NOTES ON HOTELLING'S GENERALIZED T

By P. L. Hsu

1. Frequency Distribution When the Hypothesis Tested is Not True

a. The Problem. Let the simultaneous elementary probability law of the $k(f+1)$ variables z_i and z'_{ir} ($i = 1, 2, \cdots, k$; $r = 1, 2, \cdots, f$) be

$$(1) \quad p(z, z') = (\sqrt{2\pi})^{-k(f+1)} | C |^{\frac{1}{2}(f+1)} \exp\left[-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij}\{(z_i - \zeta_i)(z_j - \zeta_j) + v'_{ij}\} \right],$$

where

$$v'_{ij} = \sum_{r=1}^{f} z'_{ir} z'_{jr} \qquad (i, j = 1, 2, \cdots, k)$$

C stands for the matrix $\| c_{ij} \|$ and $| C |$, the corresponding determinant. It is required to find the elementary probability law of the statistic

$$T = | V' |^{-1} \sum_{i,j=1}^{k} V'_{ij} z_i z_j,$$

where $| V' | = | v'_{ij} |$ and V'_{ij} denotes the cofactor of the element v'_{ij} in the matrix $\| v'_{ij} \|$.

The quantity fT is a generalization of "Student's" t considered by Hotelling [1]*. It is an appropriate criterion to test the hypothesis, say H_0, that the ζ_i in the parent population as given by (1) all vanish. The distribution of T when the hypothesis H_0 is true has already been obtained by Hotelling. But our knowledge of the test is hardly complete unless we know also the distribution of T when the ζ_i do not all vanish. Indeed, only such a knowledge can enable us to control the risk of error of the second kind, i.e. of failure to detect that H_0 is untrue [3, 4].

b. The Solution. Let H be a $k \times k$ non-singular matrix such that $H'CH = I$, the unit matrix, where H' denotes the transposed matrix of H. Let the sets of variables $(z_1, z_2, \cdots, z_k)$ and $(z'_{1r}, z'_{2r}, \cdots, z'_{kr})$ $(r = 1, 2, \cdots f)$ be subject to the same collineation by means of H, so that

$$\| z_1, z_2, \cdots, z_k \| = \| t_1, t_2, \cdots, t_k \| \cdot H'$$

$$\| z'_{1r}, z'_{2r}, \cdots, z'_{kr} \| = \| t'_{1r}, t'_{2r}, \cdots, t'_{kr} \| \cdot H' \quad (r = 1, 2, \cdots, f)$$

where the t_i and t'_{ir} are the new variables. Let further the quantities τ_i be defined by

* References are given at the end of the paper.

(2) $$\|\zeta_1, \zeta_2, \cdots, \zeta_k\| = \|\tau_1, \tau_2, \cdots, \tau_k\| \cdot H'.$$

Then, as is easy to verify, the simultaneous distribution of the new variables will be given by

(3) $$p_1(t, t') = (\sqrt{2\pi})^{-k(f+1)} \exp\left[-\frac{1}{2}\sum_{i=1}^{k}\{(t_i - \tau_i)^2 + u'_{ii}\}\right],$$

while the statistic T, as a function of the t's, retains the original form:

(4) $$T = |U|^{-1}\sum_{i,j=1}^{k} U'_{ij} t_i t_j$$

where

$$u'_{ij} = \sum_{r=1}^{f} t'_{ir} t'_{jr} \qquad (i, j = 1, 2, \cdots, k),$$

$|U'| = |u'_{ij}|$, and U'_{ij} is the cofactor of the element u'_{ij} in the matrix $\| u'_{ij} \|$. By virtue of (2) we have the following relation between the old and new parametric constants:

(5) $$\sum_{i=1}^{k} \tau_i^2 = \sum_{i,j=1}^{k} c_{ij}\zeta_i\zeta_j.$$

Our problem is thus reduced to finding the derived distribution of T defined by (4) from the parent population given by (3).

We solve this problem by obtaining an expression for the Laplace integral $E(e^{-\beta T})$, i.e. the mathematical expectation of $e^{-\beta T}$ for real non-negative β. A few words are perhaps needed to explain the fact that the Laplace transform of an elementary probability law determines the latter uniquely except on a null set of points. If $f(x)$ is an elementary probability law which vanishes for all negative x and if

$$g(\beta) = \int_0^\infty e^{-\beta x} f(x)\, dx \quad \text{for} \quad \beta \geq 0,$$

then, letting c be any fixed positive constant, we have

$$g(c - \beta) = \int_0^\infty e^{\beta x} e^{-cx} f(x)\, dx$$

for all $\beta \leq c$. We get therefore

$$m_h = \int_0^\infty x^h e^{-cx} f(x)\, dx = \frac{d^h}{d\beta^h} g(c - \beta)\Big|_{\beta=0}, \quad (h = 0, 1, 2, \cdots)$$

the definite integral being obviously finite for all $h \geq 0$. Now a sufficient condition that the set of numbers m_h determines the function $e^{-cx}f(x)$ uniquely, with the exception of a null set at most, is that the latter multiplied by $e^{k\sqrt{x}}$ be summable $(0, \alpha)$ for some positive k (cf. [6], p. 320). Since this condition is trivially satisfied by the function $e^{-cx}f(x)$, this function, and therefore $f(x)$ itself, must be uniquely determined by the m_h. In other words, $f(x)$ is uniquely determined by its Laplace transform $g(\beta)$. We now proceed to find the Laplace integral $E(e^{-\beta T})$.

Introduce the function

$$g(t, t', \theta, \alpha) = (\sqrt{2\pi})^k \mid U' \mid^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left\{\sum_{i,j=1}^{k} u'_{ij}\theta_i\theta_j + 2i\alpha \sum_{i=1}^{k} t_i\theta_i\right\}\right]$$

and write

$$F(t, t', \theta, \alpha) = p_1(t, t')g(t, t', \theta, \alpha),$$

where all the arguments take real values only. For any functions $\varphi(\theta)$ and $\psi(t, t')$ let us write

$$\int \varphi(\theta)\, d\theta = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(\theta)\, d\theta_1 \cdots d\theta_k$$

$$\int \psi(t, t')\, d(t, t') = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi(t, t')\, dt_1 \cdots dt_k\, dt'_{11} \cdots dt'_{kf}.$$

We have

$$\int d(t, t') \int \mid F(t, t', \theta, \alpha) \mid d\theta = \int p_1(t, t')\, d(t, t') \int g(t, t', \theta, 0)\, d\theta = 1$$

whence we know that

$$\int d(t, t') \int F\, d\theta = \int d\theta \int F\, d(t, t') \tag{6}$$

On the right-hand side of (6) we find

$$\int p_1(t, t')\, d(t, t') \int g(t, t', \theta, \alpha)\, d\theta = \int e^{-\frac{1}{2}\alpha^2 T} p_1(t, t')\, d(t, t') = E(e^{-\frac{1}{2}\alpha^2 T})$$

while for the integral on the right-hand side of (6) we have

$$\begin{aligned} &\int F\, d(t, t') \\ &= (\sqrt{2\pi})^{-k(f+2)} \exp\left(-\frac{1}{2}\sum_{i=1}^{k} \tau_i^2\right) \int \exp\left[-\frac{1}{2}\sum_{i=1}^{k} \{t_i^2 + 2(i\alpha\theta_i - \tau_i\}\right] dt \\ &\times \int \mid U' \mid^{\frac{1}{2}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{k} (\theta_i\theta_j + \delta_{ij})\, u'_{ij}\right] dt', \end{aligned} \tag{7}$$

where we mean by the δ_{ij} the quantities

$$\left.\begin{aligned} \delta_{ij} &= 0 \qquad \text{for } i \neq j \\ \delta_{ii} &= 1 \end{aligned}\right\} (i, j = 1, 2, \cdots, k)$$

In the equation (7) the integral with respect to the t_i is immediately written down as

$$(\sqrt{2\pi})^k \exp\left[\frac{1}{2}\sum_{i=1}^{k}(\tau_i - i\alpha\theta_i)^2\right]$$

As to the integral with respect to the t'_{ir}, we may evaluate it by the method by which Wilks [7] evaluated the moments of the generalized variance. The result is

$$2^{\frac{1}{2}k}(\sqrt{2\pi})^{kf}\,|\,\theta_i\theta_j + \delta_{ij}\,|^{-\frac{1}{2}(f+1)}\,\frac{\Gamma\frac{1}{2}(f+1)}{\Gamma(\frac{1}{2}(f+1-k))}$$

Making the substitution into (7) we get, after necessary reductions,

$$\int F\,d(t, t') = \frac{(\sqrt{\pi})^{-k}\Gamma(\frac{1}{2}(f+1))}{\Gamma(\frac{1}{2}(f+1-k))}\,|\,\theta_i\theta_j + \delta_{ij}\,|^{-\frac{1}{2}(f+1)} \times \exp\left[-\sum_{i=1}^{k}\{\tfrac{1}{2}\alpha^2\theta_i^2 + i\alpha\tau_i\theta_i\}\right]$$

whence, noticing that $|\,\theta_i\theta_j + \delta_{ij}\,| = 1 + \sum_{i=1}^{k}\theta_i^2$

$$E(e^{-\frac{1}{2}\alpha^2 T}) = \frac{(\sqrt{\pi})^{-k}\Gamma(\frac{1}{2}(f+1))}{\Gamma(\frac{1}{2}(f+1-k))}\int\left(1 + \sum_{i=1}^{k}\theta_i^2\right)^{-\frac{1}{2}(f+1)} \exp\left[-\sum_{i=1}^{k}\{\tfrac{1}{2}\alpha^2\theta_i^2 + i\alpha\tau_i\theta_i\}\right]d\theta \tag{8}$$

Equation (8) gives the Laplace transform of the elementary probability law, $p(T)$, of T. There is no essential difficulty in getting $p(T)$ by inversion directly from (8). Nevertheless, it may be of interest to get $p(T)$ indirectly by identifying the right-hand side of (8) with the Laplace transform of another elementary probability law which is otherwise *known*. For this purpose consider the simultaneous elementary probability law

$$p(x, y) = (\sqrt{2\pi})^{-(f_1+f_2)}\exp\left[-\frac{1}{2}\sum_{i=1}^{f_1}(x_i - \xi_i)^2 - \frac{1}{2}\sum_{j=1}^{f_2}y_j^2\right]$$

and let us find the derived distribution of the statistic

$$L = \sum_{i=1}^{f_1}x_i^2 \Big/ \sum_{j=1}^{f_2}y_j^2$$

As before, we introduce the function

$$g(x, y, \theta, \alpha) = (\sqrt{2\pi})^{-f_1}\left(\sum_{j=1}^{f_2}y_j^2\right)^{\frac{1}{2}f_1}\exp\left[-\frac{1}{2}\left(\sum_{j=1}^{f_2}y_j^2\sum_{i=1}^{f_1}\theta_i^2 + 2i\alpha\sum_{i=1}^{f_1}x_i\theta_i\right)\right]$$

write

$$F(x, y, \theta, \alpha) = p(x, y)g(x, y, \theta, \alpha)$$

and ascertain that

$$\int d(x, y) \int F\, d\theta = \int d\theta \int F\, d(x, y) \tag{9}$$

On the left-hand side of (9) we find

$$\int e^{-\frac{1}{2}\alpha^2 L} p(x, y)\, d(x, y) = E(e^{-\frac{1}{2}\alpha^2 L})$$

while for the integral on the right-hand side of (9), we have

$$\begin{aligned}\int F\, d(x, y) &= (\sqrt{2\pi})^{-(2f_1+f_2)} \exp\left(-\frac{1}{2}\sum_{i=1}^{f_1} \xi_i^2\right) \\ &\quad \times \int \exp\left[-\frac{1}{2}\sum_{i=1}^{f_1} \{x_i^2 + 2(i\alpha\theta_i - \xi_i)x_i\}\right] dx \\ &\quad \times \int \left(\sum_{j=1}^{f_2} y_j^2\right)^{\frac{1}{2}f_1} \exp\left[-\frac{1}{2}\left(1 + \sum_{i=1}^{f_1} \theta_i^2\right)\sum_{j=1}^{f_2} y_j^2\right] dy \\ &= \frac{(\sqrt{\pi})^{-f_1}\Gamma(\frac{1}{2}(f_1+f_2))}{\Gamma(\frac{1}{2}f_2)} \left(1 + \sum_{i=1}^{f_1} \theta_i^2\right)^{-\frac{1}{2}(f_1+f_2)} \\ &\quad \exp\left[-\frac{1}{2}\sum_{i=1}^{f_1} (\alpha^2\theta_i^2 + 2i\alpha\xi_i\theta_i)\right]\end{aligned}$$

Writing

$$f_1 = k, \qquad f_2 = f + 1 - k \tag{10}$$

we get finally

$$\begin{aligned}E(e^{-\frac{1}{2}\alpha^2 L}) &= \frac{(\sqrt{\pi})^{-k}\Gamma(\frac{1}{2}(f+1))}{\Gamma(\frac{1}{2}(f+1-k))} \int \left(1 + \sum_{i=1}^{k} \theta_i^2\right)^{-\frac{1}{2}(f+1)} \\ &\quad \exp\left[-\frac{1}{2}\sum_{i=1}^{k} (\alpha^2\theta_i^2 + 2i\alpha\xi_i\theta_i)\right]\end{aligned} \tag{11}$$

From the identity of (8) and (11) we conclude that T is distributed exactly the same as L with the appropriate "degrees of freedom" f_1 and f_2 given by (10). But the elementary probability law of L has already been derived by P. C. Tang [5]. Using his result we immediately write down the elementary probability law of T:

$$p(T) = e^{-\lambda} \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \frac{1}{B(\frac{1}{2}f_1 + h, \frac{1}{2}f_2)} T^{\frac{1}{2}f_1+h-1}(1 + T)^{-\frac{1}{2}(f_1+f_2)-h} \tag{12}$$

where f_1 and f_2 are given by (10) and

$$\lambda = \frac{1}{2}\sum_{i=1}^{k} \tau_i^2 = \frac{1}{2}\sum_{i,j=1}^{k} c_{ij}\zeta_i\zeta_j \tag{13}$$

in accordance with (5). The tables of probability integrals prepared by Tang can, of course, be used to suit our purpose.

2. An Optimum Property of the T-Test. To any reader familiar with the Neyman-Pearson theory of testing statistical hypotheses [3, 4], the theorem stated below may be of considerable interest.

Denote by W the $k(f+1)$-dimensional space of the z_i and z'_{ir} and let w be any region in W which may possibly serve as a critical region for the rejection of the hypothesis H_0. Let us speak of a critical region w as belonging to the class D if w satisfies the following condition:

$$\int_w p(z, z')d(z, z') = \epsilon + \frac{\alpha}{2} \sum_{i,j=1}^{k} c_{ij}\zeta_i\zeta_j + R \tag{14}$$

where $\epsilon < 1$ is a positive constant independent of the ζ_i, c_{ij} and the region w, α is a constant depending on w only, but not on the ζ_i or c_{ij}, and where R for any given set of values of the c_{ij} is an infinitesimal of at least the third order as all the ζ_i tend to zero.

THEOREM. *Of all the regions belonging to the class D, the particular region which gives the largest possible value to the coefficient α in the equation* (14) *is the region defined by $T \geq T_\epsilon$, where T_ϵ is a constant so determined that the probability, when all ξ_i vanish, of the observed T being not less than T_ϵ is exactly ϵ.*

The significance of the theorem is clear. Every critical region belonging to the class D serves as an unbiased exact test of the hypothesis H_0, ϵ being the preassigned chance of rejecting H_0 if it is true. Further, as is seen from (14), as the ζ_i start to depart from zero, the increased chance of rejecting H_0 due to its falsehood is approximately proportional to the quantity $\Sigma c_{ij}\zeta_i\zeta_j$. The coefficient α therefore measures the power of the critical region w to detect the falsehood of H_0, at least when the departure of the ζ_i from zero is small. Our theorem asserts that in this particular sense the T-test is the most powerful of its kind.

The method of proof is very much the same as that by which Neyman and Pearson proved some of their general theorems concerning unbiased tests. However, as the present theorem has not yet been contained in their more general results, we shall give it a full proof without referring, save in one occasion, to these authors.

PROOF. Write

$$v'_{ij} + z_iz_j = s_{ij} \qquad (i, j = 1, 2, \cdots, k)$$

$$p_0(z, z') = (\sqrt{2\pi})^{-k(f+1)}C^{\frac{1}{2}(f+1)} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{k} c_{ij}s_{ij}\right] \tag{15}$$

and denote by $p_0(s)$ the simultaneous elementary probability law of the variables s_{ij} derived from (15). Let W_1 be the domain of all possible positions of the point $(s_{11}, s_{12}, \cdots, s_{kk})$ in the $\frac{1}{2}k(k+1)$-dimensional space.

We know, although we omit the proof of it, that there is no elementary probability law of the variables s_{ij} other than $p_0(s)$ which has the same moments of all orders as those derived from $p_0(z, z')$. It then follows that if $g(s)$ be any summable function of the s_{ij} and if

$$\int_{W_1} \left(\prod_{i,j=1}^{k} s_{ij}^{r_{ij}} \right) g(s) p_0(s)\, ds = 0 \tag{16}$$

for all positive integers r_{ij} or zero, then we must have $g(s) \equiv 0$ except perhaps on a null set of points.

It follows therefore that the identity

$$\int_{W_1} g(s) p_0(s)\, ds \equiv 0 \tag{17}$$

implies the identity $g(s) \equiv 0$ provided $g(s)$ does not involve the parameters c_{ij}. For, substituting for $p_0(s)$ its expression as given by Wishart [8] we shall have

$$K \int_{W_1} g(s) p_0(s)\, ds \equiv \int_{W_1} g(s) \mid S \mid^{\frac{1}{2}(f-k)} \exp\left[-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij} s_{ij} \right] ds \equiv 0 \tag{18}$$

where $\mid S \mid = \mid s_{ij} \mid$ and K is some constant. Differentiating (18) successively with respect to the c_{ij} and dividing the results by K, we shall regain the equations (16). Hence it follows that $g(s) \equiv 0$.

This being established, let w be any region belonging to D and rewrite the equation (14), so that

$$\begin{aligned} (\sqrt{2\pi})^{-k(f+1)} C^{\frac{1}{2}(f+1)} \int_{w} \exp\left[-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij} \{ (z_i - \zeta_i)(z_j - \zeta_j) + v'_{ij} \} \right] d(z, z') \\ = \epsilon + \frac{\alpha}{2} \sum_{i,j=1}^{k} c_{ij} \zeta_i \zeta_j + R \end{aligned} \tag{19}$$

Setting all the ζ_i to zero in both sides of (19), we have

$$\int_{w} p_0(z, z')\, d(z, z') \equiv \epsilon \tag{20}$$

identically in the c_{ij}. Differentiating (19) once with respect to ζ_i and afterwards setting all the ζ_i to zero, we easily get

$$\int_{w} z_i p_0(z, z')\, d(z, z') \equiv 0 \qquad (i = 1, 2, \cdots k) \tag{21}$$

for all possible values of the c_{ij}.

Finally, differentiating (19) with respect to ζ_i and then to ζ_j and putting all $\zeta_i = 0$ in the result we obtain

$$\int_{w} \left\{ \left(\sum_{h=1}^{k} c_{ih} z_h \right) \left(\sum_{h=1}^{k} c_{jh} z_h \right) - c_{ij} \right\} p_0(z, z')\, d(z, z') \equiv \alpha c_{ij} \qquad (i, j = 1, 2, \cdots k)$$

whence, renumbering (20)

$$\sum_{h,l=1}^{k} c_{ih} c_{jl} q_{hl} \equiv \beta c_{ij} \qquad (i, j = 1, 2, \cdots, k) \tag{22}$$

in which we denote by $\beta = \alpha + \epsilon$ and

$$q_{hl} = \int_w z_h z_l p_0(z, z')\, d(z, z') \qquad (h, l = 1, 2, \cdots, k)$$

If we denote by Q the matrix of order k formed of the elements q_{hl}, we see that (22) may be written as

$$CQC \equiv \beta C,$$

whence, since C has its inverse matrix, C^{-1},

$$Q \equiv \beta C^{-1}$$

i.e.,

$$q_{ij} \equiv \beta c_{ij}^{(-1)} \qquad (i, j = 1, 2, \cdots, k) \tag{23}$$

where $c_{ij}^{(-1)}$ denotes the element in the matrix C^{-1} which corresponds to the element c_{ij} in the matrix C.

Conditions (20), (21) and (23) are necessary for the region w to belong to the class D. They are evidently also sufficient.

Let us evaluate the integrals in (20), (21) and the q_{ij} by first evaluating the surface integrals on any surface, say $G(s)$, on which all the s_{ij} have constant values, and then integrating the results with respect to the s_{ij} over a region, say w_1, of the s_{ij} contained in W_1. Thus we may write (20), (21) and (23) in the form

$$\int_{w_1} f(s) p_0(s)\, ds \equiv \epsilon, \qquad \int_{w_1} g_i(s) p_0(s)\, ds \equiv 0, \qquad \int_{w_1} \varphi_{ij}(s) p_0(s) \equiv \beta c_{ij}^{(-1)}, \qquad (i, j = 1, 2, \cdots, k), \tag{24}$$

where

$$f(s) = \frac{1}{p_0(s)} \int_{G(s)} p_0(z, z')\, dG(s)$$

$$g_i(s) = \frac{1}{p_0(s)} \int_{G(s)} z_i p_0(z, z')\, dG(s)$$

$$\varphi_{ij}(s) = \frac{1}{p_0(s)} \int_{G(s)} z_i z_j p_0(z, z')\, dG(s)$$

It is readily verified that the function $p_0(z, z')/p_0(s)$ is free from the parameters c_{ij}, and consequently so are the functions $f(s)$, $g_i(s)$, $\varphi_{ij}(s)$. Besides, we can extend the definition of these functions in the whole domain W_1 by assigning them the value zero outside of the region w_1. Doing this we can now write the equations (24) as

$$\int_{W_1} (f(s) - \epsilon) p_0(s)\, ds \equiv 0, \qquad \int_{W_1} g_i(s) p_0(s)\, ds \equiv 0,$$
$$\int_{W_1} [\varphi_{ij}(s) - \gamma s_{ij}]\, p_0(s)\, ds \equiv 0 \qquad (i, j = 1, 2, \cdots, k) \tag{25}$$

where $\gamma = \dfrac{1}{f + 1} \beta$.

Now all the equations (25) are of the form (17); consequently, according to the already established result and remembering the definitions of the functions $f(s)$, $g_i(s)$ and $\varphi_{ij}(s)$, we must have

$$\int_{G(s)} p_0(z, z')\, dG(s) = \epsilon p_0(s) \tag{26}$$

$$\int_{G(s)} z_i p_0(z, z')\, dG(s) = 0 \tag{27}$$

$$\int_{G(s)} z_i z_j p_0(z, z')\, dG(s) = \gamma s_{ij} p_0(s) \tag{28}$$

in the whole domain W_1.

Hence the most general region belonging to the class D is constructed as follows. On any surface $s_{ij} = \text{const.}$ $(i, j = 1, 2, \cdots k)$ we take an areal region such that it satisfies the equations (26)–(28); we then allow the s_{ij} to vary in the whole domain W_1. Equations (28) may now be replaced by

$$\int_{G(s)} \left(\frac{z_1^2}{s_{11}} - \frac{z_i z_j}{s_{ij}}\right) p_0(z, z') = 0, \qquad (i, j = 1, 2, \cdots, k) \tag{28'}$$

Let us call w_0 the region defined by $T \geq T_\epsilon$. Since w_0 belongs to the class D (cf. (12)), its cross section, say $G_0(s)$, by any surface $s_{ij} = \text{const.}$ $(i, j = 1, 2, \cdots, k)$ must satisfy the equations (26), (27) and (28'). Since $\gamma = \dfrac{1}{f+1}(\alpha + \epsilon)$, all we have to prove now is that among all the areal regions $G(s)$ satisfying the equations (26), (27) and (28') it is the region $G_0(s)$ that gives the largest possible value to $\gamma p_0(s)$. Now

$$\gamma p_0(s) = \int_{G(s)} \frac{z_1^2}{s_{11}} p_0(z, z')\, d(z, z') \tag{29}$$

and, according to a Lemma of Neyman and Pearson, [3, p. 10] the right-hand side of (29) will attain its maximum value if $G(s)$ is defined by an inequality of the form

$$\frac{z_1^2}{s_{11}} \geq \sum_{i,j=1}^{k} a_{ij}\left(\frac{z_1^2}{s_{11}} - \frac{z_i z_j}{s_{ij}}\right) + \sum_{i=1}^{k} b_i z_i + c \tag{30}$$

where the a_{ij}, b_i and c are constants so determined as to enable the region $G(s)$ to satisfy the equations (26)–(28). We shall show presently that the region $G_0(s)$ is defined by such an inequality.

The inequality $T \geq T_\epsilon$ may be written as

$$\frac{|v'_{ij}|}{|v'_{ij} + z_i z_j|} \leq \frac{1}{1 + T_\epsilon}$$

f.e.

$$\frac{|s_{ij} - z_i z_j|}{|s_{ij}|} \leq \frac{1}{1 + T_\epsilon},$$

or

$$\sum_{i,j=1}^{k} s_{ij}^{(-1)} z_i z_j \geq \frac{T_\epsilon}{1+T_\epsilon} \tag{31}$$

where $s_{ij}^{(-1)}$ denotes the (i, j)th element in the inverse matrix of $\| s_{ij} \|$. The region $G_0(s)$ is therefore defined by the same inequality (31) in which we regard the s_{ij} as constants.

If we put

$$a_{ij} = \frac{1}{k} s_{ij} s_{ij}^{(-1)}, \qquad b_i = 0, \qquad c = \frac{1}{k} \frac{1}{1+T_\epsilon} \qquad (i, j = 1, 2, \cdots, k)$$

in (30) we can easily reduce the inequality (30) into (31).

The proof is now complete.

3. Note on Applications of T. It is already known that the T-test may be used for the following purposes (a) and (b):

(a) Given a k-variate normal surface

$$p(x) = (\sqrt{2\pi})^{-k} C^{\frac{1}{2}} \exp\left[-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij}(x_i - \xi_i)(x_j - \xi_j) \right]$$

with the unknown ξ_i and c_{ij}. n observations

$$(x_{1l}, x_{2l}, \cdots, x_{kl}), \qquad (l = 1, 2, \cdots, n)$$

having been made, it is required to test the hypothesis that the ξ_i have the particular values ξ_i^0 for $i = 1, 2, \cdots, k$.

Here we use the T-test with

$$\left.\begin{aligned} z_i = \sqrt{n}(\bar{x}_i - \xi_i^0), \qquad v'_{ij} &= \sum_{l=1}^{m} (x_l - \bar{x}_i)(\bar{x}_{jl} - \bar{x}_j) \\ \zeta_i = \sqrt{n}\,(\xi_i - \xi_i^0), \qquad f &= n - 1 \end{aligned}\right\} \quad (i, j = 1, 2, \cdots, k)$$

where

$$\bar{x}_i = \frac{1}{n} \sum_{l=1}^{n} x_{il}$$

(b) Given two k-variate normal surfaces

$$p_1(x) = (\sqrt{2\pi})^{-k} C^{\frac{1}{2}} \exp\left(-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij}(x_i - \xi_i)(x_j - \xi_j) \right)$$

$$p_2(x) = (\sqrt{2\pi})^{-k} C^{\frac{1}{2}} \exp\left(-\frac{1}{2} \sum_{i,j=1}^{k} c_{ij}(y_i - \eta_i)(y_j - \eta_j) \right)$$

where the c_{ij} are common to the two surfaces while all the ξ_i, ξ_j, c_{ij} are unknown. Samples of n_1 and n_2 having been drawn respectively from the two populations, to test the hypothesis that $\xi_i = \eta_i$ for all i.

Let the samples be

$$(x_{1l}, x_{2l}, \cdots, x_{kl}), \qquad (l = 1, 2, \cdots, n_1)$$

and

$$(y_{1h}, y_{2h}, \cdots, y_{kh}), \qquad (h = 1, 2, \cdots, n_2)$$

Let

$$\bar{x}_i = \frac{1}{n_1}\sum_{i=1}^{n1} x_{il}, \qquad \bar{y}_i = \frac{1}{n_2}\sum_{h=1}^{n2} y_{ih} \qquad (i = 1, 2, \cdots, k)$$

We use the T-test with

$$z_i = \sqrt{\frac{n_1 n_2}{n_1 + n_2}}(\bar{x}_i - \bar{y}_i), \qquad v'_{ij} = \sum_{l=1}^{n1}(x_{il} - \bar{x}_i)(x_{jl} - \bar{x}_j) + \sum_{h=1}^{n2}(y_{ih} - \bar{y}_i)(y_{ih} - \bar{y}_j)$$

$$\zeta_i = \sqrt{\frac{n_1 n_2}{n_1 + n_2}}(\xi_i - \eta_i), \qquad f = n_1 + n_2 - 2$$

$$(i, j = 1, 2, \cdots, k)$$

A third application of T, which appears to be novel, is the following:

(c) Given a $(k + 1)$-variate normal surface

$$p(x) = (\sqrt{2\pi})^{-(k+1)} D^{\frac{1}{2}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{k+1} d_{ij}(x_i - \xi_i)(x_j - \xi_j)\right], \qquad D = |d_{ij}|,$$

where the ξ_i and d_{ij} are all unknown. n observations

$$(x_{1l}, x_{2l}, \cdots, x_{k+1,l}) \qquad (l = 1, 2, \cdots, n)$$

having been made, to test the hypothesis that all the ξ_i are equal.

If we put

$$y_i = x_i - x_{k+1} \qquad (i = 1, 2, \cdots, k),$$

then we have a k-variate normal surface for the variables y_i.

$$p(y) = (\sqrt{2\pi})^{-k} C^{\frac{1}{2}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{k} c_{ij}(y_i - \eta_i)\right]$$

where $\eta_i = \xi_i - \xi_{k+1}$ $(i = 1, 2, \cdots, k)$. Thus the problem is reduced to testing the hypothesis that $\eta_i = 0$ for $i = 1, 2, \cdots, k$ and therefore belongs to the type (a). Write

$$y_{il} = x_{il} - x_{k+1,l} \qquad (i = 1, 2, \cdots, k;\ l = 1, 2, \cdots, n)$$

and

$$\bar{y}_i = \frac{1}{2}\sum_{l=1}^{n} y_{il}, \qquad (i = 1, 2, \cdots, k).$$

We use the T-test with

$$\left.\begin{aligned} z_i = \sqrt{n}\,\bar{y}_i, \qquad v'_{ij} = \sum_{l=1}^{n} (y_{il} - \bar{y})(y_{jl} - \bar{y}_j) \\ \zeta_i = \sqrt{n}\,\eta_i, \qquad f = n + 1 \end{aligned}\right\} (i, j = 1, 2, \cdots, k)$$

Although there are no simple expressions for the c_{ij}, there is one for the parameter $\Sigma\, c_{ij}\eta_i\eta_j$, on which alone the distribution of T depends. We have indeed

$$\sum_{i,j=1}^{k} c_{ij}\eta_i\eta_j = \frac{1}{D} \begin{vmatrix} \sigma_{11} & \cdots & \sigma_{1,\,k+1} & \xi_1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{k+1,1} & \cdots & \sigma_{k+1,\,k+1} & \xi_{k+1} & 1 \\ \xi_1 & \cdots & \xi_{k+1} & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{vmatrix}$$

where

$$D = \begin{vmatrix} \sigma_{11} & \cdots & \sigma_{1,\,k+1} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{k+1,1} & \cdots & \sigma_{k+1,\,k+1} & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix}$$

where σ_{ij} is the covariance between x_i and x_j.

Expressing T in terms of the original variables x, we have

$$T = -\frac{1}{D'} \begin{vmatrix} s_{11}\; s_{12} & \cdots & s_{1,\,k+1} & \bar{x}_1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{k+1,1} & \cdots & s_{k+1,\,k+1} & \bar{x}_{k+1} & 1 \\ \bar{x}_1 & \cdots & \bar{x}_{k+1} & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{vmatrix}$$

where

$$D' = \begin{vmatrix} s_{11} & \cdots & s_{1,\,k+1} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ s_{k+1,1} & \cdots & s_{k+1,\,k+1} & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix}$$

and where

$$\bar{x}_i = \frac{1}{n}\sum_{l=1}^{n} x_{il}, \qquad s_{ij} = \frac{1}{n}\sum_{l=1}^{n} (x_{il} - \bar{x}_i)(x_{jl} - \bar{x}_j), \qquad (i, j = 1, 2, \cdots, k+1)$$

Therefore T is independent of which variable has been taken as the $(k + 1)$st.

UNIVERSITY COLLEGE, LONDON.

REFERENCES

[1] H. HOTELLING, *Ann. Math. Statist.*, Vol. 2 (1931) pp. 359–378.

[2] J. NEYMAN, *Bull. Soc. Math. France*, Vol. 63 (1935) pp. 246–266.
[3] J. NEYMAN AND E. S. PEARSON, *Statist. Res. Mem.*, Vol. 1 (1936) pp. 1–37.
[4] J. NEYMAN AND E. S. PEARSON, *Statist. Res. Mem.*, Vol. 2 (1938).
[5] P. C. TANG, *Statist. Res. Mem.*, Vol. 2 (1938).
[6] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*. Oxford Univ. Press (1937).
[7] S. S. WILKS, *Biometrika*, Vol. 24 (1932) pp. 471–494.
[8] J. WISHART, *Biometrika*, Vol. 20A (1928) pp. 32–52.