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Notes on Fourier Analysis (XXIX): An Extrapolation Theorem

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1. Introduction. Let $f(x)$ be a real measurable function defined in the interval $(0,2\pi)$ we write $f(x)\in L^{p}(p>0)$ when $|f(x)|^{p}$ is integrable in $(0,2\pi)$, and $f(x)\in L^{*k}(a>0)$ when $|f(x)|\log^{k}(1+f^{2}(x))$ is integrable in $(0,2\pi)$. L^{*1} is the function class which was introduced by A. Zygmund [1].

In the theory of Fourier series, the transformations of the following type play an important rôle: that is, $T[f(x)]=g(x)$ transforms every integrable function $f(x)$ to another $g(x)$, both being defined in $(0,2\pi)$, such that the inequalities

$$\{\int_{0}^{2\pi}T[f(x)]|^{p}\,dx\}^{1/p}\leqq A_{p}\{\int_{0}^{2\pi}|f(x)|^{p}\,dx\}^{2/p}\qquad(p>1), \tag{1.1}$$

and

$$\int_{0}^{2\pi}|T[f(x)]|\,dx\leqq A_{k}|\int_{0}^{2\pi}f(x)|\,\log^{k}(1+f^{2}(x))\,dx+B_{k} \tag{1.2}$$

hold where A_{p}, A_{k}, B_{k} are constants depending only on p, k and k, respectively.

The inequalities (1.1) and (1.2) are usually proved independently. We shall now give a general principle to deduce the inequality of the type (12) from that of the type (1.1). That is,

Theorem. *Let T be a transformation which transforms every integrable function to a measurable function, both being defined in a finite interval (a,b), such that* (i)

$$f(x)=\sum_{\nu=0}^{\infty}f_{\nu}(x)\ \text{implies}\ |T[f]|\leqq\sum_{\nu=0}^{\infty}|T[f_{\nu}]| \tag{1.3}$$

and

$$|T[f]=|T[-f]|, \tag{1.4}$$

(ii) *the inequality*

$$\{\int_{a}^{b}|T[f]|^{p}\,dx\}^{1/p}\leqq A_{p}\{\int_{a}^{b}|f(x)|^{p}\,dx\}^{1/p} \tag{1.5}$$

holds with the constant A_{p} satifying the inequality

$$A_{p}\leqq A/(p-1)^{k} \tag{1.6}$$

for all p, $1<p\leqq 2$, for some $k>0$, and for a constant A depending only on *the length of the interval* (a, b). *Tnen we have*

$$\int_a^b |T[f]|dx \leqq A_k \int_a^b |f(x)| \log^k (1+f^2(x))\, dx + B_k, \tag{1.7}$$

where A_k *and* B_k *are constants depending only on* k *and* $b-a$.

In § 2 we shall prove this theorem and in the remaining part we shall show that, by means of this key theorem, almost all inequalities of the form (1.2) already known can be deduced from the inequalies of the form (1.1).

2. Proof of theorem. The idea of the proof is essentially due to E. C. Titchmarsh [2] and J. Marcinkiewicz [3].

It is sufficient to prove (1.7) for $f(x) \geqq 1$ in (a, b) for the general case, if we put

$$\begin{aligned} f(x) &= f_1(x) + f_2(x) \\ &= (f_1(x)+1) - (-f_2(x)+1) \\ &= g_1(x) - g_2(x), \end{aligned}$$

where $f_1(x) \geqq 0$, $f_2(x) < 0$, then $g_1(x) \geqq 1$, $g_2(x) \geqq 1$. If inequality (1.7) is proved for g_1 and g_2, it holds also for $f(x)$ by (1.3) and (1.4).

Let us define $f_\nu(x)$ by

$$\begin{aligned} f_\nu(x) &= f(x), \qquad && (x \in E_\nu), \\ &= 0 && (x \bar{\in} E_\nu), \end{aligned}$$

where $E_\nu = \{x\,;\, 2^\nu \leqq f(x) < 2^{\nu+1}\}$[1], $\nu = 0,1,2,\cdots$ Then

$$f(x) = \sum_{\nu=0}^{\infty} f_\nu(x) = \sum_{\nu=0}^{\infty} 2^\nu \varphi_\nu(x),$$

where $\varphi_\nu(x) = f_\nu(x)/2^\nu$. By (1.3),

$$|T[f]| \leqq \sum_{\nu=0}^{\infty} |T[f_\nu]| \sum_{\nu=0}^{\infty} 2^\nu |T[\varphi_\nu]|,$$

Integrating both sides over (a, b), we have

$$\begin{aligned} \int_a^b |T[f]|dx &\leqq \sum_{\nu=0}^{\infty} 2^\nu \int_a^b |T[\varphi_\nu]|\, dx \\ &\leqq \sum_{\nu=3}^{\infty} 2^\nu \{\int_a^b |T[\varphi_\nu]|^{1/p_\nu} dx\}^{1/p_\nu} \{\int_a^b dx\}^{1/q_\nu}, \end{aligned}$$

where $2 \geqq p_\nu > 1$, $\dfrac{1}{p_\nu} + \dfrac{1}{q_\nu} = 1$. Applying (1.4) for every $\varphi_\nu(x)$, we have

1) We suppose that all sets which we consider in this section are contained in the interval (a, b).

$$\int_a^b | T[f]| \, dx \leq A' \sum_{\nu=0}^{\infty} 2^\nu A_{p_\nu} \{\int_a^b \varphi_\nu^{p_\nu}(dx)\}^{1/p_\nu}$$

$$\int_a^b |T[f]| \, dx \leq A'\sum_{\nu=0}^{\infty} 2^\nu A_{p_\nu} \{\int_a^b \varphi_\nu^{p_\nu}(x) \, dx\}^{1/p_\nu}$$

$$\leq A''\sum_{\nu=0}^{\infty} 2^\nu \frac{1}{(p_\nu-1)^k} \{\int_a^b \varphi_\nu(x) dx\}^{1/p_\nu},$$

A' and A'' depending only on $b-a$. If we put $p_\nu = 1+1/\nu$, then

$$\int_a^b | T[f]| dx \leq A''' \sum_{\nu=0}^{\infty} 2^\nu \nu^k \{\int_a^b \varphi_\nu(x) dx\}^{\frac{\nu}{\nu+1}}$$

$$\leq A'''' \sum_{\nu=0}^{\infty} 2^\nu \nu^k \int_a^b \varphi_\nu(x) \, dx + B_k^{[2)]}$$

$$\leq A_k \int_b^b |f(x)| \log^k f^2(x) \, dx + B_k,$$

where A_k and B_k depend only on k and $b-a$. This proves the theorem.

Remark. In the formulation of our theorem, we have supposed that $f(x)$ is defined in a finite interval (a, b) and that $T[f]$ is a function on the same interval, but it is obvious from the above proof that our theorem remains true even when T *transforms a real integrable function defined on an abstract measure-space* Ω *with a finite total measure into a real integrable function in another space* Ω' *with the same property as* Ω.

3. Applications. We shall give some applications of our theorem.

3.1. **Conjugate functions.** Let $f(x)$ be an integrable function in $(0,2\pi)$ with period 2π and its conjugate function be

$$\bar{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2tg\,\frac{1}{2}\,t}\, dt. \tag{3.1.1}$$

As is well known, *the inequalities*

$$\int_0^{2\pi} |\bar{f}(x)|^p \, dx \leq A_p^p \int_0^{2\pi} |f(x)|^p \, dx \quad (p>1), \tag{3.1.2}$$

and

$$\int_0^{2\pi} |\bar{f}(x)| \, dx \leq A \int_0^{2\pi} |f(x)| \log (1+f^2(x)) \, dx + B, \tag{3.1.3}$$

hold where A_p *depends only on* p *and* A,B *are absolute constants.*

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2) From Young's inequality, we can easily see $a^{\nu/(\nu+1)} < 2a + \frac{1}{\nu^2}$ for $a > 0$ and $\nu \geq 1$, so it follows that $2^\nu \nu^k \{\int \varphi_\nu \, dx\}^{\frac{\nu}{\nu+1}} \leq A 2^\nu \nu^k \int \varphi_\nu \, dx + 1/\nu^2$.

It is also known that A_p satisfies (1.6) with $k=1$.

We can deduce (3.1.3) from (3.1.2). Since we cannot apply our extrapolation theorem directly, it requires some modifications. Let us define $\bar{f}_\varepsilon(x)$ by

$$\bar{f}_\varepsilon(x) = -\frac{1}{\pi}\int_\varepsilon^\pi \frac{f(x+t)-f(x-t)}{2tg\,\frac{1}{2}t}\,dt \quad (\pi \geqq \varepsilon > 0). \tag{3.1.4}$$

Then it is well known (cf. M. Riesz [3]) that

$$\int_0^{2\pi} |\bar{f}_\varepsilon(x)|^p\,dx \leqq A_p^p \int_0^{2\pi} |f(x)|^p dx \quad (p>1), \tag{3.1.5}$$

with A_p in (3.1.2). If we define the transformation by $T[f]=\bar{f}_\varepsilon$ and apply our theorem to (3.1.5), we have

$$\int_0^{2\pi} |\bar{f}_\varepsilon(x)|\,dx \leqq A \int_0^{2\pi} |f(x)| \log (1+f^2(x))\,dx + B,$$

with absolute constants A and B. Making $\varepsilon \to 0$, we have

$$\int_0^{2\pi} |\bar{f}(x)|\,dx \leqq A\int_0^{2\pi} |f(x)| \log (1+f^2(x))dx+B.$$

This method proving (3.1.3) is quite analogous to that of Titchimarsh [2]

Moreover, if we put

$$\tilde{f}(x) = \sup_{0<\varepsilon\leqq\pi} |\bar{f}_\varepsilon(x)|,$$

$$\int_0^{2\pi} [\tilde{f}(x)]^p\,dx \leqq A_p^p \int_0^{2\pi} |f(x)|^p\,dx \quad (p>1), \tag{3.1.6}$$

$$\int_0^{2\pi} \tilde{f}(x)\,dx \leqq A \int_0^{2\pi} |f(x)| \log (1+f^2(x))\,dx+B. \tag{3.1.7}$$

The inequality (3.1.7) can similarly be deduced from the inequality (3.1.6).

3.2. **Maximal theorems of Hardy-Littlewood** [4]. For an integrable function in an interval (a,b), let us put

$$f^*(x) = \sup_\xi \frac{1}{x-\xi}\int_\xi^x |f(t)|\,dt \quad (a \leqq \xi \leqq b), \tag{3.2.1}$$

then the following inequalities hold:

$$\int_a^b [f^*(x)]^p\,dx \leqq 2\Big(\frac{p}{p-1}\Big)^p \int_a^b |f(x)|^p\,dx \quad (p>1), \tag{3.2.2}$$

$$\int_a^b f^*(x)\,dx \leqq A\int_a^b |f(x)| \log (1+f^2(x))dx+B, \tag{3.2.3}$$

where A and B depend only on $b-a$.
In this case if we put $T[f]=f^*(x)$, T satisfies the conditions (1.3) and (1.4). Hence we can deduce the inequality (3.2.3) from (3.2.2) by our theorem.

2.3. Inequalities of Marcinkiewicz for the double Fourier series. J. Marcinkiewicz [5] has proved the following theorems: *Let $f(x,y)$ be an integrable function with period 2π for each variables, and the (m,n)-th partial sum of its Fourier series be $s_{m,n}(f,x,y)=s_{m,n}(x,y)$. Let us define*

$$(2.3.1) \qquad H_n^2(x,y)=(n+1)^{-1}\sum_{m=0}^{n}\{s_{mm}-f\}^2,$$

$$H(x,y)=\sup_n H_n(x,y).$$

Then

$$(2.3.2) \qquad \int_0^{2\pi}\int_0^{2\pi} H^p(x,y)\, dx\, dy \leqq A_p^p \int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^p\, dx\, dy, \quad (p>1)$$

$$(2.3.3) \qquad \int_0^{2\pi}\int_0^{2\pi} H^{1-\varepsilon}(x,y)\, dx\, dy \leqq A_\varepsilon \int_0^{2\pi}\int_0^{2\pi} |f(x)| \log (1+f^2(x))\, dx+B, \quad (0<\varepsilon<1),$$

where A_p depends only on p and A_ε, B_ε depend only on ε. Moreover A_p satisfies the inequality (1.6) with $k=2$.

If we put $T[f]=H(x,y)$, then we can deduce from (2.3.2) the following inequality:

$$(2.3.4) \qquad \int_0^{2\pi}\int_0^{2\pi} H(x,y)\, dx\, dy \leqq A \int_0^{2\pi}\int_0^{2\pi} |f(x,y))| \log^2 (1+f^2(x))\, dx+B,$$ where A and B are absolute constants.

2.4. Differentiation of multiple integrals. B. Jessen, J. Marcinkiewicz and A. Zygmund [6] proved the following theorems: *Let $f(x_1,\cdots,x_k)$ be an integrable function on the rectangle $(0,1\,;\,0,1\,;\cdots;\,0,1)$ in the k-dimensional Euclidean space, and let us put*

$$(2.4.1) \qquad f^*(x_1,\cdots x_k)=\sup_I \frac{1}{|I|}\int_I |f(x_1,\cdots x_k)|\, dx_1\cdots dx_k,$$

where I is an interval in $(0,1\,;\cdots;\,0,1)$ containing the point $(x_1,\cdots x_k)$, then we have the following inequalities;

$$(2.4.2) \qquad \int_0^1\int_0^1\cdots\int_0^1 f^{*p}(x_1,x_2,\cdots,x)\, dx_1\, dx_2\cdots dx_k$$

$$\leqq A_{p,k}^p \int_0^1 \int_0^1 \cdots \int_0^1 |f(x_1, x_2, \cdots, x_k)|^p \, dx_1 \, dx_2 \cdots dx, \quad (p>1),$$

(2.4.3) $$\int_0^1 \int_0^1 \cdots \int_0^1 f^*(x_1, x_2, \cdots, x) \, dx_1 \, dx_2 \cdots dx_k$$

$$\leqq A_k \int_0^1 \int_0^1 \cdots \int_0^1 |f(x_1, x_2, \cdots, x_k)| \log^k (1+f^2(x_1, x_2, \cdots, x)) \, dx_1 \, dx_2 \cdots dx_k + B_k,$$

where $A_{p,k}$ satisfies the inequality of A_p in (1.6), *and A_k, B_k depend only on k.*

The inequality (2.4.3) can be deduced from (2.4.2) by our theorem.

2.5. For an integrable function $f(x)$, defined on $(0,2\pi)$ and periodic with period 2π, let us define $\mu(x)$ by

(2.5.1) $$\mu(x) = \mu(x; f) = \left\{ \int_0^{2\pi} \frac{[F(x+t)+F(x-t)-2F(x)]^2}{t^3} dt \right.$$

where $F(x) = \int_0^x f(t) \, dt + c$, c being a constant, then we have the inequality

(2.5.2) $$\left\{ \int_0^{2\pi} \mu^p(x) dx \right\}^{1/p} \leqq A^p \left\{ \int_0^{2\pi} |f(x)|^p dx \right\}^{1/p} \quad (2 \geqq p > 1),$$

where A_p depends only on p.

This inequality was conjectured first by J. Marcinkiewicz [7] and proved by him for the special case $p=2$. Recently A. Zygmund [8] has proved it for the general case $1 < p \leqq 2$. From his proof, we can see that A_p satisfies the inequality (1.6) with $k=2$. If we put $T[f] = \mu(x)$, this transformation satisfies the conditions of the theorem in §1 and then we have immediately the inequality

(2.5.3) $$\int \mu(x) dx \leqq A \int_0^{2\pi} |f(x)| \log^2 (1+f^2(x)) dx + B.$$

2.6. **Trigonometrical interpolation.** Let $h(x)$ be a periodic continuous function with period 2π. We shall consider the trigonometrical polynomials

(2.6.1) $$U_n(h,x) = a_0^{(n)}/2 + \sum_{\nu=1}^{n} (a_\nu^{(n)} \cos \nu x + b_\nu^{(n)} \sin \nu x)$$

defined by the equalities

(2.6.2) $$U_n(h, x_i) = h(x_i) \qquad (i=0, \pm 1, \pm 2, \cdots,)$$

$$x_i = i \cdot \frac{2\pi}{2n+1}.$$

Let us put

(2.6.3) $$U_{n,i}(h, x) = a_0^{(n)}/2 + \sum_{j=1}^{i} (a_i^{(n)} \cos jx + b_j^{(n)} \sin jx),$$

(2.6.4) $$V_n(h, x) = \frac{1}{n} \sum_{i=1}^{n} U_{n,i}(h, x).$$

If $f(x)$ is a periodic integrable function with period 2π, let us write $F(f,x) = F(x) = \int_0^x f(t)dt - c,\ c = \int_0^{2\pi} f(x)dx.$

Under these notations, the following facts are known (see [10], [11]): *If $f(x) \in L^p (p>1)$ and $\{n_k\}$ is a sequence of positive integers satisfying the inequalities $n_{k+1}/n_k > a > 1$ for $k=1,2,\cdots$, then*

(2.6.5) $$\int_0^{2\pi} |U'_n(F, x) - f(x)|^p\, dx \leqq A_p^p \int_0^{2\pi} |f(x)|^p\, dx,$$

(2.6.6) $$\int_0^{2\pi} (\sum_{n=1}^{\infty} |U'(F, x) - V'(F, x)|^2/n)^{p/2}\, dx \leqq A_p^p \int_0^{2\pi} |f(x)|^p\, dx$$

(2.6.7) $$\int_0^{2\pi} (\sum_{k=1}^{\infty} |U'_{n_k}(F, x) - V'_{n_k}(F, x)|^2)^{p/2}\, dx \leqq A_p^p B_a^p \int_0^{2\pi} |f(x)|^p dx,$$

where the constant A^p's satisfy the inequality (1.6) *with $k=2$, and B_a depends only on a. From these inequalities it follows*

$$\int_0^{2\pi} |U_n'(F, x) - f(x)|^p\, dx \to 0,\ \frac{1}{n}\sum_{\nu=1}^{n} |U_\nu'(F, x) - f(x)|^2 \to 0,\ U'_{n_k}(F, x) \to f(x)$$

almost everywhere.

Using our key-theorem, we can easily deduce the following theorem from above.

If $f(x) \in L^{*2}$ and $\{n_k\}$ is same as above, then

(2.6.8) $$\int_0^{2\pi} |U_n'(F, x) - f(x)|dx \leqq A\int_0^{2\pi} |f(x)| \log^2 (1+f^2(x)dx + B,$$

(2.6.9) $$\int_0^{2\pi} (\sum_{n=1}^{\infty} |U_n'(F, x) - V'(F, x)|^2/n)\, dx \leqq A \int_0^{2\pi} |f(x)| \log^2 (1+f^2(x))\, dx + B,$$

(2.6.10) $$\int_0^{2\pi} (\sum^{\infty} |U'_{k_k}(F, x) - V'_{n_k}(F, x)|\, dx$$

$$\leqq A\int_0^{2\pi}|f(x)|\ \log^2(1+f^2(x))\ dx+B,$$

where A and B are absolute constants and so it follows

$$\int_0^{2\pi}|U_n(F,x)-f(x)|\ dx\to 0, \frac{1}{n}\sum_{\nu=1}^{n}|U_\nu'(F,x)-f(x)|^2\to 0, \text{ and } U_{n_k}'(F,x)\to f(x)$$

almost everywhere.

Corresponding results for Fourier series may be found in Zygmund [12].

2.7. **Walsh-Kacmarz series.** Let $\{\psi_n(x)\}$ be the Walsh-Kacmarz system in the interval (0.1), and $f(x)$ be integrable with the Fourier expansion by the system $\{\psi_n(x)\}$

(1) $$f(x)\sim\sum_{n=0}^{n}c_n\ \psi_n(x).$$

Let us put

(2) $$S_n(f,x)\equiv S_n(x)\equiv\sum_{m=0}^{n-1}C_m\ \psi_m(t),$$

(3) $$\sigma_n^{(\alpha)}(f,x)\equiv\sigma_n^{(\alpha)}(x)\equiv\frac{1}{A_n^{(\alpha)}}\sum_{m=0}^{n}S_m(x)\ A_{n-m}^{(\alpha-1)},\ A_n^{(\alpha)}=\binom{n+\alpha}{n},$$

(4) $$\Delta_{n+1}(f,x)\equiv\Delta_n(x)=\sum_{m=2^n}^{2^{n+1}-1}c_m\ \psi_m(x)\quad(n\geqq o),\ \Delta_0(f,x)$$
$$\equiv\Delta^0(x)=c_0,$$

(5) $$\Delta^*(f,x)\equiv\Delta^*(x)=\sum_{n=0}^{\infty}\varepsilon_n\Delta_n(x),$$

where $\{\varepsilon_n\}$ is a system of arbitrary unit factors. Then the following inequalities are known:

(6) $$\int_0^1\sup_n|S_{2^n}(x)|^p\ dx\leqq A_p^p\int_0^1|f(x)|^p\ dx\quad(p>1),$$
$$\int_0^1\sup_n|S_{2^n}(x)|\ dx\geqq A\int_0^1|f(x)|\ \log(1+f^2(x))dx+B,$$

(7) $$\frac{1}{A_p^p}\int_0^1\left[\sum_{n=0}^{\infty}\Delta_n^2(x)\right]^{p/2}dx\leqq\int_0^1|f(dx)|^p dx\leqq A_p^p\int_0^1\left[\sum_{n=0}^{\infty}\Delta_n^2(x)\right]^{p/2}dt\quad(p>1),$$

(8) $$\frac{1}{A_p^p}\int_0^1|\Delta^*(x)|^p dx\leqq\int_0^1|f(x)|^p dx\leqq A_p^p\int_0^1|\Delta^*(x)|^p dx\quad(p>1),$$

(9) $$\int_0^1|S_n(x)|^p dx\leqq A_{p'}^p\int_0^1|f(x)|^p\ dx\quad(p>1),$$

(10) $$\int_0^1 \sup_n |\sigma_n^{(1)}(t)|^p \, dt \leqq A_p^p \int_0^1 |f(x)|^p \, dx \quad (p>1),$$

(11) $$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{|S_n(x)-\sigma_n^{(1)}(x)|}{n}\right)^{p/2} dx \leqq A_p^p \int_0^1 |f(x)|^p \, dx \quad (p>1),$$

(12) $$\int_0^1 \left(\sum_{n=0}^{\infty} |S_2 \,(x)-\sigma_{2^n}(x)|^2\right)^{p/2} dx \leqq A_p^p \int_0^1 |f(x)|^p dx \quad (p>1)$$

$$\int_0^1 \left(\sup \frac{1}{n+1} \sum_{m=0}^{n} |S_m(t)-f(x)|^2\right)^{p/2} dx \leqq A_n^p \int_0^{2\pi} |f(x)|^p dx \quad (p>1).$$

Among the above inequalities (6)—(10) were proved by R.F.A.C. Paley [7] and the others were proved by G. Sunouchi [10]. The constants A_p in (6)—(8) satisfies the inequality (1.6) for $k=1$, and those in (9)—(13) satisfies it for $k=2$, so we have immediately by our theorem the following inequalities:

(7)′ $$\int_0^1 \left[\sum_{n=0}^{\infty} \Delta_n^2(x)\right]^{1/2} dx \leqq A\int_0^1 |f(x)| \ (1+f^2(x)) \ dx+B,$$

(8′) $$\int_0^1 |\Delta^*(x)|dx \leqq A\int_0^1 |f(x)|\log(1+f^2(x))\,dx+B,$$

(9′) $$\int_0^1 |S_n(x)|dx \leqq A\int_0^1 |f(x)| \log^2(1+f^2(x)) \ dx+B,$$

(10′) $$\int_0^1 \sup_n |\sigma_n^{(1)}(x)|dx \leqq A\int_0^1 |f(x)| \ \log^2 \ (1+f^2(x)) \ dx+B,$$

(11′) $$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{[S_n(x)-\sigma_n(x)|^2}{n}\right]^{1/2} dx \leqq A\int_0^1 |f(x)| \ \log^2 \ (1+f^2(x)) \ dx+B,$$

(12₁) $$\int_0^1 \left[\sum_{n=1}^{\infty} |S_{2n}(x)-\sigma_{2n}^{(1)}(x)|^2\right]^{1/2} dx \leqq A\int_0^1 |f(x)| \ \log^2 \ (1+f^2(x))dx+B,$$

(13′) $$\int_0^1 \left[\sup \frac{1}{n+1} \sum_{m=0}^{n} |S_m(x)-f(x)|^2\right]^{1/2} dx \leqq A\int_0^1 |f(x)| \log^2 (1+f^2(x))dx+B.$$

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