NOTES ON TAUBERIAN THEOREMS FOR RIEMANN SUMMABILITY

KENJI YANO

(Received February 1, 1956 and in revised forms November 12, 1956 and September 13, 1957)

1. Introduction. In a given series

$$\sum a_n = \sum_{n=1}^{\infty} a_n \qquad (a_0 = 0),$$

if

$$\sum_{n=1}^{\infty} s_n \frac{\sin nt}{n} \qquad \left(\text{where } s_n = \sum_{\nu=0}^{n} a_\nu\right) \tag{1.1}$$

converges for every value of t $(0 < t < \delta \leqq 2\pi)$, and

$$\lim_{t \to +0} \frac{2}{\pi} \sum_{n=1}^{\infty} s_n \frac{\sin nt}{n} = s,$$

then we call that the series $\sum a_n$ is summable (R_1) to s, and write

$$\sum a_n = s\ (R_1).$$

Similarly, we call that the series $\sum a_n$ is summable $(R, 1)$ to s, and write $\sum a_n = s(R, 1)$, if the series

$$\sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} \tag{1.2}$$

converges for every value of t $(0 < t < \delta \leqq 2\pi)$, and

$$\lim_{t \to +0} \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} = s.$$

The summabilities (R_1) and $(R, 1)$ have been studied by many writers, O. Szász [4, 5, 6, 7], G. Sunouchi [1], H. Hirokawa and G. Sunouchi [3] and others. In this note we shall unify and extend the theorems due to the above writers. Generally speaking, the Riemann summabilities are near to the convergence of series and so Tauberian conditions of the above authors may be replaced by the conditions on s_n^{-s} $(s > 0)$, which will be defined in a moment.

We denote by s_n^γ the n-th Cesàro sum of order γ of the series $\sum a_n$, i. e.

$$s_n^\gamma = \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma} a_\nu \qquad (a_0 = 0),$$

where A_n^γ is defined by the identity

$$(1 - x)^{-\gamma-1} = \sum_{n=0}^{\infty} A^\gamma x^n \qquad (|x| < 1),$$

for all real γ, then paritcularly we have $s_0^\gamma = 0$, $s_n^0 = s_n$ and

$$s_n^{-1} = a_n, \quad s_{n+1}^{-2} = a_{n+1} - a_n = -\Delta a_n.$$

We restrict each a_n to be real throughout this paper.

THEOREM 1. *Let $r > 0$, $0 < s < 1$ (or $s = 1, 2, \ldots.$), and $0 < \alpha \leqq 1$. If*

$$\text{(a)} \qquad \frac{1}{n}\sum_{\nu=1}^{n} |s_\nu^r| = o(n^{r\alpha}) \qquad (n \to \infty),$$

and

$$\text{(b)} \qquad \frac{1}{n}\sum_{\nu=n+1}^{2n} (|s_\nu^{-s}| - s_\nu^{-s}) = O(n^{-s\alpha}) \qquad (n \to \infty),$$

then $\Sigma a_n = 0\,(R_1)$. Indeed the series (1.1) converges uniformly in $0 \leqq t \leqq \pi$.

THEOREM 2. *Under the same assumptions as Theorem 1, $\Sigma a_n = 0\,(R, 1)$. Moreover, the series (1.2) converges uniformly in $0 \leqq t \leqq \pi$ if and only if Σa_n converges. Here we suppose that $\left(\dfrac{\sin nt}{nt}\right)_{t=0} = \lim\limits_{t\to 0} \dfrac{\sin nt}{nt} = 1$.*

2. Preliminary lemmas. We require a number of lemmas.

LEMMA 1. *If $\alpha > 0$ and $u_n \geqq 0$ then we have the following equivalent relations:*

$$\text{(I)} \qquad \sum_1^n \frac{u_\nu}{\nu} = o(n^\alpha) \Leftrightarrow \sum_{n+1}^{2n} u_\nu = o(n^{\alpha+1}),$$

$$\text{(II)} \qquad \sum_1^n u_\nu = o(n^\alpha) \Leftrightarrow \sum_{n+1}^{2n} u_\nu = o(n^\alpha),$$

$$\text{(III)} \qquad \sum_n^\infty \frac{u_\nu}{\nu} = O(n^{-\alpha}) \Leftrightarrow \sum_{n+1}^{2n} u_\nu = O(n^{-\alpha+1}),$$

$$\text{(IV)} \qquad \sum_n^\infty u_\nu = O(n^{-\alpha}) \Leftrightarrow \sum_{n+1}^{2n} u_\nu = O(n^{-\alpha}),$$

as $n \to \infty$. O may be replaced by o, and conversely.

We prove (I). Clearly, the latter equation follows from the former. Inversely, if the latter holds then

$$\sum_{n+1}^{2n} \frac{u_\nu}{\nu} = o(n^\alpha).$$

Hence, taking the integer k such as $2^{k-1} \leqq n < 2^k$,

$$\sum_1^n \frac{u_\nu}{\nu} = \sum_{[n/2]+1}^{n} + \sum_{[n/4]+1}^{[n/2]} + \ldots. + \sum_{[n/2^k]+1}^{[n/2^{k-1}]}$$
$$< (2^\alpha - 1)\varepsilon[(n/2)^\alpha + \ldots. + (n/2^k)^\alpha] + G$$

$$< \varepsilon n^{\alpha} + G \qquad (n > n_0),$$

where $G = \sum_{\nu=1}^{n_0} u_\nu/\nu$ is a constant depending on n_0. From this follows the former, and (I) is proved. The proofs of (II), (III) and (IV) are analogous.

LEMMA 2.1. Let $r > 0$, $-1 < b \leqq a + r, C > 0$ *and* $H > 0$. *If*

(2.1) $$s_n^r = o(n^b) \qquad (n \to \infty),$$

(2.2) $$s_{n+m} - s_n > -Cn^a \text{ for } m = 1, 2, \ldots, [Hn] \qquad (n > n_0),$$

then

(2.3) $$s_n = O(n^a) \qquad (n \to \infty),$$

and $$s_n^\mu = o(n^{a(1-\mu/r)+b\mu/r}) \qquad (n \to \infty),$$

for every μ *such that* $0 < \mu < r$.

This lemma is due to Prof. S. Izumi. He pointed out that this may be obtained immediately by replacing the condition $s_{n+m} - s_n > -Kn^\alpha m$ in Theorem 6 of L. S. Bosanquet [9] by $s_{n+m} - s_n > -Kn^\alpha$, and the proof is quite analogous as the Theorem 6.

Lemma 2.1 is an improvement of the well-known convexity theorem due to Dixson and Ferrar [8] (cf. G. Sunouchi [2]). It must be noticed that (2.3) is a result from the two conditions (2.1) and (2.2).

LEMMA 2. *Let* $r > 0, s > 0$ *and* $0 < \alpha \leqq 1$. *Then, the two conditions* (a) *and* (b) *in Theorem* 1 *imply the following relations*:

(2.4) $$s_n^{1+r} = o(n^{1+r\alpha})$$

(2.5) $$\sum_{1}^{n} \frac{|s_\nu^r|}{\nu} = o(n^{r\alpha}),$$

(2.6) $$s_n^{1-s} = O(n^{1-s\alpha}),$$

(2.7) $$s_n^{1+\mu} = o(n^{1+\mu\alpha}) \qquad (-s < \mu < r),$$

(2.8) $$\sum_{n}^{\infty} \frac{|s_\nu^{-s}|}{\nu} = O(n^{-s\alpha}),$$

(2.9) $$\sum_{1}^{n} |s_\nu^{-s}| = O(n^{1-s\alpha}) \qquad (s\alpha < 1),$$

as $n \to \infty$.

Indeed, (2.4) follows immediately from (a). (2.5) does from (a) and Lemma 1. Next, from (b) we have

(2.10) $$s_{n+m}^{1-s} - s_n^{1-s} > -Cn^{1-s\alpha} \text{ for } m = 1, 2, \ldots, n \qquad (n > n_0).$$

Hence, applying Lemma 2.1 to (2.4) and (2.10) we have (2.6) and (2.7), and then (b) is equivalent to

$$\sum_{n+1}^{2n} |s_\nu{}^{-s}| = O(n^{1-s\alpha}) \qquad (n \to \infty),$$

which yields (2.8) and (2.9) by Lemma 1.

3. Proof of Theorem 1. We denote by the capital letter S_n the n-th partial sum of the series (1.1), then

$$S_n = \sum_{\nu=1}^{n} s_\nu \frac{\sin \nu t}{\nu} = \Re\Big(\int_0^t \sum_1^n s_\nu e^{i\nu u}\, du\Big) = -\Re\Big(\int_t^\pi \sum_1^n s_\nu e^{i\nu u}\, du\Big).$$

From the identity

$$s_\nu = \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{s-1} s_\mu^{-s} \qquad (\nu = 1, 2, \ldots.),$$

we have

$$U_n = \sum_{\nu=1}^{n} s_\nu e^{i\nu u} = \sum_{\nu=1}^{n} e^{i\nu u} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{s-1} s_\mu^{-s},$$

and

$$S_n = \Re\Big(\int_0^t U_n\, du\Big) = -\Re\Big(\int_t^\pi U_n\, du\Big). \tag{3.1}$$

Changing the order of summation we get, in the case $0 < s < 1$

$$U_n = \sum_{\mu=1}^{n} s_\mu^{-s} e^{i\mu u} \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{s-1} e^{i(\nu-\mu)u}$$
$$= \sum_{\mu=1}^{n} s_\mu^{-s} e^{i\mu u} \Big(\sum_{\nu=\mu}^{\infty} - \sum_{\nu=n+1}^{\infty}\Big)$$

i.e.

$$U_n = (1 - e^{iu})^{-s} \sum_{\mu=1}^{n} s_\mu^{-s} e^{i\mu u} - Q_n, \tag{3.2}$$

where

$$Q_n = \sum_{\mu=1}^{n} s_\mu^{-s} \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{s-1} e^{i\nu u} = \sum_{\mu=1}^{m} + \sum_{\mu=m+1}^{n} = Q_n' + Q_n''.$$

Here, we define m such as

$$n - m = [\varepsilon n^\alpha] \qquad (0 < \varepsilon < 1/2). \tag{3.3}$$

If s is an integer, then applying Abel's transformation s times to $U_n = \sum_{\nu=1}^{n} s_\nu e^{i\nu u}$ we see easily that

$$Q_n = \sum_{j=1}^{s} s_n^{1-j} (1 - e^{iu})^{-j} e^{i(n+1)u}. \tag{3.4}$$

If $0 < s < 1$, then by Abel's transformation

$$Q_n' = (1 - e^{iu})^{-1}\Big[\sum_{\mu=1}^{m} s_\mu^{-s} \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{s-2} e^{i\nu u} + \sum_{\mu=1}^{m} A_{n-\mu}^{s-1} s_\mu^{-s} e^{i(n+1)u}\Big],$$

$$Q''_n = \sum_{\mu=m+1}^{n} s_\mu^{1-s} \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{s-2} e^{i\nu u} + s_m^{1-s} \sum_{\nu=m+1}^{n} A_{\nu-1-m}^{s-1} e^{i\nu u}$$
$$+ (1-e^{iu})^{-s} [s_n^{1-s} e^{i(n+1)u} - s_m^{1-s} e^{i(m+1)u}],$$

and by (2.6) and (2.9) in Lemma 2

$$s_n^{1-s} = O(n^{1-s\alpha}) \text{ and } \sum_{\mu=1}^{n} |s_\mu^{-s}| = O(n^{1-s\alpha}). \tag{3.5}$$

Therefore, integrating by parts

$$\int_t^\pi Q'_n du = \sum_{\mu=1}^{m} s_\mu^{-s} \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{s-2} \int_t^\pi (1-e^{iu})^{-1} e^{i\nu u} du$$
$$+ \sum_{\mu=1}^{m} A_{n-\mu}^{s-1} s_\mu^{-s} \int_t^\pi (1-e^{iu})^{-1} e^{i(n+1)u} du$$
$$= O\Big(\sum_{\mu=1}^{m} |s_\mu^{-s}| \sum_{\nu=n+1}^{\infty} |A_{\nu-\mu}^{s-2}|(\nu t)^{-1}\Big) + O\Big(\sum_{\mu=1}^{m} A_{n-\mu}^{s-1} |s_\mu^{-s}| (nt)^{-1}\Big)$$
$$= O\Big(\sum_{\mu=1}^{m} |s_\mu^{-s}|(nt)^{-1}(n+1-\mu)^{s-1}\Big) + O\Big((n-m)^{s-1}(nt)^{-1} \sum_{\mu=1}^{m} |s_\mu^{-s}|\Big)$$
$$= O\Big((nt)^{-1}(n-m)^{s-1} \sum_{\mu=1}^{n} |s_\mu^{-s}|\Big)$$
$$= O((nt)^{-1} (\varepsilon n^\alpha)^{s-1} n^{1-s\alpha}) \qquad \text{by (3.3), (3.5),}$$
$$= O(\varepsilon^{s-1} (n^\alpha t)^{-1}).$$

Similarly, integrating by parts and observing that $m \sim n$ we have

$$\int_t^\pi Q''_n du = O\Big(\sum_{\mu=m+1}^{n} |s_\mu^{1-s}| \sum_{\nu=n+1}^{\infty} |A_{\nu-\mu}^{s-2}| \nu^{-1}\Big)$$
$$+ O\Big(|s_m^{1-s}| \sum_{\nu=m+1}^{n} A_{\nu-1-m}^{s-1} \nu^{-1}\Big) + O(|s_n^{1-s}| nt^s)$$
$$= O\Big(\sum_{\mu=m+1}^{n} |s_\mu^{1-s}| n^{-1}(n+1-\mu)^{s-1}\Big)$$
$$+ O(|s_m^{1-s}| n^{-1}(n-m)^s) + O(|s_n^{1-s}|/nt^s)$$
$$= O(n^{1-s\alpha} n^{-1}(n-m)^s) + O(n^{1-s\alpha}/nt^s) \qquad \text{by (3.5),}$$
$$= O(n^{-s\alpha}(\varepsilon n^\alpha)^s) + O(n^\alpha t)^{-s} \qquad \text{by (3.3),}$$
$$= O(\varepsilon^s) + O(n^\alpha t)^{-s}.$$

Thus, when $0 < s < 1$ we have

$$\int_t^\pi Q_n du = O(n^\alpha t)^{-s} + O(\varepsilon^{s-1}(n^\alpha t)^{-1}) + O(\varepsilon^s). \tag{3.6}$$

When s is an integer, from (3.4) and from $s_n^{1-j} = O(n^{1-j\alpha})$ for $j = 1, 2, \ldots, s$, which is (2.6) and (2.7) in Lemma 2, it follows

$$\int_t^\pi Q_n\,du = \sum_{j=1}^{s} O(n^\alpha t)^{-j}. \tag{3.6$'$}$$

Further, we see that by $\sum_{\mu=n}^\infty |s_\mu^{-s}|/\mu = O(n^{-s\alpha})$ which is (2.8),

$$\int_t^\pi (1 - e^{iu})^{-s} \sum_{\mu=n+1}^{n+p} s_\mu^{-s} e^{i\mu u}\,du = O\Big(\sum_{\mu=n+1}^{\infty} \frac{|s_\mu^{-s}|}{\mu t^s}\Big) = O(n^\alpha t)^{-s}$$

for every integer $p > 0$. Hence, by (3.1) and (3.2) it holds

$$S_{n+p} - S_n = O(n^\alpha t)^{-s} + \Re\Big(\int_t^\pi (Q_{n+p} - Q_n)\,du\Big), \tag{3.7}$$

for every $p > 0$. Now, we put

$$R_n = \sum_{\nu=n+1}^{\infty} s_\nu \frac{\sin \nu t}{\nu}.$$

Then by (3.7), (3.6) and (3.6)$'$ we have

$$R_n = \begin{cases} O(n^\alpha t)^{-s} + O(\varepsilon^{s-1}(n^\alpha t)^{-1}) + O(\varepsilon^s) & (0 < s < 1) \\ \sum_{j=1}^{s} O(n^\alpha t)^{-j} & (s = 1, 2, \ldots), \end{cases} \tag{3.8}$$

where O's are uniform in $0 < t < 2\pi$. In particular, (3.8) shows that the series (1.1) converges uniformly in $0 < \delta \leqq t \leqq 2\pi - \delta$.

Next, from the expression (3.2) of U_n replaced s by $-r$ we have

$$U_n = (1 - e^{iu})^r \sum_{\mu=1}^{n} s_\mu^r e^{i\mu u} - P_n,$$

where

$$P_n = \sum_{\mu=1}^{n} s_\mu^r \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{-r-1} e^{i\nu u}.$$

Clearly, integrating by parts and using $\sum_{\mu=1}^n |s_\mu^r|/\mu = o(n^{r\alpha})$ which is (2.5) in Lemma 2, we have

$$\int_0^t (1 - e^{iu})^r \sum_{\mu=1}^{n} s_\mu^r e^{i\mu u}\,du = O\Big(t^r \sum_{\mu=1}^{n} \frac{|s_\mu^r|}{\mu}\Big) = o(n^\alpha t)^r,$$

and then

$$\int_0^t U_n\,du = o(n^\alpha t)^r - \int_0^t P_n\,du. \tag{3.9}$$

Here, we suppose that r is not an integer, since the case r an integer may be proved by the same arguments. Applying Abel's transformation successively $[r]+1$ times to the second sum of P_n we have

$$(3.10)\qquad P_n=(1-e^{iu})^{[r]+1}\sum_{\mu=1}^{n} s_\mu^r \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{[r]-r} e^{i\nu u}-\sum_{j=0}^{[r]} s_n^{j+1}(1-e^{iu})^j e^{i(n+1)u}.$$

Concerning the first term of the last expression

$$(1-e^{iu})^{[r]+1}\Big(\sum_{\mu=1}^{m}+\sum_{\mu=m+1}^{n}\Big)=P_n'+P_n''$$

say, where we define m such as

$$(3.3)'\qquad n-m=[n^\alpha/2].$$

Then again by Abel's transformation

$$P_n'=(1-e^{iu})^{[r]}\Big[\sum_{\mu=1}^{m} s_\mu^r \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{[r]-r-1} e^{i\nu u}+\sum_{\mu=1}^{m} A_{n-\mu}^{[r]-r} s_\mu^r e^{i(n+1)u}\Big],$$

$$P_n''=(1-e^{iu})^{[r]+1}\Big[\sum_{\mu=m+1}^{n} s_\mu^{1+r} \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{[r]-r-1} e^{i\nu u}+s_m^{1+r}\sum_{\nu=m+1}^{n} A_{\nu-1-m}^{[r]-r} e^{i\nu u}\Big]$$
$$+(1-e^{iu})^r[s_n^{1+r}e^{i(n+1)u}-s_m^{1+r}e^{i(m+1)u}].$$

Observing that

$$(3.11)\qquad \sum_{\mu=1}^{n}|s_\mu^r|=o(n^{1+r\alpha})\quad\text{and}\quad s_n^{1+r}=o(n^{1+r\alpha}),$$

which is (a) and (2.4) in Lemma 2, we have

$$\int_0^t P_n' du=\sum_{\mu=1}^{m} s_\mu^r \sum_{\nu=n+1}^{\infty} A_{\nu-\mu}^{[r]-r-1}\int_0^t (1-e^{iu})^{[r]} e^{i\nu u}\,du$$
$$+\sum_{\mu=1}^{m} A_{n-\mu}^{[r]-r} s_\mu^r\int_0^t (1-e^{iu})^{[r]} e^{i(n+1)u}\,du$$
$$=O\Big(\sum_{\mu=1}^{m}|s_\mu^r|\sum_{\nu=n+1}^{\infty}|A_{\nu-\mu}^{[r]-r-1}|t^{[r]}/\nu\Big)+O\Big(\sum_{\mu=1}^{m}A_{n-\mu}^{[r]-r}|s_\mu^r|t^{[r]}/n\Big)$$
$$=O\Big((n-m)^{[r]-r}t^{[r]}n^{-1}\sum_{\mu=1}^{m}|s_\mu^r|\Big)=o\,((n^\alpha)^{[r]-r}t^{[r]}n^{r\alpha})=o(n^\alpha t)^{[r]},$$

since $n-m=[n^\alpha/2]$ by (3.3)'. Again, using (3.3)' and (3.11)

$$\int_0^t P_n'' du=O\Big(\sum_{\mu=m+1}^{n}|s_\mu^{1+r}|\sum_{\nu=n+1}^{\infty}|A_{\nu-\mu}^{[r]-r-1}|t^{[r]+1}/\nu\Big)$$
$$+O\Big(|s_m^{1+r}|\sum_{\nu=m+1}^{n}A_{\nu-1-m}^{[r]-r}t^{[r]+1}/\nu\Big)+(O|s_n^{1+r}|t^r/n)$$
$$=O\Big(\sum_{\mu=m+1}^{n}|s_\mu^{1+r}|t^{[r]+1}n^{-1}(n+1-\mu)^{[r]-r}\Big)$$

$$+ O(|s_m^{1+r}|t^{[r]+1}n^{-1}(n-m)^{[r]-r+1}) + O(|s_n^{1+r}|t^r/n)$$

$$= o\left(n^{1+r\alpha}t^{[r]+1}\, n^{-1} \sum_{\mu=m+1}^{n} (n+1-\mu)^{[r]-r} \right)$$

$$+ o(n^{1+r\alpha}\, t^{[r]+1}\, n^{-1}(n-m)^{[r]-r+1}) + o(n^{1+r\alpha}t^r/n)$$

$$= o(n^{r\alpha}t^{[r]+1}\,(n^\alpha)^{[r]-r+1}) + o(n^\alpha t)^r$$

$$= o(n^\alpha t)^{[r]+1} + o(n^\alpha t)^r.$$

Further, observing that $s_n^{1+\mu} = o(n^{1+\mu\alpha})$ for $-s<\mu\leqq r$, which is (2.7) in Lemma 2, we have

$$\int_0^t s_n^{j+1}(1-e^{iu})^j e^{i(n+1)u}\, du = o(n^\alpha t)^j \qquad j = 0, 1, \ldots, [r].$$

Hence using (3.9), (3.10) and (3.1) we get

$$S_n = o(n^\alpha t)^r + \sum_{j=0}^{[r]+1} o(n^\alpha t)^j, \tag{3.12}$$

where o's are uniform in $0 \leqq t \leqq \pi$.

Here also we may suppose that $0 < s < 1$ since the conclusion is unchanged when s is an integer. Then, from (3.12) and (3.8) it follows

$$|S_n| < \varepsilon^{r+2}[(n^\alpha t)^r + \sum_{j=0}^{[r]+1}(n^\alpha t)^j]$$

and $$(n > n_0),$$

$$|R_n| < C[(n^\alpha t)^{-s} + \varepsilon^{s-1}(n^\alpha t)^{-1} + \varepsilon^s]$$

for all values of t such as $0 < t \leqq \pi$, where $\varepsilon = \varepsilon(n_0) > 0$ is arbitrary and C is an absolute constant.

Therefore for a fixed $n > n_0$, if $t \geqq 1/n^\alpha\varepsilon$ then

$$|R_n| < 3C\varepsilon^s,$$

and if $t < 1/n^\alpha\varepsilon$ then from the identity $R_n = (S_m - S_n) + R_m$, where $m = [(1/\varepsilon t)^{1/\alpha}]$, and from the above inequalities for the absolute value of S_n and R_n we have

$$|R_n| < 2(r+3)\varepsilon + 3C\varepsilon^s.$$

Hence. the series (1.1) converges uniformly in $0 < t \leqq \pi$, and so does in $0 \leqq t \leqq \pi$, since its terms are continuous in this closed interval. Thus the theorem is established completely.

4. **Proof of Theorem 2.** We denote by S_n' the n-th partial sum of the series (1.2), and by R_n' the remainder. Then, by Abel's transformation

$$S_n' = {}_n s \frac{\sin(n+1)t}{(n+1)t} + \sum_{\nu=1}^{n} s_\nu \Delta_\nu(t)$$

and

$$R_n = -s_n \frac{\sin (n+1)t}{(n+1)t} + \sum_{\nu=n+1}^{\infty} s_\nu \Delta_\nu(t),$$

where

$$\Delta_\nu(t) = \frac{\sin \nu t}{\nu t} - \frac{\sin (\nu+1)t}{(\nu+1)t} = \Re\left(\frac{1}{t}\int_0^t (1-e^{iu})\, e^{i\nu u}\, du\right).$$

We see that the series $S_n' + R_n'$ converges uniformly in $0 < t \leqq \pi$ by the same arguments as the proof of Theorem 1, and in particular $\Sigma a_n = 0\,(R, 1)$ since, observing that $s_n = O(n^{1-\alpha})$ by (2.6) in Lemma 2,

$$s_n \frac{\sin (n+1)t}{(n+1)t} = O(1/n^\alpha t) = O(\varepsilon)$$

when $t \sim 1/n^\alpha \varepsilon$, i. e, $n = [(1/\varepsilon t)^{1/\alpha}]$.

Further, clearly by the arguments used at the end of the last article the series (1.2) converges uniformly in $0 \leqq t \leqq \pi$ if and only if

$$s_n \frac{\sin(n+1)t}{(n+1)t} = o(1) \qquad \text{uniformly in this interval,}$$

which is equivalent to $s_n = o(1)$. Thus, we get the theorem.

5. **Corollaries.** We consider again the two series

$$(1.1) \qquad \sum_{n=1}^{\infty} s_n \frac{\sin nt}{n} \qquad \text{and} \quad (1.2) \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt}.$$

COROLLARY 1. *Let* $0 < \alpha < 1$ *and* $r > 0$. *If*

$$s_n^r = o(n^{r\alpha}) \quad \text{and} \quad \sum_{\nu=n}^{\infty} |a_\nu|/\nu = O(n^{-\alpha}) \qquad (n \to \infty),$$

then $\Sigma a_n = 0(R_1)$ *and* $\Sigma a_n = 0(R, 1)$.

This follows from Theorems 1, 2 and Lemma 1. This is due to H. Hirokawa and G. Sunouchi [3], of which the case $r = 1$ does to G. Sunouchi [1].

COROLLARY 2. *If* Σa_n *is summable* $(C, 1-\delta)$ *for some positive* $\delta < 1$, *and if* $\sum_{\nu=1}^{n} |s_\nu^{-\delta}| = O(n^{1-\delta})$, *then the series* Σa_n *is summable* (R_1) *and* $(R, 1)$ *to the same sum.*

This is due to O. Szász [6, 7].

For the sake of simplicity we assumed that $a_0 = 0$ and $\sigma = 0$ in $\Sigma a_n = \sigma$ (R_1). But, generally when $a_0 = \sigma \neq 0$ also Theorems 1 and 2 are valid provided that (1.1), (1.2) and s_n^γ are replaced by

$$\sigma + \frac{2}{\pi}\sum_{n=1}^{\infty} (s_n - \sigma)\frac{\sin nt}{n}, \quad \sigma + \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} \quad \text{and} \quad s_n^\gamma - A_n^\gamma \sigma$$

respectively. Thus, the above Corollary 2 follows from Theorems 1 and 2 when $r = 1-\delta$, $s = \delta$ and $\alpha = 1$.

COROLLARY 3. *If* $\sum a_n$ *converges and*

$$\sum_{\nu=n}^{2n}(|a_\nu|-a_\nu)=O(1) \qquad (n\to\infty), \tag{5.1}$$

then the series (1.1) *and* (1.2) *converge uniformly in* $(0,\pi)$.

This is due to O. Szász [5, 7], and follows from Theorems 1 and 2 when $s=1$ and $\alpha=1$.

COROLLARY 4. *If* $\sum a_n$ *is summable in Abel sense and* (5.1) *holds, then* (1.1) *converges uniformly in* $(0,\pi)$.

This follows immediately from Theorem 1 with $r=s=\alpha=1$ and the following lemma due to O. Szász [4]:

LEMMA 3. *If* $\sum a_n$ *is summable in Abels sense and* (5.1) *holds, then* $\sum a_n$ *is summable* $(C,1)$ *to the same sum. And, moreover* $s_n=O(1)$.

The last corollary holds for arbitrary $s>0$:

COROLLARY 5. *Theorems* 1 *and* 2 *holds also if* $\sum a_n=0\,(A)$ *and if for some positive* s

$$\sum_{\nu=n}^{2n}(|s_\nu^{-s}|-s_\nu^{-s})=O(n^{1-s}) \qquad (n\to\infty). \tag{5.2}$$

Moreover, the series (1.2) *converges uniformly in* $(0,\pi)$ *when* $s>1$.

For the proof, when $0<s\leqq 1$ it is sufficient to show that $\sum a_n=0\,(A)$ and (5.2) imply $\sum a_n=0\,(C)$, and the case $s=1$ is immediate by Lemma 3. In the case $0<s<1$, (5.2) implies

$$s_n^{1-s}>-Cn^{1-s},\quad C>0, \tag{5.3}$$

by Lemma 1 since $1-s>0$.

On the other hand, letting $\sigma_n^{1-s}=s_n^{1-s}/A_n^{1-s}$ we see that by a theorem due to O. Szász [10], $\sum a_nx^n=o(1)$ implies

$$(1-x)\sum_{n=0}^{\infty}\sigma_n^{1-s}x^n=o(1) \qquad (x\to 1-0),$$

which and $\sigma_n^{1-s}>-C$ yield

$$\frac{1}{n+1}\sum_{\nu=0}^{n}\sigma_\nu^{1-s}=o(1) \qquad (n\to\infty).$$

And, this is equivalent to $\sigma_n^{2-s}=o(1)$ by the well-known relation between Cesàro's summability and Hölder's. Hence, in this case $\sum a_n=0(A)$ and (5.2) imply $\sigma_n^{2-s}=o(1)$.

Next, we suppose that $1<s\leqq 2$. Then, applying Lemma 1, (IV) to (5.2) we get

$$s_{n+p}^{1-s}-s_n^{1-s}>-Cn^{1-s}\quad\text{for}\quad n>n_0,\ p=1,2,\dots, \tag{5.4}$$

which implies the existence of $\lim s_n^{1-s}$, and this limit must vanish by the condition $\sum a_n x^n = o(1)$ which is equivalent to

$$(1-x)^{2-s}\sum_{n=0}^{\infty} s_n^{1-s} x^n = o(1) \qquad (x \to 1-0).$$

Hence, again from (5.4) we see in turn that $s_n^{1-s} \leqq Cn^{1-s}$, i.e. $s_n^{-(s-1)} \leqq Cn^{-(s-1)}$, and

$$\sum_{\nu=n}^{2n} (|s_\nu^{-(s-1)}| + s_\nu^{-(s-1)}) = O(n^{1-(s-1)}) \qquad (0 < s-1 \leqq 1).$$

Thus, the above case is reduced to the case $0 < s \leqq 1$, and so on.

Further, applying Lemma 2.1 to (5.4) and $\sum a_n = 0\,(C)$ we see that $s_n = o(1)$ when $s > 1$. Hence, we get the corollary.

6. THEOREM 3. (I) *If*

$$(6.1) \qquad \frac{1}{n}\sum_{\nu=n+1}^{2n} |s_\nu| = o(1/\log n) \qquad (n \to \infty),$$

and if for two some positive s and $\delta < 1$

$$(6.2) \qquad \frac{1}{n}\sum_{\nu=n+1}^{2n} (|s_\nu^{-s}| - s_\nu^{-s}) = O(n^{-\delta}) \qquad (n \to \infty),$$

then $\sum a_n = 0\,(R_1)$, and indeed the series (1.1) *converges uniformly in* $(0, \pi)$.

(II) *Under the same assumptions as in (I) $\sum a_n = 0\,(R, 1)$, and moreover the series* (1.2) *converges uniformly in* $(0, \pi)$ *if and only if $\sum a_n$ converges.*

Here, we notice that s may be as large as we wish.

In fact, (6.1) yields $s_n^1 = o(n)$. On the other hand, letting $\delta = s\alpha$ we may suppose that α is small and $0 < \alpha < 1$ since δ may be replaced by any smaller positive number. Then, $s_n^1 = o(n)$ and (6.2) imply

$$(6.3) \qquad s_n^{1-s} = O(n^{1-s\alpha}) \quad \text{and} \quad \sum_{\nu=1}^{n} |s_\nu^{-s}| = O(n^{1-s\alpha})$$

by Lemmas 1 and 2.1 since $1 - s\alpha = 1 - \delta > 0$. Further, in these circumstances we may suppose that s is an integer, for (6.3) implies

$$s_n^{1-\sigma} = O(n^{1-\sigma\alpha'}) \quad \text{and} \quad \sum_{\nu=1}^{n} |s_\nu^{-\sigma}| = O(n^{1-\sigma\alpha'})$$

for every σ and α' such that $\sigma > s$ and $\sigma\alpha' = s\alpha = \delta$, and of course $0 < \alpha' < 1$.

Hence, observing that (6.3) implies (2.6), (2.7), (2.8) and (2.9) in Lemma 2 we have (3.8) in the proof of Theorem 1, i.e.

$$(6.4) \qquad R_n = \sum_{\nu=n+1}^{\infty} s_\nu \frac{\sin \nu t}{\nu} = \sum_{j=1}^{s} O(n^\alpha t)^{-j},$$

where O's are uniform in $0 < t \leqq \pi$.

Now, we proceed our arguments by O. Szász's method which was used to prove Theorem 5, [5]. Observing that (6.1) is equivalent to $\sum_{\nu=1}^{n} |s_\nu| = o(n/\log n)$ as (I) in Lemma 1, let

$$\sum_{\nu=1}^{n} |s_\nu| < \varepsilon^2 n/\log n \qquad (n > n_0), \tag{6.5}$$

where $\varepsilon > 0$ is given. And, we define the number $\lambda = \lambda(n, t)$ such that

$$\lambda = 1 + 1/\varepsilon nt, \tag{6.6}$$

corresponding to $n > n_0$ and positive $t \leqq \pi$.

Letting $m = [n^{1/\alpha}]$, we have from (6.4) $R_m = \sum_{j=1}^{s} O(nt)^{-j}$. On the other hand, if $\lambda < 2$ then (6.6) yields $(nt)^{-1} < \varepsilon$, and so

$$R_m = \sum_{j=1}^{s} O(\varepsilon^j) = O(\varepsilon) \qquad (m > m_0 = [n_0^{1/\alpha}]). \tag{6.7}$$

If $\lambda \geqq 2$ then dividing R_n into three parts

$$R_n = \sum_{n+1}^{n_1} + \sum_{n_1+1}^{n_2} + \sum_{n_2+1}^{\infty} = R' + R'' + R''',$$

where $n_1 = [\lambda n]$ and $n_2 = [n_1^{\exp(1/\varepsilon)}]$, we have

$$|R'| \leqq t \sum_{n+1}^{n_1} |s_\nu| < t\varepsilon^2 n_1/\log n_1 \leqq \frac{\varepsilon}{(\lambda-1)n} \lambda n/\log(\lambda n) < 2\varepsilon,$$

since $\lambda \geqq 2$. And

$$|R''| < \sum_{n_1+1}^{n_2} |s_\nu|/\nu.$$

Applying Abel's transformation to the right hand side we see easily that by (6.5)

$$|R''| < \varepsilon^2 \log(\log n_2/\log n_1) + \varepsilon^2/\log n_2,$$

from which we have by the definition of n_1 and n_2

$$|R''| < \varepsilon + \varepsilon^2.$$

Further, from (6.4) it follows

$$R''' = \sum_{j=1}^{s} O(n_2^{\alpha} t)^{-j} = \sum_{j=1}^{s} O(\varepsilon^j) = O(\varepsilon),$$

by the definition of n_2 provided that $\alpha e^{1/\varepsilon} \geqq 1$. Hence, $R_n = O(\varepsilon)$ for $n > n_0$.

From this and (6.7) we see that $R_n = O(\varepsilon)$ for $n > m_0 = [n_0^{1/\alpha}]$ uniformly in $0 < t \leqq \pi$ when $\alpha e^{1/\varepsilon} \geqq 1$. This proves (I). The proof of (II) is analogous as above and the proof of Theorem 2.

COROLLARY 6. *If* $\sum_{\nu=1}^{n} |s_\nu| = o(n/\log n)$, *and if for some poistive* $\delta < 1$,

$$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = (n^{1-\delta}) \qquad (n \to \infty), \tag{6.8}$$

then $\Sigma a_n = 0\,(R_1)$ *and* $\Sigma a_n = 0\,(R, 1)$

This is a theorem of O. Szász [4] when (6. 8) is replaced by $\sum_{\nu=n}^{2n} |a_\nu| = O(n^{1-\delta})$.

7. REMARK. Disregarding the uniformity of convergence we have in the place of Theorems 1 and 2 the following theorem not depending on the value of $s > 0$, the case $\alpha = 1$ is Corollary 5.

THEOREM 4. *Let* $r > 0$, $s > 0$ *and* $0 < \alpha < 1$. *If*

$$\frac{1}{n}\sum_{\nu=n+1}^{2n} |s_\nu^r| = o(n^{r\alpha}) \qquad (n \to \infty),$$

and

$$\frac{1}{n}\sum_{\nu=n+1}^{2n} (|s_\nu^{-s}| - s_\nu^{-s}) = O(n^{-s\alpha}) \qquad (n \to \infty),$$

then $\Sigma a_n = 0(R_1)$ *and* $\Sigma a_n = 0(R, 1)$.

Finally, I wish to express my heartfelt gratitude to Professors S. Izumi, G. Sunouchi and Mr. H. Hirokawa for their suggestions and kind advices.

REFERENCES

[1] G. SUNOUCHI, Tauberian theorems for Riemann summability, Tôhoku Math. J., (2), 5(1953), 34-42.

[2] G. SUNOUCHI, Notes on Fourier analysis (XLVII), Convexity theorem and Cesàro summability of Fourier series, Journ. of Math., 1(1953), 104-109.

[3] H. HIROKAWA and G. SUNOUCHI, Two theorems on the Riemann summability, Tôhoku Math. J., (2), 5(1954), 261-267.

[4] O. SZÁSZ, On convergence and summability of trigonometric series, Amer. J. Math., 64(1942), 575-591.

[5] O. SZÁSZ, On Abel and Lebesgue summability, Bull. Amer. Math. Soc., 49 (1943), 885-893.

[6] O. SZÁSZ, On Lebesgue summability and its generalization to integrals, Amer. J. Math., 67(1945), 389-396.

[7] O. SZÁSZ, Tauberian theorems for summability (R_1), Amer. J. Math., 73(1951), 779-791.

[8] A. L. DIXON and W. L. FERRAR, On Cesàro sums, J. London Math. Soc., 7(1932), 87-93.

[9] L. S. BOSANQUET, Notes on convexity theorems, J. London Math. Soc., 18(1943), 239-248.

[10] O. SZÁSZ, On products of summability methods, Proc. Amer. Math. Soc., 3(1952), 257-263.

MATHEMATICAL INSTITUTE, NARA WOMEN'S UNIVERSITY.