Chin. Ann. of Math.
21B: 1(2000),33-38.

NOTES ON GLAISHER'S CONGRUENCES**

HONG Shaofang*

Abstract

Let p be an odd prime and let $n \geq 1, k \geq 0$ and r be integers. Denote by B_k the k-th Bernoulli number. It is proved that (i) If $r \geq 1$ is odd and suppose $p \geq r+4$, then $\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv -\frac{(2n+1)r(r+1)}{2(r+2)} B_{p-r-2}p^2 \pmod{p^3}$. (ii) If $r \geq 2$ is even and suppose $p \geq r+3$, then $\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv \frac{r}{r+1} B_{p-r-1}p \pmod{p^2}$. (iii) $\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv -(2n+1)p \pmod{p^2}$. This result generalizes the Glaisher's congruence. As a corollary, a generalization of the Wolstenholme's theorem is obtained.

Keywords Glaisher's congruence, kth Bernoulli number, Teichmuller character, p-adic L function
1991 MR Subject Classification 11A41, 11S80
Chinese Library Classification O156.1

§1. Introduction

Several authors (see [2,pp.95-103]) have studied the sums

$$\sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{1}{j^r} \tag{1.1}$$

modulo powers of the prime p, especially in the cases where $r = 1$ or $n = 1$. The well-known Wolstenholme's theorem (see [5]) asserts that if $p \geq 5$ is prime, then

$$\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}.$$

Define the Bernoulli numbers $B_k (k = 0, 1, 2, \cdots)$ by the series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{1.2}$$

Glaisher in 1900 found the following strengthened congruences.

Theorem A(see [3],[4], or [7]). *Let r be an integer and let p be an odd prime.*

Manuscript received January 18, 1999. Revised September 20, 1999.
*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.
**Project supported by the Postdoctoral Foundation of China.

(i) *If* $r \geq 1$ *is odd and suppose* $p \geq r+4$*, then*

$$\sum_{j=1}^{p-1} \frac{1}{j^r} \equiv -\frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}.$$

(ii) *If* $r \geq 2$ *is even and suppose* $p \geq r+3$*, then*

$$\sum_{j=1}^{p-1} \frac{1}{j^r} \equiv \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}.$$

(iii)

$$\sum_{j=1}^{p-1} \frac{1}{j^{p-2}} \equiv -p \pmod{p^2}.$$

Boyd[1] gave an explicit p-adic expansion of the sum (1.1) in the case $r=1$. Recently, Washington[8] obtained an explicit p-adic expansion of the sum (1.1) as a power series in n and the coefficients are values of p-adic L functions (see Theorem B).

In the present paper we will generalize the Glaisher's results by using the Washington's p-adic expansion of the sum (1.1).The main result in this paper is as follows:

Theorem 1.1. *Let* p *be an odd prime and let* $n \geq 0$ *and* r *be integers.*

(i) *If* $r \geq 1$ *is odd and suppose* $p \geq r+4$*, then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv -\frac{(2n+1)r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}.$$

(ii) *If* $r \geq 2$ *is even and suppose* $p \geq r+3$*, then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}.$$

(iii)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv -(2n+1)p \pmod{p^2}.$$

If let $n=0$, then Theorem 1.1 becomes Theorem A.

§2. Preliminaries on P-adic L Functions

Let p be a prime and let $L_p(s,\chi)$ be the p-adic function attached to a character χ. In this section we introduce some facts about p-adic-valued L functions.

Let ω be the p-adic-valued Teichmuller character, so $\omega(a) \equiv a \pmod p$ and $\omega(a)^p = \omega(a)$ when $p \geq 3$. If $p \dagger a$, let $\langle a \rangle = a/\omega(a)$. If $x \in \mathbf{Z}_p$(= the ring of the p-adic integers), let $\binom{x}{k} = (x)(x-1)\cdots(x-k+1)/k!$. When p is odd, or when $p=2$ and $\omega^t = 1$, the p-adic L function for the character ω^t satisfies

$$L_p(s,\omega^t) = \frac{1}{s-1}\frac{1}{p}\sum_{a=0}^{p-1} \omega(a)^t \langle a \rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} (B_j) \Big(\frac{p}{a}\Big)^j$$

for $s \in \mathbf{Z}_p$. This is a p-adic analytic function. In order to prove Theorem 1.1, we need the following results.

Lemma 2.1.[9] (i) *If t is odd, then $L_p(s,\omega^t)$ is identically 0;*
(ii) *If $t \not\equiv 0 \pmod{p-1}$, then for all $s \in \mathbf{Z}_p, L_p(s,\omega^t) \in \mathbf{Z}_p$;*
(iii) *If $t \not\equiv 0 \pmod{p-1}$, then for all $s_1, s_2 \in \mathbf{Z}_p$, we have*

$$L_p(s_1,\omega^t) \equiv L_p(s_2,\omega^t) \pmod{p};$$

(iv) *If $1 \le k \equiv t \pmod{p-1}$, then*

$$L_p(1-k,\omega^t) = -\frac{1-p^{k-1}}{k}B_k.$$

Lemma 2.2.[8] *Assume $p \ge 5, p \ge r$, and $k \ge 3$. If either $r \ne p-3$ or $k \ne 3$, then $L_p(r+k,\omega^{1-k-r})p^k \equiv 0 \pmod{p^3}$. In the case $r=p-3$ and $k=3$, we have $L_p(p,1)p^3 \equiv p^2 \pmod{p^3}$.*

§3. Proof of the Main Result

In order to prove our main result, we need the following p-adic expansion of the sum (1.1) as a power series in n.

Theorem B.[8] *Let p be an odd prime and let $n, r \ge 1$ be integers. Then*

$$\sum_{\substack{j=1\\(j,p)=1}}^{np} \frac{1}{j^r} = -\sum_{k=1}^{\infty}\binom{-r}{k} L_p(r+k,\omega^{1-k-r})(pn)^k.$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem A, we only need to consider the case $n \ge 1$.In the following let $n \ge 1$. Clearly we have

$$\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} = \sum_{\substack{j=1\\(j,p)=1}}^{(n+1)p}\frac{1}{j^r} - \sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{1}{j^r}.$$

It then follows from Theorem B that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} &= -\sum_{k=1}^{\infty}\binom{-r}{k}L_p(r+k,\omega^{1-k-r})(p(n+1))^k \\ &\quad + \sum_{k=1}^{\infty}\binom{-r}{k}L_p(r+k,\omega^{1-k-r})(pn)^k \\ &= \sum_{k=1}^{\infty}\binom{-r}{k}L_p(r+k,\omega^{1-k-r})p^k(n^k-(n+1)^k). \qquad (3.1)\end{aligned}$$

(i) Let $r \ge 1$ be odd and suppose $p \ge r+4$. Since $r \le p-4$, by Lemma 2.2 we have that for $k \ge 3$, $L_p(r+k,\ \omega^{1-k-r})p^k \equiv 0 \pmod{p^3}$. Note that r is odd. By Lemma 2.1(i) the summand for $k=1$ vanishes in Equation (3.1). Therefore

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} &\equiv \binom{-r}{2}L_p(r+2,\omega^{-1-r})p^2(n^2-(n+1)^2)) \\ &\equiv -\frac{(2n+1)r(r+1)}{2}L_p(r+2,\omega^{-1-r})p^2 \pmod{p^3}. \qquad (3.2)\end{aligned}$$

Using Lemma 2.1(iii)(note that $1+r \not\equiv 0 \pmod{p-1}$), we have

$$L_p(r+2,\omega^{-1-r}) \equiv L_p(r+2-p+1,\omega^{-1-r}) \pmod{p}. \qquad (3.3)$$

By Lemma 2.1(iv) we have

$$L_p(r+2-p+1,\omega^{-1-r}) = -\frac{1-p^{p-r-3}}{p-r-2}B_{p-r-2}. \tag{3.4}$$

It can be deduced from Equations (3.2)–(3.4) that

$$\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} \equiv \frac{(2n+1)r(r+1)}{2(p-r-2)}(1-p^{p-r-3})B_{p-r-2}p^2 \pmod{p^3}.$$

Since $\frac{1}{p-r-2} \equiv -\frac{1}{r+2} \pmod{p}$, we have that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} &\equiv -\frac{(2n+1)r(r+1)}{2(r+2)}(1-p^{p-r-3})B_{p-r-2}p^2\\ &\equiv -\frac{(2n+1)r(r+1)}{2(r+2)}B_{p-r-2}p^2 \pmod{p^3}\end{aligned}$$

as desired.

(ii) Let $r \geq 2$ be even and suppose $p \geq r+3$. By Lemma 2.2 we have that for $k \geq 3, L_p(r+k,\omega^{1-k-r})p^k \equiv 0 \pmod{p^2}$. Since r is even, by Lemma 2.1(i) the summand for $k=2$ vanishes in Equation (3.1). By Lemma 2.1(iii) and (iv), it follows from Equation (3.1) that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^r} &\equiv \binom{-r}{1}L_p(r+1,\omega^{-r})p(n-(n+1))\\ &\equiv rL_p(r+1,\omega^{-r})p\\ &\equiv rL_p(r+1-p+1,\omega^{-r})p\\ &\equiv -r(1-p^{p-r-2})\frac{B_{p-r-1}}{p-r-1}p\\ &\equiv \frac{r}{r+1}B_{p-r-1}p \pmod{p^2}.\end{aligned}$$

(iii) Let $r = p-2$. Then for $k = 1, 1-k-r$ is odd. Thus $L_p(r+k,\omega^{1-k-r}) = 0$. For $k \geq 3$, by Lemma 2.2 we have that $L_p(r+k,\omega^{1-k-r})p^k \equiv 0 \pmod{p^2}$. For $k=2$, from Lemma 2.2 we deduce that $L_p(p,1)p^2 \equiv p \pmod{p^2}$. Then it follows from Equation (3.1) that

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{1}{(np+j)^{p-2}} &\equiv \binom{-p+2}{2}L_p(p,1)p^2(-2n-1)\\ &\equiv -\frac{(2n+1)(p-1)(p-2)}{2}p\\ &\equiv -(2n+1)p \pmod{p^2}.\end{aligned}$$

The proof is complete.

§4. Corollaries

In the present section, we give some corollaries of the main result.

Corollary 4.1. *Let p be an odd prime and let $n \geq 0$ and $r \geq 1$ be integers. Suppose that r is odd and $p \geq r+4$. Then the following congruences hold:*

(i)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv 0 \pmod{p^2}.$$

(ii)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{r+1}} \equiv 0 \pmod{p}.$$

(iii)

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv 0 \pmod{p}.$$

Proof. By the von Staudt-Clausen Theorem (see [6,9]), we have

$$B_{p-r-2} + \sum_{\substack{(l-1)|(p-r-2) \\ l \text{ prime}}} \frac{1}{l} \in \mathbf{Z}. \tag{4.1}$$

Since $p \geq r+4$, we have $\frac{1}{l} \in \mathbf{Z}_p$ for all $1 \leq l \leq p-r-2$. Then it follows from Equation (4.1) that $B_{p-r-2} \in \mathbf{Z}_p$. Note that $p \geq r+4$ implies $\frac{1}{r+2} \in \mathbf{Z}_p$. Thus the result follows from Theorem 1.1. This completes the proof.

Remark 4.1. If let $n=0$ and $r=1$, then Corollary 4.1(i) reduces to the Wolstenholme's theorem (see [5]).

Lemma 4.1.[6] *Let m be even and p a prime such that $(p-1) \dagger m$. Let $S_m(p) = 1^m + 2^m + \cdots + (p-1)^m$. Then $S_m(p) \equiv pB_m \pmod{p^2}$.*

Corollary 4.2. *Let p be an odd prime and let $n \geq 0$ and $r \geq 1$ be integers. Suppose that r is odd and $p \geq r+4$. Then each of the following is true:*

(i) $\sum_{j=1}^{p-1} \frac{1}{(np+j)^{r+1}} \not\equiv 0 \pmod{p^2}$;

(ii) *If* $2n \equiv -1 \pmod{p}$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \equiv 0 \pmod{p^3}$$

and

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \equiv 0 \pmod{p^2};$$

(iii) *If* $2n \not\equiv -1 \pmod{p}$, *then*

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^r} \not\equiv 0 \pmod{p^3}$$

and

$$\sum_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}} \not\equiv 0 \pmod{p^2}.$$

Proof. We claim that $B_{p-r-2} \not\equiv 0 \pmod{p}$. Otherwise we have $B_{p-r-2} \equiv 0 \pmod{p}$. Since $r \geq 1$ and $p \geq r+4$, we have $(p-1) \dagger (p-r-2)$. By Lemma 4.1 we have

$pB_{p-r-2} \equiv S_{p-r-2}(p) \pmod{p^2}$. Thus one deduces that

$$S_{p-r-2}(p) \equiv 0 \pmod{p^2}. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} 2^{p-r-2}S_{p-r-2}(p) = 2^{p-r-2} + 4^{p-r-2} + \cdots + (p-1)^{p-r-2} \\ + (p+1)^{p-r-2} + (p+3)^{p-r-2} + \cdots + (p+(p-2))^{p-r-2} \\ \equiv S_{p-r-2}(p) + p(p-r-2)(1+3+\cdots+(p-2)) \\ \equiv S_{p-r-2}(p) + p(p-r-2)\Big(\frac{p-1}{2}\Big)^2 \pmod{p^2}. \end{aligned} \tag{4.3}$$

Thus Equations (4.2) and (4.3) imply that

$$p-r-2 \equiv 0 \pmod{p}. \tag{4.4}$$

Since $2 \le p-r-2 \le p-3$, Equation (4.4) does not hold and the assertion is true. Note that $r, r+1, \frac{1}{2}$ and $\frac{1}{r+2} \not\equiv 0 \pmod{p}$. Then the result follows from Theorem 1.1. The proof is complete.

For a ρ-adic integer $n \in \mathbf{Z}_p$, let $\mathrm{ord}_p n$ denote the integer m such that $p^m | n$ and $p^{m+1} \dagger n$. Combining Corollaries 4.1 and 4.2, we then have the following theorem.

Theorem 4.1. *Let p be an odd prime and let $n \ge 0$ and $r \ge 1$ be integers. Suppose that r is odd and $p \ge r+4$. Then each of the following is true:*

(i) $\mathrm{ord}_p\Big(\sum\limits_{j=1}^{p-1} \frac{1}{(np+j)^{r+1}}\Big) = 1$;

(ii) *If* $2n \not\equiv -1 \pmod{p}$, *then* $\mathrm{ord}_p\Big(\sum\limits_{j=1}^{p-1} \frac{1}{(np+j)^{r}}\Big) = 2$ *and* $\mathrm{ord}_p\Big(\sum\limits_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}}\Big) = 1$;

(iii) *If* $2n \equiv -1 \pmod{p}$, *then* $\mathrm{ord}_p\Big(\sum\limits_{j=1}^{p-1} \frac{1}{(np+j)^{r}}\Big) \ge 3$ *and* $\mathrm{ord}_p\Big(\sum\limits_{j=1}^{p-1} \frac{1}{(np+j)^{p-2}}\Big) \ge 2$.

Remark 4.2. By Theorem 4.1, one can see that the Wolstenholme's theorem is the best possible in the sense of power divisibled by p.

Acknowledgement. The author wishes to thank Professor FENG Keqin for his encouragement and help in this paper.

References

[1] Boyd, D., A p-adic study of the partial sums of the harmonic series, *Experiment Math.* 3(1994), 287–302.
[2] Dickson, L. E., History of the Theory of Numbers, Vol.I, Chelsea, New York, 1952 (especially Chapter 3).
[3] Glaisher, J. W. L., On the residues of the sums of products of the first $p-1$ numbers,and their powers, to modulus p^2 or p^3, *Quart. J. Pure Appl. Math.,* **31**(1900),321–353.
[4] Glaisher, J. W. L., On the residues of the inverse powers of numbers in arithmetic progression, *Quart. J. Pure Appl. Math.,* **32**(1901), 271–305.
[5] Hardy, G. H. & Wright, E. M., An Introduction to the Theory of Numbers, 4th ed. Oxford Univ. Press, London, 1960.
[6] Ireland, K. & Rosen, M., A Classical Introduction to Modern Number Theory, Springer-Verlag, New York, 1982.
[7] Lehmer, E., On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. Math.,* **39**(1938), 350–360.
[8] Washington, L., p-adic L-functions and sums of powers, *J. Number Theory,* **69**(1998), 50–61.
[9] Washington, L., Introduction to Cyclotomic Fields, Springer-Verlag, New York, 1982.