

Notes on infinitesimal variations of submanifolds

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§ 0. Introduction.

In a previous paper [5], the present author studied variations of the metric tensor, the Christoffel symbols and the second fundamental tensors of submanifolds of a Riemannian manifold under infinitesimal variations of the submanifolds.

In this paper, we assume that submanifolds under consideration are compact and orientable and we obtain, using integral formulas, some global results on infinitesimal isometric, affine and conformal variations of the submanifolds.

§ 1. Preliminaries [1].

We consider an m -dimensional Riemannian manifold M^m covered by a system of coordinate neighborhoods $\{U; x^h\}$ and denote by g_{ji} , Γ_{ji}^h and ∇_j the metric tensor, the Christoffel symbols formed with g_{ji} and the operator of covariant differentiation with respect to Γ_{ji}^h of M^m respectively, where, here and in the sequel, the indices h, i, j, k, \dots run over the range $\{1', 2', \dots, m'\}$.

We then consider an n -dimensional compact orientable Riemannian manifold M^n covered by a system of coordinate neighborhoods $\{V; y^a\}$ and denote by g_{cb} , Γ_{cb}^a , ∇_c , $K_{dc}{}^a$ and K_{cb} the metric tensor, the Christoffel symbols formed with g_{cb} , the operator of covariant differentiation with respect to Γ_{cb}^a , the curvature tensor and the Ricci tensor of M^n respectively, where, here and in the sequel, the indices a, b, c, \dots run over the range $\{1, 2, \dots, n\}$.

We assume that M^n is isometrically immersed in M^m by the immersion: $M^n \rightarrow M^m$ and represent the immersion by

$$x^h = x^h(y^a).$$

Since the immersion is isometric, we have

$$(1.1) \quad g_{cb} = B_c{}^j B_b{}^i g_{ji},$$

where we have put $B_c{}^j = \partial_c x^j$ ($\partial_c = \partial/\partial y^c$).

We can assume that $[B_b{}^h]$ gives the positive orientation of M^n .

We choose $m-n$ mutually orthogonal unit normals C_y^h to M^n , where, here and in the sequel, the indices x, y, z run over the range $\{n+1, \dots, m\}$. The metric tensor of the normal bundle of M^n is given by

$$(1.2) \quad g_{zy} = C_z^j C_y^i g_{ji}.$$

Now, the equations of Gauss for M^n are written as

$$(1.3) \quad \nabla_c B_b^h = h_{cb}^x C_x^h,$$

where

$$\nabla_c B_b^h = \partial_c B_b^h + \Gamma_{ji}^h B_c^j B_b^i - \Gamma_{cb}^a B_a^h$$

is the van der Waerden-Bortolotti covariant derivative of B_b^h and h_{cb}^x are components of the second fundamental tensor with respect to the normal C_x^h .

On the other hand, the equations of Weingarten for M^n are written as

$$(1.4) \quad \nabla_c C_y^h = -h_c^a{}_y B_a^h,$$

where

$$\nabla_c C_y^h = \partial_c C_y^h + \Gamma_{ji}^h B_c^j C_y^i - \Gamma_{cy}^x C_x^h$$

is the van der Waerden-Bortolotti covariant derivative of C_y^h , Γ_{cy}^x being components of the linear connection induced in the normal bundle, that is,

$$\Gamma_{cy}^x = (\partial_c C_y^h + \Gamma_{ji}^h B_c^j C_y^i) C_h^x$$

and $C_h^x = C_y^i g^{yx} g_{ih}$, g^{yx} being contravariant components of the metric tensor of the normal bundle of M^n and

$$h_c^a{}_y = h_{cb}^z g^{ba} g_{zy},$$

g^{ba} being contravariant components of the metric tensor of M^n .

§ 2. Infinitesimal variations [2] [3] [5].

We now consider an infinitesimal variation of M^n given by

$$(2.1) \quad \bar{x}^h = x^h + \xi^h(y)\varepsilon,$$

where $\xi^h(y)$ is a vector field defined along M^n and ε is an infinitesimal.

Under the infinitesimal variation (2.1) the vectors B_b^h tangent to M^n are transformed into $\bar{B}_b^h = \partial_b \bar{x}^h = B_b^h + \partial_b \xi^h \varepsilon$ tangent to the deformed submanifold.

Carrying \bar{B}_b^h at (\bar{x}^h) back to (x^h) parallelly, we obtain

$$\tilde{B}_b^h = \bar{B}_b^h + \Gamma_{ji}^h(x + \xi\varepsilon)\xi^j \bar{B}_b^i \varepsilon,$$

that is,

$$(2.2) \quad \hat{B}_b^h = B_b^h + (\nabla_b \xi^h) \varepsilon,$$

neglecting terms of order higher than one with respect to ε , where

$$(2.3) \quad \nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}^h B_b^j \xi^i.$$

In the sequel we always neglect the terms of order higher than one with respect to ε .

Thus putting $\delta B_b^h = \hat{B}_b^h - B_b^h$, we have

$$(2.4) \quad \delta B_b^h = (\nabla_b \xi^h) \varepsilon.$$

If we decompose ξ^h as

$$(2.5) \quad \xi^h = \xi^a B_a^h + \xi^x C_x^h,$$

equation (2.4) can be written as

$$(2.6) \quad \delta B_b^h = [(\nabla_b \xi^a - h_b^a \xi^x) B_a^h + (\nabla_b \xi^x + h_{ba}^x \xi^a) C_x^h] \varepsilon.$$

Thus, from (1.1) and (2.6), we find [5]

$$(2.7) \quad \delta g_{cb} = (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx} \xi^x) \varepsilon,$$

where $\xi_b = \xi^c g_{cb}$ and $h_{cbx} = h_{cb}^y g_{yx}$.

From (2.7) we find [5]

$$(2.8) \quad \delta g^{ba} = -(\nabla^b \xi^a + \nabla^a \xi^b - 2h^{ba} \xi^x) \varepsilon,$$

where $\nabla^b = g^{ba} \nabla_a$ and $h^{ba} = h_{edax} g^{eb} g^{da}$.

When $\delta g_{cb} = 0$, we say that the infinitesimal variation is *isometric* and when $\delta g_{cb} = 2\lambda g_{cb} \varepsilon$, λ being a certain scalar, we say that the infinitesimal variation is *conformal*. If the variation is conformal and λ is a constant, we say that the infinitesimal variation is *homothetic*.

From (2.7) we have

THEOREM A. [5] *In order for an infinitesimal variation (2.1) of a submanifold to be isometric, it is necessary and sufficient that*

$$(2.9) \quad \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx} \xi^x = 0.$$

THEOREM B. [5] *In order for an infinitesimal variation (2.1) of a submanifold to be conformal, it is necessary and sufficient that*

$$(2.10) \quad \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx} \xi^x = 2\lambda g_{cb},$$

λ being a certain scalar.

Using (2.7) and (2.8), we calculate the infinitesimal variation of the

Christoffel symbols

$$\Gamma_{cb}^a = \frac{1}{2}(\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{cb})g^{ea}$$

and obtain

$$(2.12) \quad \delta\Gamma_{cb}^a = \frac{1}{2}[\nabla_c(\delta g_{be}) + \nabla_b(\delta g_{ce}) - \nabla_e(\delta g_{cb})]g^{ea},$$

which, using (2.7), we can write as

$$(2.13) \quad \delta\Gamma_{cb}^a = [\nabla_c \nabla_b \xi^a + K_{dcb}{}^a \xi^d - \nabla_c(h_b{}^a{}_x \xi^x) - \nabla_b(h_c{}^a{}_x \xi^x) + \nabla^a(h_{cbx} \xi^x)]\varepsilon.$$

If $\delta\Gamma_{cb}^a = 0$, we say that the infinitesimal variation is *affine* and if $\delta\Gamma_{cb}^a = (\delta_c^a p_b + \delta_b^a p_c)\varepsilon$ for some 1-form p_b , we say that the variation is *projective*.

Thus we have

THEOREM C. [5] *In order for an infinitesimal variation (2.1) of a submanifold to be affine, it is necessary and sufficient that*

$$(2.14) \quad \nabla_c \nabla_b \xi^a + K_{dcb}{}^a \xi^d - \nabla_c(h_b{}^a{}_x \xi^x) - \nabla_b(h_c{}^a{}_x \xi^x) + \nabla^a(h_{cbx} \xi^x) = 0.$$

Now we have the following integral formula [4]

$$(2.15) \quad \int \left[(g^{cb} \nabla_c \nabla_b \xi^a + K_d{}^a \xi^d) \xi_a + \frac{1}{2} (\nabla_c \xi_b + \nabla_b \xi_c) (\nabla^c \xi^b + \nabla^b \xi^c) - (\nabla_a \xi^a)^2 \right] dV = 0,$$

which is valid for an arbitrary vector field ξ^a in a compact orientable Riemannian manifold M^n , dV being the volume element of the manifold.

From (2.15), we can easily derive

$$(2.16) \quad \int \left[\{g^{cb} \nabla_c \nabla_b \xi^a + K_d{}^a \xi^d - 2\nabla^c(h_c{}^a{}_x \xi^x) + \nabla^a(h_c{}^c{}_x \xi^x)\} \xi_a \right. \\ \left. + \frac{1}{2} (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb}{}^y \xi^y) (\nabla^c \xi^b + \nabla^b \xi^c - 2h^{cb}{}^x \xi^x) \right. \\ \left. + (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb}{}^y \xi^y) h^{cb}{}^x \xi^x \right. \\ \left. - (\nabla_c \xi^c - h_c{}^c{}_x \xi^x) (\nabla_b \xi^b) \right] dV = 0,$$

which is valid for arbitrary ξ^a and ξ^x .

Now suppose that an infinitesimal variation (2.1) of the submanifold is isometric. Then since it is affine, we have (2.14), from which, we have, by transvection with g^{cb} ,

$$(2.17) \quad g^{cb} \nabla_c \nabla_b \xi^a + K_d{}^a \xi^d - 2\nabla^c(h_c{}^a{}_x \xi^x) + \nabla^a(h_c{}^c{}_x \xi^x) = 0.$$

We also obtain from (2.9),

$$(2.18) \quad (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb} \xi^y) h^c{}_x \xi^x = 0$$

and

$$(2.19) \quad (\nabla_c \xi^c - h_c{}^c{}_x \xi^x)(\nabla_b \xi^b) = 0.$$

Conversely if (2.17), (2.18) and (2.19) are satisfied, we have, from (2.16),

$$\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb} \xi^y = 0,$$

which shows that the infinitesimal variation is an isometry. Thus we have

THEOREM D. [5] *In order for an infinitesimal variation of a compact orientable submanifold of a Riemannian manifold to be isometric, it is necessary and sufficient that we have (2.17), (2.18) and (2.19).*

§ 3. Infinitesimal isometries.

Suppose that the infinitesimal variation (2.1) is an isometry. Then we have (2.17). Thus substituting (2.17) into

$$(3.1) \quad \begin{aligned} \frac{1}{2} \Delta(\xi^a \xi_a) &= \frac{1}{2} g^{cb} \nabla_c \nabla_b (\xi^a \xi_a) \\ &= (g^{cb} \nabla_c \nabla_b \xi^a) \xi_a + (\nabla^c \xi^b)(\nabla_c \xi_b), \end{aligned}$$

we find

$$\frac{1}{2} \Delta(\xi^a \xi_a) = -K_{cb} \xi^c \xi^b + \{2\nabla^c (h_c{}^a{}_x \xi^x) - \nabla^a (h_c{}^c{}_x \xi^x)\} \xi_a + (\nabla^c \xi^b)(\nabla_c \xi_b),$$

from which, by integration over M^n ,

$$0 = \int [-K_{cb} \xi^c \xi^b - 2h^c{}_x \xi^x \nabla_c \xi_b + (h_c{}^c{}_x \xi^x)(\nabla_b \xi^b) + (\nabla^c \xi^b)(\nabla_c \xi_b)] dV$$

or

$$\int [-K_{cb} \xi^c \xi^b - h^c{}_y \xi^y (\nabla_c \xi_b + \nabla_b \xi_c) + (h_c{}^c{}_x \xi^x)(\nabla_b \xi^b) + (\nabla^c \xi^b)(\nabla_c \xi_b)] dV = 0.$$

Since (2.1) is an isometry, we have

$$\nabla_c \xi_b + \nabla_b \xi_c = 2h_{cb} \xi^x, \quad \nabla_a \xi^a = h_c{}^c{}_x \xi^x.$$

Thus substituting these into the above equation, we find

$$(3.2) \quad \int [-K_{cb} \xi^c \xi^b - 2(h^c{}_y h_{cb} \xi^y) \xi^x \xi^x + (h_c{}^c{}_x \xi^x)^2 + (\nabla^c \xi^b)(\nabla_c \xi_b)] dV = 0.$$

From (3.2), we have

THEOREM 3.1. *If an infinitesimal isometric variation of a compact orientable submanifold M^n of a Riemannian manifold M^m satisfies*

$$(3.3) \quad K_{cb}\xi^c\xi^b + 2(h^{cb}h_{cbx})\xi^y\xi^x \leq 0,$$

then ξ^a satisfies

$$\nabla_c\xi^a = 0 \quad \text{and consequently} \quad K_{cb}\xi^c\xi^b = 0$$

and

$$h_{cbx}\xi^x = 0,$$

that is, M^n is geodesic in the direction ξ^x . Moreover if M^n is irreducible, then $\xi^a = 0$, that is, the variation is normal and the submanifold is geodesic in the direction of the variation.

§ 4. Infinitesimal affine variations.

For an infinitesimal affine variation (2.1), we have (2.14), from which, by transvection with g^{cb} , we obtain

$$(4.1) \quad g^{cb}\nabla_c\nabla_b\xi^a + K_d^a\xi^d - 2\nabla^c(h_c^a{}_x\xi^x) + \nabla^a(h_c^c{}_x\xi^x) = 0$$

and, by contraction with respect to a and b , we have

$$(4.2) \quad \nabla_c(\nabla_a\xi^a - h_a^a{}_x\xi^x) = 0$$

and consequently we obtain

$$(4.3) \quad \nabla_a\xi^a - h_a^a{}_x\xi^x = \text{constant}.$$

Thus (2.16) becomes

$$\begin{aligned} & \int \left[\frac{1}{2}(\nabla_c\xi_b + \nabla_b\xi_c - 2h_{cb}{}_y\xi^y)(\nabla^c\xi^b + \nabla^b\xi^c - 2h^{cb}{}_x\xi^x) \right. \\ & \quad \left. + (\nabla_c\xi_b + \nabla_b\xi_c - 2h_{cb}{}_y\xi^y)h^{cb}{}_x\xi^x \right] dV = 0 \end{aligned}$$

because of $\int (\nabla_b\xi^b) dV = 0$.

Thus if

$$(4.4) \quad (\nabla_c\xi_b + \nabla_b\xi_c - 2h_{cb}{}_y\xi^y)h^{cb}{}_x\xi^x = 0,$$

then we have, from the equation above,

$$(4.5) \quad \nabla_c\xi_b + \nabla_b\xi_c - 2h_{cb}{}_y\xi^y = 0.$$

The converse being evident, we have

THEOREM 4.1. *If an infinitesimal affine variation of a compact orientable submanifold of a Riemannian manifold satisfies (4.4) then the variation is an isometry.*

§ 5. Infinitesimal conformal variations.

If an infinitesimal variation (2.1) is conformal, we have

$$(5.1) \quad \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx} \xi^x = 2\lambda g_{cb}$$

for a certain function λ , from which, transvecting with g^{cb} , we find

$$(5.2) \quad \lambda = \frac{1}{n} (\nabla_a \xi^a - h_a^a{}_x \xi^x).$$

Thus we can write (5.1) as

$$(5.3) \quad \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx} \xi^x - \frac{2}{n} (\nabla_e \xi^e - h_e^e{}_x \xi^x) g_{cb} = 0.$$

On the other hand, for an infinitesimal conformal variation, we have $\delta g_{cb} = 2\lambda g_{cb} \varepsilon$ and consequently from (2.12),

$$(5.4) \quad \delta \Gamma_{cb}^a = (\delta_c^a \lambda_b + \delta_b^a \lambda_c - g_{cb} \lambda^a) \varepsilon,$$

where $\lambda_b = \nabla_b \lambda$ and $\lambda^a = \nabla^a \lambda$, that is,

$$(5.5) \quad \begin{aligned} & \nabla_c \nabla_b \xi^a + K_{dcb}{}^a \xi^d - \nabla_c (h_b^a{}_x \xi^x) - \nabla_b (h_c^a{}_x \xi^x) + \nabla^a (h_{cbx} \xi^x) \\ & = \delta_c^a \lambda_b + \delta_b^a \lambda_c - g_{cb} \lambda^a, \end{aligned}$$

from which, transvecting with g^{cb} , we find

$$g^{cb} \nabla_c \nabla_b \xi^a + K_d{}^a \xi^d - 2\nabla^c (h_c^a{}_x \xi^x) + \nabla^a (h_c^c{}_x \xi^x) = -(n-2)\lambda^a,$$

or, λ being given by (5.2),

$$(5.6) \quad \begin{aligned} & g^{cb} \nabla_c \nabla_b \xi^a + K_d{}^a \xi^d - 2\nabla^c (h_c^a{}_x \xi^x) + \nabla^a (h_c^c{}_x \xi^x) \\ & + \frac{n-2}{n} \nabla^a (\nabla_e \xi^e - h_e^e{}_x \xi^x) = 0. \end{aligned}$$

Now, we can transform (2.16) into

$$(5.7) \quad \begin{aligned} & \int \left[\left\{ g^{cb} \nabla_c \nabla_b \xi^a + K_d{}^a \xi^d - 2\nabla^c (h_c^a{}_x \xi^x) + \nabla^a (h_c^c{}_x \xi^x) \right. \right. \\ & \left. \left. + \frac{n-2}{n} \nabla^a (\nabla_e \xi^e - h_e^e{}_x \xi^x) \right\} \xi_a \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left\{ \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb} \xi^y - \frac{2}{n} (\nabla_e \xi^e - h_e^e \xi^x) g_{cb} \right\} \\
& \quad \times \left\{ \nabla^c \xi^b + \nabla^b \xi^c - 2h^{cb} \xi^x - \frac{2}{n} (\nabla_a \xi^a - h_a^a \xi^x) g^{cb} \right\} \\
& \quad + \left\{ \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb} \xi^y - \frac{2}{n} (\nabla_e \xi^e - h_e^e \xi^y) g_{cb} \right\} h^{cb} \xi^x \Big] dV = 0.
\end{aligned}$$

Thus if the infinitesimal variation is conformal, we have (5.3) and consequently

$$(5.8) \quad \left\{ \nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb} \xi^y - \frac{2}{n} (\nabla_e \xi^e - h_e^e \xi^y) g_{cb} \right\} h^{cb} \xi^x = 0$$

and also (5.6).

Conversely if (5.6) and (5.8) are satisfied, we have from (5.7)

$$\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cb} \xi^y - \frac{2}{n} (\nabla_e \xi^e - h_e^e \xi^y) g_{cb} = 0,$$

which shows that the infinitesimal variation is conformal. Thus we have

THEOREM. 5.1. *In order for an infinitesimal variation (2.1) to be conformal, it is necessary and sufficient that (5.6) and (5.8) hold.*

Substituting (5.6) into (3.1), we find

$$\begin{aligned}
\frac{1}{2} \Delta(\xi^a \xi_a) &= -K_{cb} \xi^c \xi^b + 2\xi_a \nabla^c (h_c^a \xi^x) - \xi_a \nabla^a (h_c^c \xi^x) \\
&\quad - \frac{n-2}{n} \xi_a \nabla^a (\nabla_e \xi^e - h_e^e \xi^x) + (\nabla^c \xi^b) (\nabla_c \xi_b),
\end{aligned}$$

from which, integrating over M^n ,

$$\begin{aligned}
& \int \left[-K_{cb} \xi^c \xi^b - (h^{cb} \xi^x) (\nabla_c \xi_b + \nabla_b \xi_c) + h_c^c \xi^x (\nabla_a \xi^a) \right. \\
& \quad \left. + \frac{n-2}{n} (\nabla_e \xi^e - h_e^e \xi^x) (\nabla_a \xi^a) + (\nabla^c \xi^b) (\nabla_c \xi_b) \right] dV = 0.
\end{aligned}$$

Since the variation is conformal, we have

$$\nabla_c \xi_b + \nabla_b \xi_c = 2h_{cb} \xi^x + \frac{2}{n} (\nabla_e \xi^e - h_e^e \xi^x) g_{cb}.$$

Substituting this into the above integral formula, we find

$$\begin{aligned}
& \int \left[-K_{cb} \xi^c \xi^b - 2h^{cb} \xi^y h_{cb} \xi^x - \frac{2}{n} h_c^c \xi^y (\nabla_e \xi^e - h_e^e \xi^x) + h_c^c \xi^x (\nabla_a \xi^a) \right. \\
& \quad \left. + \frac{n-2}{n} (\nabla_e \xi^e - h_e^e \xi^x) (\nabla_a \xi^a) + (\nabla^c \xi^b) (\nabla_c \xi_b) \right] dV = 0,
\end{aligned}$$

from which

$$(5.9) \quad \int \left[-K_{cb} \xi^c \xi^b - 2 \left(h^{cb}{}_{,y} - \frac{1}{n} g^{cb} h_e{}^e{}_y \right) \left(h_{cbx} - \frac{1}{n} g_{cb} h_a{}^a{}_x \right) \xi^y \xi^x \right. \\ \left. + \frac{n-2}{n} (\nabla_a \xi^a)^2 + (\nabla^c \xi^b)(\nabla_c \xi_b) \right] dV = 0.$$

Thus from (5.9), we have

THEOREM 5.2. *If an infinitesimal conformal variation of a compact orientable submanifold M^n of a Riemannian manifold M^m satisfies*

$$(5.10) \quad K_{cb} \xi^c \xi^b + 2 \left(h^{cb}{}_{,y} - \frac{1}{n} g^{cb} h_e{}^e{}_y \right) \left(h_{cbx} - \frac{1}{n} g_{cb} h_a{}^a{}_x \right) \xi^y \xi^x \leq 0,$$

then, ξ^a satisfies

$$\nabla_c \xi^a = 0 \quad \text{and consequently} \quad K_{cb} \xi^c \xi^b = 0$$

and

$$h_{cbx} \xi^x = \frac{1}{n} g_{cb} h_a{}^a{}_x \xi^x,$$

that is, M^n is umbilical in the direction ξ^x . Moreover if M^n is irreducible, then $\xi^a = 0$, that is, the variation is normal and the submanifold is umbilical in the direction of the variation.

Bibliography

- [1] Bang-yen Chen, *Geometry of Submanifolds*, Marcel Dekker, Inc., 1975.
- [2] J. A. Schouten, *Ricci-Calculus*, Springer-Verlag, 1954.
- [3] K. Yano, *Sur la théorie des déformations infinitésimales*, J. Fac. Sci. Univ. Tokyo, **6** (1949), 1-75.
- [4] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, Inc., 1970.
- [5] K. Yano, *Infinitesimal variations of submanifolds*, Kodai Math. J., **1** (1978), 30-44.

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