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George P. H. Styan

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NOTES ON THE DISTRIBUTION OF QUADRATIC FORMS IN SINGULAR NORMAL VARIABLES

By George P. H. Styan

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SUMMARY

Conditions for a quadratic form in singular normal variables to be distributed as χ^2 are obtained, correcting and extending previous results by Good (1969). Connections with the non-central case and independence of two quadratic forms are also given, simplifying earlier results by Khatri (1963). Finally, a further generalization of the well-known theorem of Cochran (1934) is presented.

SHORT TITLE

DISTRIBUTIONS OF SINGULAR QUADRATIC FORMS

1. CENTRAL CHI-SQUARE.

Throughout this section let $\underset{\sim}{x}$ follow a multivariate normal distribution with mean vector $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{C}$, possibly singular, and let $\underset{\sim}{A}$ be a symmetric matrix, not necessarily semi-definite. Then Good (1969) obtained conditions for $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ to be distributed as chi-square; these conditions simplify those given earlier by Khatri (1963), Rayner and Livingstone (1965), Rao (1966), and Shanbhag (1968). While sufficient, Good's conditions are here shown not to be necessary, as claimed.

The conditions depend on the following theorem given by Good (1969).

THEOREM 1. <u>A necessary and sufficient condition for $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ to follow a χ^2-distribution with r degrees of freedom is that $\underset{\sim}{A}\underset{\sim}{C}$ have r unit characteristic roots, the rest zero.</u>

Proof follows by equating characteristic functions of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ and χ^2_r, i.e., $|\underset{\sim}{I} - 2it\underset{\sim}{A}\underset{\sim}{C}|^{-\frac{1}{2}} = (1-2it)^{-r/2}$.

Good (1969) claims that if $\underset{\sim}{A}\underset{\sim}{C}$ has $\underline{r}$ unit characteristic roots, the rest zero, then $\underset{\sim}{A}\underset{\sim}{C}$ must be idempotent, rank $\underline{r}$. A counter-example was given by Khatri (1963); another is given by

$$\underset{\sim}{A} = \begin{pmatrix} 2,-1, & 0 \\ -1, & 0, & 0 \\ 0, & 0, & 1 \end{pmatrix} ; \qquad \underset{\sim}{C} = \begin{pmatrix} 1, & 1, & 0 \\ 1, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix} . \tag{1}$$

Here $\underset{\sim}{A}\underset{\sim}{C}$ has 1 unit root, 2 zero roots, trace 1 and rank 2, and is not idempotent for

$$\underset{\sim}{A}\underset{\sim}{C} = \begin{pmatrix} 1, & 1, & 0 \\ -1, & -1, & 0 \\ 0, & 0, & 1 \end{pmatrix} \neq (\underset{\sim}{A}\underset{\sim}{C})^2 = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{pmatrix} , \tag{2}$$

but $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ $(\equiv x_3^2)$ follows a χ^2-distribution with one degree of freedom.

Thus Corollaries (i) and (ii) of Good (1969),

$$(\underset{\sim}{AC})^2 = \underset{\sim}{AC}, \qquad \text{rank}(\underset{\sim}{AC}) = r, \tag{3}$$

$$\text{tr}\{(\underset{\sim}{AC})^2\} = \text{tr}(\underset{\sim}{AC}) = r = \text{rank}(\underset{\sim}{AC}), \tag{4}$$

do not provide necessary conditions for $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ to be distributed as χ_r^2, though they are each sufficient. Khatri (1963), Rayner and Livingstone (1965), Rao (1966), and Shanbhag (1968) showed that

THEOREM 2. <u>A set of necessary and sufficient conditions for</u> $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ <u>to follow a</u> χ^2<u>-distribution with</u> r <u>degrees of freedom is</u>

$$\underset{\sim}{CACAC} = \underset{\sim}{CAC}; \qquad \text{rank}(\underset{\sim}{CAC}) = \text{tr}(\underset{\sim}{AC}) = r. \tag{5}$$

Proof follows from writing $\underset{\sim}{C} = \underset{\sim}{TT}'$ where $\underset{\sim}{T}$ has full column rank. Since $\underset{\sim}{AC} = \underset{\sim}{ATT}'$ and $\underset{\sim}{T}'\underset{\sim}{AT}$ have the same nonzero characteristic roots, we find that $\underset{\sim}{T}'\underset{\sim}{AT}$ is idempotent, it being symmetric. Because $\text{rank}(\underset{\sim}{CAC}) \le \text{rank}(\underset{\sim}{T}'\underset{\sim}{AT}) = \text{rank}\{(\underset{\sim}{T}'\underset{\sim}{AT})^3\} \le \text{rank}(\underset{\sim}{CAC})$, we obtain $\text{tr}(\underset{\sim}{AC}) = \text{tr}(\underset{\sim}{T}'\underset{\sim}{AT}) = \text{rank}(\underset{\sim}{T}'\underset{\sim}{AT}) = \text{rank}(\underset{\sim}{CAC}) = r$, the number of nonzero roots of $\underset{\sim}{AC}$. Thus (5) follows. Notice that rank $(\underset{\sim}{AC})$ may exceed r as in (2). This leads to

THEOREM 3. <u>A necessary and sufficient condition for</u> $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ <u>to follow a</u> χ^2<u>-distribution with</u> r <u>degrees of freedom is</u>

$$(\underset{\sim}{AC})^2 = \underset{\sim}{AC} \tag{6}$$

<u>if and only if</u> $\text{rank}(\underset{\sim}{AC}) = \text{tr}(\underset{\sim}{AC}) = r$, <u>or</u> $\text{rank}(\underset{\sim}{AC}) = \text{rank}(\underset{\sim}{CAC}) = r$.

Proof follows from the following

LEMMA 1. <u>A square matrix</u> $\underset{\sim}{X}$, <u>not necessarily symmetric, satisfying</u> $\underset{\sim}{X}^2 = \underset{\sim}{X}^3$, <u>is idempotent if and only if</u> $\text{rank}(\underset{\sim}{X}) = \text{tr}(\underset{\sim}{X})$ <u>or</u> $\text{rank}(\underset{\sim}{X}) = \text{rank}(\underset{\sim}{X}^2)$.

Proof. If $X = X^2$, $\mathrm{rank}(X) = \mathrm{tr}(X) = \mathrm{rank}(X^2)$ follows immediately. Conversely, if $X^2 = X^3$, then $Z'Y = (Z'Y)^2$, where $X = YZ'$ with with Y and Z of full column rank equal to the rank of X. If $\mathrm{rank}(X) = \mathrm{tr}(X) = \mathrm{tr}(YZ') = \mathrm{tr}(Z'Y) = \mathrm{rank}\ (Z'Y)$, then $Z'Y$ is nonsingular and equals I. Hence $X = YZ'YZ' = X^2$. If $\mathrm{rank}(X) = \mathrm{rank}(X^2) = \mathrm{rank}\ (YZ'YZ')$, then $\mathrm{rank}(Z'Y) \geq \mathrm{rank}(X)$ and so $Z'Y = I$ and $X=X^2$. (qed)

Proof of Theorem 3. From (5), $(AC)^3 = (AC)^2$ and $\mathrm{rank}(CAC) = \mathrm{rank}(CACAC) \leq \mathrm{rank}\{(AC)^2\} \leq \mathrm{rank}(CAC)$. Thus substituting $X = AC$ in Lemma 1 gives the result. (qed)

Rayner and Livingstone (1965) and Shanbhag (1968) proved that (5) implied (6) when A is semi-definite. In such a case $\mathrm{rank}(AC)$ equals the number of its nonzero characteristic roots and so also $\mathrm{tr}(AC)$ and $\mathrm{rank}(CAC)$. The condition of AC symmetric, also given by Shanbhag (1968), similarly implies $\mathrm{rank}(AC) = \mathrm{tr}(AC) = \mathrm{rank}(CAC) = r$. Good (1969) states that Shanbhag (1968) proved that it was necessary that AC be symmetric for (5) to imply (6). Not only is this not so, but Shanbhag (1968) claimed only sufficiency, which is true. Mitra (1968) shows that when $\mathrm{rank}(X) = \mathrm{rank}(X^2)$, there exists a matrix W such that $WX^2 = X$. Thus if $X^2 = X^3$, we obtain $WX^2 = WX^2X = X^2 = X$. The conditions are equivalent to X (or AC) having rank equal to the number of its nonzero characteristic roots.

Shanbhag (1968) proved that

$$(AC)^2 = (AC)^3; \qquad \mathrm{tr}(AC) = r, \tag{7}$$

$$\mathrm{tr}\{(AC)^2\} = \mathrm{tr}(AC) = r = \mathrm{rank}(CAC), \tag{8}$$

are each equivalent to (5). We now give slightly simpler proofs. Writing

$\underset{\sim}{C} = \underset{\sim}{T}\underset{\sim}{T}'$ as before, we find from (7) that $(\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T})^2 = (\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T})^3$. Since $\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T}$ is symmetric it is idempotent and so $\underset{\sim}{CACAC} = \underset{\sim}{CAC}$. Hence (5) follows because we have already shown that $\mathrm{rank}(\underset{\sim}{CAC}) = \mathrm{tr}(\underset{\sim}{AC}) = r$. The converse is immediate. When (8) holds, substitute λ_i, the $\underline{i}$-th characteristic root of $\underset{\sim}{AC}$, to obtain, as in Good (1969), $\Sigma\lambda_i^2 = \Sigma\lambda_i = \underline{r}$. The summation is over $\underline{i}$ from 1 to $\underline{r}$ since $\mathrm{rank}(\underset{\sim}{CAC}) = \mathrm{tr}(\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T}) = \underline{r}$ is the number of nonzero roots of $\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T}$ or $\underset{\sim}{ATT}' = \underset{\sim}{AC}$. Hence $\Sigma(\lambda_i - 1)^2 = 0$ and $\underset{\sim}{AC}$ has $\underline{r}$ unit roots, the rest zero. Thus (8) implies (5) using Theorem 1. But (8) implies (4) only under the conditions of Theorem 3. Conversely (5) implies (8) since $\mathrm{tr}\{(\underset{\sim}{AC})^2\} = \mathrm{tr}(\underset{\sim}{ATT}'\underset{\sim}{ATT}') = \mathrm{tr}\{(\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T})^2\} = \mathrm{tr}(\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T}) = \mathrm{tr}(\underset{\sim}{AC})$.

We now turn to the non-central case and obtain similar conditions.

2. NON-CENTRAL CHI-SQUARE.

Unless stated to the contrary, we will assume in what follows that the mean vector of $\underset{\sim}{x}$ is $\underset{\sim}{\mu}$, not necessarily $\underset{\sim}{0}$. Khatri (1963) and Rayner and Livingstone (1965) proved

THEOREM 4. *A set of necessary and sufficient conditions for* $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ *to follow a non-central* $\chi_r^2(\delta^2)$ *distribution is (5) and*

$$\underset{\sim}{\mu}'(\underset{\sim}{AC})^2 = \underset{\sim}{\mu}'\underset{\sim}{AC}, \tag{9}$$

and

$$\delta^2 = \underset{\sim}{\mu}'\underset{\sim}{ACA}\underset{\sim}{\mu} = \underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu}. \tag{10}$$

Applying Lemma 1 to Theorem 4, we obtain in parallel to Theorem 3

COROLLARY 1. *A necessary and sufficient condition for* $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ *to follow a non-central* $\chi_r^2(\delta^2)$ *distribution is*

$$(AC)^2 = AC; \qquad \delta^2 = \mu' ACA\mu = \mu' A\mu \tag{11}$$

if and only if $\operatorname{rank}(AC) = \operatorname{tr}(AC) = r$ *or* $\operatorname{rank}(AC) = \operatorname{rank}(CAC) = r$.

We notice that when $\operatorname{rank}(AC) = \operatorname{tr}(AC)$ or $\operatorname{rank}(AC) = \operatorname{rank}(CAC)$ the non-central case requires the additional restriction of $\mu' ACA\mu = \mu' A\mu$ over the central situation, cf. (11) and (6). When C is nonsingular this extra condition is automatically satisfied for then (5) implies $ACA = A$, or C is a generalized inverse of A as noted by Rao (1962). A parallel result proved by Khatri (1968) is the necessary and sufficient condition for $x'Ax$ to follow a χ^2_r-distribution where $\operatorname{rank}(C) = r$. Here $T'AT = I$ so (5) implies $CAC = C$, or A is a generalized inverse of C. Shanbhag (1968) observed that this reduction also applies when A is positive definite. In the non-central case, however, we still require $\mu' ACA\mu = \mu' A\mu$ in addition, though (9) is automatically satisfied. When C is nonsingular and of order (and rank) r, then $A = C^{-1}$ is a single necessary and sufficient condition.

Another reduction of Theorem 4 occurs when $\mu' AC = 0'$. This generalizes a special case given by Rayner and Livingstone (1965).

COROLLARY 2. *When* $\mu' AC = 0'$, *a necessary and sufficient condition for* $x'Ax - \mu' A\mu$ *to follow a* χ^2_r*-distribution is (5).*

Proof. The quadratic form $(x-\mu)'A(x-\mu)$ follows a χ^2_r-distribution if and only if (5) holds. When $\mu' AC = 0'$, $\mu' Ax$ has variance 0 and so $\mu' Ax \equiv \mu' A\mu$. Hence $x'Ax - \mu' A\mu = (x-\mu)'A(x-\mu)$ and so the result follows. (qed)

3. INDEPENDENCE.

The above results extend the properties originated by Cochran

(1934) and Craig (1943) with the case $\underset{\sim}{C} = \underset{\sim}{I}$. Parallel results for independence of two quadratic forms were then also obtained, and recently extended to the singular case by Khatri (1963), for $\underset{\sim}{\mu}$ not necessarily $\underset{\sim}{0}$, and by Good (1963) and Shanbhag (1966) for $\underset{\sim}{\mu} = \underset{\sim}{0}$ only.

THEOREM 5. <u>A set of necessary and sufficient conditions for $\underset{\sim}{x}'\underset{\sim}{Ax}$ and $\underset{\sim}{x}'\underset{\sim}{Bx}$ to be independently distributed is</u>

$$\underset{\sim}{CACBC} = \underset{\sim}{0}, \tag{12}$$

$$\underset{\sim}{CACB\mu} = \underset{\sim}{CBCA\mu} = \underset{\sim}{0}, \tag{13}$$

<u>and</u>

$$\underset{\sim}{\mu}'\underset{\sim}{ACB\mu} = 0, \tag{14}$$

<u>where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are symmetric matrices, not necessarily semi-definite, and $\underset{\sim}{x}$ follows a multivariate normal distribution with mean vector $\underset{\sim}{\mu}$, not necessarily $\underset{\sim}{0}$, and covariance matrix $\underset{\sim}{C}$, possibly singular</u>.

Theorem 5 was proved by Khatri (1963) for the more general case of a Wishart distribution. Shanbhag (1966) showed that when $\underset{\sim}{A}$ is semi-definite (12), (13), and (14) reduce to

$$\underset{\sim}{ACBC} = \underset{\sim}{0}; \qquad \underset{\sim}{ACB\mu} = \underset{\sim}{0}, \tag{15}$$

while for $\underset{\sim}{A}$ and $\underset{\sim}{B}$ both semi-definite (15) reduces to

$$\underset{\sim}{ACB} = \underset{\sim}{0}. \tag{16}$$

We note that (16) is the same necessary and sufficient condition for independence when $\underset{\sim}{C}$ is nonsingular.

Condition (12) does not always reduce to $\underset{\sim}{ACBC} = \underset{\sim}{0}$ when $\operatorname{rank}(\underset{\sim}{AC}) = \operatorname{rank}(\underset{\sim}{CAC})$, nor is this restriction necessary for the reduction, since $\underset{\sim}{CBC} = \underset{\sim}{0}$ may hold throughout. Thus we do not obtain a parallel result to Theorem 3 for independence. Also semi-definiteness of $\underset{\sim}{A}$ reduces $\underset{\sim}{CACAC} = \underset{\sim}{CAC}$

only to $\underset{\sim}{ACAC} = \underset{\sim}{AC}$ and not necessarily further to $\underset{\sim}{ACA} = \underset{\sim}{A}$. The last equation can never hold if $\text{rank}(\underset{\sim}{A}) > \text{rank}(\underset{\sim}{C})$. This is, however, not relevant in (16). Also in the case of independence, semi-definiteness yields a condition not involving $\underset{\sim}{\mu}$, which is not so for chi-square.

4. COCHRAN'S THEOREM.

Let $\underset{\sim}{x}$ follow a p-variate normal distribution with mean vector $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{I}_p$, and let $\underset{\sim}{A}_i$ $(i = 1, \ldots, k)$ be symmetric matrices of ranks r_i $(i = 1, \ldots, k)$, such that $\sum_{i=1}^{k} \underset{\sim}{A}_i = \underset{\sim}{I}_p$. Then Cochran (1934) proved that $\underset{\sim}{x}'\underset{\sim}{A}_i\underset{\sim}{x}$ are each independently distributed as $\chi^2_{r_i}$ if and only if $\sum_{i=1}^{k} r_i = p$. Madow (1940) generalized the result for $\underset{\sim}{x}$ with mean vector $\underset{\sim}{\mu}$, not necessarily $\underset{\sim}{0}$, while Chipman and Rao(1964) extended Cochran's Theorem to $\underset{\sim}{x}$ with positive definite covariance matrix $\underset{\sim}{\Sigma}$. Elegant matrix proofs of Cochran's original result and related interrelationships have been given by James (1952), Banerjee (1964), Chipman and Rao (1964), Loynes (1966), and Khatri (1968). We use the results of sections 1 to 3 of this paper to prove a further generalization.

Let $\underset{\sim}{x}$ follow a multivariate normal distribution with mean vector $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{C}$, possibly singular. Let $Q = \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ and $Q_i = \underset{\sim}{x}'\underset{\sim}{A}_i\underset{\sim}{x}$ $(i = 1, \ldots, k)$ be quadratic forms such that $Q = \sum_{i=1}^{k} Q_i$ and $r = \text{rank}(\underset{\sim}{CAC})$, $r_i = \text{rank}(\underset{\sim}{CA}_i\underset{\sim}{C})$, $i = 1, \ldots, k$. When $\underset{\sim}{C}$ is nonsingular we have $r = \text{rank}(\underset{\sim}{A})$ and $r_i = \text{rank}(\underset{\sim}{A}_i)$, $i = 1, \ldots, k$. Consider the following propositions:

(a) Q follows a $\chi^2_r(\underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{\mu})$ distribution,

(b) Q_i follows a $\chi^2_{r_i}(\underset{\sim}{\mu}'\underset{\sim}{A}_i\underset{\sim}{\mu})$ distribution, $i = 1, \ldots, k$,

(c) Q_i, Q_j are independently distributed, for all $i \neq j$ $(i, j = 1, \ldots, k)$,

(d) $r = \sum_{i=1}^{k} r_i$.

THEOREM 6. *If* (I) *$\mathbf{C}$ is nonsingular, or* (II) *$\mathbf{C}$ is singular and $\boldsymbol{\mu} = \mathbf{0}$, or* (III) *$\mathbf{C}$ is singular, $\boldsymbol{\mu}$ not necessarily $\mathbf{0}$, and $\mathbf{A}_i$ positive semi-definite* ($i = 1, \ldots, k$) *then*

(a) and (d) imply (b) and (c), (17)

(a) and (b) imply (c) and (d), (18)

(a) and (c) imply (b) and (d), (19)

and (b) and (c) imply (a) and (d). (20)

Proof. Let $\mathbf{C} = \mathbf{T}\mathbf{T}'$, where $\mathbf{T}$ has full column rank equal to the rank of $\mathbf{C}$. When $\mathbf{C}$ is nonsingular so is $\mathbf{T}$. Then under (I) or (II), we may rewrite (a) to (c) as

(a') $\mathbf{T}'\mathbf{A}\mathbf{T} = (\mathbf{T}'\mathbf{A}\mathbf{T})^2$

(b') $\mathbf{T}'\mathbf{A}_i\mathbf{T} = (\mathbf{T}'\mathbf{A}_i\mathbf{T})^2$, $i=1, \ldots, k$,

(c') $\mathbf{T}'\mathbf{A}_i\mathbf{T}\mathbf{T}'\mathbf{A}_j\mathbf{T} = \mathbf{0}$, for all $i \neq j$ ($i, j=1, \ldots, k$),

using the results of section 1 to 3 of this paper. For (III) we need in addition to (a') and (b'),

(a'') $\boldsymbol{\mu}'\mathbf{A}\mathbf{C}\mathbf{A}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$,

(b'') $\boldsymbol{\mu}'\mathbf{A}_i\mathbf{C}\mathbf{A}_i\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{A}_i\boldsymbol{\mu}$, $i=1, \ldots, k$.

Notice that $\mathbf{T}'\mathbf{A}_i\mathbf{T}$ has rank r_i ($i=1, \ldots, k$ or absent). We first prove (17) to (20) with (a), (b), and (c) replaced by (a'), (b'), and (c'). We then prove that when these hold, (a'') and (b'') are equivalent.

Write $\mathbf{T}'\mathbf{A}_i\mathbf{T} = \mathbf{U}_i\mathbf{V}_i'$, where $\mathbf{U}_i$ and $\mathbf{V}_i$ have full column rank r_i($i=1, \ldots, k$ or absent). Then (a') and (d) imply $\mathbf{T}'\mathbf{A}\mathbf{T} = \mathbf{U}\mathbf{V}'$, where $\mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_k)$ and $\mathbf{V} = (\mathbf{V}_1, \ldots, \mathbf{V}_k)$, following Chipman and Rao (1964). Since $\mathbf{U}\mathbf{V}'$ is idempotent from (a'), $\mathbf{V}'\mathbf{U} = \mathbf{I}_r$ and so $\mathbf{V}_i'\mathbf{U}_i = \mathbf{I}_{r_i}$ ($i=1, \ldots, k$) and and $\mathbf{V}_i'\mathbf{U}_j = \mathbf{0}$ for all $i \neq j$ ($i, j=1, \ldots, k$). Thus (b') and (c') follow.

Moreover, (a') and (b') imply $r = tr(\underset{\sim}{T}'\underset{\sim}{A}\underset{\sim}{T}) = \sum_{i=1}^{k} tr(\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T}) = \sum_{i=1}^{k} r_i$ or (d). Hence (c') follows from the previous reasoning. When (a') and (c') hold, we obtain $\sum_{i=1}^{k}(\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T}) = \sum_{i=1}^{k}(\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T})^2$. Multiplying through by $\underset{\sim}{T}'\underset{\sim}{A}_j\underset{\sim}{T}$ ($\underline{j}$ fixed) yields $(\underset{\sim}{T}'\underset{\sim}{A}_j\underset{\sim}{T})^2 = (\underset{\sim}{T}'\underset{\sim}{A}_j\underset{\sim}{T})^3$ and so (b') follows, since $\underset{\sim}{T}'\underset{\sim}{A}_j\underset{\sim}{T}$ is symmetric, and hence (d). Finally, (b') and (c') imply (a') since $(\sum_{i=1}^{k}\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T})^2 = \sum_{i=1}^{k}(\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T})^2 + \sum_{i\neq j}(\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T})(\underset{\sim}{T}'\underset{\sim}{A}_j\underset{\sim}{T}) = \sum_{i=1}^{k}(\underset{\sim}{T}'\underset{\sim}{A}_i\underset{\sim}{T})$, and (d) follows as before.

When (a'), (b'), (c'), and (d) hold, (b'') directly implies (a'') upon summation over $\underline{i}$ from 1 to $\underline{k}$. To show the converse let $\underset{\sim}{A}_i = \underset{\sim}{B}_i\underset{\sim}{B}_i'$, where $\underset{\sim}{B}_i$ has full column rank (i=1,...,k). This is possible when we assume $\underset{\sim}{A}_i$ positive semi-definite. Then (a'') implies

$$\sum_{i=1}^{k}\underset{\sim}{\mu}'\underset{\sim}{B}_i(\underset{\sim}{I}-\underset{\sim}{B}_i'\underset{\sim}{C}\underset{\sim}{B}_i)\underset{\sim}{B}_i'\underset{\sim}{\mu} = 0. \tag{21}$$

But $\underset{\sim}{B}_i'\underset{\sim}{C}\underset{\sim}{B}_i$ has the same nonzero characteristic roots as $\underset{\sim}{B}_i\underset{\sim}{B}_i'\underset{\sim}{C} = \underset{\sim}{A}\underset{\sim}{C}$ which from (a') and Theorem 1 are all unity. Since $\underset{\sim}{B}_i'\underset{\sim}{C}\underset{\sim}{B}_i$ is symmetric it follows that it and $\underset{\sim}{I}-\underset{\sim}{B}_i'\underset{\sim}{C}\underset{\sim}{B}_i$ are positive semi-definite, i=1, . . .,k. Hence each component in (21) must be nonnegative and so zero. Thus (b'') follows. (qed)

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