

NOTES ON THE JUMP OF A STRUCTURES

ANTONIO MONTALBÁN

ABSTRACT. We introduce the notions of a *complete set of computably infinitary Π_n^0 relations* on a structure, of the *jump of a structure*, and of *admitting n th jump inversion*.

INTRODUCTION

This paper is part of the study of the interactions between structural properties of a structure and computational properties of its presentations. We concentrate on the Turing jumps of the presentations of a structure, and the main notion defined in this paper is the one of the jump of a structure. Even if the definitions and theorems of this paper are all new, many of the ideas were already in the air, but they were not concretely formulated.

We start by defining what it means for a set of relations in a structure to be a complete set of computably infinitary Π_n^0 relations. The idea is that a set of relations is computably infinitary Π_n^0 complete if it captures the whole Π_n^0 structural information of the structure. In the second section we look at classes of structures which have a finite complete set of computably infinitary Π_n^0 relations. In the first section, we also define the n th jump of a structure to be the structure together with a complete set of computably infinitary Π_n^0 relations. In the last section we look at the degree spectrum of the jump of a structure. We also define the notion of a structure \mathcal{A} admitting n th jump inversion and prove it is equivalent to saying whenever \mathcal{A} has an X -low $_n$ copy for some X , then it has an X -computable copy.

Throughout this paper we use \mathcal{L} to denote a computable first order language. We use Π_n^c to denote the set of computably infinitary Π_n^0 \mathcal{L} -formulas (see [AK00, Ch 7] for background on this language), and we use $\Pi_n^{c,Z}$ to denote the class of Z -computably infinitary Π_n^0 \mathcal{L} -formulas.

1. MAIN DEFINITIONS

Definition 1.1. Let \mathcal{A} be an \mathcal{L} -structure. Let $\{P_0, P_1, \dots\}$ be a finite or infinite set of uniformly Π_n^c relations on \mathcal{A} . That is, there is a c.e. list of Π_n^c -formulas defining the relations P_i on \mathcal{A} . We say that $\{P_0, P_1, \dots\}$ is a *complete set of Π_n^c relations on \mathcal{A}* if every Π_n^c \mathcal{L} -formula $\psi(\bar{x})$ is equivalent to a $\Sigma_1^{c,0^{(n)}}$ $(\mathcal{L} \cup \{P_0, \dots\})$ -formula, and there is a computable procedure to find this equivalent formula. In other words, $\{P_0, P_1, \dots\}$ is a *complete set of Π_n^c relations on \mathcal{A}* if for every Π_n^c \mathcal{L} -formula $\psi(\bar{x})$ we can uniformly produce a $0^{(n)}$ -computable list $\varphi_0(\bar{x}), \varphi_1(\bar{x}), \varphi_2(\bar{x}), \dots$ of finitary existential \mathcal{L} -formulas that may mention the relations P_0, P_1, \dots such that

$$\mathcal{A} \models \psi(\bar{x}) \iff \bigvee_i \varphi_i(\bar{x}).$$

Observe that if $\{P_0, P_1, \dots\}$ is a complete set of Π_n^c relations on \mathcal{A} then every Σ_{n+1}^c \mathcal{L} -formula is also equivalent to a $\Sigma_1^{c,0^{(n)}}$ $(\mathcal{L} \cup \{P_0, \dots, P_k\})$ -formula.

⁰ Saved: January 29, 2009

Compiled: February 5, 2009

This research was partially supported by NSF grant DMS-0600824.

Definition 1.2. If $\{P_0, P_1, \dots\}$ is a complete set of Π_n^c relations on \mathcal{A} , we say that $(\mathcal{A}, P_0, P_1, \dots)$ is an *nth jump* of \mathcal{A} and write

$$\mathcal{A}^{(n)} = (\mathcal{A}, P_0, P_1, \dots).$$

Note that being the *nth* jump of a structure is a property of isomorphism types of structures and not of presentations of structures.

The second thing to notice is that every structure always has an *nth* jump: Just let $\{P_0, P_1, \dots\}$ be a computable list of all the Π_n^c relations on \mathcal{A} .

Observation 1.3. If \mathcal{A} has a copy $\leq_T X$, then $\mathcal{A}^{(n)}$ has a copy $\leq_T X^{(n)}$.

Even though a structure might have many *nth* jumps, the following lemma show that, from a complexity viewpoint, all these jumps are the same.

Lemma 1.4. *Let P_0, P_1, \dots and R_0, R_1, \dots be complete sets of Π_n^c relations on \mathcal{A} . For every $Y \geq_T 0^{(n)}$ we have that*

$$(\mathcal{A}, R_0, R_1, \dots) \text{ has copy } \leq_T Y \iff (\mathcal{A}, P_0, P_1, \dots) \text{ has copy } \leq_T Y.$$

Proof. Fix a presentation of \mathcal{A} and assume that Y can compute \mathcal{A} and all the relations R_i uniformly. We will show that Y uniformly computes all the relations P_i . Since each P_i is defined by a Π_n^c formula, and this formula is equivalent to a $\Sigma_1^{c,0^{(n)}}(\mathcal{L} \cup \{R_0, \dots\})$ -formula, we have that P_i is c.e. in $Y \oplus 0^{(n)} \equiv_T Y$. The complement of P_i is Σ_n^c , and in particular Σ_{n+1}^c . Hence it is also equivalent to a $\Sigma_1^{c,0^{(n)}}(\mathcal{L} \cup \{R_0, \dots\})$ -formula and is also c.e. in Y . So P_i is computable in Y , uniformly in i . \square

Recall that for non-trivial structures (using Knight's theorem)

$$\text{degSp}(\mathcal{A}) = \{X \in \mathcal{D} : \mathcal{A} \text{ has a copy } \leq_T X\},$$

where \mathcal{D} is the set of Turing degrees. So, the lemma above can be restated as

$$\text{degSp}(\mathcal{A}, R_0, R_1, \dots) \cap \mathcal{D}_{(\geq_T 0^{(n)})} = \text{degSp}(\mathcal{A}, P_0, P_1, \dots) \cap \mathcal{D}_{(\geq_T 0^{(n)})}$$

(where $\mathcal{D}_{(\geq_T 0^{(n)})}$ is the set of Turing degrees above $0^{(n)}$). Therefore, we have that the degree spectrum of $\mathcal{A}^{(n)}$ on the degrees $\geq_T 0^{(n)}$ is independent of the possible choices of $\mathcal{A}^{(n)}$.

2. EXAMPLES

We now turn into looking at examples of jumps of structures. We start with linear orderings.

Lemma 2.1. *Let $\mathcal{A} = (A, <)$ be a linear ordering and let*

$$\text{Succ}^{\mathcal{A}} = \{(a, b) \in A^2 : \exists c (a < c < b)\}.$$

Then

$$\mathcal{A}' = (A, <, \text{Succ}^{\mathcal{A}}).$$

Sketch of the proof. We need to show that $\text{Succ}^{\mathcal{A}}$ is Π_1^c -complete. We will show that every Σ_1^c formula is equivalent to a finitary universal formula that uses the predicate Succ , and that $0'$ can uniformly find this formula. By taking complements, we will then get that every Π_1^c formula is equivalent to a $\Sigma_1^{c,0'}$ ($\mathcal{L} \cup \{\text{Succ}\}$)-formula.

Suppose that \mathcal{A} has a first and a last element called $-\infty$ and ∞ ; the general case is very similar. First, we note that every finitary existential sentence ψ is equivalent to a sentence that says that there are least n many different elements in \mathcal{A} . Let $\psi_n(x, y)$ be the formula

that says that there are at least n many different elements in between x and y . Observe that $\psi_n(x, y)$ is equivalent to a finitary universal formula over the Successor predicate:

$$\psi_n(x, y) \iff \bigwedge_{j < n} \exists z_1, \dots, z_j \left(x = z_1 \leq \dots \leq z_j = y \wedge \left(\bigwedge_{i=1}^{j-1} (\text{Succ}(z_i, z_{i+1})) \right) \right).$$

Second, if we have a formula with free variables $\psi(x_1, \dots, x_k)$, we can write ψ as a disjunction over all the permutations (τ_1, \dots, τ_k) of $(1, \dots, k)$ of the formulas $\left(\bigwedge_{j < k} x_{\tau_j} < x_{\tau_{j+1}} \right) \wedge \psi(x_1, \dots, x_k)$. Third, for every finitary existential formula $\psi(x_1, \dots, x_k)$, we have that $x_1 < x_2 < \dots < x_k \wedge \psi(x_1, \dots, x_k)$ is equivalent to a finite disjunction of formulas of the form

$$x_1 < x_2 < \dots < x_k \wedge \psi_{n_0}(-\infty, x_{\tau_1}) \wedge \psi_{n_1}(x_{\tau_1}, x_{\tau_2}) \wedge \dots \wedge \psi_{n_k}(x_{\tau_k}, \infty),$$

for some $n_0, \dots, n_k \in \omega$. Therefore, we have that any Σ_1^c formula $\psi(x_1, \dots, x_k)$ is equivalent to the disjunction over all the permutations (τ_1, \dots, τ_k) of $(1, \dots, k)$ of formulas of the form

$$x_{\tau_1} < \dots < x_{\tau_k} \wedge \left(\bigvee_j \left(\psi_{n_0^j}(-\infty, x_{\tau_1}) \wedge \psi_{n_1^j}(x_{\tau_1}, x_{\tau_2}) \wedge \dots \wedge \psi_{n_k^j}(x_{\tau_k}, \infty) \right) \right).$$

Fourth, it can be shown that in the infinite disjunction in the formula above all but finitely many of the disjuncts are redundant. Furthermore, $0'$ can find these finitely many disjuncts. \square

Notice that in all the linear orderings the same relation $\text{Succ}^{\mathcal{A}}$ that is Π_1^c complete. This motivates the following definition.

Definition 2.2. Let \mathcal{K} be a class of \mathcal{L} -structures. A c.e. set $\varphi_0, \varphi_1, \dots$ of Π_n^c formulas is a *complete set of Π_n^c formulas for \mathcal{K}* , if for each structure $\mathcal{A} \in \mathcal{K}$ we have that $\{\varphi_0^{\mathcal{A}}, \varphi_1^{\mathcal{A}}, \dots\}$ is a complete set of Π_n^c formulas on \mathcal{A} .

The Boolean algebra predicates considered by Downey, Jockusch [DJ94], Thurber [Thu95], Knight and Stob [KS00] are exactly the ones needed to define the first four jumps of a Boolean algebra.

Lemma 2.3 (Harris, Montalbán [HM]). *Let \mathcal{B} be a Boolean algebra.*

- $\mathcal{B}' = (\mathcal{B}, \text{atom}^{\mathcal{B}})$,
- $\mathcal{B}'' = (\mathcal{B}, \text{atom}^{\mathcal{B}}, \text{inf}^{\mathcal{B}}, \text{atomless}^{\mathcal{B}})$.
- $\mathcal{B}''' = (\mathcal{B}, \text{atom}^{\mathcal{B}}, \text{inf}^{\mathcal{B}}, \text{atomless}^{\mathcal{B}}, \text{atomic}^{\mathcal{B}}, 1\text{-atom}^{\mathcal{B}}, \text{atominf}^{\mathcal{B}})$.
- $\mathcal{B}^{(4)} = (\mathcal{B}, \text{atom}^{\mathcal{B}}, \text{inf}^{\mathcal{B}}, \text{atomless}^{\mathcal{B}}, \text{atomic}^{\mathcal{B}}, 1\text{-atom}^{\mathcal{B}}, \text{atominf}^{\mathcal{B}}, \sim\text{inf}^{\mathcal{B}}, \text{Int}(\omega + \eta)^{\mathcal{B}}, \text{infatomicless}^{\mathcal{B}}, 1\text{-atomless}^{\mathcal{B}}, \text{nomaxatomless}^{\mathcal{B}})$.

Furthermore, for every n there is a finite set of Π_n^c formulas which are Π_n^c complete for the class of Boolean algebras. (Definitions of the relations above can be found in [KS00] and [HM].)

We note that not all the predicates in the four items above are Π_n^c for the corresponding n , but they are Boolean combinations of Π_n^c predicates. There is no problem relaxing our definition of n th jump to allow the predicates to be Boolean combinations of Π_n^c predicates so long they still generate all other Π_n^c predicates.

Proof. Harris and the author [HM] proved that the unary predicates R_σ for $\sigma \in \mathbf{BF}_n$ are Π_n^c complete for the class of Boolean algebras. They showed that for $n \leq 4$, the relations in the four items mentioned above are Boolean combinations of R_σ for $\sigma \in \mathbf{BF}_n$ and vice versa. \square

The main lemmas in Downey, Jockusch [DJ94], Thurber [Thu95], and Knight, Stob [KS00] can now be stated as follows

Lemma 2.4. *Let X be any set, and \mathcal{B} a Boolean algebra.*

- (1) [DJ94] \mathcal{B} has a copy $\leq_T X$ if and only if \mathcal{B}' has a copy $\leq_T X'$
- (2) [Thu95] \mathcal{B}' has a copy $\leq_T X$ if and only if \mathcal{B}'' has a copy $\leq_T X'$
- (3) [KS00] \mathcal{B}'' has a copy $\leq_T X$ if and only if \mathcal{B}''' has a copy $\leq_T X'$
- (4) [KS00] \mathcal{B}''' has a copy $\leq_T X$ if and only if $\mathcal{B}^{(4)}$ has a copy $\leq_T X'$

Corollary 2.5 (Knight and Stob [KS00]). *Suppose $Y^{(4)} \leq_T X^{(4)}$. Every Y -computable Boolean algebra has a X -computable copy.*

Proof. Suppose \mathcal{B} has a copy computable in Y . Then, by Observation 1.3, $\mathcal{B}^{(4)}$ has a copy computable in $Y^{(4)} \leq_T X^{(4)}$. Applying the four items of the previous lemma one at the time starting from the last one, we get that \mathcal{B} has a X -computable copy. \square

3. JUMP INVERSIONS

The following theorem is a sort of jump inversion theorem for structures. The proof is just an applications of the ideas about generic copies structures developed by Ash, Knight, Mennasse and Slaman [AKMS89] and Chisholm [Chi90]. Essentially we prove that the jump \mathcal{A}' of a structure \mathcal{A} can compute a 1-generic copy of \mathcal{A} .

Theorem 3.1. *If $Y \geq_T 0'$ and \mathcal{A}' has a copy $\leq_T Y$, then for some X with $X' \leq_T Y$, \mathcal{A} has a copy computable in X .*

Proof. We will build a copy \mathcal{B} of \mathcal{A} with domain $B = \{b_0, b_1, \dots\}$. Let $D(\mathcal{B}) \in 2^\omega$ be the diagram of \mathcal{B} . So, for some list of atomic formulas ψ_i with variables among x_0, x_1, \dots we have that $D(\mathcal{B}) = 1$ if and only if $\mathcal{B} \models \psi_i$ where x_j is interpreted as b_j . Assume that ψ_i only uses variables among x_0, \dots, x_i . Therefore, to know the first n bits of $D(\mathcal{B})$ we only need to use the atomic relations among b_0, \dots, b_n . For each $\sigma \in 2^n$, let $\psi_\sigma(x_0, \dots, x_n)$ be the formula $\bigwedge_{i:\sigma(i)=1} \psi_i \wedge \bigwedge_{i:\sigma(i)=0} \neg\psi_i$. So, we have that $\mathcal{B} \models \psi_\sigma(b_0, \dots, b_n) \iff \sigma \subseteq D(\mathcal{B})$.

We will build a bijection $F: B \rightarrow A$ and then define \mathcal{B} by pulling back the structure of \mathcal{A} , and we will let $X = D(\mathcal{B})$. We need to define F computably in Y and we will also make sure that Y computes the Turing jump of $D(\mathcal{B})$. At each stage s we define a finite one-to-one partial map $p_s: B \rightarrow A$ with domain $\{b_0, \dots, b_{n_s}\}$, and then we will let $F = \bigcup_s p_s$. Given a finite one-to-one partial map p that maps b_0, \dots, b_n to a_0, \dots, a_n , let $D(p)$ be the $\sigma \in 2^n$ such that $\mathcal{A} \models \psi_\sigma(a_0, \dots, a_n)$. Note that $D(\mathcal{B}) = \bigcup_n D(p_n)$.

Construction:

- Let p_0 map b_0 to a_0 .
- At stage $s + 1 = 2e$ extend p_s to p_{s+1} in any way so that b_e is in the domain of p_{s+1} and a_e is in the image.
- At stage $s + 1 = 2e + 1$ we want to decide the jump of $D(\mathcal{B})$. Suppose p_s maps b_0, \dots, b_{n_s} to a_0, \dots, a_{n_s} . Using Y , decide whether there exists $q \supseteq p_s$ such that $\{e\}^{D(q)}(e) \downarrow$. Note that Y can decide this because, since it computes \mathcal{A}' , it knows whether

$$\mathcal{A} \models \bigvee_{\sigma \supseteq p_s, \{e\}^\sigma(e) \downarrow} \exists \bar{y} \psi_\sigma(a_0, \dots, a_{n_s}, \bar{y}).$$

If the answer is positive, Y can search for witnesses σ and \bar{y} and use them to define p_{s+1} adding \bar{y} to the range of p_s . In this case Y knows that $e \in D(\mathcal{B})'$. Otherwise, we let $p_{s+1} = p_s$ and Y knows that $e \notin D(\mathcal{B})'$.

We have build \mathcal{B} so that $D(\mathcal{B})' \leq_T Y$ as wanted. \square

Corollary 3.2. *For every structure \mathcal{A} ,*

$$\text{degSp}(\mathcal{A}') \cap \mathcal{D}_{(\geq T^0')} = \{X' : X \in \text{degSp}(\mathcal{A})\}.$$

Proof. That $\{X' : X \in \text{degSp}(\mathcal{A})\} \subseteq \text{degSp}(\mathcal{A}') \cap \mathcal{D}_{(\geq T^0')}$ follows from Observation 1.3. That $\text{degSp}(\mathcal{A}') \cap \mathcal{D}_{(\geq T^0')} \subseteq \{X' : X \in \text{degSp}(\mathcal{A})\}$ follows from the previous theorem. \square

Now we consider a stronger version of jump inversion.

Definition 3.3. We say that \mathcal{A} admits n th jump inversion if for every set X , we have that

$$A^{(n)} \text{ has copy } \leq_T X^{(n)} \iff A \text{ has copy } \leq_T X.$$

Observation 3.4. If \mathcal{A} admits jump inversion, then

$$\text{degSp}(\mathcal{A}) = \{X : X' \in \text{degSp}(\mathcal{A}')\}.$$

Structures which admit n th jump inversion have, in some sense, already been considered in the literature before. Lemma 2.3 above shows that Boolean algebras admit 4th jump inversion. In [HM], Harris and the author asked the following question (stated in a different way): does every Boolean algebra admit n th jump inversion? It follows from Theorem 3.5 below that this question is equivalent to the well-known question of whether for every X , every X -low $_n$ Boolean algebra has an X -computable copy. For linear orderings, the following results are known. Downey [DK92] proved that every linear ordering of the form $(\mathbb{Q} + 2 + \mathbb{Q}) \cdot \mathcal{A}$ admits jump inversion. Ash [Ash91] showed that linear orderings of the form $\omega^n \cdot \mathcal{A}$ admit $2n$ th-jump inversion. Kach and the author [KM] then used these results to prove that all the linear orderings with finitely many descending cuts admit n th jump inversion for every n . Graphs which admit α th jump inversion have been used [GHK+05] and [CFG+] to show that there exists Δ_α^0 -categorical structures which are not relatively Δ_α^0 -categorical, lifting earlier results of Goncharov and others on computably categorical structures.

Theorem 3.5. *Let \mathcal{A} be an \mathcal{L} -structure. The following are equivalent.*

- (1) \mathcal{A} admits n th jump inversion;
- (2) For every sets X, Y with $X^{(n)} \equiv_T Y^{(n)}$, we have that

$$A \text{ has copy } \leq_T X \iff A \text{ has copy } \leq_T Y.$$

Proof. To prove that (1) implies (2), we have that if \mathcal{A} has a copy computable in X , then $\mathcal{A}^{(n)}$ has a copy computable in $X^{(n)} \equiv_T Y^{(n)}$, and hence by (1) \mathcal{A} has a copy computable in Y .

For the other direction, suppose that $\mathcal{A}^{(n)}$ has a $Y^{(n)}$ -computable copy. Then, using n iterations of Theorem 3.1, for some X with $X^{(n)} \equiv_T Y^{(n)}$, \mathcal{A} has a copy computable in X . But, then, by (2), \mathcal{A} has a Y -computable copy. \square

4. QUESTIONS

- (1) What are other examples of structures with finite sets of complete Π_n^c relations?
- (2) What are other examples of structures that admit jump inversion?
- (3) Is there a structural characterization of the structures that admit jump inversion?

REFERENCES

- [AK00] C.J. Ash and J. Knight. *Computable Structures and the Hyperarithmetical Hierarchy*. Elsevier Science, 2000.
- [AKMS89] Chris Ash, Julia Knight, Mark Manasse, and Theodore Slaman. Generic copies of countable structures. *Ann. Pure Appl. Logic*, 42(3):195–205, 1989.
- [Ash91] C. J. Ash. A construction for recursive linear orderings. *J. Symbolic Logic*, 56(2):673–683, 1991.
- [CFG+] J. Chisholm, E. Fokina, S. Goncharov, J. Knight, and S. Miller. Intrinsic bounds on complexity at limit levels. to appear in the *Journal of Symbolic Logic*.

- [Chi90] John Chisholm. Effective model theory vs. recursive model theory. *J. Symbolic Logic*, 55(3):1168–1191, 1990.
- [DJ94] Rod Downey and Carl G. Jockusch. Every low Boolean algebra is isomorphic to a recursive one. *Proc. Amer. Math. Soc.*, 122(3):871–880, 1994.
- [DK92] Rodney Downey and Julia F. Knight. Orderings with α th jump degree $\mathbf{0}^{(\alpha)}$. *Proc. Amer. Math. Soc.*, 114(2):545–552, 1992.
- [GHK⁺05] Sergey Goncharov, Valentina Harizanov, Julia Knight, Charles McCoy, Russell Miller, and Reed Solomon. Enumerations in computable structure theory. *Ann. Pure Appl. Logic*, 136(3):219–246, 2005.
- [HM] Kenneth Harris and Antonio Montalbán. On the n -back-and-forth types of Boolean algebras. Submitted for publication.
- [KM] A. Kach and A. Montalbán. Linear orders with finitely many descending cuts. in preparation.
- [KS00] Julia F. Knight and Michael Stob. Computable Boolean algebras. *J. Symbolic Logic*, 65(4):1605–1623, 2000.
- [Thu95] John J. Thurber. Every low₂ Boolean algebra has a recursive copy. *Proc. Amer. Math. Soc.*, 123(12):3859–3866, 1995.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, 5734 S. UNIVERSITY AVE., CHICAGO, IL 60637, USA

E-mail address: antonio@math.uchicago.edu

URL: www.math.uchicago.edu/~antonio