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Notes on higher-derivative conformal theory with nonprimary energy-momentum tensor that applies to the Nambu-Goto string

Yuri Makeenko

NRC "Kurchatov Institute" — ITEP,
B. Cheremushkinskaya 25, 117218 Moscow, Russia

E-mail: makeenko@itep.ru

Abstract: I investigate the higher-derivative conformal theory which shows how the Nambu-Goto and Polyakov strings can be told apart. Its energy-momentum tensor is conserved, traceless but does not belong to the conformal family of the unit operator. To implement conformal invariance in such a case I develop the new technique that explicitly accounts for the quantum equation of motion and results in singular products. I show that the conformal transformations generated by the nonprimary energy-momentum tensor form a Lie algebra with a central extension which in the path-integral formalism gives a logarithmically divergent contribution to the central charge. I demonstrate how the logarithmic divergence is canceled in the string susceptibility and reproduce the previously obtained deviation from KPZ-DDK at one loop.

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Contents

1 Introduction

I continue in these Notes the investigation [1, 2] of conformal symmetry of the Nambu-Goto string. My original motivation for this study was the celebrated formula

$$\gamma_{\rm str} = (1-h)\left[\frac{d-25-\sqrt{(25-d)(1-d)}}{12}\right]+2 \tag{1.1}$$

by Knizhnik-Polyakov-Zamolodchikov [3] and David-Distler-Kawai [4, 5] (abbreviated as KPZ-DDK) for the string susceptibility index $\gamma_{\rm str}$ of surfaces of genus h. It was derived for the Polyakov string in d target-space dimensions using its conformal invariance and suffers the so-called $d=1$ barrier above which (1.1) is not real and thus unacceptable. This is sometimes formulated as a no-go theorem for the existence of bosonic string at $d>1$.

The results of ref. [1] suggest that a potential way out of this problem could be the fact that the Nambu-Goto and the Polyakov strings are in fact not equivalent despite of original Polyakov's argument [6]. The origin of this non-equivalence can be seen already at the level of the simplest higher-derivative beyond Lioville action

$$\mathcal{S}[\varphi] = \frac{1}{16\pi b_0^2}\int\sqrt{g}\left[g^{ab}\partial_a\varphi\partial_b\varphi+2m_0^2+\varepsilon R\left(R+Gg^{ab}\,\partial_a\varphi\partial_b\varphi\right)\right]. \tag{1.2}$$

Here R is the scalar curvature for the two-dimensional metric tensor g_{ab} and $\varphi = -\Delta^{-1}R$ (with Δ being the two-dimensional Laplacian) becomes a local field in the conformal gauge

$$g_{ab} = \hat{g}_{ab}\, \mathrm{e}^{\varphi}, \tag{1.3}$$

where φ is the dynamical variable often called the Liouville field and $\hat{g}_{ab}$ is a slowly varying background (fiducial) metric tensor. The term R^2 appears already for the Polyakov string but the second higher-derivative term with $G \neq 0$ is specific to the Nambu-Goto string [2], as will be reviewed in the next section.

When the action (1.2) emerges as an effective action from a string by path-integrating over the target-space coordinates, ghosts (and the Lagrange multiplier in the case of the Nambu-Goto string), the parameter ε is proportional to the ultraviolet cutoff in the target space. Thus the higher-derivative terms in the action (1.2) are suppressed for smooth metrics as εR. However, typical metrics which are essential in the path integral over the metrics g_{ab} are not smooth and have $R \sim \varepsilon^{-1}$, so the higher-derivative terms revive [2] after doing uncertainties like $\varepsilon \times \varepsilon^{-1}$. I shall consider in these Notes the action (1.2) as such not assuming that ε is infinitesimally small.

A very interesting property of the action (1.2) is that the associated energy-momentum tensor $T^{(\varphi)}_{ab}$ is conserved and traceless [1] owing to the classical equation of motion

$$-\Delta\varphi + m_0^2 + \varepsilon\Delta^2\varphi - \frac{\varepsilon}{2}(\Delta\varphi)^2 + \frac{1}{2}G\varepsilon\partial^a\varphi\partial_a\varphi\Delta\varphi - \frac{1}{2}G\varepsilon\partial^a\partial_a(\partial^b\varphi\partial_b\varphi) + G\varepsilon\partial_a(\partial^a\varphi\Delta\varphi) = 0 \tag{1.4}$$

despite m_0 and ε are dimensionful. The model described by the action (1.2) thus possesses conformal symmetry at least at the classical level.

For this reason a great simplification of the formulas occurs when using the conformal coordinates z and $\bar{z}$, where the flat metric tensor becomes

$$\hat{g}_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad \hat{g}^{ab} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{1.5}$$

The T_{zz} and $T_{z\bar{z}}$ components of the energy-momentum tensor $T^{(\varphi)}_{ab}$ then read

$$\begin{aligned} -4b_0^2 T^{(\varphi)}_{zz} = {} & (\partial\varphi)^2 - 2\varepsilon\partial\varphi\partial\Delta\varphi - 2\partial^2(\varphi - \varepsilon\Delta\varphi) - G\varepsilon(\partial\varphi)^2\Delta\varphi + 4G\varepsilon\partial\varphi\partial(\,\mathrm{e}^{-\varphi}\partial\varphi\bar{\partial}\varphi) \\ & -4G\varepsilon\partial^2(\,\mathrm{e}^{-\varphi}\partial\varphi\bar{\partial}\varphi) + G\varepsilon\partial(\partial\varphi\Delta\varphi) + G\varepsilon\frac{1}{\bar{\partial}}\partial^2(\bar{\partial}\varphi\Delta\varphi), \qquad \Delta = 4\,\mathrm{e}^{-\varphi}\partial\bar{\partial} \end{aligned} \tag{1.6}$$

and $b_0^2 T^{(\varphi)}_{z\bar{z}}\mathrm{e}^{-\varphi}$ given by the left-hand side of eq. (1.4) in the conformal gauge. We used the notation $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial\bar{z}$. Notice the nonlocality of the last term in (1.6) which is inherited from a nonlocality of the action (1.2). The presence of this nonlocal term plays a crucial role in the computation of the central charge at one loop [1].

T_{zz} given by (1.6) obeys the conservation law[1]

$$\bar{\partial}T^{(\varphi)}_{zz} = 0 \tag{1.7}$$

[1]This can be verified using the Mathematica program from appendix A.

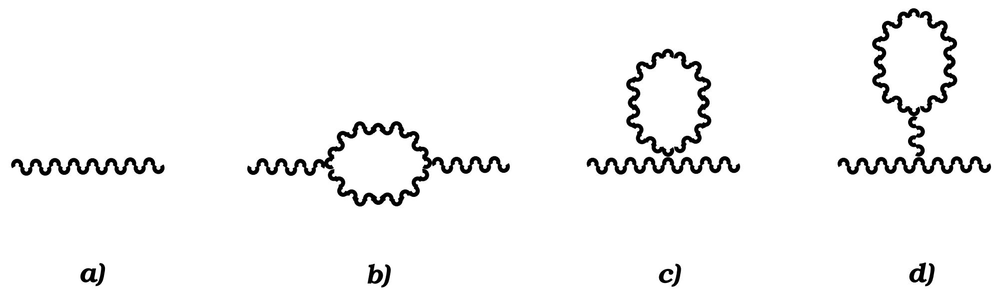

Figure 1. One-loop diagrams for the propagator $\langle\varphi(z)\varphi(0)\rangle$.

because $T^{(\varphi)}_{z\bar z}$ vanishes owing to the classical equation of motion (1.4). In the quantum case eq. (1.4) is replaced by the Schwinger-Dyson equation

$$\text{left-hand side of eq. (1.4)} \overset{\text{w.s.}}{=} 8\pi b_0^2 \frac{\delta}{\delta\varphi}, \tag{1.8}$$

where the equality is undersood in the weak sense, i.e. under the sign of averages in the path-integral formalism. Equation (1.4) is recovered in the classical limit $b_0^2 \to 0$.

The infrared limit of our model is described by an effective action, governing smooth fluctuations of φ, and the effective energy-momentum tensor

$$T^{(\text{eff})}_{zz} = \frac{1}{2b^2}\left(q\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2\right) \tag{1.9}$$

which is quadratic in φ. The arguments are similar to DDK [4, 5]. Here b^2 describes the renormalization of φ, i.e. the change $b_0^2 \to b^2$ in the action (1.2) and q characterizes the theory. In the usual case of the Liouville action where $\varepsilon = 0$ they obey the DDK equation

$$\frac{6q^2}{b^2} + 1 = \frac{6}{b_0^2} \tag{1.10}$$

derived from the background independence. The left-hand side of eq. (1.10) is the central charge of φ. For the Polyakov string eq. (1.10) provides the vanishing of the total central charge leading for $b_0^2 = 6/(26-d)$ to eq. (1.1).

In ref. [1] I computed b^2 and q/b^2 at the one-loop order (the first correction in b_0^2 to the classical values) by a straightforward evaluation of the associated one-loop QFT diagrams shown in figure 1 and figure 2, respectively. Both b^2 and q/b^2 look ugly for the given regularization and involve linear and logarithmic divergences, but the product

$$\frac{q^2}{b^2} = \frac{q^2}{b^4} \times b^2 = \frac{1}{b_0^2} - \frac{1}{6} - G + \mathcal{O}(b_0^2) \tag{1.11}$$

is simple and remarkably depends on G in contrast to the usual DDK result for the Liouville action. The additional factor of b^2 in eq. (1.11) is due to the renormalization of φ by b.

In connection with the $d = 1$ barrier it would be most interesting to understand how the DDK equation (1.10) is to be modified to all orders in b_0^2 for the action (1.2). An attempt

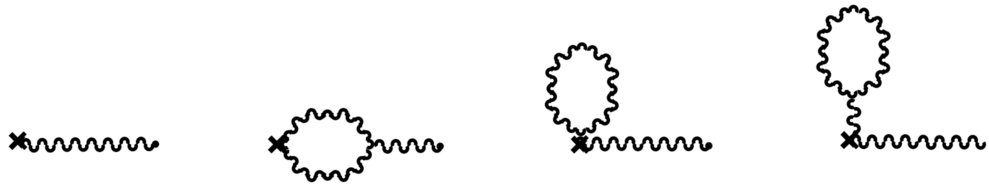

Figure 2. One-loop diagrams for q/b^2 in $T_{zz}^{(\mathrm{eff})}$.

to find out what happens at one loop is undertaken in ref. [1], applying the methods of conformal field theory (CFT). The computed averaged operator product reads

$$\langle T_{zz}(z)T_{zz}(0)\rangle = \frac{c}{2z^4}, \tag{1.12}$$

where c is the central charge in the path-integral formalism. It is given at one loop by the diagrams in figure 3. The diagrams a) to j) contribute $6q^2/b^2$ to the central charge, while the diagram k) has the finite part exactly coinciding with that in eq. (1.11) but also an additional logarithmic divergence

$$c = \frac{6q^2}{b^2} + 1 + 6G\left(1 - 2\int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2}\right) + \mathcal{O}(b_0^2). \tag{1.13}$$

Both $\propto G$ finite and divergent parts come from the nonlocal (last) term in (1.6).

I shall describe in these Notes how the logarithmic divergence of the central charge is canceled in the computation of a physical observable — the string susceptibility. The reason for the appearance of that divergence are subtleties in the realization of conformal symmetry generated by the energy-momentum tensor (1.6) which is not primary. I shall also confirm the final part in (1.13) by another technique.

These Notes are organized as follows. After a brief review of the setup in section 2 I describe in section 3 massive conformal fields which can serve as Pauli-Villars' regulators preserving conformal symmetry. Section 4 is devoted to some unusual properties of the energy-momentum tensor (1.6). In section 5 I introduce a new form of the generator of conformal transformations based on the quantum equation of motion, which drastically simplifies computations with the action (1.2). By computing the commutator of two conformal transformations I show they form the Lie algebra with a certain central extension. In section 6 I compute the central charge at one loop by using the formulas derived in section 5 and reproduce eq. (1.13) including the logarithmic divergence. Finally, I demonstrate in section 7 how this logarithmic divergence cancels in the calculation of a physical observable — the string susceptibility. Some Conclusions are drawn in section 8. Appendix A contains the Mathematica program demonstrating the conservation and tracelessness of the energy-momentum tensor (1.6). Appendix B is devoted to the derivation of formulas for the singular products involved in the computations.

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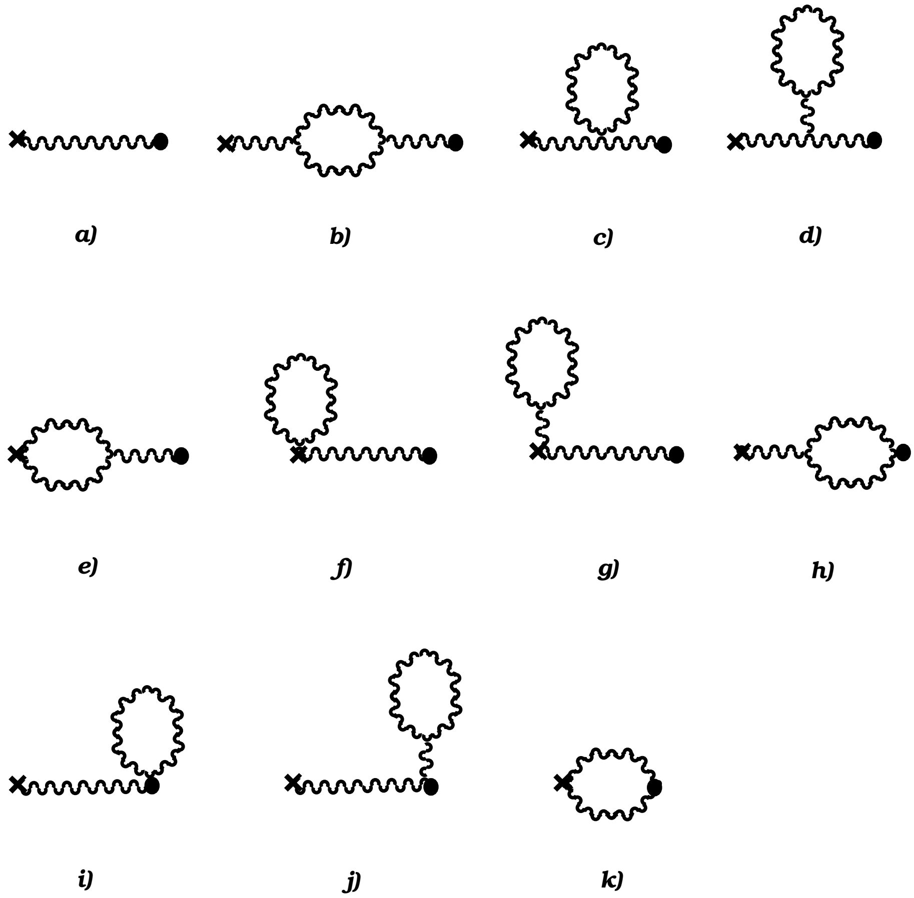

Figure 3. One-loop diagrams for the averaged operator product $\langle T_{zz}(z)T_{zz}(0)\rangle$. The points z and 0 are depicted by the cross and the dot, respectively.

2 The setup

There are two ways of describing a quantum string: the Nambu-Goto or Polyakov formulations which are commonly believed to be equivalent except the dimension of the target-space coordinate X^μ is shifted $d \to d-1$ because the metric g_{ab} is independent for the Polyakov string described by the action

$$S = \frac{1}{4\pi\alpha'}\int \sqrt{g}g^{ab}\partial_a X\cdot\partial_b X \tag{2.1}$$

which is quadratic in X^μ that makes it easy to integrate it out in the path integral. Alternatively, the Nambu-Goto action is the area of the string worldsheet which is highly nonlinear in X^μ but can be made quadratic, introducing the Lagrange multiplier λ^{ab} and an independent metric tensor g_{ab},

$$S_{\rm NG} = \frac{1}{2\pi\alpha'}\int\sqrt{\det\left(\partial_a X\cdot\partial_b X\right)} = \frac{1}{2\pi\alpha'}\int\left[\sqrt{g}+\frac{1}{2}\lambda^{ab}(\partial_a X\cdot\partial_b X-g_{ab})\right]. \tag{2.2}$$

The equivalence of the two string formulations is evident at the classical level. The general argument in favor of the equivalence in the quantum case is based [6] on the fact that λ^{ab} propagates only to the distances of the order of the ultraviolet cutoff and therefore should not affect the theory at large distances. The statement relies on an effective action, governing fluctuations of g_{ab} and λ^{ab}, which emerges after the path integration over the target-space coordinates of the string (including possible Pauli-Villars' regulators) and the ghosts associated with fixing the conformal gauge (1.3). I shall not describe here this issue which is carefully reviewed in ref. [7].

Because the Lagrange multiplier λ^{ab} does not ropagate to macroscopic distances it is tempting to additionally path-integrate over λ^{ab} about the ground-state value $\bar\lambda^{ab} = \bar\lambda\sqrt{g}g^{ab}$ to obtain an effective action describing fluctuations of g_{ab} in the infrared regime. Notice that $\bar\lambda^{ab}$ does not depend on φ in the conformal gauge (1.3). As was shown in ref. [2], we then arrive at the action (1.2) with $b_0^2 = 6/(26-d)$ and a certain value of G plus additional terms of the order ε^2 or higher. I did not mention herein the ground-state value of the metric tensor $\bar g_{ab} = \bar\rho\hat g_{ab}$ all of which are physically equivalent because of the so-called background independence that stays in the given case that $\bar\rho$ is not observable thanks to the Weyl invariance.

A few comments concerning the derivation of the action (1.2) are in order. If we set $\lambda^{zz} = \lambda^{\bar z\bar z} = 0$ and path-integrate over $\lambda^{z\bar z}$, then only the R^2 term in (1.2) would appear like for the Polyakov string where it comes from the Seeley expansion of the heat kernel. The first term in (1.2) is associated with the conformal anomaly while the R^2 term comes from the next order of the Seeley expansion in the worldsheet UV cutoff ε. It is familiar from the studies [8–10] of the R^2 two-dimensional gravity, where the first term in the action (1.2) was missing. The second higher-derivative term with $G \neq 0$ emerges alternatively after integrating out λ^{zz} and $\lambda^{\bar z\bar z}$. The actual value of G was not computed in [2] but it was shown to be nonvanishing for the Nambu-Goto string. Only these two terms are independent to order ε. The others can be reduced to them modulo boundary terms integrating by parts.

Of course the two higher-derivative terms in the action (1.2) are negligible classically for smooth metrics as powers of $\varepsilon R \ll 1$, reproducing the Liouville action. However, the term with quartic derivative provides both a UV cutoff and also an interaction whose coupling constant is ε. We thus encounter uncertainties like $\varepsilon \times \varepsilon^{-1}$ so the higher-derivative terms quantumly revive [2] after doing these uncertainties. In other words the typical metrics which are essential in the path integral over φ are not smooth and have $R \sim \varepsilon^{-1}$. The first term in the action (1.2) is thus only an effective action governing smooth classical configuration, while the higher-derivative terms result in nontrivial interactions that cause an eventful private life of φ which occurs at the very small distances of order $\sqrt{\varepsilon}$ but nevertheless can be observable like the finite G term in eq. (1.13) (see eq. (7.7) below).

There are also other higher-derivative terms of the order ε^2 and higher additionally to (1.2) which are also not suppressed quantumly by $\varepsilon \to 0$. One may expect they do not change the results owing to the universality which often takes place near the critical point. The arguments are presented in ref. [2] where the universality was explicitly demonstrated for a six-order high-derivative term $\sim \varepsilon^2$. It looks like an appearance of anomalies in QFT. This issue of yet higher-derivative terms refers however to the strings as a whole while the action (1.2) by itself may be considered as a toy model of how to tell the Nambu-Goto and Polyakov strings apart.

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3 Pauli-Villars regulators as massive conformal fields

To regularize divergences in the quantum case I implement the Pauli-Villars regularization, adding to (1.2) the following action for the regulator field Y:

$$\mathcal{S}[Y] = \frac{1}{16\pi b_0^2} \int \sqrt{g} \left[g^{ab} \partial_a Y \partial_b Y + M^2 Y^2 + \varepsilon (\Delta Y)^2 + G\varepsilon g^{ab} \partial_a Y \partial_b Y R \right]. \tag{3.1}$$

The field Y has a very large mass M and obeys wrong statistics to produce the minus sign for every loop, regularizing devergences coming from the loops of φ.

To be precise, the introduction of one regulator is not enough to regularize all the divergences. Some logarithmic divergences still remain. As was pointed out in [11], the correct procedure is to introduce two regulators of mass squared M^2 with wrong statistics, which can be viewed as anticommuting Grassmann variables, and one regulator of mass squared $2M^2$ with normal statistics. Then all diagrams including quadratically divergent tadpoles will be regularized. However, for the purposes of computing final parts just one regulator Y would be enough. The contribution of the two others is canceled being mass independent.

The regulators also contribute to the energy-momentum tensor. The total one

$$T_{ab} = T_{ab}^{(\varphi)} + T_{ab}^{(Y)} \tag{3.2}$$

is conserved and traceless thanks to the classical equations of motion for φ and Y

$$\text{l.h.s. side of eq. (1.4)} + \frac{M^2}{2} Y^2 - \frac{\varepsilon}{2} (\Delta Y)^2 + G\frac{\varepsilon}{2} \partial^a Y \partial_a Y \Delta\varphi - G\frac{\varepsilon}{2} \Delta(\partial^a Y \partial_a Y) = 0, \tag{3.3}$$

$$-\Delta Y + M^2 Y + \varepsilon \Delta^2 Y + G\varepsilon \partial_a (\partial^a Y \Delta\varphi) = 0, \tag{3.4}$$

respectively. Thus the Pauli-Villars regulators are classically conformal fields in spite of they are massive.

This situation seems to be different from the usual one in QFT, where an anomaly emerges if the regularization breaks the classical symmetry. We may thus expect that conformal symmetry of the classical action (1.2) will be maintained at the quantum level for the Pauli-Villars regularization. This can be confirmed by explicit computations at one loop and partially (for $\varepsilon = 0$) at two loops.

For the contribution of the regulator to T_{zz} we find

$$-4b_0^2 T_{zz}^{(Y)} = \partial Y \partial Y - 2\varepsilon \partial Y \partial \Delta Y - G\varepsilon \partial Y \partial Y \Delta\varphi + 4G\varepsilon \partial\varphi \partial(\,\mathrm{e}^{-\varphi} \partial Y \bar\partial Y) - 4G\varepsilon \partial^2(\,\mathrm{e}^{-\varphi} \partial Y \bar\partial Y). \tag{3.5}$$

When I say "regulators" I mean large-mass fields with wrong statistics to provide minus signs for every loop. But the above eqs. (3.1) to (3.5) also apply to the case of a usual massive field with normal statistics interacting with two-dimensional gravity. Thus our consideration below also applies to such a model of the massive conformal field.

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4 Properties of nonprimary energy-momentum tensor

The usual definition of the central charge c is linked to the transformation law

$$\delta_\xi T_{zz} = \frac{c}{12}\xi''' + 2\xi' T_{zz} + \xi\partial T_{zz} \tag{4.1}$$

of the energy-momentum tensor T_{zz} under an infinitesimal conformal transformation $\delta z = \xi(z)$. It is prescribed for the conserved tensorial field, which is the descendant of the primary unit operator, as was pointed out in the original paper by Belavin-Polyakov-Zamolodchikov (BPZ) [12]. Equation (4.1) is easily derivable for T_{zz} which is quadratic in φ. Let us check how it works for T_{zz} given by eq. (1.6) which is not quadratic in φ.

It is easy to calculate how the energy-momentum tensor (1.6) changes under the infinitesimal conformal transformation. Substituting

$$\delta_\xi \varphi = \xi' + \xi\partial\varphi \tag{4.2}$$

into (1.6), we find

$$\begin{aligned}\delta_\xi T^{(\varphi)}_{zz} = &\frac{1}{2b_0^2}\xi''' + 2\xi' T^{(\varphi)}_{zz} + \xi\partial T^{(\varphi)}_{zz} + \frac{1}{b_0^2} G\varepsilon\, \mathrm{e}^{-\varphi}\Big\{\xi''''\bar\partial\varphi + \xi'''(\partial\bar\partial\varphi - 3\partial\varphi\bar\partial\varphi)\\ &+\xi''\Big[2\bar\partial\varphi(\partial\varphi)^2 - \partial\varphi\partial\bar\partial\varphi - \bar\partial\varphi\partial^2\varphi\Big] - \mathrm{e}^{\varphi}\frac{1}{\bar\partial}[\xi''\partial(\mathrm{e}^{-\varphi}\bar\partial\varphi\partial\bar\partial\varphi)]\Big\}.\end{aligned} \tag{4.3}$$

It deviates from eq. (4.1) by the presence of the additional terms which arise because the action (1.2) involves the structure

$$L = g^{ab}\partial_a\varphi\partial_b\varphi = 4\,\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi \tag{4.4}$$

(familiar from the Lagrangian of a free field) which is scalar but not primary and transforms as

$$\delta_\xi L = \xi\partial L + 4\xi''\,\mathrm{e}^{-\varphi}\bar\partial\varphi. \tag{4.5}$$

The additional terms do not appear for $G = 0$ because the scalar curvature $R = -4\,\mathrm{e}^{-\varphi}\partial\bar\partial\varphi$ is a primary scalar:

$$\delta_\xi R = \xi\partial R. \tag{4.6}$$

Then the action is not quadratic in φ but the usual CFT technique perfectly works.

It is also easy to repeat the calculation for the total energy-momentum tensor (3.2) which is the sum of (1.6) and that of the regulators (3.5). Accounting for the transformation of Y as a primary scalar

$$\delta_\xi Y = \xi\partial Y \tag{4.7}$$

and noting that

$$\delta_\xi T^{(Y)}_{zz} = 2\xi' T^{(Y)}_{zz} + \xi\partial T^{(Y)}_{zz}, \tag{4.8}$$

we arrive again at eq. (4.3) with $T^{(\varphi)}_{zz}$ substituted by the total T_{zz}.

The additional terms in (4.3) vanish as $\varepsilon \to 0$, so they do not affect then the classical limit if φ is smooth. However, averaging (4.3) over φ with the quadratic weight, we get the following ξ''' term:

$$\left\langle \delta_\xi T^{(\varphi)}_{zz}(0) \right\rangle = \xi'''(0) \left(\frac{1}{2b_0^2} - G \int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2} \right), \tag{4.9}$$

where we substituted

$$\langle \varepsilon L \rangle = \langle \varepsilon R \rangle = 2 \int \mathrm{d}k^2 \frac{\varepsilon}{1+\varepsilon k^2}. \tag{4.10}$$

All other terms in (4.3) do not contribute to the average. The logarithmically divergent second term in (4.9) is just the same as in eq. (1.13) what may help to understand its appearance there. While in the calculation of (1.13) in ref. [1] I used $\varepsilon \to 0$, it has been now derived for an arbitrary ε. It is worth noting that the logarithmic divergence would not appear in the operator formalism where the operators are normal-ordered and the vacuum expectation value $\langle 0| {:}L{:} |0\rangle = 0$.

Several questions immediately arise as to the definition of the central charge, which is linked to the Virasoro algebra, for a nonprimary energy-momentum tensor. I shall answer part of them in the next two sections.

5 Conformal symmetry at the quantum level

For a general action $S[\varphi]$ the generator of the conformal transformation can be written as

$$\hat{\delta}_\xi \equiv \int_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \xi(z) T_{zz}(z) = \frac{1}{\pi} \int_{D_1} \xi \bar{\partial} T_{zz} \overset{\text{w.s.}}{=} \int_{D_1} \left(\xi' \frac{\delta}{\delta\varphi} + \xi \partial\varphi \frac{\delta}{\delta\varphi} \right) \tag{5.1}$$

with

$$\frac{\delta\varphi(0)}{\delta\varphi(z)} = \delta^{(2)}(z), \qquad \delta^{(2)}(z) = \bar{\partial} \frac{1}{\pi z}. \tag{5.2}$$

We have used the (quantum) equation of motion

$$\bar{\partial} T_{zz} = -\pi \partial \frac{\delta S}{\delta\varphi} + \pi \partial\varphi \frac{\delta S}{\delta\varphi}, \qquad \frac{\delta S}{\delta\varphi} \overset{\text{w.s.}}{=} \frac{\delta}{\delta\varphi} \tag{5.3}$$

and integrated by parts. The domain D_1 includes the singularities of $\xi(z)$ leaving outside possible singularities of the function $X(\omega_i)$ on which (5.2) acts and C_1 bounds D_1. In the classical limit $b_0^2 \to 0$ we reproduce (4.2) by acting with the right-hand side of eq. (5.1) on φ.

An advantage of the proposed form of $\hat{\delta}_\xi$ on the right-hand side of eq. (5.1) over the standard one on the left is that it takes into account a tremendous cancellation of the diagrams in the quantum case we now proceed, while there are subtleties associated with singular products. The results of computing the variational derivative may seem to be the same as in a free theory (with the quadratic action) but their averages can differ. I would classify the calculation below as a pragmatic mixture of the CFT and QFT methods.

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In the quantum case we have an additional effect of the regulator

$$\left\langle \hat{\delta}_\xi X(\omega_i) \right\rangle = \left\langle \int_{D_1} \mathrm{d}^2 z \left(\xi'(z) \frac{\delta}{\delta\varphi(z)} + \xi(z)\partial\varphi(z) \frac{\delta}{\delta\varphi(z)} + \xi(z)\partial Y(z) \frac{\delta}{\delta Y(z)} \right) X(\omega_i) \right\rangle . \tag{5.4}$$

Averaging over the regulators, we arrive at the effective action, governing fluctuations of φ, and the effective energy-momentum tensor, which in the infrared limit becomes quadratic in φ as shown in eq. (1.9). The arguments are similar to DDK. Equation (5.4) is then substituted by

$$\left\langle \hat{\delta}_\xi X(\omega_i) \right\rangle = \left\langle \int_{D_1} \mathrm{d}^2 z \left(q\xi'(z) \frac{\delta}{\delta\varphi(z)} + \xi(z)\partial\varphi(z) \frac{\delta}{\delta\varphi(z)} \right) X(\omega_i) \right\rangle . \tag{5.5}$$

As is demonstrated in ref. [1] by explicit calculations at one loop, these two ways of computing the central charge are complementary: we either add at $q = 1$ the contributions from φ and the regulators or consider at $q \neq 1$ only the contribution from φ à la DDK.

Given eq. (5.5) it is instructive to reproduce

$$\hat{\delta}_\xi \, \mathrm{e}^{\varphi(\omega)} \overset{\text{w.s.}}{=} (q - b^2)\xi'(\omega) \, \mathrm{e}^{\varphi(\omega)} + \xi(\omega)\partial\varphi(\omega) \, \mathrm{e}^{\varphi(\omega)} \tag{5.6}$$

for the quadratic action. We obtain

$$\begin{aligned} \left\langle \hat{\delta}_\xi \, \mathrm{e}^{\varphi(\omega)} X \right\rangle &= \int_{D_1} \mathrm{d}^2 z \left\langle [q\xi'(z) + \xi(z)\partial\varphi(z)] \frac{\delta}{\delta\varphi(z)} \, \mathrm{e}^{\varphi(\omega)} X \right\rangle \\ &= q\xi'(\omega) \left\langle \mathrm{e}^{\varphi(\omega)} X \right\rangle + \int_{D_1} \mathrm{d}^2 z \, \xi(z) \left\langle \partial\varphi(z) \, \mathrm{e}^{\varphi(\omega)} X \right\rangle \delta^{(2)}(z-\omega) \\ &= q\xi'(\omega) \left\langle \mathrm{e}^{\varphi(\omega)} X \right\rangle + \int_{D_1} \mathrm{d}^2 z \, \xi(z) \langle \partial\varphi(z)\varphi(\omega) \rangle \delta^{(2)}(z-\omega) \left\langle \mathrm{e}^{\varphi(\omega)} X \right\rangle \\ &\quad + \xi(\omega) \left\langle \partial\varphi(\omega) \, \mathrm{e}^{\varphi(\omega)} X \right\rangle . \end{aligned} \tag{5.7}$$

The most interesting is the second term on the right-hand side, where the singular product equals

$$\int_{D_1} \mathrm{d}^2 z \, \xi(z) \langle \partial\varphi(z)\varphi(\omega) \rangle \delta^{(2)}(z-\omega) = -b^2 \xi'(\omega) \tag{5.8}$$

as shown in eq. (B.3) of appendix B, reproducing (5.6).

If we repeat this computation for the case of the higher-derivative action (1.2), we still infer the second line in eq. (5.7) from the first one but now the average in the second line does not factorize in general because of the interaction, unless the diagrams with interaction are mutually canceled. Nevertheless, the factorization holds at the one-loop order where the passage from the second to the third line in eq. (5.7) works and the averages are to be calculated in the non-interacting higher-derivative theory. We thus obtain to order b_0^2 the same conformal weight of e^{φ} as in (5.6) because eq. (5.8) still holds in this case as shown in appendix B.

Using eq. (5.5) we have for the commutator of two conformal transformations

$$\begin{aligned} \left\langle (\hat{\delta}_\eta \hat{\delta}_\xi - \hat{\delta}_\xi \hat{\delta}_\eta) X \right\rangle &= \left\langle \hat{\delta}_\zeta X \right\rangle \\ &\quad + \int_{D_1} \mathrm{d}^2 z \int_{D_z} \mathrm{d}^2 \omega \left\langle [q\xi'(z) + \xi(z)\partial\varphi(z)] \, [q\eta'(\omega) + \eta(\omega)\partial\varphi(\omega)] \frac{\delta^2 S}{\delta\varphi(z)\delta\varphi(\omega)} X \right\rangle \end{aligned} \tag{5.9}$$

with $\zeta = \xi\eta' - \xi'\eta$ as it should. Here the domain D_1 includes the singularities of $\xi(z)$ and $\eta(z)$, leaving outside possible singularities of X, and D_z comprises z. This represents (see e.g. [6]) the commutator in the operator formalism. The first term on the right-hand side is linked to the classical eq. (5.1) while the second term can be written in the form

$$\left\langle (\hat{\delta}_\eta \hat{\delta}_\xi - \hat{\delta}_\xi \hat{\delta}_\eta) X \right\rangle = \left\langle \hat{\delta}_\zeta X \right\rangle + \frac{1}{24} \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} [\xi'''(z)\eta(z) - \xi(z)\eta'''(z)] \left\langle cX \right\rangle \tag{5.10}$$

which is usually linked to the central charge of the Virasoro algebra. As we shall momentarily see, c in eq. (5.10) is the usual c-number for the quadratic action but it will be φ-dependent for the higher-derivative action (1.2) with $G \neq 0$, as we observed already for the term with ξ''' in eq. (4.3).

6 The central charge at one loop

Let us begin by showing how to reproduce from eq. (5.10) the usual results for the quadratic action. Noting that

$$\frac{\delta^2 S}{\delta\varphi(z)\delta\varphi(\omega)} = \frac{1}{8\pi b^2}(-4\partial\bar{\partial})\delta^{(2)}(z-\omega) \tag{6.1}$$

for the quadratic action, we find

$$\int_{D_1} \mathrm{d}^2 z \int_{D_z} \mathrm{d}^2\omega\, q^2 \xi'(z)\eta'(\omega) \frac{\delta^2 S}{\delta\varphi(z)\delta\varphi(\omega)} = \frac{q^2}{4b^2} \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} [\xi'''(z)\eta(z) - \xi(z)\eta'''(z)] \tag{6.2}$$

and

$$\begin{aligned} &\int_{D_1} \mathrm{d}^2 z \int_{D_z} \mathrm{d}^2\omega\, \xi(z)\eta(\omega) \left\langle \partial\varphi(z)\partial\varphi(\omega)\right\rangle \frac{\delta^2 S}{\delta\varphi(z)\delta\varphi(\omega)} \\ &= -\frac{1}{2b^2} \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \xi(z) \int_{D_z} \mathrm{d}^2\omega \left[\eta'(z) \left\langle \partial^2\varphi(\omega)\varphi(z)\right\rangle + \eta(z) \left\langle \partial^3\varphi(\omega)\varphi(z)\right\rangle \right] \\ &= \frac{1}{24} \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left[\xi'''(\omega)\eta(z) - \xi(\omega)\eta'''(z)\right], \end{aligned} \tag{6.3}$$

where we have used the formula

$$\begin{aligned} &\int_{D_z} \mathrm{d}^2\omega\, f(\omega) \left\langle \partial^n\varphi(\omega)\varphi(z)\right\rangle \delta^{(2)}(\omega - z) = (-1)^n b^2 H_n f^{(n)}(z), \\ &H_1 = 1, \qquad H_2 = \frac{1}{3}, \qquad H_3 = \frac{1}{6}, \qquad H_n = \frac{2}{n(n+1)} \end{aligned} \tag{6.4}$$

for the singular products in CFT with the quadratic action. It generalizes eq. (5.8) and is derived in appendix B. The sum of (6.2) and (6.3) gives $c = 6q^2/b^2 + 1$, reproducing DDK.

It is more lengthy to deal with the higher-derivative terms in the action (1.2) which we expand in the powers of φ as

$$\mathcal{S} = \mathcal{S}^{(2)} + \mathcal{S}^{(3)} + \mathcal{S}^{(4)} + \mathcal{O}(\varphi^5) \tag{6.5}$$

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with

$$
\begin{aligned}
\mathcal{S}^{(2)} &= \frac{1}{4\pi b^2}\int \Big[\partial\varphi\bar\partial\varphi + 4\varepsilon(\partial\bar\partial\varphi)^2\Big], &(6.6a)\\
\mathcal{S}^{(3)} &= -\frac{1}{\pi b^2}\int \Big[\varepsilon\varphi(\partial\bar\partial\varphi)^2 + G\varepsilon\partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\Big], &(6.6b)\\
\mathcal{S}^{(4)} &= \frac{1}{\pi b^2}\int \left[\frac{1}{2}\varepsilon\varphi^2(\partial\bar\partial\varphi)^2 + G\varepsilon\varphi\partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\right]. &(6.6c)
\end{aligned}
$$

The next orders will not contribute at one loop.

The average in the second term on the right-hand side of eq. (5.9) factorizes at one loop and we obtain for the nonvanishing terms with G

$$
\begin{aligned}
&\int_{D_1} \mathrm{d}^2 z \int_{D_z} \mathrm{d}^2\omega\, \xi'(z)\eta'(\omega)\left\langle \frac{\delta^2 S^{(4)}}{\delta\varphi(z)\delta\varphi(\omega)}\right\rangle\\
&= \frac{G\varepsilon}{b^2}\int \mathrm{d}^2z\mathrm{d}^2\omega\mathrm{d}^2t\,\xi'(z)\eta'(\omega)\left\langle\partial\varphi(t)\bar\partial\varphi(t)\right\rangle[\delta^{(2)}(t-z)\partial\bar\partial\delta^{(2)}(t-\omega)+\partial\bar\partial\delta^{(2)}(t-z)\delta^{(2)}(t-\omega)]\\
&= \frac{1}{4}\oint_{C_1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}[\xi'''(z)\eta(z)-\xi(z)\eta'''(z)]\left(-2G\int\frac{\varepsilon\mathrm{d}k^2}{1+\varepsilon k^2}\right) \qquad (6.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{D_1}\mathrm{d}^2z\int_{D_z}\mathrm{d}^2\omega\left\langle[\xi'(z)\eta(\omega)\partial\varphi(\omega)+\xi(z)\eta'(\omega)\partial\varphi(z)]\frac{\delta^2 S^{(3)}}{\delta\varphi(z)\delta\varphi(\omega)}\right\rangle\\
&= -\frac{G\varepsilon}{b^2}\int \mathrm{d}^2z\mathrm{d}^2\omega\mathrm{d}^2t\left\langle[\xi'(z)\eta(\omega)\partial\varphi(\omega)+\xi(z)\eta'(\omega)\partial\varphi(z)]\partial\bar\partial\varphi(t)\right\rangle\\
&\quad\times[\partial\delta^{(2)}(t-z)\bar\partial\delta^{(2)}(t-\omega)+\bar\partial\delta^{(2)}(t-z)\partial\delta^{(2)}(t-\omega)]\\
&= \frac{G\varepsilon}{b^2}\oint_{C_1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\int_{D_z}\mathrm{d}^2\omega[\xi''(z)\eta(\omega)+\xi(z)\eta''(\omega)]\left\langle\partial^2\bar\partial\varphi(\omega)\varphi(z)\right\rangle\delta^{(2)}(\omega-z)\\
&= \frac{1}{4}\oint_{C_1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}[\xi'''(z)\eta(z)-\xi(z)\eta'''(z)]\,G. \qquad (6.8)
\end{aligned}
$$

Here we have used eq. (4.10) and substituted

$$
\int_{D_z}\mathrm{d}^2\omega\, f(z)\left\langle -4\varepsilon\partial^2\bar\partial\varphi(\omega)\varphi(z)\right\rangle\delta^{(2)}(\omega-z) = -b^2 f'(z) \qquad (6.9)
$$

as calculated in appendix B for the propagator in a higher-derivative theory which is just what we have in eq. (6.9) in the one-loop approximation.

The sum of (6.7) and (6.8) remarkably reproduces the $\propto G$ addition to the central charge in eq. (1.13). I also mention that the same final part can be obtained additionally to (4.9) in a slightly different way by a direct computation of $\left\langle\hat\delta T^{(\varphi)}_{zz}\right\rangle$ at one loop.

7 Cancellation of the logarithmic divergence

For the usual Polyakov string the central charge of φ is observable, e.g. via the string susceptibility index γ_{str}. I show in this section that it remains finite at one loop for the

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higher-derivative action (1.2) in spite of the logarithmic divergence of the central charge which cancels in $\gamma_{\rm str}$.

The gauge (1.3) implies the relation

$$\sqrt{g}R \Rightarrow \sqrt{\hat{g}}\left(q\hat{R} - \hat{\Delta}\varphi\right) \tag{7.1}$$

between the scalar curvatures R and $\hat{R}$ for the metrics g_{ab} and $\hat{g}_{ab}$, respectively, and analogously

$$\Delta \equiv \frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b = \mathrm{e}^{-\varphi}\frac{1}{\sqrt{\hat{g}}}\partial_a\sqrt{\hat{g}}\hat{g}^{ab}\partial_b \equiv \mathrm{e}^{-\varphi}\hat{\Delta} \tag{7.2}$$

between the two-dimensional Laplacians. The factor q is as before to appear in eq. (1.9).

The string susceptibility index can be derived [4, 5] from the response of the system to a uniform dilatation of space, which means adding a constant to φ. The quadratic part of the action then results in the topological Gauss-Bonnet term producing the Euler characteristic $2-2h$ in

$$\gamma_{\rm str} = (1-h)\frac{q}{b^2} + \gamma_1 = (1-h)\frac{q^2}{2b^2}\left(1+\sqrt{1-\frac{4b^2}{q^2}}\right) + \gamma_1 \tag{7.3}$$

where $\gamma_1 = 2$ for a closed string in R_d. This reproduces the usual $\gamma_{\rm str}$ (1.1) of KPZ-DDK.

For the higher-derivative action (1.2) we have an additional contribution from the term with G which reads

$$\mathcal{S}_G = -\frac{G\varepsilon}{16\pi b^2}\int \mathrm{e}^{-\varphi}\sqrt{\hat{g}}\hat{g}^{ab}\partial_a\left(\varphi - \frac{q}{\hat{\Delta}}\hat{R}\right)\partial_b\left(\varphi - \frac{q}{\hat{\Delta}}\hat{R}\right)(\hat{\Delta}\varphi - q\hat{R}). \tag{7.4}$$

The relation (7.1) is used in the derivation. While this term is negligible for smooth configurations as $\varepsilon \to 0$, it revives after averaging over φ as is already explained.

For a constant infinitesimal $\delta\varphi$ the variation of the action is expressed via the left-hand side of the classical equation of motion (1.4) where we substitute $\varphi \Rightarrow \varphi + q\hat{\varphi}$ to account for the conformal background metric. The first term on the left-hand side of eq. (1.4) immediately gives the Gauss-Bonnet term. It is also easy to show the cancellation at one loop between the third and fourth terms which are present at $G=0$. Therefore, they do not contribute to $\gamma_{\rm str}$.

Concerning the terms with $G \neq 0$, only the last one in eq. (1.4) gives a nonvanishing contribution at one loop after averaging over φ. We find

$$\langle\delta\mathcal{S}_G\rangle = -\frac{Gq\varepsilon}{8\pi b^2}\delta\varphi\int\sqrt{\hat{g}}\hat{g}^{ab}\left\langle \mathrm{e}^{-\varphi}\partial_a\varphi\partial_b\varphi\right\rangle\hat{R} \tag{7.5}$$

which gives an addition to the right-hand side of eq. (7.3). Expanding to the one-loop order where our results coincide with DDK except for the additional terms in (1.13) at $G \neq 0$, we find

$$\gamma_{\rm str} = (1-h)\left[\frac{q^2}{b^2} - 1 - 2G\int \mathrm{d}k^2\frac{\varepsilon}{1+\varepsilon k^2} + \mathcal{O}(b_0^2)\right] + \gamma_1. \tag{7.6}$$

Extracting q^2/b^2 from eq. (1.13), we see the cancellation between the logarithmic divergence in (7.6) and the logarithmic divergence in the central charge. The contribution of the final part in (1.13) remains and we obtain from (7.6)

$$\gamma_{\rm str} = (1-h)\left(\frac{1}{b_0^2} - \frac{7}{6} - G + \mathcal{O}(b_0^2)\right) + \gamma_1 \tag{7.7}$$

showing for $G \neq 0$ a deviation from the one-loop result [13–15] for the Polyakov string in d target-space dimensions for which $b_0^2 = 6/(26-d)$ and $G = 0$, thus confirming the results of ref. [1]. In the operator formalism the logarithmic divergence does not appear both in the central charge, as is already mentioned in section 4, and in eq. (7.6) because of the normal ordering of the operators in S_G, resulting in the vanishing of the vacuum expectation value $\langle 0|{:}L{:}|0\rangle$ in eq. (7.5).

8 Conclusion

As I already mentioned in the Introduction, my main motivation for these Notes was to confirm the results of ref. [1] and to understand the origin of the logarithmic divergence in the one-loop central charge of φ for the conformal theory with the higher-derivative action (1.2) at $G \neq 0$. This happens because its energy-momentum tensor (1.6) is not a descendant of the primary unit operator owing to the presence of the structure (4.4) which is not a primary scalar. Nevertheless, this logarithmic divergence is what a doctor ordered to have finite $\gamma_{\rm str}$ which would diverge otherwise in the path-integral formalism.

I have developed a kind of the general technique to deal with the conformal transformation generated by a nonprimary energy-momentum tensor. It is based on using the representation (5.1) of the generator, which takes into account the quantum equation of motion and involves singular products in the quantum case. Using this technique, I have calculated the commutator of two conformal transformations with the central extension shown in eq. (5.9). For the quadratic action it reproduces the usual results, while for the higher-derivative action (1.2) the central extension being φ-dependent still results at one loop in the Virasoro algebra with the central charge given by the sum of (6.7) and (6.8).

The most important task will be of course to go beyond the one-loop approximation for the higher-derivative action (1.2) to understand how $\gamma_{\rm str}$ may depend on d for the Nambu-Goto string represented by $G \neq 0$ in our toy model. A very interesting problem will be to find out what Mathematical structures may be encoded in (5.10) with a φ-dependent c. This is beyond the subject of these Notes dealing with the one-loop order.

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A Mathematica program showing conservation and tracelessness of T_{ab}

This Mathematica program can be copied and pasted in Mathematica to demonstrate the conservation and tracelessness of the energy-momentum tensor (1.6).

```
(* checking the conservation and traceless of T_ab for any G *)
(* the Liouville field *)
v = vp[z, bz]
G =.
(* Tzz for minimal interaction *)
Tmzz = D[v,
     z]^2 + (1 - G) (-2 eps D[v, z] D[4 Exp[-v] D[v, z, bz], z]) +
   4 eps G (D[v, z] D[Exp[-v], z, z, bz] +
      D[Exp[-v], z] D[v, z, z, bz]);
(* minimal T_ab is not traceless *)
Tmzbz = -4 G eps D[v, z, bz] D[Exp[-v], z, bz] +
   4 (1 - G) eps Exp[-v] D[v, z, bz]^2;
Expand[%]
(* classical equation of motion -> cem=0 *)
cem = -D[v, z, bz] +
   eps  (1 - G) (D[4 Exp[-v] D[v, z, bz], z, bz] -
      1/2 Exp[-v] 4 D[v, z, bz]^2) +
   2 eps G (Exp[-v] D[v, z, z, bz, bz] - D[Exp[-v], z, z, bz, bz]);
(* conservation of minimal Tzz *)
Expand[D[Tmzz, bz] + D[Tmzbz, z] + 2 D[v, z] cem]
(* contribution from nonminimal interaction except nonlocal term *)
delTzz = -2 (1 - G) D[v - 4 eps Exp[-v] D[v, z, bz], z, z] -
   2 G D[v - 4 eps Exp[-v] D[v, z, bz] +
      2 eps Exp[-v] D[v, z] D[v, bz], z, z] +
   4 eps  G D[Exp[-v] D[v, z, bz] D[v, z], z];
(* additional contribution from nonlocal term *)
addi = eps G D[D[v, bz] 4 Exp[-v] D[v, z, bz], z, z];
(* final calculation of d_bz T_zz *)
Expand[D[Tmzz + delTzz, bz] + addi + 2 D[v, z] cem - 2 D[cem, z]]
(* =0 -> it works! *)
```

B Web of formulas for the singular products

Let me begin by mentioning a very important role played in CFT by the formula

$$\delta^{(2)}(z) = \bar{\partial} \frac{1}{\pi z}. \tag{B.1}$$

Its big brothers can be derived using the obvious identity

$$\frac{1}{z^n}\bar{\partial}\frac{1}{z} = (-1)^n \frac{1}{(n+1)!}\partial^n\bar{\partial}\frac{1}{z}. \tag{B.2}$$

Proceeding this way, we arrive at the formula

$$\begin{aligned} &8\pi\int \mathrm{d}^2 z\, \xi(z)\partial^n G_0(z)\delta^{(2)}(z) = (-1)^n H_n \xi^{(n)}(0), \\ &H_1 = 1, \qquad H_2 = \frac{1}{3}, \qquad H_3 = \frac{1}{6}, \qquad H_n = \frac{2}{n(n+1)}. \end{aligned} \tag{B.3}$$

for the singular products in a free CFT with the propagator

$$G_0(z) = -\frac{1}{2\pi}\log\left(\sqrt{z\bar{z}}\mu\right), \tag{B.4}$$

where μ represents a IR cutoff.

I derive in this appendix eq. (B.3) and other similar formulas explicitly taking into account a regularization of the singular products.

To derive (B.3) and its generalizations, we introduce a UV regularization a for example by

$$G_a(z) = -\frac{1}{4\pi}\log\left[(z\bar{z}+a^2)\mu^2\right] \tag{B.5}$$

and regularize the delta function by

$$\delta_a^{(2)}(z) = -4\partial\bar{\partial}G_a(z) = \frac{a^2}{\pi(z\bar{z}+a^2)^2} \tag{B.6}$$

as is prescribed by the formula of the type

$$-\frac{1}{2b_0^2\pi}\int \mathrm{d}^2 z\,\xi(z)\left\langle (\partial\varphi(z)\partial\bar{\partial}\varphi(z))\,\mathrm{e}^{\varphi(0)}\right\rangle = 8\pi b_0^2\int \mathrm{d}^2 z\,\xi(z)\partial G_a(z)(-4\partial\bar{\partial})G_a(z)\left\langle\,\mathrm{e}^{\varphi(0)}\right\rangle \tag{B.7}$$

for the conformal transformation of a free field. We then derive

$$8\pi\int \mathrm{d}^2 z\,\xi(z)\partial^n G_a(z)\delta_a^{(2)}(z) = (-1)^n H_n\xi^{(n)}(0), \tag{B.8}$$

reproducing eq. (B.3) in the limit $a \to 0$.

It is instructive to repeat the computation for the proper-time regularization, where the lower limit of the integrals over the proper time is $\tau > 0$. We then have

$$G_\tau(z) = \frac{1}{4\pi}\left[\mathrm{Ei}\Big(-\frac{z\bar{z}}{4\tau}\Big) - \log(z\bar{z}\mu^2)\right], \qquad \delta_\tau^2(z) = -4\partial\bar{\partial}G_\varepsilon(z) = \frac{\mathrm{e}^{-z\bar{z}/4\tau}}{4\pi\tau} \tag{B.9}$$

with Ei being the exponential integral, which gives

$$8\pi\int \mathrm{d}^2 z\,\xi(z)\partial^n G_\tau(z)\delta_\tau^{(2)}(z) = (-1)^n\frac{1}{2^{n-1}n}\xi^{(n)}(0). \tag{B.10}$$

Rather surprisingly the numbers are not universal except for $n = 1$ and depend on the regularization applied. However, for the central charge in eq. (6.3) we have

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{6}\right) = \frac{1}{12} \tag{B.11}$$

from eq. (B.8) and

$$\frac{1}{2}\left(\frac{1}{4}-\frac{1}{12}\right)=\frac{1}{12} \tag{B.12}$$

from eq. (B.10) that gives $c=1$ in both cases. We thus may expect that observables are universal.

Equation (B.8) is to be compared with an analogous formula for the free massive field Y with the regularized propagator

$$\langle Y(z)Y(0)\rangle = 8\pi b_0^2 G_{M,a}(z), \qquad G_{M,a}(z)=\frac{1}{2\pi}K_0(M\sqrt{z\bar z+a^2}). \tag{B.13}$$

and the regularized delta-function

$$\delta^{(2)}_{M,a}(z)=(-4\partial\bar\partial+M^2)\frac{1}{2\pi}K_0(M\sqrt{z\bar z+a^2})=\frac{a^2M^2K_2(M\sqrt{z\bar z+a^2})}{2\pi(z\bar z+a^2)}. \tag{B.14}$$

In the limit $aM\to 0$, which is justified by small a (and/or small M), we find

$$\int \mathrm{d}^2z\,\xi(z)\,\langle\partial^n Y(z)Y(0)\rangle\,\delta^{(2)}(z)=(-1)^n b_0^2 H_n \xi^{(n)}(0) \tag{B.15}$$

reproducing the numbers in eq. (B.3). This shows the vanishing of the quantum correction to the total central charge of φ plus the requlator in the quadratic case if the regulators are not path-integrated out.

One more sequence of the numbers emerges for the regularization by higher-derivatives like in eq. (1.2). We then have for the propagator of the higher-derivative massless field

$$\langle\varphi(z)\varphi(0)\rangle = 8\pi b_0^2 G_\varepsilon(z), \qquad G_\varepsilon(z)=-\frac{1}{2\pi}\left[K_0\Big(\sqrt{\frac{z\bar z}{\varepsilon}}\Big)+\log\big(\sqrt{z\bar z}\mu\big)\right] \tag{B.16}$$

in coordinate space. We also introduce

$$\delta^{(2)}_\varepsilon(z)=-4\partial\bar\partial G_\varepsilon(z)=\frac{1}{2\pi\varepsilon}K_0\left(\sqrt{\frac{z\bar z}{\varepsilon}}\right)\overset{\varepsilon\to 0}{\to}\delta^{(2)}(z) \tag{B.17}$$

to obtain

$$8\pi\int \mathrm{d}^2z\,\xi(z)\partial^n G_\varepsilon(z)\delta^{(2)}_\varepsilon(z)=(-1)^n b_0^2 H_n\xi^{(n)}(0) \tag{B.18}$$

which is the same as for the regularization (B.5) and

$$\begin{gathered}8\pi\int \mathrm{d}^2z\,\xi(z)(-4\varepsilon\partial^{n+1}\bar\partial)G_\varepsilon(z)\delta^{(2)}_\varepsilon(z)=(-1)^n b_0^2 J_n\xi^{(n)}(0),\\ J_1=1,\qquad J_2=\frac{2}{3},\qquad J_3=\frac{1}{2},\qquad J_n=\frac{2}{n+1}.\end{gathered} \tag{B.19}$$

For the higher-derivative action (1.2) the (unregularized) propagator is given by (B.16) but now we consider ε as a finite parameter. To be more precise, the consideration in the previous paragraph is associated with the usual BPZ case $\sqrt{\varepsilon}\ll z$ when ε is a regularizing parameter, while now we introduce a UV cutoff a and consider $a\ll z\ll\sqrt{\varepsilon}$. Speaking

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another way, we now consider the limit $\varepsilon \to \infty$ which is described for $G = 0$ by a free conformal theory with two scalar fields [8].

Introducing the UV cutoff a as before

$$\langle\varphi(z)\varphi(0)\rangle = 8\pi b_0^2 G_{\varepsilon,a}(z), \qquad G_{\varepsilon,a}(z) = -\frac{1}{2\pi}\left[K_0\Big(\sqrt{\frac{z\bar z + a^2}{\varepsilon}}\Big) + \log\big(\sqrt{z\bar z + a^2}\,\mu\big)\right] \tag{B.20}$$

and also introducing

$$\delta^{(2)}_{\varepsilon,a}(z) = \Big(-4\partial\bar\partial + 16\varepsilon\partial^2\bar\partial^2\Big)\, G_{\varepsilon,a}(z) \overset{a\to 0}{\to} \delta^{(2)}(z), \tag{B.21}$$

we find after a little computation

$$8\pi \int \mathrm{d}^2 z\, \xi(z)\partial^n G_{\varepsilon,a}(z)\delta^{(2)}_{\varepsilon,a}(z) \overset{a\to 0}{\to} 0 \tag{B.22}$$

and

$$\begin{aligned} &8\pi \int \mathrm{d}^2 z\, \xi(z)(-4\varepsilon\partial^{n+1}\bar\partial)G_{\varepsilon,a}(z)\delta^{(2)}_{\varepsilon,a}(z) \overset{a\to 0}{\to} (-1)^n b_0^2 Q_n \xi^{(n)}(0), \\ &Q_1 = 1, \qquad Q_2 = \frac{3}{10}, \qquad Q_3 = \frac{2}{15}, \qquad Q_n = \frac{12}{n(n+2)(n+3)}. \end{aligned} \tag{B.23}$$

This set of numbers differs from those in eq. (B.19) except for $n = 1$ which was the case in eq. (6.9).

The most general case is the higher-derivative massive field, whose (unregularized) propagator reads

$$\langle Y(-p)Y(p)\rangle = \frac{8\pi b_0^2}{p^2 + M^2 + \varepsilon p^4}. \tag{B.24}$$

in momentum space or, introducing the regularization a as before,

$$\begin{aligned} \langle Y(z)Y(0)\rangle &= 8\pi b_0^2 G_{\varepsilon,M,a}(z), \\ G_{\varepsilon,M,a}(z) &= \frac{1}{2\pi\varepsilon(M_+ - M_-)}\left[K_0(M_-\sqrt{z\bar z + a^2}) - K_0(M_+\sqrt{z\bar z + a^2})\right], \\ M_\pm &= \frac{1 \pm \sqrt{1 - 4\varepsilon M^2}}{2\varepsilon} \end{aligned} \tag{B.25}$$

in coordinate space. The associated set of numbers coincides for $a \to 0$ with the one in eqs. (B.22), (B.23). This again provides the cancellation of the (finite part of the) quantum correction to the central charge coming from φ and the regulators if the regulators are not integrated out. The sum of two logarithmically divergent parts is then regularized as $\log(M^2\varepsilon)$.

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