minkowski situation for a massive scalar field before going on to the Schwarzschild case. Writing flat spacetimes in Rindler[16] coordinates leads to the metric

$$ds^2 = \rho d\tau^2 - \frac{d\rho^2}{\rho} - dx^2 - dy^2\,, \tag{2.1}$$

an obviously static metric. By the coordinate transformation

$$t = 2\sqrt{\rho}\,\sinh(\tfrac{1}{2}\tau)\,,\quad z = 2\sqrt{\rho}\,\cosh(\tfrac{1}{2}\tau) \tag{2.2}$$

one obtains the Minkowski metric

$$ds^2 = dt^2 - dz^2 - dx^2 - dy^2\,. \tag{2.3}$$

Let us examine the scalar wave equation

$$\phi^{;\alpha}{}_{;\alpha} - \mu^2\phi = 0\,. \tag{2.4}$$

The normal-mode solutions to this equation in Minkowski coordinates are

$${}_M\phi_{\omega k} = \frac{e^{-i\omega t}}{[(2\pi)^3|\omega|]^{1/2}}\,e^{-i\vec{k}\cdot\vec{x}}\,, \tag{2.5}$$

where

$$\vec{x} = (x, y, z)\,,\quad \vec{k} = (k_1, k_2, k_3)$$

and

$$\omega = (\mu^2 + \vec{k}\cdot\vec{k})^{1/2}\,.$$

One can define a second quantization by forming the wave operator $\Phi(x)$ in the usual way and expanding in these normal modes:

$$\Phi = \sum_{\substack{\omega>0\\ \vec{k};}} ({}_M a_{\omega\vec{k}}\,{}_M\phi_{\omega\vec{k}} + {}_M a^\dagger_{\omega\vec{k}}\,{}_M\phi^*_{\omega\vec{k}})\,, \tag{2.6}$$

with a Minkowski vacuum $|0\rangle_M$ defined by

$${}_M a|0\rangle_M = 0\,. \tag{2.7}$$

On the other hand, one can define normal modes in the Rindler system

$${}_R\phi_{\tilde{\omega}k} = \frac{e^{-i\tilde{\omega}\tau}}{[(2\pi)^3|\tilde{\omega}|]^{1/2}}\,g(\rho)\,e^{-i(k_1x + k_2y)}\,, \tag{2.8}$$

where $g(\rho)$ satisfies the equation

$$\left[\rho\frac{d}{d\rho}\rho\frac{d}{d\rho} + \tilde{\omega}^2 - (\mu^2 + k_1{}^2 + k_2{}^2)\rho\right]g(\rho) = 0\,. \tag{2.9}$$

These modes are complete in the region of that spacetime covered by the Rindler coordinate system (i.e., $\rho > 0$ and $z > |t|$). We can expand Φ in terms of this basis in this half of spacetime:

$$\Phi = \sum_{k;\tilde{\omega}>0} ({}_R a_{\tilde{\omega}k}\,{}_R\phi_{\omega k} + {}_R a^\dagger_{\tilde{\omega}k}\,{}_R\phi^*_{\omega k})\,. \tag{2.10}$$

The Fulling-Rindler vacuum is defined by

$${}_R a|0\rangle_F = 0\,. \tag{2.11}$$

What is the relationship between these two systems of quantization? To begin with, we also define a complete set of functions in the other half Rindler space ($\rho > 0$ and $z < -|t|$). Designating this side by R^- and the other half by R^+ we can write

$$\Phi = \sum_{k;\tilde{\omega}>0} ({}_{R^+}a_{\tilde{\omega}k}\,{}_{R^+}\phi_{\tilde{\omega}k} + {}_{R^-}a^\dagger_{\tilde{\omega}k}\,{}_{R^-}\phi_{\tilde{\omega}k} + \text{H.c.})\,. \tag{2.12}$$

${}_{R^+}\phi$ is defined to be zero in the region R^-, whereas ${}_{R^-}\phi$ is zero in the region R^+. I.e., ${}_{R^-}\phi_{\omega k}(\tau, \rho,$

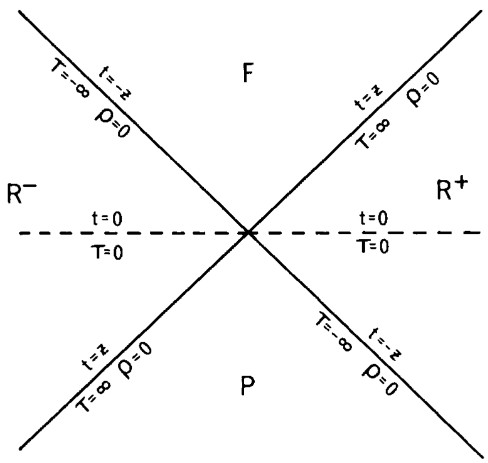

FIG. 1. The various regions of Minkowski spacetime in Rindler accelerated coordinates.

$x, y)$ is functionally equal to ${}_{R^+}\phi_{\omega k}(\tau,\rho,x,y)$, where ρ, τ are taken as lying in their respective half spaces. The choice of which operators to treat as creation operators and which as annihilation operators is forced on us by the commutation relations obeyed by these operators.

The functions ${}_{R^+}\phi_{\tilde{\omega}k}$ and ${}_{R^-}\phi_{\omega k}$ can be analytically extended across the future and past horizons $(t=\pm z)$ into the regions F (i.e., $t>|z|$) and P $(t<-|z|)$ of Minkowski spacetime (see Fig. 1), such that ${}_{R^+}\phi$ are zero only in the region R^- and ${}_{R^-}\phi$ are zero only in R^+. The expansion of Eq. (2.12) is then valid in the full Minkowski spacetime. (This follows from the fact that the union of the two spacelike hypersurfaces given by $\tau=0$ in both R^+ and R^- is a Cauchy surface for the full Minkowski spacetime.)

I will now show that the second quantization defined by this expansion of Φ is not equivalent to the usual Minkowski second quantization, and will find a linear combination of the ${}_{R^+}a$ and ${}_{R^-}a^\dagger$ operators which are annihilation operators of the usual Minkowski second quantization.

Finding the relationship between this expansion and the more normal Minkowski expansion of Φ is equivalent to finding the relationship between the positive-frequency spaces defined by $\omega>0$ and $\tilde{\omega}>0$. This is most easily found by going to the null hypersurface $\rho=0$ or $t=-z$. Define the null coordinates

$$\begin{aligned} &U=t-z\,, \quad V=t+z\,, \\ &u=\tau-\ln\rho\,, \quad v=\tau+\ln\rho\,. \end{aligned} \tag{2.13}$$

Then we find

$$U=\mp 2e^{-u/2}\,, \quad V=\pm 2e^{v/2}\,,$$

with the top sign for u, v in R^+ and the lower sign for u, v in R^-.

By a suitable choice of phase, the Rindler mode ${}_R\phi_{\omega k}$ goes to

$$\frac{(e^{-i\tilde{\omega}u}+\alpha e^{-i\tilde{\omega}v})e^{-i(k_1x+k_2y)}}{[(2\pi)^3|\tilde{\omega}|]^{1/2}} \tag{2.14}$$

near $\rho=0$. α is a constant of unit modulus.

Along the surface $v=-\infty$, $V=0$, the past horizon of the Rindler coordinates, we obtain

$$\begin{aligned} {}_{R^\pm}\phi_{\tilde{\omega}k} &= \frac{e^{-i\tilde{\omega}u}e^{-i(k_1x+k_2y)}}{[(2\pi)^3|\tilde{\omega}|]^{1/2}} \\ &= \frac{(\frac{1}{2}|U|)^{+i2\tilde{\omega}}e^{-i(k_1x+k_2y)}}{[(2\pi)^3|\tilde{\omega}|]^{1/2}}\,. \end{aligned} \tag{2.15}$$

(I have dropped the $e^{-i\omega v}$ term, a procedure which can be justified by constructing wave packets.)

A positive-frequency Minkowski mode on this surface becomes

$${}_M\phi_\omega(U, V=0, x, y)=\frac{e^{-i(\omega+k_3)U/2}e^{-i(k_1x+k_2y)}}{[(2\pi)^3|\tilde{\omega}|]^{1/2}}\,. \tag{2.16}$$

An important point to note is that for $\omega>0$,

$$\omega+k_3=(\mu^2+k_1{}^2+k_2{}^2+k_3{}^2)^{1/2}+k_3>0 \tag{2.17}$$

for all values of k_3, positive or negative, and vice versa for $\omega<0$, i.e., $\mathrm{sign}(\omega+k_3)=\mathrm{sign}(\omega)$.

The positive-frequency ω Minkowski modes are therefore characterized by the condition that they are analytic and bounded in the lower half complex U plane $[\mathrm{Im}(U)<0]$ on $V=0$.

Similarly, note that the combination

$$\begin{aligned} {}_R\hat{\phi}_{\tilde{\omega}k} = &\frac{e^{\pi\tilde{\omega}}}{|2\sinh(2\pi\tilde{\omega})|^{1/2}}\;{}_{R^+}\phi_{\tilde{\omega}k} \\ &+\frac{e^{-\pi\tilde{\omega}}}{|2\sinh(2\pi\tilde{\omega})|^{1/2}}\;{}_{R^-}\phi_{\tilde{\omega}k} \end{aligned}$$

becomes, on the $V=0$ surface,

$${}_R\hat{\phi}_{\tilde{\omega}k} \approx \frac{e^{-i(k_1x+k_2y)}}{|(2\pi)^3|\tilde{\omega}|2\sinh(2\pi\tilde{\omega})|^{1/2}} \times \begin{cases} e^{\pi\tilde{\omega}}|U|^{+i2\tilde{\omega}}, & U<0 \\ e^{-\pi\tilde{\omega}}|U|^{+i2\tilde{\omega}}, & U>0 \end{cases} \tag{2.18}$$

which is the restriction to real U of a function analytic in the entire lower half complex U plane. ${}_R\hat{\phi}_{\tilde{\omega}}$ is therefore a positive-frequency function in the Minkowski sense for *all* values of $\tilde{\omega}$, positive and negative. The expression $|(2\pi)^3\tilde{\omega}2\sinh(2\pi\tilde{\omega})|^{-1/2}$ has been chosen to normalize ${}_R\hat{\phi}_{\tilde{\omega}}$ under the usual scalar inner product.

Note that for positive $\tilde{\omega}$, the function ${}_R\hat{\phi}_{\tilde{\omega}}$ is almost entirely concentrated in the R^+ region, whereas for $\tilde{\omega}<0$ it is concentrated in the R^- region. This is understandable, as the Rindler time τ is a function which increases with time t in R^+ but decreases with t in R^-. The positive-frequency Rindler mode would be expected to be an almost positive-frequency Minkowski mode in R^+ but almost negative-frequency Minkowski mode in R^-, as is actually found to be the case.

Also note in passing that the probability that a positive-frequency Rindler function ${}_{R^+}\phi_{\tilde{\omega}}$ is in a negative-frequency Minkowski state is just

$$\frac{e^{-\pi\tilde{\omega}}}{2\sinh(2\pi\tilde{\omega})},$$

which is exactly the Boltzmann factor for a Bose gas with temperature $1/4\pi$. From this, the analysis of Davies[17] on the Hawking process in flat spacetime can be obtained.

The Minkowski vacuum state can be reexpressed as a many-particle Fulling-Rindler (FR) state. By comparing the expansion of Φ in terms of the Minkowski positive-frequency basis of Eq. (2.17), and comparing the resultant creation and annihilation operator with the FR set of (2.12) one obtains

$$\begin{aligned}(e^{\pi\tilde{\omega}}\,{}_{R^+}a_{\tilde{\omega}k} + e^{-\pi\tilde{\omega}}\,{}_{R^-}a^\dagger_{\tilde{\omega}k})|0\rangle_M &= 0\,,\\ (e^{-\pi\tilde{\omega}}\,{}_{R^+}a^\dagger_{\tilde{\omega}k} + e^{\pi\tilde{\omega}}\,{}_{R^-}a_{\tilde{\omega}k})|0\rangle_M &= 0\,.\end{aligned} \tag{2.19a}$$

With the FR vacuum defined by

$${}_{R^+}a|0\rangle_F = {}_{R^-}a|0\rangle_F = 0$$

one obtains

$$|0\rangle_M = Z\left[\prod_{\tilde{\omega},k}\exp(e^{-2\pi\omega}\,{}_{R^+}a^\dagger_{\tilde{\omega}k}\,{}_{R^-}a^\dagger_{\tilde{\omega}k})\right]|0\rangle_F\,, \tag{2.19b}$$

where Z is a normalization constant.

In the next section we shall find that an accelerated detector will respond to the presence of these FR particles in the Minkowski vacuum.

But what has this digression to do with black holes? A similar analysis is possible for an Schwarzschild black hole. For the virtual states of a quantum field which emanate from the past horizon of a black hole, two possible Foch quantization are also possible, corresponding exactly to the FR-Minkowski possibilities in flat spacetime. I will take the Schwarzschild metric as the example, but any other black-hole metric would also do as well. By going through the analysis I shall describe, one readily obtains the Hawking-Gibbons[18] results for charged-rotating-black-hole evaporation. In fact, this technique was noticed by me before I was aware of the original published Hawking result. It was only after becoming aware of Hawking's result, however, that I realized that this mathematical procedure had physical significance and was related to the question of black-hole evaporation.

The Schwarzschild metric is given by

$$\begin{aligned}ds^2 = (1-2M/r)dt^2 - (1-2M/r)^{-1}dr^2\\ - r^2(d\theta^2 - \sin^2\theta\, d\phi^2).\end{aligned} \tag{2.20}$$

The massless scalar wave equation is given by

$$(-g)^{-1/2}[\phi_{,\mu}g^{\mu\nu}(-g)^{1/2}]_{,\nu} = 0. \tag{2.21}$$

The normal-mode solutions to this may be written as

$$\phi_{\omega lm} = (2\pi|\omega|)^{-1/2}e^{-i\omega t}f_{\omega l}(r)Y_{lm}(\theta,\phi), \tag{2.22}$$

where $f_{\omega l}$ obeys the equation

$$\left[\frac{1}{r^2}\frac{d}{dr^*}r^2\frac{d}{dr^*} + \omega^2 - \left(1-\frac{2M}{r}\right)\frac{l(l+1)}{r^2}\right]f_{\omega l}(r) = 0, \tag{2.23}$$

and where $dr^* = (1-2M/r)^{-1}dr$.

This has two solutions, one representing waves incoming from infinity, and one representing waves coming out of the past horizon of the black hole. These solutions are designated by ${}^+f$ and ${}^-f$, respectively. The latter, near the horizon, behaves as

$${}^-f_{\omega l}(r) \sim e^{+i\omega r^*} + {}^-A_{\omega l}e^{-i\omega r^*}. \tag{2.24}$$

The standard naive prescription for defining the positive-frequency states in this case is to define the $\omega>0$ states as the positive-frequency states. This prescription I shall call the η definition, as it is related to the timelike Killing vector $\eta = \partial/\partial t$. I.e., one defines positive frequency via the modes which are eigenfunctions of the Lie derivative of the scalar field in the η direction.

On the other hand, the Schwarzschild metric (and all the black-hole metrics) has another vector field ξ which is of Killing type on the past horizon, H^- (i.e., $\mathcal{L}_\xi g_{\mu\nu} = 0$ on H^-). Going to Kruskal coordinates

$$\begin{aligned}&ds^2 = 2M\frac{e^{-r/2M}}{r}dU\,dV - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),\\ &U = -4Me^{-u/4M},\quad V = 4Me^{v/4M},\\ &v = t + r + 2M\ln(r/2M - 1),\\ &u = t - r - 2M\ln(r/2M - 1),\end{aligned} \tag{2.25}$$

the vector $\xi = \partial/\partial U$ defines this Killing vector on H^-. Furthermore, the integral curves of ξ, unlike those of η, are geodesics on H^-.

If we define normal modes such that on H^- they are eigenfunctions of the Lie derivative with respect to ξ,

$$\mathcal{L}_\xi(\tilde{\phi}_{\tilde{\omega}}) = -i\tilde{\omega}\tilde{\phi}_{\tilde{\omega}}, \tag{2.26}$$

we obtain a complete set of modes which go as

$$\tilde{\phi}_{\tilde{\omega}} = e^{-i\tilde{\omega}U}Y_{lm}(\theta,\phi)/(2\pi|\tilde{\omega}|)^{1/2} \tag{2.27}$$

on H^-. The usual η modes on H^- become

$$ {}^{-}\phi_{\omega} = (2\pi|\omega|)^{-1/2} \frac{|U|^{+i4M\omega}}{4M} Y_{lm}(\theta,\phi) \,. \tag{2.28} $$

As in the flat-spacetime Rindler case, the horizon divides the spacetime into two separate exterior spacetimes. Outgoing η modes can again be defined in both halves of the spacetime. These will be designated by + or − presubscripts. I.e.,

$$ {}^{-}_{+}\phi_{\omega lm} = e^{-i\omega t}\, {}^{-}f_{\omega l}(r^*) Y_{lm}(\theta,\phi)/(2\pi\omega)^{1/2} $$

for t, r located in the right exterior spacetime and zero in the left exterior region, and similarly for ${}^{-}_{-}\phi_{\omega}$ which is zero in the right exterior region. Then the outgoing mode

$$ {}^{-}\hat{\phi}_{\omega} = (e^{2\pi M\omega}\, {}^{-}_{+}\phi_{\omega} + e^{-2\pi M\omega}\, {}^{-}_{-}\phi_{\omega})[2\sinh(4\pi M\omega)]^{-1/2} \tag{2.29} $$

is positive frequency in the ξ sense for all values of ω. I.e., either this set, or the set of Eq. (2.27) for $\tilde{\omega} > 0$ are a complete set of ξ positive-frequency waves.

There are a number of reasons why I favor the ξ definition over the η definition for positive frequency for modes emanating from the past horizon of the Schwarzschild black hole.

Firstly, the next section will indicate that a particle detector freely falling near the future horizon will respond to the presence of ξ, not η, positive-frequency modes—i.e., these ξ frequency modes are, to my mind, the best definition of what one would mean in saying a particle is coming out of the past horizon of the black hole.

Secondly, if one examines the stellar collapse situation, a mode which starts off as $e^{-i\omega v}/(2\pi\omega)^{1/2}$ at infinity goes as

$$ \exp[-i\omega(1-2m/R_0)^{1/2} e^{-u/4M}(1-\nu)(R_0 - 2M)/\nu] \approx e^{-i\omega\alpha U} $$

when it exits from the collapsing star (α is a positive constant depending on the details of the collapse). This equation is obtained from Eqs. (1.8) and (1.15). I.e., a mode which emerges from the inside of the star just as it is about to cross its horizon, and which was a positive-frequency mode when it came toward the star from infinity, will be a ξ positive-frequency mode as it leaves the star. In this sense, the ξ definition of positive frequency does correspond to replacing the collapse process by boundary conditions on H^-.

Thirdly, as I shall soon indicate, the Feynman propagator obtained from the η-definition of positive frequency has singular physical (i.e., not related to coordinate singularities) derivatives on the future horizon whereas that using the ξ definition does not. As most physical properties of the scalar field (e.g., charge density, energy, etc.) involve derivatives, this suggests that the physics under the η definition becomes singular there, even if the propagator itself does not increase without bound.

Fourthly, the ξ definition of positive frequency is invariant under ordinary time translation, as can be seen from Eq. (1.29). Furthermore, it is also invariant under the additional symmetry of U translation along the past horizon. Under the condition that the vacuum state should be invariant under the largest symmetry group available, the ξ definition for modes leaving H^- would be the preferred definition. (In flat spacetime it is the property that the Minkowski definition of positive frequency is invariant under the full Poincaré group, whereas the Rindler definition is not, that singles it out as the preferred quantization.)

To summarize, positive frequency for those states which enter the exterior Schwarzschild spacetime through H^-, the past horizon of the black hole, is defined via the Killing vector ξ on H^-. For those states which originate at infinity, however, the usual η definition will be used. On $\mathscr{I}^-$, the η definition has all of the advantages that the ξ definition has on H^-.

In second-quantizing the field Φ in the exterior of the Schwarzschild black hole we expand it as follows (neglecting terms equal to zero there):

$$ \Phi = \sum_{lm} \left[\int_0^\infty d\omega ({}^{+}a_{\omega lm}\, {}^{+}\phi_{\omega lm} + {}^{+}a^\dagger_{\omega lm}\, {}^{+}\phi^*_{\omega lm}) + \int_0^\infty d\tilde{\omega} (\tilde{a}_{\tilde{\omega} lm} \tilde{\phi}_{\tilde{\omega} lm} + \tilde{a}^\dagger_{\tilde{\omega} lm} \tilde{\phi}_{\tilde{\omega} lm}) \right]. \tag{2.30a} $$

Alternately, the second part could have been expanded in terms of the ${}^{-}\hat{\phi}_{\omega lm}$ modes of Eq. (2.29). In the exterior region of the Schwarzschild spacetime this becomes

$$ \sum_{lm} \int_{-\infty}^{\infty} d\omega \frac{e^{2\pi M\omega}}{[2\sinh(4\pi M\omega]^{1/2}} (\hat{a}_{\omega lm}\, {}^{-}\hat{\phi}_{\omega lm} + \hat{a}^\dagger_{\omega lm}\, {}^{-}\hat{\phi}^*_{\omega lm}) \,. \tag{2.30b} $$

The vacuum state is defined by

$$ 0 = {}^{+}a_{\omega lm}|0\rangle = \tilde{a}_{\tilde{\omega} lm}|0\rangle \;(= \hat{a}_{\omega lm}|0\rangle) \,. $$

The alternative η definition of positive frequency leads to the expansion for Φ in terms of the modes ${}^{\pm}_{\pm}\phi_{\omega lm}$ in both halves of the full Kruskal extension:

$$ \Phi = \sum_{lm} \int_0^\infty d\omega \,({}^{+}_{+}b_{\omega lm}\, {}^{+}_{+}\phi_{\omega lm} + {}^{+}_{-}b^\dagger_{\omega lm}\, {}^{+}_{-}\phi_{\omega lm} + {}^{-}_{+}b_{\omega lm}\, {}^{-}_{+}\phi_{\omega lm} + {}^{-}_{-}b^\dagger_{\omega lm}\, {}^{-}_{-}\phi_{\omega lm}) + \text{H.c.} \tag{2.31} $$

The $\omega > 0$ modes are associated with creation operators in the left exterior region because coordinate time t runs backward there. The vacuum state $|0\rangle_\eta$ associated with this expansion is given by

$$ {}^{\pm}_{\pm}b_{\omega lm}|0\rangle_\eta = 0\,. $$

We can relate the two vacuum states $|0\rangle$ and $|0\rangle_\eta$ by noticing that

$$\hat{a}_{\omega lm} = \begin{cases} p(\omega)\,{}^{-}_{+}b_{\omega lm} + p(-\omega)\,{}^{-}_{-}b^{\dagger}_{\omega lm}, & \omega > 0 \\ p(\omega)\,{}^{-}_{+}b^{\dagger}_{\omega lm} + p(-\omega)\,{}^{-}_{-}b_{\omega lm}, & \omega < 0 \end{cases}$$

$$p(\omega) = e^{-2\pi M\omega}/[2\sinh(4\pi M\omega)]^{1/2}. \qquad (2.32)$$

This then gives

$$|0\rangle = Z \prod_{\omega, l, m} \exp(e^{-4\pi M\omega}\,{}^{-}_{+}b^{\dagger}_{\omega lm}\,{}^{-}_{-}b^{\dagger}_{\omega lm})|0\rangle_\eta\,, \qquad (2.33)$$

where Z is again an (infinite) normalization constant.

In terms of the η vacuum, the state $|0\rangle$ has an infinite number of pairs of particles. As has been suggested by Hawking, these particles occur in pairs, one particle in the right-hand exterior region and the other on the other side of the horizon. (In the context of the collapse situation, Wald[19] has come to a similar conclusion.)

By inserting the metric

$$ds^2 = \frac{dU dV}{e} - (2M)^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (2.34)$$

between the two past horizons of the Kruskal extension of the Schwarzschild metric, one finds that the ξ definition of positive frequency is exactly the definition one would naturally use for this interior metric. By this trick one can also see that the ξ definition of positive frequency can be made symmetric for both exterior regions of the Kruskal extension of spacetime. In the previous definition, I defined positive frequency along the $V=0$ hypersurface, which is the past horizon for the one exterior region but becomes the future horizon for the other. The above trick shows that procedure is not the only possible one, and that positive frequency could equally well have been defined over the past horizon of both extensions. In the one exterior region of concern to us, the change will make no difference.

Let us now briefly examine the Feynman propagator $G(x,x')$. One could try to define it using DeWitt's suggestions of writing the equation

$$(\nabla_\alpha \nabla^\alpha - \mu^2 + i\epsilon)G(x,x',\epsilon) = i\delta(x,x'), \qquad (2.35)$$

solving for $G_\epsilon(x,x',\epsilon)$, taking the limit as $\epsilon \to 0$. This prescription, however, does not lead to an unambiguous $G(x,x')$ for incomplete spacetimes. We go to Rindler spacetime for the example.

For two points x, x' both within one half of the Rindler spacetime, we can solve Eq. (2.34) by

$$G(x,x',\epsilon) = \int d\omega\, d\tilde{\omega}\, dk_1 dk_2 \frac{e^{-i\omega(t-t')} f_{\tilde{\omega}}(\rho) f_{\tilde{\omega}}(\rho')}{\omega^2 - \tilde{\omega}^2 + \epsilon} \times e^{-i[k_1(x-x') + k_2(y-y')]}.$$

Performing the ω integration first one obtains not the usual Minkowski Feynman propagator, but the Rindler propagator (i.e., the propagator takes Rindler, not Minkowski positive frequencies forward in time). This again demonstrates that in addition to Eq. (2.34) one needs boundary conditions on $G(x,x')$ as the two points become widely separated and on the horizon at $\rho = 0$, $t = \pm\infty$ to obtain the correct propagator. [As Boulware[20] has shown, if one does supply the correct boundary conditions in the fully extended Rindler metric (i.e., full Minkowski spacetime), the usual propagator is obtained.]

The Feynman propagator may be defined as

$$G(x,x') = \langle 0|T(\Phi(x)\Phi(x'))|0\rangle\,, \qquad (2.36)$$

where $T(\ldots)$ designates the time-ordered product. This equation immediately leads to an alternative method of finding the Feynman propagator. Regard x' as fixed. Let $G_A(x,x')$ be the advanced Green's function for the field (which is invariantly defined). Take the point x back to the hypersurface on which one has defined what one means by positive frequencies. (E.g., $\mathscr{I}^-$ or the past horizon.) Subtract from $G_A(x,x')$ a free positive-frequency wave exactly equal to the positive-frequency component of $G_A(x,x')$ on the hypersurface. This sum is then the Feynman propagator.

I would like to briefly compare the behavior of the Feynman propagators obtained by using the ξ definition and the η definition of positive frequency. Take the point x of the advanced Green's function $G_A(x,x')$ to lie on either $\mathscr{I}^-$ or H^-, the past horizon. As positive frequency for a wave on $\mathscr{I}^-$ is defined in the same way for both η and ξ definitions, only the behavior on H^-, the past horizon, will cause any difference. The projection kernel $K(x,\tilde{x})$ for positive η frequencies on H^- can be given by

$$K_\eta(x,\tilde{x}) = \frac{1}{(u - \tilde{u} + i0)} \sum_{lm} Y_{lm}(\theta,\phi) Y^*_{lm}(\tilde{\theta},\tilde{\phi}), \qquad (2.37)$$

where both u, $\tilde{u}$ lie on the lower half ($U<0$) of H^-. The η positive-frequency component of $G_A(x,x')$ on H^- is then given by

$$P_\eta(G_A(x,x'))=\frac{i}{2}\int_{H^-}\sqrt{-g}\,g^{\mu\nu}G_A(\tilde{x},x')\frac{\overleftrightarrow{\partial}}{\partial u}K_\eta(x,\tilde{x})d\tilde{u}\,d\tilde{\theta}\,d\tilde{\phi}$$
$$=-i4M^2\int_{H^-}\left(\frac{\partial}{\partial\tilde{u}}G_A(\tilde{x},x')\right)\frac{\kappa(\theta\phi,\tilde{\theta}\tilde{\phi})}{u-\tilde{u}+i0}d\tilde{u}\,d\cos\tilde{\theta}\,d\tilde{\phi}$$
$$=-i4M^2\int_{H^-}\left(\frac{\partial}{\partial\tilde{U}}G_A(\tilde{x},x')\right)\frac{\kappa(\theta\phi,\tilde{\theta}\tilde{\phi})}{4M\ln(U/\tilde{U})+i0}d\tilde{U}\,d\cos\tilde{\theta}\,d\tilde{\phi}, \tag{2.38}$$
$$\kappa(\theta\phi,\tilde{\theta}\tilde{\phi})=\sum_{lm}Y_{lm}(\theta\phi)Y^*_{lm}(\tilde{\theta}\tilde{\phi}).$$

On the other hand, the projection kernel for ξ positive frequency is given by

$$K_\xi(x,\tilde{x})=\frac{1}{U-\tilde{U}+i0}\kappa(\theta\phi,\tilde{\theta}\tilde{\phi}), \tag{2.39}$$

which gives a projection of

$$P_\xi(G_A(x,x'))=-i4M^2\times\int_{H^-}\left(\frac{\partial}{\partial\tilde{U}}G_A(\tilde{x},x')\right)\frac{\kappa(\theta\phi,\tilde{\theta}\tilde{\phi})}{U-\tilde{U}+i0}\times d\tilde{U}\,d\cos\tilde{\theta}\,d\tilde{\phi}. \tag{2.40}$$

Note that $P_\eta(G(x,x'))$ will have a cusp at $U=0$. Although $P_\eta(G(x,x'))$ goes to zero as $U\to 0$ (i.e., as x approaches the future horizon), the first derivative of $P_\eta(G_A(x,x'))$ diverges as $[U(\ln U)^2]^{-1}$ as $U\to 0$. Since U is a well-defined null coordinate near the horizon, this implies that the gradient of $G_\eta(x,x')$, the Feynman propagator in the η system, will diverge as the future horizon $U=0$ is approached.

Boulware[20] has also examined the Feynman propagator in the full extended Kruskal metric. He arrived at the η definition for his propagator. This result was, however, obtained because his definition of particles coming out of the past horizon depended on an implicit use of the η definition of positive frequency.

The singularity in the Feynman propagator on the future horizon when the η definition is used is nonphysical. Such a singularity would locally single out the horizon, whereas the spacetime structure and casual structure of the future horizon gives no reason why it should be singled out.

III. PARTICLE DETECTORS

This section will examine the behavior of particle detectors under accelerated states of motion of the detector. The main conclusions I will draw will be the following:

(a) A particle detector will react to states which have positive frequency with respect to the detectors proper time, not with respect to any universal time.

(b) The process of detection of a field quanta by a detector, defined as the exciting of the detector by the field, may correspond to either the absorption or the emission of a field quanta when the detector is an accelerated one.

Both of these results, although surprising at first glance, are, using hindsight, very reasonable, but they make the generalization of what is meant by a vacuum state to a nonflat spacetime extremely difficult.

The model detectors I will investigate are of two types. One is a box, containing a Schrödinger particle in its ground state. The detector is said to have detected a quanta of the massless scalar field Φ, if the detector is found in a state other than its ground state at some time.

The other model is a fully relativistic one. Here I assume that the detector, described by the scalar field Ψ with mass μ is coupled via the field Φ to an "excited" state described by the scalar field φ with mass $M>\mu$. Both Ψ and φ are assumed to be complex fields, but for simplicity Φ is taken to be a real field. The detector is said to have detected a Φ quanta if the detector is found in the excited state φ at some time. This detector could be further complicated by coupling φ back to Ψ via some "dial" fields (representing states of a computer memory, or readings on an instrument dial) but we shall leave it in its most primitive form.

Let us examine a box detector being uniformly accelerated. The simplest way to treat this system is to go to a Rindler coordinate system, in which the box is at rest at ρ coordinate ρ_0, and at $x=y=0$. The equation of the particle in the box is given by

$$i\frac{\partial\Psi}{\partial\tau}=\frac{1}{2m\sqrt{\rho_0}}\times\left[\rho_0^2\frac{\partial^2}{\partial\rho^2}+\rho_0\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+m^2(\rho-\rho_0)\right] \tag{3.1a}$$

or

$$i\frac{\partial\psi}{\partial(\sqrt{\rho_0}\tau)}=\frac{1}{2m}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{m^2\xi}{\sqrt{\rho_0}}\right)\psi, \tag{3.1b}$$

where $\xi=(\rho-\rho_0)/\sqrt{\rho_0}$ is the proper length coordinate in the ρ direction, and $\sqrt{\rho_0}\,\tau$ is the proper time. I have assumed that ξ is small everywhere within the box.

This equation is just the Schrödinger equation for a particle in a potential $m\xi/2\sqrt{\rho_0}$. As $\sqrt{\rho_0}=1/2a$, with a the acceleration of the box, this is exactly the expected result.

The eigenfunctions of the right-hand side of (3.1b) will be denoted by ψ_j with eigenvalue E_j. The lowest state will have $j=0$. The τ dependence of ψ_j will therefore be $\exp(-iE_j\tau\sqrt{\rho_0})=\exp(-iE_j\tau/2a)$.

One now introduces a coupling to the field Φ by introducing an interaction of the form $\epsilon\Phi\psi$ into the equation.

The field is assumed to be in the ordinary Minkowski vacuum state $|0\rangle_M$. The lowest-order probability that the detector now goes from the state ψ_0 to an excited state ψ_j per unit proper time is[21]

$$P_j=\lim_{T\to\infty}\frac{\epsilon^2}{\sqrt{\rho_0 T}}\times\sum_{|p\rangle}\left|\int_0^T d\tau\int_{\text{box}}\sqrt{-g}\,d^3x\,\psi_j^*\langle p|\Phi|0\rangle_M\psi_0\right|^2,$$

where $\{|p\rangle\}$ is a complete set of states of the free Φ field. The factor $(\rho_0)^{-1/2}$ is introduced to make P_j the transition rate per unit proper time of the detector. Performing the integral over the time first and remembering that ψ_j has a time dependence of the form $e^{-iE_j\tau/2a}$ we obtain

$$P_j=\lim_{T\to\infty}\frac{\epsilon^2}{\sqrt{\rho_0}\,T}\sum_{|p\rangle}\left|\int_{\text{box}}\sqrt{-g}\,d^3x\left[\int_0^T d\tau\, e^{-i(E_j-E_0)\tau/2a}\langle p|\Phi|0\rangle_M\right]h_j^*h_0\right|^2, \tag{3.2}$$

where h_j^*, h_0 are the spatial eigenfunctions of the right-hand side of Eq. (3.1). If we now expand Φ in terms of the Rindler modes of Eqs. (2.8) and (2.12), we obtain

$$P_j=\lim_{T\to\infty}\frac{\epsilon^2}{\sqrt{\rho_0}\,T}\sum_{|p\rangle}\left|\sum_{k;\tilde\omega>0}\left\{\int_0^T d\tau\, e^{i\{[(E_j-E_0)/2a]-\tilde\omega\}\tau}\langle p|_{R^+}a_{\tilde\omega k}|0\rangle_M\int_{\text{box}}\sqrt{-g}\,\frac{d^3x}{[(2\pi)^3\tilde\omega]^{1/2}}h_0h_j^*g_{\tilde\omega k}(\rho)e^{-i(k_1x+k_1y)}\right.\right.$$
$$\left.\left.+\int_0^T d\tau\, e^{i\{[(E_j-E_0)/2a]+\tilde\omega\}\tau}\langle p|_{R^+}a^\dagger_{\tilde\omega k}|0\rangle\int_{\text{box}}\sqrt{-g}\,\frac{d^3x}{[(2\pi)^3\tilde\omega]^{1/2}}h_0h_j^*g^*_{\tilde\omega}(\rho)e^{i(k_1x+k_2y)}\right\}\right|^2. \tag{3.3}$$

Evaluating the time integrals, using the relationship that

$$\lim_{T\to\infty}\frac{4[\sin(\alpha T/2)]^2}{\alpha^2 T}=2\pi\delta(\alpha)$$

and letting $\omega_j=(E_j-E_0)/2a$ one obtains

$$P_j=2\pi\epsilon^2\sum_{|p\rangle}\left|\sum_k\langle p|_{R^+}a_{\omega_j k}|0\rangle_M\times\int_{\text{box}}\sqrt{-g}\,\frac{d^3x}{[(2\pi)^3\omega_j]^{1/2}}h_0h_j^*\times g^*_{\omega_j k}(\rho)e^{-i(k_1x+k_2y)}\right|^2. \tag{3.4}$$

The first factor represents the destruction of one of the Rindler particles which exist in the state $|0\rangle_M$ by the detector in the detection process. The second factor represents the sensitivity of the detector to a FR mode of frequency ω_j. Note that if $(k_1^2+k_2^2)>(E_j-E_0)^2$, $g_{\omega_j k}(\rho)$ falls to zero rapidly, as can be seen from Eq. (2.9).

One can evaluate the term $\langle p|_{R^+}a_{\tilde\omega k}|0\rangle$ by rewriting $_{R^+}a_{\tilde\omega k}$ in terms of the Minkowski creation and annihilation operators $\hat a_{\tilde\omega k}$, $\hat a^\dagger_{\tilde\omega k}$ corresponding to the Minkowski modes $\hat\phi_{\tilde\omega}$ of (2.17):

$$_{R^+}a_{\tilde\omega k}=[2\sinh(2\pi\tilde\omega)]^{-1/2}(e^{\pi\omega}\hat a_{\tilde\omega k}+e^{-\pi\omega}\hat a^\dagger_{\tilde\omega k}). \tag{3.5}$$

As $|0\rangle_M$ is the Minkowski vacuum, we have

$$\langle p|_{R^+}a_{\tilde\omega k}|0\rangle=[2\sinh(2\pi\tilde\omega)]^{-1/2}\langle p|\hat a^\dagger_{\tilde\omega k}|0\rangle e^{-\pi\tilde\omega}$$

or

$$|\langle p|_{R^+}a_{\omega_j k}|0\rangle|^2=\frac{e^{-2\pi\omega_j}}{2\sinh(2\pi\omega_j)}\,\delta(p;\omega_j,k). \tag{3.6}$$

We finally obtain

$$P_j=\frac{2\pi e^{-2\pi\omega_j}}{\sqrt{\rho_0}\,2\sinh(2\pi\omega_j)}\times\sum_k\left|\int_{\text{box}}\sqrt{-g}\,\frac{d^3x}{[(2\pi)^3\omega_j]^{1/2}}h_j^*h_0\times g_{\omega_j k}(\rho)e^{-(k_1x+k_2y)}\right|^2. \tag{3.7}$$

To evaluate this expression one needs an estimate for $g_{\omega_j k}$. Although an exact solution may be found in terms of Hankel functions, it is sufficient (and more transparent) to use a WKB approximation in solving Eq. (2.9) for ρ near ρ_0.

For $\omega_j^2 < (k_1^2+k_2^2)\rho_0$, the solution will be essentially zero owing to the boundary conditions as ρ goes to infinity. For $\omega_j^2 > (k_1^2+k_1^2)\rho_0$, we obtain

$$g_{\omega_j k}(\rho) = [1-(k_1^2+k_2^2)\rho_0/\tilde{\omega}^2]^{-1/4} \times \exp\left(\int [\tilde{\omega}^2 - (k_1^2+k_2^2)\rho]^{1/2} d\rho\right) + \text{complex conjugate}$$
$$= \frac{e^{ik_3\zeta+\delta}}{v_3^{1/2}} + \text{complex conjugate}, \tag{3.8}$$

with

$$k_3 = [\omega_j^2/\rho_0 - (k_1^2+k_2^2)]^{1/2},$$
$$v_3 = k_3\sqrt{\rho_0}/\omega_j.$$

v_3 is the velocity in the ρ direction of the wave as measured by physical coordinates, and δ is a phase depending on ω_j.

Defining the detector cross section for a plane wave with wave numbers k_1, k_2, k_3 in flat spacetime by

$$\sigma_j(k_1,k_2,k_3) = 2\pi \left| \int_{\text{box}} \frac{h_j^* h_0 e^{i(k_1x+k_2y+k_3z)}}{\sqrt{\nu}}\, dx\,dy\,dz \right|^2, \tag{3.9}$$

where $\nu = (k_1^2+k_2^2+k_3^2)^{1/2}$, one finds

$$P_j = \frac{1}{(e^{2\pi(E_j-E_0)/a}-1)} \int \frac{dk_1 dk_2 dk_3\, \delta(\nu-(E_j-E_0))\sigma_j}{(2\pi)^3 v_3}. \tag{3.10}$$

I have assumed a sufficient line width for the detector so that the detector will average over the phase δ of Eq. (3.8). This result is exactly what one would expect of a detector immersed in a thermal bath of scalar photons of temperature $a/2\pi$.

The essential reason for this result is that the detector measures frequencies with respect to its own proper time. For an accelerated observer, this definition of positive frequency is not equivalant to that of a nonaccelerated observer.

In flat spacetime, positive frequency defined with respect to any geodesic detector is equivalent to that of any other geodesic detector. One can therefore demand that any flat spacetime detector be a geodesic detector, and no contradictions will arise. However, the generalization of this to a nonflat spacetime results in the possibility that two equally valid geodesic detectors will disagree on whether there are field quanta present. What to one is a vacuum state, to the other will be a many-particle state, and vice versa. The generalization of the concept of particles in a nonflat spacetime therefore becomes very difficult.

Before going on to this problem, however, let us continue examining the accelerated detector in flat spacetime. We note that the state $\langle p|$ in which the Φ field is left by the detection process is a state with fewer FR particles than the $|0\rangle_M$ state. On the other hand, the $\langle p|$ state is a one-particle state as far as the usual Minkowski observer is concerned. He sees the detector jumping up to its excited state ψ_j by emission of a Φ quantum, not by absorption. What the detector regards as the detection (and thus absorption) of a Φ quantum, the Minkowski observer sees as the emission by the detector of such a quantum. The energy for this emission, as far as the Minkowski observer is concerned, comes from the external field accelerating the detector. However, it is not this external field which couples the ground state ψ_0 to the excited state ψ_j. If $\epsilon=0$, the detector stays forever in the state ψ_0. These results are independent of the means used to accelerate the detector, but depend only on the acceleration itself. It is the field Φ which is producing the excitation of the accelerated detector.

This process is not simply the result of our semiclassical model for the detector. The fully relativistic model for a detector mentioned at the beginning of this section displays the same phenomena.

The detector Ψ, its excited state φ, and the field Φ are assumed to obey equations of motion derived from an action of the form

$$\int d^4x\sqrt{-g}\,[\Psi^*_{,\nu}\Psi^{,\nu} - \mu^2\Psi^*\Psi + \varphi^*_{,\nu}\varphi^{,\nu} - M^2\varphi^*\varphi + \Phi_{,\nu}\Phi^{,\nu} + \epsilon(\Psi^*\varphi+\varphi^*\Psi)\Phi]. \tag{3.11}$$

The mass of the detector, μ, can be taken to be very large (e.g., kilograms for a bubble chamber) and $(M-\mu)$ may be very small. Note that in the absence of the coupling with the Φ field (i.e., $\epsilon=0$) a Ψ particle will never spontaneously become a φ particle

Similarly, in flat spacetime, where Ψ does not interact with any external field (i.e., is unaccelerated) and where Φ is in its vacuum state, no transition from Ψ to φ ever occurs. The presence of a Φ quantum could however produce such a transition which one would regard as the detection of that quantum.

If we now accelerate the detector field Ψ, for example by an external electromagnetic field, one can have a transition to a φ state, even when the Φ field is initially in the vacuum state.

Regarding Ψ, φ, Φ as quantum operators, we have the equation

$$\Psi(x) = \Psi_0(x) + \epsilon\int G_\Psi(x,x')\varphi(x')\Phi(x')d^4x', \tag{3.12}$$

and similar equations for the φ and Φ states.

Here $G_\Psi(x,x')$ is the retarded Green's function for the free Ψ field, and Ψ_0 is the free ($\epsilon=0$) field operator.

The fields $\Psi_0, \varphi_0, \Phi_0$ can be expanded in normal modes of the free field equations as

$$\begin{aligned}\Psi_0 &= \sum_\lambda (a_\lambda \psi_\lambda + b_\lambda^\dagger \psi_\lambda^*),\\ \varphi_0 &= \sum_\kappa (\tilde{a}_\kappa \varphi_\kappa + \tilde{b}_\kappa \varphi_\kappa^*),\\ \Phi_0 &= \sum_\sigma (\hat{a}_\sigma \phi_\sigma + \hat{a}_\sigma^\dagger \phi_\sigma^*).\end{aligned} \tag{3.13}$$

Defining the vacuum state $|0\rangle$ such that

$$0 = a_\lambda|0\rangle = \tilde{a}_\kappa|0\rangle = \hat{a}_\sigma|0\rangle$$

we are interested in the initial state $a_\lambda^\dagger|0\rangle$ (i.e., a single detector in its ground state). The probability that this state will be transformed to the final state

$$\tilde{a}_\kappa^\dagger \hat{a}_\sigma|0\rangle$$

with the detector in its excited state φ_κ will be given by the standard reduction formulas[22]:

$$\begin{aligned}|(\langle 0|\hat{a}_\sigma\tilde{a}_\kappa)_{\text{out}}(a_\lambda^\dagger|0\rangle)_{\text{in}}|^2 &= \left|\int d^4x (\langle 0|\hat{a}_\sigma \tilde{a}_\kappa)_{\text{out}} \Psi(x)|0\rangle(\overleftarrow{\Box} - \mu^2)\psi_\lambda(x)\right|^2\\ &= \epsilon^2\left|\int d^4x (\langle 0|\hat{a}_\sigma\tilde{a}_\kappa)_{\text{out}}\varphi(x)\Phi(x)|0\rangle\psi_\lambda(x)\right|^2\\ &\approx \epsilon^2\left|\int d^4x \langle 0|\hat{a}_\sigma\tilde{a}_\kappa\varphi_0(x)\Phi_0(x)|0\rangle\,\psi_\lambda(x)\right|^2\\ &= \epsilon^2\left|\int d^4x\, \varphi_\kappa^*(x)\phi_\sigma^*(x)\psi_\lambda(x)\right|^2.\end{aligned} \tag{3.14}$$

The third line on the right-hand side comes from keeping only the lowest-order term in ϵ for Φ and φ.

If all three of φ, ϕ, ψ are free field modes, this integral will be zero (the conservation of both energy and momentum prevent the decay of a lower-mass Ψ particle into a higher-mass φ particle plus a Φ quantum).

However, if the modes ψ_λ and φ_κ are modes under the influence of some background field, this integral need no longer be equal to zero. Although a Minkowski observer will say that the accelerated Ψ particle has emitted a Φ quantum and gone to the excited state φ, with the external field supplying the necessary energy and momentum, one could equally interpret this as the detection of a Φ quantum by the accelerated Ψ detector.

A well-known example of such a process is the formation of a neutron by the inverse beta decay of a proton under the influence of strong or electromagnetic interactions. Although we are used to saying that the proton has emitted a positron and a neutrino, one could also say that the accelerated proton has detected one of the many high-energy neutrinos which are present in the Minkowski vacuum in the proton's accelerated frame of reference.

I would now like to return to the question of particle detectors in a curved spacetime. In flat spacetime one could say that the only valid detectors are geodesic detectors, detectors unaccelerated by any external forces. One would now like to apply the same reasoning to a curved spacetime. However, one immediately runs into trouble. As mentioned before, not all geodesics are equivalent in a curved spacetime. The simplest example of this problem is to consider three detectors near a massive body. One detector is fixed at constant radius r, the other two detectors are orbiting the mass in a circular orbit of radius r, but in opposite directions.

Consider the modes of the Φ field. They may be written as

$$\begin{aligned}\phi &= e^{-i\omega t} f(r)\, Y_{lm}(\theta,\phi)\\ &= e^{-i\omega t} e^{-im\phi} f(r) P_{lm}(\theta).\end{aligned} \tag{3.15}$$

For the fixed detector at constant r, a positive frequency mode is one for which $\omega > 0$. For the orbiting detectors, however, their geodesics are defined by $\phi = +\omega_r t$ with $\omega_r \approx (M/r^3)^{1/2}$. Their definition of positive frequency will be that $\omega + m\omega_r > 0$, not $\omega > 0$. Therefore the state of the Φ field which the first regards as the vacuum, neither of the other two detectors will regard as the vacuum. Furthermore, they will never be able to agree with each other as to what they mean by the no-particle state. As $r \to \infty$, the three definitions will become identical, but not for any finite r.

This also throws into confusion the association we make in flat spacetime between the presence of a particle and the carrying of energy and momentum by that particle. Which of the possible definitions of particle in a curved spacetime is the one that corresponds to the real world, and in particular, which are the particles whose stress-energy tensor contributes to the gravitational field? Answering this question is, of course, the key problem in understanding the effects of matter quantization on the gravitational field.

Applying these results on particle detectors to the black-hole evaporation problem, one finds that for a detector stationed near the horizon of the black hole, the transition probability of the detector per unit time can be calculated in a similar way to that for a static detector in Rindler coordinates.

Assume that the detector is placed at a radius $r=R$ very near the horizon of the black hole. The acceleration experienced by the detector is then given by

$$a=M/[R^2(1-2M/R)^{1/2}]$$

Using the technique of Sec. II of replacing the collapse by boundary conditions on the past horizon, and proceeding exactly as for the flat-spacetime accelerated detector, one finally obtains a transition rate per unit proper time:

$$P_j=\frac{2\pi\epsilon^2}{(1-2M/R)^{1/2}(e^{8\pi M\omega_j}-1)} \times\sum_{lm}\left|\int_{\text{box}}\sqrt{-g}\,h_j^* h_0\, {}^-f_{\omega_j l}(r)Y_{lm}(\theta,\phi)\right|^2. \tag{3.16}$$

where

$$\begin{aligned}\omega_j&=(E_j-E_0)(1-2M/R)^{1/2}\\&=M(E_j-E_0)/(aR^2)\\&\approx(E_j-E_0)/(4Ma)\end{aligned}$$

and ${}^-f_{\omega_j l}(r)$ is the outgoing radial function defined in (2.22) and (2.24).

Again we use a WKB approximation to solve Eq. (2.23). Definiting a proper radius coordinate by

$$d\zeta=dr/(1-2M/r)^{1/2} \tag{3.17}$$

such that $\zeta=0$ at the center of the detector, we obtain

$${}^-f_{\omega_j l}(r(\zeta))\approx v_r^{-1/2}\left(e^{i[(E_j-E_0)v_r\zeta+\varphi]} + {}^-A_{\omega_j l}e^{-i[(E_j-E_0)v_r\zeta+\varphi]}\right),$$

$$v_r=\{1-[l(l+1)+1]/[R^2(E_j-E_0)^2]\}^{1/2}.$$

v_r is the proper velocity in the radial direction of the wave, while $A_{\omega_j l}$ is approximately unity for $\omega_j<l(l+1)/9M^2$ and falls rapidly to zero for larger ω_j.

This approximation for ${}^-f_{\omega_j l}$ holds as long as $(E_j-E_0)^2\gtrsim l(l+1)/R^2$. For smaller energies (or large angular momenta), ${}^-f_{\omega_j l}$ is approximately zero.

Furthermore, define the cross section for detection of a wave with energy ν and angular momentum l,m in flat spacetime as

$$\sigma_\pm(\nu,l,m) =2\pi\left|\int_{\text{box}}r^2dr\,d\cos\theta\, d\phi\,\frac{h_j^*h_0e^{\pm i\nu v_r r}Y_{lm}(\theta,\phi)}{(2\pi\nu)^{1/2}v_r}\right|^2,$$

where the $\pm$ refers to outgoing or ingoing waves, respectively. I have used the WKB approximation to the Bessel function with

$$v_r=[1-l(l+1)/r^2\nu^2]^{1/2}.$$

This leads to a black-hole transition amplitude for the detector of

$$\begin{aligned}P_j&=\frac{2\pi}{e^{8\pi M\omega_j}-1}\sum_{lm}\left|\int_{\text{box}}R^2\frac{dr\,d\cos\theta\, d\phi}{(1-2M/R)^{1/2}}\frac{h_j^*h_0}{(2\pi\nu_j)^{1/2}}\left(e^{i(\nu_j v_r\zeta+\varphi)}+A_{\omega_j l}e^{-i(\nu_j v_r\zeta+\varphi)}\right)\right|^2\\&\approx\frac{1}{(e^{2\pi\nu_j/a}-1)}\sum_{lm}\left[\sigma_+(\nu_j,l,m)+|A_{\omega_j l}|^2\sigma_-(\nu_j,l,m)\right].\end{aligned}$$

with

$$\begin{aligned}\nu_j&=E_j-E_0\\&=4Ma\omega_j\end{aligned}$$

and where I have again assumed the line width of the detector is broad enough to average over the phase φ.

This expression is again the detector transition rate for a detector immersed in a thermal scalar photon bath of temperature $a/2\pi$ in flat spacetime. Note that the temperature and number of detectable particles diverge as $R\to 2M$ in precisely the same way as for an accelerating detector in flat spacetime. In both cases the temperature diverges as $a/2\pi$.

The number of particles seen by a freely falling detector is much more difficult to calculate. If the detector is far from the horizon, its motion toward the black hole would be expected to lead to particle detection. Furthermore, near the horizon, the detector has a lifetime of only of the order of M and cannot, therefore, measure particles less energetic than that. However, the particles seen by the static detector have energies of the order of $a\simeq(R-2M)^{-1}$. Furthermore, if they were real particles, the Doppler shift from the static frame to the freely falling frame would increase this energy for those particles which seem to be coming out of the black hole from the viewpoint of the static observer. If they were real, these particles should present no difficulty in observation to the freely falling observer.

Near the horizon, the Schwarzschild metric can be approximated by

$$ds^2 = \frac{r-2m}{2M}dt^2 - \frac{2M}{r-2M}dr^2 - (2M)^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (3.18)$$

which can be transformed into the cylindrical metric

$$ds^2 = d\tau^2 - d\rho^2 - (2M)^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (3.19)$$

The geodesics near the horizon are essentially the geodesics of this cylindrical metric, and the positive frequencies seen by the geodesic detectors are essentially the positive frequencies with respect to τ of this metric.

We can attach this metric to the inside of the black hole at the horizon, in order to eliminate the singularities within the black hole. As the Hawking process is insensitive to what goes on within the black hole, this procedure should not alter the Hawking process. The main purpose in doing so is to ease the analysis of the response of a geodesic detector near the black hole.

On the past horizon of the black hole we have already decided that the ξ definition of positive frequency is the most appropriate. Similarly, near the future horizon, the definition using the vector field $\partial/\partial V = \xi'$, which is of Killing type, on the future horizon will correspond to positive frequency in the sense naturally associated with the cylindrical metric (3.19) and thus also with a geodesic particle detector near the horizon.

The definition of positive frequency for states coming out of the past horizon makes them also of positive frequency in the metric of (3.19), and a geodesic detector will therefore see no particles coming out of the black hole. However, when these states reflect off the curvature of the Schwarzschild exterior the positive-frequency states will pick up negative-frequency components in the sense of this cylindrical metric (the ξ' sense) and, similarly, the states coming in from infinity which are positive frequency at infinity will have negative-frequency components in the ξ' sense. This implies that a geodesic detector near the horizon will see particles flowing into the black hole (i.e., will be excited by states which are coming from the exterior region). The geodesic detector therefore sees no outflow of particles, but does see an influx of particles into the black hole.

The key question is: Which of these definitions, if any, corresponds to what the gravitational field sees as particles, i.e., as carriers of energy and momentum? If one accepts the "geodesic" particles, both near the horizon and at infinity, as the true particles, one obtains the paradoxical situation that particles both flow into the horizon and out through infinity—i.e., the "black hole" increases in area, and loses mass to infinity. The only place where the "energy" for such a process could originate would be in the vacuum polarization of the field in the region outside the black hole.

In this context it would be of interest to push the fermion-boson cancellation scheme of Sec. I, or some other renormalization technique for the energy-momentum tensor, to four dimensions to examine the behavior of the energy-momentum tensor near the black hole.

IV. SPHERICALLY SYMMETRIC SCALAR MINISUPERSPACE

As a beginning of another approach to this problem, the Hamiltonian for a spherically symmetric scalar field coupled to the gravitational field is presented. This work is basically a correction of that done by Berger, Chitre, Nutku, and Moncrief.

Berger *et al.* (hereafter referred to as BCNM) used the ADM (Arnowitt-Deser-Misner) formalism to derive a Hamiltonian for the coupled scalar gravitational fields under the restriction of spherical symmetry.

Using spatial coordinates adapted to the spherical symmetry (i.e., the usual θ, ϕ coordinates on a sphere), they chose the radial coordinate r such that the area of the sphere of constant t, r has area $4\pi r^2$. Under these restrictions, the ADM action[23]

$$I = \int \left(\pi^{ij}\frac{\partial g_{ij}}{\partial t} - N_\alpha H^\alpha + \pi_\Phi \frac{\partial \Phi}{\partial t}\right) dt\, dr\, d\theta\, d\phi \quad (4.1)$$

becomes

$$I = 4\pi \int (\pi_\mu \dot{\mu} + \pi_\Phi \dot{\Phi} - N_0 H^0)\, dt\, dr. \quad (4.2)$$

Here, Φ is the scalar field and π_Φ its conjugate momentum; μ is defined by requiring g_{ij} to be diagonal and to be given by

$$g_{ij} = \mathrm{diag}(e^{2\mu}, r^2, r^2\sin^2\theta). \quad (4.3)$$

π_μ is the conjugate momentum to μ giving a diagonal momentum tensor

$$\pi^{ij} = \mathrm{diag}\left(\frac{\pi_\mu}{2}e^{-2\mu}, \frac{\pi_\lambda}{4r^2}, \frac{\pi_\lambda}{4r^2\sin^2\theta}\right), \quad (4.4)$$
$$\pi_\lambda = r[e^{\mu}(e^{-\mu}\pi_\mu)' + \pi_\Phi \Phi'].$$

The function H^0 is given by

$$H^0 = \frac{e^{-\mu}}{r^2}\left[\frac{\pi_\mu^2}{8} - \frac{r\pi_\mu}{4}(\pi_\mu' - \mu'\pi_\mu - \pi_\Phi\Phi') + 2r^2(1 - 2r\mu' - e^{2\mu}) + \frac{\pi_\Phi^2}{4} + r^4\Phi'^2\right]. \quad (4.5)$$

The overdot represents $\partial/\partial t$ and the prime represents $\partial/\partial r$. The shift vector is given by

$$\begin{aligned} N_i &= \left(\frac{\pi_\mu}{4r}e^{\mu}N_0, 0, 0\right) \\ &= (N_r, N_\theta, N_\phi). \end{aligned} \tag{4.6}$$

The equations which result from variation of the action, Eq. (4.2), with respect to $N_0, \pi_\mu, \mu, \pi_\Phi, \Phi$ respectively are

$$H^0 = 0, \tag{4.7a}$$

$$4r^2 e^{2\mu}\dot{\mu} = r(N_0 e^{\mu})'\pi_\mu + N_0 e^{\mu}\Phi'\pi_\Phi, \tag{4.7b}$$

$$\dot{\pi}_\mu = 4N_0 e^{\mu} - 4(N_0 r e^{-\mu})' + \frac{1}{4}\left(\frac{1}{r}N_0 e^{-\mu}\pi_\mu{}^2\right)', \tag{4.7c}$$

$$2r^2 e^{\mu}\dot{\Phi} = N_0\pi_\Phi + \tfrac{1}{2}rN_0\Phi'\pi_\mu, \tag{4.7d}$$

$$\dot{\pi}_\Phi = \left(\frac{1}{4r}N_0 e^{-\mu}\pi_\Phi\pi_\mu + 2r^2 N_0 e^{-\mu}\Phi'\right)'. \tag{4.7e}$$

One still has one final coordinate condition to choose. We choose the parameter t such that $\pi_\mu = 0$ at all times. In particular, if $\tilde{t}$ is a coordinate such that $\tilde{\pi}_\mu \neq 0$, one can make a coordinate change such that $\pi_\mu = 0$. From Eq. (4.6) we see that $\pi_\mu = 0$ is equivalent to choosing t such that $N_1 = 0$, i.e., the metric in the t coordinates is

$$N_0{}^2 dt^2 - (g_{ij}dx^i dx^j), \tag{4.8}$$

the relation between t and $\tilde{t}$ will be given by

$$t = f(\tilde{t}, r), \tag{4.9}$$

and the metric in $\tilde{t}$ coordinates is

$$N_0{}^2\dot{f}^2 d\tilde{t}^2 + 2N_0\dot{f}f' d\tilde{t}\,dr + N_0{}^2 f'^2 dr^2 - (g_{ij}dx^i dx^j). \tag{4.10}$$

But in the ADM formalism, the metric is

$$(\tilde{N}_0{}^2 - g^{rr}\tilde{N}_1{}^2)d\tilde{t}^2 - \tilde{N}_1 d\tilde{t}\,dr - N_1{}^2 dr^2 - g_{ij}\,dx^i dx^j. \tag{4.11}$$

Therefore

$$\frac{-\tilde{N}_1}{\tilde{N}_0{}^2 - g_{rr}\tilde{N}_1{}^2} = \frac{f'}{\dot{f}}. \tag{4.12}$$

With (4.6), (4.3) this gives the first-order partial-differential equation for f:

$$\frac{\partial f}{\partial \tilde{t}} + \frac{\tilde{\pi}_\mu}{4re^{-\mu}\tilde{N}_0(1 - \tilde{\pi}_\mu{}^2/16r^2)}\frac{\partial f}{\partial r} = 0. \tag{4.13}$$

From (4.7c) this becomes

$$\frac{\partial f}{\partial \tilde{t}} + \frac{\tilde{\pi}_\mu}{\int dr(4\tilde{N}_0 e^{\mu} - \dot{\tilde{\pi}})}\frac{\partial f}{\partial r} = 0. \tag{4.14}$$

The surfaces of constant f (i.e., constant t) in $\tilde{t}, r$ space are given by the characteristic equation

$$\frac{dr}{d\tilde{t}} = \frac{4rN_0 e^{-\mu}}{\pi_\mu}(1 - \tilde{\pi}_\mu{}^2/16r^2). \tag{4.15}$$

Where $1 - \tilde{\pi}_\mu{}^2/16r^2$ is zero, $\tilde{t}$ is a null coordinate, and that surface is a null surface.

Equation (4.13), together with the boundary condition that

$$f(\tilde{t}, r) \sim \tilde{t} \text{ as } r \to \infty,$$

will completely specify f. (Note that the only condition on $\tilde{\pi}_\mu$, N_0, and e^{μ} necessary to satisfy the boundary condition is that $\tilde{\pi}_\mu \to 0$ as $r \to \infty$, and that $N_0 e^{-\mu}$ approach a constant.)

These equations are different from those stated by BCNM and those found by Kuchař for cylindrical waves.[24] The presence of the factor $N_0 e^{\mu}$ in (4.7c) prevents us from using the Kuchař transformation here.

Let us assume the spacetime is such that the transformation $\pi_\mu = 0$ everywhere is possible. Choosing N_0 so that $N_0 = 1$ at $r = \infty$ and such that Eq. (4.7c) is satisfied, one finds

$$N_0 = \frac{\exp[+\mu + \int_\infty^r (e^{2\mu}/\bar{r})d\bar{r}]}{r}. \tag{4.16}$$

Furthermore, the condition $\pi_\mu = 0$ allows Eq. (4.7a) to be solved for μ, giving

$$\begin{aligned} e^{-2\mu} &= \frac{\exp(-\int_\infty^r S_\Phi d\bar{r})}{r}\int_0^r d\bar{r}\exp\left(\int_0^{\bar{r}} S_\Phi d\tilde{r}\right), \\ S_\Phi &= \frac{\pi_\Phi{}^2}{8r^3} + r\frac{\Phi'^2}{2} \end{aligned} \tag{4.17}$$

Using (4.7a) and (4.8) we obtain

$$rN_0 e^{-\mu} = re^{-2\mu}\exp\left(\int_\infty^r S_\Phi d\bar{r}\right). \tag{4.18}$$

Equation (4.7b) is automatically satisfied because of the Bianchi identities. With these solutions for N, μ, Eqs. (4.7d) and (4.7e) may both be derived from the Hamiltonian

$$\mathcal{H} = -2\int_0^\infty dr\left[\exp\left(\int_\infty^r S_\Phi(\bar{r})d\bar{r}\right) - 1\right]. \tag{4.19}$$

This Hamiltonian differs considerably from the BCNM result. An examination of their result, however, shows that their reduced Hamiltonian does not reproduce Eqs. (4.7).

The reason for this difference is instructive. After solving (4.7a) for μ, a variation of π_Φ and Φ will lead to a nonzero variation of μ on the boundary of the region of integration. This leads to nonzero boundary terms when the action is varied, with respect to π_Φ or Φ. Compensating for these boundary terms leads to the Hamiltonian as given by Eq. (4.19) for the reduced problem.

Solve Eq. (4.7a) for μ as a function of π_μ, π_Φ, and Φ, and choose π_μ to be zero. (I assume this can be done by an appropriate choice of N_0.) The action, Eq. (4.2), becomes

$$\int \left[\pi_\Phi \dot{\Phi} - N_0 H^0(\mu, \pi_\Phi, \Phi) \right] dt\, dr, \tag{4.20}$$

with $\mu = \mu(\pi_\Phi, \Phi)$ such that $H^0 \equiv 0$. The variation of this action with respect to π_Φ, for example, leads to

$$\int (\dot{\Phi}\delta\pi_\Phi - N_0 \delta H^0) dt\, dr. \tag{4.21}$$

But

$$\delta H^0 = \frac{\delta' H^0}{\delta' \mu} \delta\mu + \frac{\delta' H^0}{\delta' \pi_\Phi} \delta\pi_\Phi, \tag{4.22}$$

where the prime denotes the variation with μ regarded as an independent function, and $\delta\mu$ is the variation in μ caused by a variation of π_Φ. Expression (4.21) becomes

$$\int \left[\left(\dot{\Phi} - N_0 \frac{\delta' H^0}{\delta' \pi_\Phi} \right) \delta\pi_\Phi - N_0 \frac{\delta' H^0}{\delta' \mu} \delta\mu \right] dt\, dr. \tag{4.23}$$

The first term is exactly the expression one obtains by varying the original action (4.2) with respect to π_Φ. The latter term is essentially the term one obtains by varying the original action with respect to μ except that now it does not vanish on the boundary. The nonboundary terms of $N_0 \delta' H^0 / \delta' \mu$ are equal to $-\dot{\pi}_\mu$ and thus are zero. The boundary terms, however, are given by

$$4 \int_{r \text{ on boundary}} (r N_0 e^{-\mu} \delta\mu) dt, \tag{4.24}$$

which is not equal to zero. Furthermore, as $r \to \infty$, Eqs. (4.16)–(4.18) imply that

$$4 r N_0 e^{-\mu} \delta\mu \to + \delta\mathcal{H}, \tag{4.25}$$

where $\mathcal{H}$ is given by (4.19).

In order that the action (4.2) give the same equations after solving and substituting for μ, one must add a term $\int \mathcal{H} dt$ to the action to cancel out the boundary terms produced by the boundary variation of μ. This leads to (4.19) as the Hamiltonian density.

The importance of such boundary terms has also been analyzed by Regge and Teitelboim.[25]

The Hamiltonian (4.19) may be rewritten by an integration by parts as

$$\mathcal{H} = \int_0^\infty \left(\frac{\pi_\Phi^2}{4r^2} + r^2 \Phi'^2 \right) \exp\left(\int_\infty^r S_\Phi(\bar{r}) d\bar{r} \right) dr. \tag{4.26}$$

The first factor in the integrand is just the flat-spacetime Hamiltonian density for the scalar field. Furthermore, one finds that $\mathcal{H}$ is just equal to the gravitational mass as measured at infinity; i.e., near infinity $g_{00} \sim 1 - 2\mathcal{H}/r$.[26]

In quantizing the Hamiltonian (4.19), the fields π_Φ, Φ are interpreted as conjugate fields obeying the usual commutation relations

$$\delta(t, t')[\pi_\Phi(t, r), \Phi(\bar{t}, \bar{r})] = i\delta(r, \bar{r})\delta(t, \bar{t}). \tag{4.27}$$

Finding solutions for this problem is, however, extremely difficult. For the unquantized fields, it is obvious that $\Phi = 0$, $\pi_\Phi = 0$ represents a minimum for $\mathcal{H}$, corresponding to flat spacetime. No proof has been found to demonstrate that flat spacetime ($\langle\mathcal{H}\rangle = 0$) represents the quantum-mechanical minimum expectation value, however. If flat spacetime were a solution to this quantum problem, one could, for example, form a quantum wave packet at infinity representing a collapsing shell of scalar "photons." One could then calculate whether there was any possibility of forming a black hole, or whether any structure formed by such a collapse always eventually evaporated. Classically, Christodoulou[27] has shown that a collapsing scalar "photon" cloud can form a black hole. The absence of such a possibility in the quantum regime would indicate that the Hawking process is a realistic possibility.

Unfortunately, no progress has as yet been made on this approach and I present it here in the hope that someone else may be able to do something with it.

ACKNOWLEDGMENTS

There are many people I have greatly benefitted from in this work. In particular, I would like to thank S. Hawking, G. Gibbons, and B. DeWitt for helpful conversations, the relativity group at Princeton for inviting me to give a talk which was the impetus for my examination of accelerated detectors, C. Misner for pointing out the BCNM paper to me, and V. Moncrief for helping me to understand the ADM formalism. I would also like to thank the Miller Foundation of the University of California for generous support while part of this work was being done and the Department of Physics, Princeton University, where some of the preparation of the manuscript was done.

APPENDIX

To demonstrate the validity of the approximation (1.53) to Eq. (1.52), let us examine the expression

$$\beta(\omega,\omega')=\frac{1}{2\pi}\int_{-\infty}^{0}dv\left|\frac{\omega'}{\omega}\right|^{1/2}\exp[i\omega(v-4M\ln|v|)]\times e^{i\omega' v}, \tag{A1}$$

which is an intermediate between (1.52) and (1.53). (One could presumably specify a collapse for which this would be the exact expression.) For simplicity v_0 has been taken to be zero.

This leads to

$$\beta(\omega,\omega')=\frac{1}{2\pi}\left|\frac{\omega'}{\omega}\right|^{1/2}\Gamma(1-i4M\omega)\times[i(\omega+\omega')]^{4i\omega M-1}. \tag{A2}$$

Define

$$\begin{aligned}K(\omega,\tilde\omega)&=\int_{\omega'>0}d\omega'\beta(\omega,\omega')^*\beta(\tilde\omega,\omega')\\&=i^{-4i(\omega+\tilde\omega)M}\Gamma(1+i4M\omega)\Gamma(1-i4M\tilde\omega)\int_0^\infty d\omega'\omega'(\omega'+\omega)^{-1-4i\omega M}(\omega'+\tilde\omega)^{-1+4i\tilde\omega M}.\end{aligned} \tag{A3}$$

Defining $\Delta=(\omega-\tilde\omega)$, the integral may be written as

$$\begin{aligned}\frac{\omega}{\Delta}\int_0^\infty d\omega'\left[\frac{(\omega'+\omega)^{-4i\omega M}(\omega'+\omega-\Delta)^{4i(\omega-\Delta)M}}{\omega'+\omega}\right]&=\frac{\omega^{-4i\Delta M+1}}{\Delta}\int_1^\infty\frac{dx}{x}\left[x^{-4i\Delta M}\left(1-\frac{\Delta}{\omega x}\right)^{4i(\omega-\Delta)M}\right]\\&\quad-\frac{\tilde\omega^{-4i\Delta M+1}}{\Delta}\int_1^\infty\frac{dy}{y}\left[y^{-4i\Delta M}\left(1+\frac{\Delta}{\tilde\omega y}\right)^{4i(\tilde\omega+\Delta)M}\right]\\&\simeq\frac{i}{4M\Delta-i0^+}+O(1),\end{aligned} \tag{A4}$$

where $O(1)$ means of order unity in Δ near $\Delta=0$.

As the expression $K(\omega,\tilde\omega)$ enters into the calculation of the energy flow through infinity by being multiplied by $e^{-i(\omega-\tilde\omega)u}=e^{-i\Delta u}$ and integrated over $\omega,\tilde\omega$ [or equivalently over Δ and $\nu=\frac{1}{2}(\omega+\tilde\omega)$], and since we are interested only in the behavior for large u, only the behavior of $K(\omega,\tilde\omega)$ near $\Delta=0$ will contribute. The terms of $O(1)$ in Δ will die off at least as fast as $1/u$. The only term which contributes to the steady state energy flow through infinity is therefore the term equal to $i/(4m\Delta-i0^+)$.

For large positive u, $e^{-i\Delta u}$ is zero for contours going through $\Delta=i\infty$. Such contours will include the pole at $\Delta=i0^+$.

Therefore we obtain

$$K(\omega,\omega')\approx\frac{e^{-4\pi M\omega}}{2\sinh(4\pi M\omega)}\delta(\omega-\tilde\omega)$$

as the only term which contributes to the steady state energy flow at infinity. This expression is the one used in the paper [Eq. (1.54)].

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*Research supported in part by the Miller Institute for Basic Research, University of California, Berkeley, California, by the National Research Council of Canada, and by McMaster University, Hamilton, Ontario, Canada.

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[11]The following argument is essentially the one given originally by Hawking (see Ref. 1).

[12]This technique was used first in Ref. 4 in a different context and was applied to this problem by Hawking (Ref. 2) and DeWitt (Ref. 6).

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$$\frac{D^{1/2}}{s^2}\, e^{i(\sigma/2s)}$$

is a representation for $\delta^4(x,x')$ in the limit as $s\to 0$. It appears that this is either not a representation except in flat spacetime, or that it is the wrong representation (i.e., does not obey the right boundary conditions).