Notes on the Schreier graphs of the Grigorchuk group

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Abstract

The paper is concerned with the space of the marked Schreier graphs of the Grigorchuk group and the action of the group on this space. In particular, we describe the invariant set of the Schreier graphs corresponding to the action on the boundary of the binary rooted tree and dynamics of the group action restricted to this invariant set.

1 Introduction

This paper is devoted to the study of two equivalent dynamical systems of the Grigorchuk group $\mathcal{G}$, the action on the space of the marked Schreier graphs and the action on the space of subgroups. The main object of study is going to be the set of the marked Schreier graphs of the standard action of the group on the boundary of the binary rooted tree and their limit points in the space of all marked Schreier graphs of $\mathcal{G}$.

Given a finitely generated group G with a fixed generating set S, to each action of G we associate its Schreier graph, which is a combinatorial object that encodes some information about orbits of the action. The marked Schreier graphs of various actions form a topological space $\mathrm{Sch}(G,S)$ and there is a natural action of G on this space. Any action of G corresponds to an invariant set in $\mathrm{Sch}(G,S)$ and any action with an invariant measure gives rise to an invariant measure on $\mathrm{Sch}(G,S)$. The latter allows to define the notion of a random Schreier graph, which is closely related to the notion of a random subgroup of G.

A principal problem is to determine how much information about the original action can be learned from the Schreier graphs. The worst case here is a free action, for which nothing beyond its freeness can be recovered. Vershik [5] introduced the notion of a totally nonfree action. This is an action such that all points have distinct stabilizers. In this case the information about the original action can be recovered almost completely. Further extensive development of these ideas was done by Grigorchuk [3].

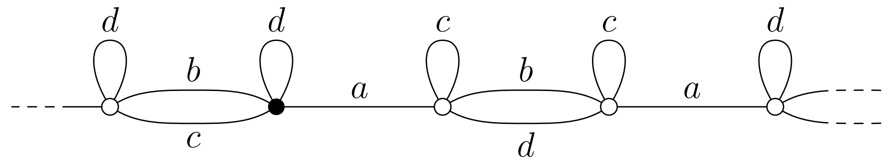

Figure 1: The marked Schreier graph of $0^\infty = 000\ldots$

The Grigorchuk group was introduced in [1] as a simple example of a finitely generated infinite torsion group. Later it was revealed that this group has intermediate growth and a number of other remarkable properties (see the survey [2]). In this paper we are going to use the branching property of the Grigorchuk group, which implies that its action on the boundary of the binary rooted tree is totally nonfree in a very strong sense.

The main results of the paper are summarized in the following two theorems. The first theorem contains a detailed description of the invariant set of the Schreier graphs. The second theorem is concerned with the dynamics of the group action restricted to that invariant set.

Theorem 1.1 *Let $F : \partial\mathcal{T} \to \mathrm{Sch}(\mathcal{G}, \{a,b,c,d\})$ be the mapping that assigns to any point on the boundary of the binary rooted tree the marked Schreier graph of its orbit under the action of the Grigorchuk group. Then*

(i) F is injective;

(ii) F is measurable; it is continuous everywhere except for a countable set, the orbit of the point $\xi_0 = 111\ldots$;

(iii) the Schreier graph $F(\xi_0)$ is an isolated point in the closure of $F(\partial\mathcal{T})$; the other isolated points are graphs obtained by varying the marked vertex of $F(\xi_0)$;

(iv) the closure of the set $F(\partial\mathcal{T})$ differs from $F(\partial\mathcal{T})$ in countably many points; these are obtained from three graphs $\Delta_0, \Delta_1, \Delta_2$ choosing the marked vertex arbitrarily;

(v) as an unmarked graph, $F(\xi_0)$ is a double quotient of each Δ_i ($i = 0,1,2$); also, there exists a graph Δ such that each Δ_i is a double quotient of Δ.

Theorem 1.2 *Using notation of the previous theorem, let Ω be the set of non-isolated points of the closure of $F(\partial\mathcal{T})$. Then*

(i) Ω is a minimal invariant set for the action of the Grigorchuk group $\mathcal{G}$ on $\mathrm{Sch}(\mathcal{G}, \{a,b,c,d\})$;

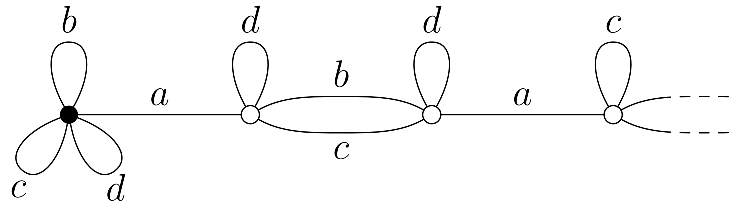

Figure 2: The marked Schreier graph of $1^\infty = 111\dots$

(ii) the action of $\mathcal{G}$ on Ω is a continuous extension of the action on the boundary of the binary rooted tree; the extension is one-to-one everywhere except for a countable set, where it is three-to-one;

(iii) there exists a unique Borel probability measure ν on $\mathrm{Sch}(\mathcal{G}, \{a, b, c, d\})$ invariant under the action of $\mathcal{G}$ and supported on the set Ω;

(iv) the action of $\mathcal{G}$ on Ω with the invariant measure ν is isomorphic to the action of $\mathcal{G}$ on $\partial\mathcal{T}$ with the uniform measure.

The paper is organized as follows. Section 2 contains a detailed construction of the space of marked graphs. The construction is more general than that in [3]. Section 3 contains notation and definitions concerning group actions. In Section 4 we introduce the Schreier graphs of a finitely generated group, the space of marked Schreier graphs, and the action of the group on that space. In Section 5 we study the space of subgroups of a countable group and establish a relation of this space with the space of marked Schreier graphs. Section 6 is devoted to general considerations concerning groups of automorphisms of a regular rooted tree and their actions on the boundary of the tree. In Section 7 we apply the results of the previous sections to the study of the Grigorchuk group and prove Theorems 1.1 and 1.2. The exposition in Sections 2–6 is as general as possible, to make their results applicable to the actions of groups other than the Grigorchuk group.

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2 Space of marked graphs

A *graph* Γ is a combinatorial object that consists of *vertices* and *edges* related so that every edge joins two vertices or a vertex to itself (in the latter case the edge is called a *loop*). The vertices joined by an edge are its *endpoints*. Let V be the *vertex set* of the graph Γ and E be the set of its edges. Traditionally E is regarded as a subset of $V \times V$, i.e., any edge is identified with the pair of its endpoints. In this paper, however, we are going to consider graphs with multiple edges joining the same vertices. Also, our graphs will carry additional structure.

To accomodate this, we regard E merely as a reference set whereas the actual information about the edges is contained in their *attributes*, which are functions on E. In a plain graph any edge has only one attribute: its endpoints, which are an unordered pair of vertices. Other types of graphs involve more attributes.

A *directed graph* has directed edges. The endpoints of a *directed edge* e are ordered, namely, there is a *beginning* $\alpha(e) \in V$ and an *end* $\omega(e) \in V$. Clearly, an undirected loop is no different from a directed one. An undirected edge joining two distinct vertices may be regarded as two directed edges e_1 and e_2 with the same endpoints and opposite directions, i.e., $\alpha(e_2) = \omega(e_1)$ and $\omega(e_2) = \alpha(e_1)$. This way we can represent any graph with undirected edges as a directed graph. Conversely, some directed graphs can be regarded as graphs with undirected edges (we shall use this in Section 7).

A *graph with labeled edges* is a graph in which each edge e is assigned a *label* $l(e)$. The labels are elements of a prescribed finite set. A *marked graph* is a graph with a distinguished vertex called the *marked vertex*.

The vertices of a graph are pictured as dots or small circles. An undirected edge is pictured as an arc joining its endpoints. A directed edge is pictured as an arrow going from its beginning to its end. The label of an edge is written next to the edge. Alternatively, one might think of labels as colors and picture a graph with labeled edges as a colored graph.

Let Γ be a graph and V be its vertex set. To any subset V' of V we associate a graph Γ' called a *subgraph* of Γ. By definition, the vertex set of the graph Γ' is V' and the edges are those edges of Γ that have both endpoints in V' (all attributes are retained). If Γ is a marked graph and the marked vertex is in V', it will also be the marked vertex of the subgraph Γ'. Otherwise the subgraph is not marked.

Suppose Γ_1 and Γ_2 are graphs of the same type. For any $i \in \{1, 2\}$ let V_i be the vertex set of Γ_i and E_i be the set of its edges. The graph Γ_1 is said to be *isomorphic* to Γ_2 if there exist bijections $f : V_1 \to V_2$ and $\phi : E_1 \to E_2$ that respect the structure of the graphs. First of all, this means that f sends the endpoints of any edge $e \in E_1$ to the endpoints of $\phi(e)$. If Γ_1 and Γ_2 are directed graphs, we additionally require that $\alpha(\phi(e)) = f(\alpha(e))$ and $\omega(\phi(e)) = f(\omega(e))$ for all $e \in E_1$. If Γ_1 and Γ_2 have labeled edges, we also require that ϕ preserve labels. If Γ_1 and Γ_2 are marked graphs, we also require that f map the marked vertex of Γ_1 to the marked vertex of Γ_2. Assuming the above requirements are met, the mapping f of the vertex set is called an *isomorphism of the graphs* Γ_1 and Γ_2. If $\Gamma_1 = \Gamma_2$ then f is also called an *automorphism* of the graph Γ_1. We call the mapping ϕ a *companion mapping* of f. If neither of the graphs Γ_1 and Γ_2 admits multiple edges with identical attributes, the companion mapping is uniquely determined by the isomorphism f. Further, we say that the graph Γ_2 is a *quotient* of Γ_1 if all of the above requirements are met except the mappings f and ϕ need not be injective. Moreover, Γ_2 is a *k-fold quotient* of Γ_1 if f is k-to-1. Finally, we say that the graphs Γ_1 and Γ_2 coincide *up to renaming edges* if they have the same vertices and there is a one-to-one correspondence between their edges that preserves all attributes. An equivalent condition is that the identity map on the common vertex set is an isomorphism of these graphs.

A *path* in a graph Γ is a sequence of vertices $v_0, v_1, \dots, v_n$ together with a sequence of edges $e_1, \dots, e_n$ such that for any $1 \le i \le n$ the endpoints of the edge e_i are v_{i-1} and v_i. We say that the vertex v_0 is the beginning of the path and v_n is the end. The path is *closed* if $v_n = v_0$. The *length* of the path is the number of edges in the sequence (counted with repetitions), which is a nonnegative integer. The path is a *directed path* if the edges are directed and, moreover, $\alpha(e_i) = v_{i-1}$ and $\omega(e_i) = v_i$ for $1 \le i \le n$. If the graph Γ has labeled edges then the path is assigned a *code word* $l(e_1)l(e_2)\dots l(e_n)$, which is a string of labels read off the edges while traversing the path.

We say that a vertex v of a graph Γ is *connected to* a vertex v' if there is a path in Γ such that the beginning of the path is v and the end is v'. The length of the shortest path with this property is the *distance* from v to v'. The connectivity is an equivalence relation on the vertex set of Γ. The subgraphs of Γ corresponding to the equivalence classes are *connected components* of the graph Γ. A graph is *connected* if all vertices are connected to each other. Clearly, the connected components of any graph are its maximal connected subgraphs.

Let v be a vertex of a graph Γ. For any integer $n \ge 0$, the *closed ball* of radius n centered at v, denoted $\overline{B}_\Gamma(v, n)$, is the subgraph of Γ whose vertex set consists of all vertices in Γ at distance at most n from the vertex v. A graph is *locally finite* if every vertex is the endpoint for only finitely many edges. If the graph Γ is locally finite then any closed ball of Γ is a *finite graph*, i.e., it has a finite number of vertices and a finite number of edges.

Let $\mathcal{MG}$ denote the set of isomorphism classes of all marked directed graphs with labeled edges. For convenience, we regard elements of $\mathcal{MG}$ as graphs (i.e., we choose representatives of isomorphism classes). It is easy to observe that connectedness and local finiteness of graphs are preserved under isomorphisms. Let $\mathcal{MG}_0$ denote the subset of $\mathcal{MG}$ consisting of connected, locally finite graphs. We endow the set $\mathcal{MG}_0$ with a topology as follows. The topology is generated by sets $\mathcal{U}(\Gamma_0, V_0) \subset \mathcal{MG}_0$, where Γ_0 runs over all finite graphs in $\mathcal{MG}_0$ and V_0 can be any subset of the vertex set of Γ_0. By definition, $\mathcal{U}(\Gamma_0, V_0)$ is the set of all isomorphism classes in $\mathcal{MG}_0$ containing any graph Γ such that Γ_0 is a subgraph of Γ and every edge of Γ with at least one endpoint in the set V_0 is actually an edge of Γ_0. In other words, there is no edge in Γ that joins a vertex from V_0 to a vertex outside the vertex set of Γ_0. For example, $\mathcal{U}(\Gamma_0, \emptyset)$ is the set of all graphs in $\mathcal{MG}_0$ that have a subgraph isomorphic to Γ_0. On the other hand, if V_0 is the entire vertex set of Γ_0 then $\mathcal{U}(\Gamma_0, V_0)$ contains only the graph Γ_0. As a consequence, every finite graph in $\mathcal{MG}_0$ is an isolated point. The following lemma implies that sets of the form $\mathcal{U}(\Gamma_0, V_0)$ constitute a base of the topology.

Lemma 2.1 *Any nonempty intersection of two sets of the form $\mathcal{U}(\Gamma_0, V_0)$ can be represented as the union of some sets of the same form.*

Proof. Let $\Gamma_1, \Gamma_2 \in \mathcal{MG}_0$ be finite graphs and V_1, V_2 be subsets of their vertex sets. Consider an arbitrary graph $\Gamma \in \mathcal{U}(\Gamma_1, V_1) \cap \mathcal{U}(\Gamma_2, V_2)$. For any $i \in \{1, 2\}$ let $f_i : W_i \to W_i'$ be an isomorphism of the graph Γ_i with a subgraph of Γ such that no edge of Γ joins a vertex from the set $f_i(V_i)$ to a vertex outside W_i'. Denote by Γ_0 the finite subgraph of Γ with the vertex set $W_0 = W_1' \cup W_2'$. Since the subgraphs of Γ with vertex sets W_1' and W_2' are both connected and both contain the marked vertex of Γ, the subgraph Γ_0 is also marked and connected. Besides, no edge of Γ joins a vertex from the set $V_0 = f_1(V_1) \cup f_2(V_2)$ to a vertex outside W_0. Hence $\Gamma \in \mathcal{U}(\Gamma_0, V_0)$. It is easy to observe that the entire set $\mathcal{U}(\Gamma_0, V_0)$ is contained in the intersection $\mathcal{U}(\Gamma_1, V_1) \cap \mathcal{U}(\Gamma_2, V_2)$. The lemma follows. ∎

Next we introduce a distance function on $\mathcal{MG}_0$. Consider arbitrary graphs $\Gamma_1, \Gamma_2 \in \mathcal{MG}_0$. Let v_1 be the marked vertex of Γ_1 and v_2 be the marked vertex of Γ_2. We let $\delta(\Gamma_1, \Gamma_2) = 0$ if the graphs Γ_1 and Γ_2 are isomorphic (i.e., they represent the same element of $\mathcal{MG}_0$). Otherwise we let $\delta(\Gamma_1, \Gamma_2) = 2^{-n}$, where n is the smallest nonnegative integer such that the closed balls $\overline{B}_{\Gamma_1}(v_1, n)$ and $\overline{B}_{\Gamma_2}(v_2, n)$ are not isomorphic.

Lemma 2.2 *The graphs Γ_1 and Γ_2 are isomorphic if and only if the closed balls $\overline{B}_{\Gamma_1}(v_1, n)$ and $\overline{B}_{\Gamma_2}(v_2, n)$ are isomorphic for any integer $n \geq 0$.*

Proof. For any $i \in \{1, 2\}$ let V_i denote the vertex set of the graph Γ_i and E_i denote its set of edges. Further, for any integer $n \geq 0$ let $V_i(n)$ and $E_i(n)$ denote the vertex set and the set of edges of the closed ball $\overline{B}_{\Gamma_i}(v_i, n)$. First assume that the graph Γ_1 is isomorphic to Γ_2. Let $f : V_1 \to V_2$ be an isomorphism of these graphs and $\phi : E_1 \to E_2$ be its companion mapping. Clearly, $f(v_1) = v_2$. It is easy to see that any isomorphism of graphs preserves distances between vertices. It follows that f maps $V_1(n)$ onto $V_2(n)$ for any $n \geq 0$. Consequently, ϕ maps $E_1(n)$ onto $E_2(n)$. Hence the restriction of f to the set $V_1(n)$ is an isomorphism of the graphs $\overline{B}_{\Gamma_1}(v_1, n)$ and $\overline{B}_{\Gamma_2}(v_2, n)$.

Now assume that for every integer $n \geq 0$ the closed balls $\overline{B}_{\Gamma_1}(v_1, n)$ and $\overline{B}_{\Gamma_2}(v_2, n)$ are isomorphic. Let $f_n : V_1(n) \to V_2(n)$ be an isomorphism of these graphs and $\phi_n : E_1(n) \to E_2(n)$ be its companion mapping. Clearly, $f_n(v_1) = v_2$. Note that the closed ball $\overline{B}_{\Gamma_i}(v_i, n)$ is also the closed ball with the same center and radius in any of the graphs $\overline{B}_{\Gamma_i}(v_i, m)$, $m > n$. It follows that the restriction of the mapping f_m to the set $V_1(n)$ is an isomorphism of the graphs $\overline{B}_{\Gamma_1}(v_1, n)$ and $\overline{B}_{\Gamma_2}(v_2, n)$ while the restriction of ϕ_m to $E_1(n)$ is its companion mapping. Since the graphs Γ_1 and Γ_2 are locally finite, the sets $V_1(n), V_2(n), E_1(n), E_2(n)$ are finite. Hence there are only finitely many distinct restrictions $f_m|_{V_1(n)}$ or $\phi_m|_{E_1(n)}$ for any fixed n. Therefore one can find nested infinite sets of indices $I_0 \supset I_1 \supset I_2 \supset \dots$ such that the restriction $f_m|_{V_1(n)}$ is the same for all $m \in I_n$ and the restriction $\phi_m|_{E_1(n)}$ is the same for all $m \in I_n$. For any integer $n \geq 0$ let $f_n' = f_m|_{V_1(n)}$ and $\phi_n' = \phi_m|_{E_1(n)}$, where $m \in I_n$. By construction, f_n' is a restriction of f_k' and ϕ_n' is a restriction of ϕ_k' whenever $n < k$. Hence there exist maps $f : V_1 \to V_2$ and

$\phi : E_1 \to E_2$ such that all f'_n are restrictions of f and all ϕ'_n are restrictions of ϕ. Since the graphs Γ_1 and Γ_2 are connected, any finite collection of vertices and edges in either graph is contained in a closed ball centered at the marked vertex. As for any $n \geq 0$ the mapping f'_n is an isomorphism of $\overline{B}_{\Gamma_1}(v_1, n)$ and $\overline{B}_{\Gamma_2}(v_2, n)$ and ϕ'_n is its companion mapping, it follows that f is an isomorphism of Γ_1 and Γ_2 and ϕ is its companion mapping. ∎

Lemma 2.2 implies that δ is a well-defined function on $\mathcal{MG}_0 \times \mathcal{MG}_0$. This is a distance function, which makes $\mathcal{MG}_0$ into an ultrametric space.

Lemma 2.3 *The distance function δ is compatible with the topology on $\mathcal{MG}_0$.*

Proof. The base of the topology on $\mathcal{MG}_0$ consists of the sets $\mathcal{U}(\Gamma_0, V_0)$. The base of the topology defined by the distance function δ is formed by open balls $\mathcal{B}(\Gamma_1, \epsilon) = \{\Gamma \in \mathcal{MG}_0 \mid \delta(\Gamma, \Gamma_1) < \epsilon\}$, where Γ_1 can be any graph in $\mathcal{MG}_0$ and $\epsilon > 0$. We have to show that any element of either base is the union of some elements of the other base.

First consider an open ball $\mathcal{B}(\Gamma_1, \epsilon)$. If $\epsilon > 1$ then $\mathcal{B}(\Gamma_1, \epsilon) = \mathcal{MG}_0$, which is the union of all sets $\mathcal{U}(\Gamma_0, V_0)$. Otherwise let n be the largest integer such that $\epsilon \leq 2^{-n}$. Clearly, $\mathcal{B}(\Gamma_1, \epsilon) = \mathcal{B}(\Gamma_1, 2^{-n})$. Let $\Gamma_0 = \overline{B}_{\Gamma_1}(v_1, n)$, where v_1 is the marked vertex of the graph Γ_1, and let V_0 be the set of all vertices of Γ_1 at distance at most $n-1$ from v_1. Consider an arbitrary graph $\Gamma \in \mathcal{MG}_0$ such that Γ_0 is a subgraph of Γ. Clearly, Γ_0 is also a subgraph of the closed ball $\overline{B}_\Gamma(v_1, n)$. If v is a vertex of Γ at distance k from the marked vertex v_1, then any vertex joined to v by an edge is at distance at most $k+1$ and at least $k-1$ from v_1. Moreover, if $k > 0$ then v is joined to a vertex at distance exactly $k-1$ from v_1. It follows that $\Gamma_0 = \overline{B}_\Gamma(v_1, n)$ if and only if no vertex from the set V_0 is joined in Γ to a vertex that is not a vertex of Γ_0. Thus $\mathcal{B}(\Gamma_1, 2^{-n}) = \mathcal{U}(\Gamma_0, V_0)$.

Now consider the set $\mathcal{U}(\Gamma_0, V_0)$, where Γ_0 is a finite graph in $\mathcal{MG}_0$ and V_0 is a subset of its vertex set. Denote by v_0 the marked vertex of Γ_0. Let n be the smallest integer such that every vertex of Γ_0 is at distance at most n from v_0 and every vertex from V_0 is at distance at most $n-1$ from v_0. Take any graph $\Gamma \in \mathcal{MG}_0$ such that Γ_0 is a subgraph of Γ and there is no edge in Γ joining a vertex from V_0 to a vertex outside the vertex set of Γ_0. Let $\Gamma_1 = \overline{B}_\Gamma(v_0, n)$ and V_1 be the set of all vertices of Γ at distance at most $n-1$ from v_0. By the above, $\mathcal{U}(\Gamma_1, V_1) = \mathcal{B}(\Gamma, 2^{-n})$. At the same time, $\mathcal{U}(\Gamma_1, V_1) \subset \mathcal{U}(\Gamma_0, V_0)$ since Γ_0 is a subgraph of Γ_1 and V_0 is a subset of V_1. Thus for any graph $\Gamma \in \mathcal{U}(\Gamma_0, V_0)$ the entire open ball $\mathcal{B}(\Gamma, 2^{-n})$ is contained in $\mathcal{U}(\Gamma_0, V_0)$. In particular, $\mathcal{U}(\Gamma_0, V_0)$ is the union of those open balls. ∎

Given a positive integer N and a finite set L, let $\mathcal{MG}(N, L)$ denote the subset of $\mathcal{MG}$ consisting of all graphs in which every vertex is the endpoint for at most N edges and every label belongs to L. Further, let $\mathcal{MG}_0(N, L) = \mathcal{MG}(N, L) \cap \mathcal{MG}_0$.

Proposition 2.4 *$\mathcal{MG}_0(N, L)$ is a compact subset of the metric space $\mathcal{MG}_0$.*

Proof. We have to show that any sequence of graphs $\Gamma_1, \Gamma_2, \dots$ in $\mathcal{MG}_0(N, L)$ has a subsequence converging to some graph in $\mathcal{MG}_0(N, L)$. For any positive integer n let V_n denote the vertex set of the graph Γ_n, E_n denote its sets of edges, and v_n denote the marked vertex of Γ_n. First consider the special case when each Γ_n is a subgraph of Γ_{n+1}. Let Γ be the graph with the vertex set $V = V_1 \cup V_2 \cup \dots$ and the set of edges $E = E_1 \cup E_2 \cup \dots$. We assume that any edge $e \in E_n$ retains its attributes (beginning, end, and label) in the graph Γ. The common marked vertex of the graphs Γ_n is set as the marked vertex of Γ. Note that any finite collection of vertices and edges of the graph Γ is already contained in some Γ_n. As the graphs $\Gamma_1, \Gamma_2, \dots$ belong to $\mathcal{MG}_0(N, L)$, it follows that $\Gamma \in \mathcal{MG}_0(N, L)$ as well. In particular, for any integer $k \geq 0$ the closed ball $\overline{B}_\Gamma(v_1, k)$ is a finite graph. Then it is a subgraph of some Γ_n. Clearly, $\overline{B}_\Gamma(v_1, k)$ is also a subgraph of the graphs $\Gamma_{n+1}, \Gamma_{n+2}, \dots$. Moreover, it remains the closed ball of radius k centered at the marked vertex in all these graphs. It follows that $\delta(\Gamma_m, \Gamma) < 2^{-k}$ for $m \geq n$. Since k can be arbitrarily large, the sequence $\Gamma_1, \Gamma_2, \dots$ converges to Γ in the metric space $\mathcal{MG}_0$.

Next consider a more general case when each Γ_n is isomorphic to a subgraph of Γ_{n+1}. This case is reduced to the previous one by repeatedly using the following observation: if a graph P_0 is isomorphic to a subgraph of a graph P then there exists a graph P' isomorphic to P such that P_0 is a subgraph of P'.

Finally consider the general case. For any graph in $\mathcal{MG}_0(N, L)$, the closed ball of radius k with any center contains at most $1+N+N^2+\dots+N^{k-1}$ vertices while the number of edges is at most N times the number of vertices. Hence for any fixed k the number of vertices and edges in the balls $\overline{B}_{\Gamma_n}(v_n, k)$ is uniformly bounded, which implies that there are only finitely many non-isomorphic graphs among them. Therefore one can find nested infinite sets of indices $I_0 \supset I_1 \supset I_2 \supset \dots$ such that the closed balls $\overline{B}_{\Gamma_n}(v_n, k)$ are isomorphic for all $n \in I_k$. Choose an increasing sequence of indices $n_0, n_1, n_2, \dots$ such that $n_k \in I_k$ for all k, and let Γ'_k be the closed ball of radius k in the graph Γ_{n_k} centered at the marked point v_{n_k}. Clearly, $\Gamma'_k \in \mathcal{MG}_0(N, L)$ and $\delta(\Gamma'_k, \Gamma_{n_k}) < 2^{-k}$. By construction, Γ'_k is isomorphic to a subgraph of Γ'_m whenever $k < m$. By the above the sequence $\Gamma'_0, \Gamma'_1, \Gamma'_2, \dots$ converges to a graph $\Gamma \in \mathcal{MG}_0(N, L)$. Since $\delta(\Gamma'_k, \Gamma_{n_k}) < 2^{-k}$ for all $k \geq 0$, the subsequence $\Gamma_{n_0}, \Gamma_{n_1}, \Gamma_{n_2}, \dots$ converges to the graph Γ as well. ∎

3 Group actions

Let M be an arbitrary nonempty set. Invertible transformations $\phi : M \to M$ form a transformation group. An *action* A of an abstract group G on the set M is a homomorphism of G into that transformation group. The action can be regarded as a collection of invertible transformations $A_g : M \to M$, $g \in G$, where A_g is the image of g under the homomorphism. The transformations are to satisfy $A_g A_h = A_{gh}$ for all $g, h \in G$. We say that A_g is the action of an element g within the action A. Alternatively, the action of the group G can be given as a mapping $A : G \times M \to M$ such that $A(g, x) = A_g(x)$ for all $g \in G$ and $x \in M$. Such a mapping defines an action of G if and only if the following two conditions hold:

- $A(gh, x) = A(g, A(h, x))$ for all $g, h \in G$ and $x \in M$;
- $A(1_G, x) = x$ for all $x \in M$, where 1_G is the unity of the group G.

A nonempty set $S \subset G$ is called a *generating set* for the group G if any element $g \in G$ can be represented as a product $g_1 g_2 \dots g_k$ where each factor g_i is an element of S or the inverse of an element of S. The elements of the generating set are called *generators* of the group G. The generating set S is *symmetric* if it is closed under taking inverses, i.e., $s^{-1} \in S$ whenever $s \in S$. If S is a generating set for G then any action A of the group G is uniquely determined by transformations A_s, $s \in S$.

Suppose G is a topological group. An action of G on a topological space M is a *continuous action* if it is continuous as a mapping of $G \times M$ to M. Similarly, an action of G on a measured space M is a *measurable action* if it is measurable as a mapping of $G \times M$ to M. A measurable action A of the group G on a measured space M with a measure μ is *measure-preserving* if the action of every element of G is measure-preserving, i.e., $\mu\big(A_g^{-1}(W)\big) = \mu(W)$ for all $g \in G$ and measurable sets $W \subset M$. In what follows, the group G will be a discrete countable group. In that case, an action A of G is continuous if and only if all transformations A_g, $g \in G$ are continuous. Likewise, the action A is measurable if and only if every A_g is measurable.

Given an action A of a group G on a set M, the *orbit* $O_A(x)$ of a point $x \in M$ under the action A is the set of all points $A_g(x)$, $g \in G$. A subset $M_0 \subset M$ is *invariant* under the action A if $A_g(M_0) \subset M_0$ for all $g \in G$. Clearly, the orbit $O_A(x)$ is invariant under the action. Moreover, this is the smallest invariant set containing x. The *restriction* of the action A to a nonempty invariant set M_0 is an action of G obtained by restricting every transformation A_g to M_0. Equivalently, one might restrict the mapping $A : G \times M \to M$ to the set $G \times M_0$. The action A is *transitive* if the only invariant subsets of M are the empty set and M itself. Equivalently, the orbit of any point is the entire set M. Assuming the action A is continuous, it is *topologically transitive* if there is an orbit dense in M, and *minimal* if every orbit of A is dense. The action is minimal if and only if the empty set and M are the only closed invariant subsets of M. Assuming the action A is measure-preserving, it is *ergodic* if any measurable invariant subset of M has zero or full measure. A continuous action on a compact space M is *uniquely ergodic* if there exists a unique Borel probability measure on M invariant under the action (the action is going to be ergodic with respect to that measure).

Given an action A of a group G on a set M, the *stabilizer* $\mathrm{St}_A(x)$ of a point $x \in M$ under the action A is the set of all elements $g \in G$ whose action fixes x, i.e., $A_g(x) = x$. The stabilizer $\mathrm{St}_A(x)$ is a subgroup of G. The action is *free* if all stabilizers are trivial. In the case when the action A is continuous, we define the *neighborhood stabilizer* $\mathrm{St}^o_A(x)$ of a point $x \in M$ to be the set of all $g \in G$ whose action fixes the point x along with its neighborhood (the neighborhood may depend on g). The neighborhood stabilizer $\mathrm{St}^o_A(x)$ is a normal subgroup of $\mathrm{St}_A(x)$.

Let $A : G \times M_1 \to M_1$ and $B : G \times M_2 \to M_2$ be actions of a group G on sets M_1 and M_2, respectively. The actions A and B are *conjugated* if there exists a bijection $f : M_1 \to M_2$ such that $B_g = fA_gf^{-1}$ for all $g \in G$. An equivalent condition is that $A(g, x) = B(g, f(x))$ for all $g \in G$ and $x \in M_1$. The bijection f is called a *conjugacy* of the action A with B. Two continuous actions of the same group are *continuously conjugated* if they are conjugated and, moreover, the conjugacy can be chosen to be a homeomorphism. Similarly, two measurable actions are *measurably conjugated* if they are conjugated and, moreover, the conjugacy f can be chosen so that both f and the inverse f^{-1} are measurable. Also, two measure-preserving actions are *isomorphic* if they are conjugated and, moreover, the conjugacy can be chosen to be an isomorphism of spaces with measure. The measure-preserving actions are *isomorphic modulo zero measure* if each action admits an invariant set of full measure such that the corresponding restrictions are isomorphic.

Given two actions $A : G \times M_1 \to M_1$ and $B : G \times M_2 \to M_2$ of a group G, the action A is an *extension* of B if there exists a mapping f of M_1 onto M_2 such that $B_gf = fA_g$ for all $g \in G$. The extension is *k-to-1* if f is k-to-1. The extension is *continuous* if the actions A and B are continuous and f can be chosen continuous.

4 The Schreier graphs

Let G be a finitely generated group. Let us fix a finite symmetric generating set S for G. Given an action A of the group G on a set M, the *Schreier graph* $\Gamma_{\mathrm{Sch}}(G, S; A)$ of the action relative to the generating set S is a directed graph with labeled edges. The vertex set of the graph $\Gamma_{\mathrm{Sch}}(G, S; A)$ is M, the set of edges is $M \times S$, and the set of labels is S. For any $x \in M$ and $s \in S$ the edge (x, s) has beginning x, end $A_s(x)$, and carries label s. Clearly, the action A can be uniquely recovered from its Schreier graph. Given another action A' of G on some set M', the Schreier graph $\Gamma_{\mathrm{Sch}}(G, S; A')$ is isomorphic to $\Gamma_{\mathrm{Sch}}(G, S; A)$ if and only if the actions A and A' are conjugated. Indeed, a bijection $f : M \to M'$ is an isomorphism of the Schreier graphs if and only if $A'_s = fA_sf^{-1}$ for all $s \in S$, which is equivalent to f being a conjugacy of the action A with A'.

Any graph of the form $\Gamma_{\mathrm{Sch}}(G, S; A)$ is called a *Schreier graph* of the group G (relative to the generating set S). Notice that any graph isomorphic to a Schreier graph is also a Schreier graph up to renaming edges. This follows from the next proposition, which explains how to recognize a Schreier graph of G.

Proposition 4.1 *A directed graph Γ with labeled edges is, up to renaming edges, a Schreier graph of the group G relative to the generating set S if and only if the following conditions are satisfied:*

(i) all labels are in S;

(ii) for any vertex v and any generator $s \in S$ there exists a unique edge with beginning v and label s;

(iii) given a directed path with code word $s_1 s_2 \dots s_k$, the path is closed whenever the reversed code word $s_k \dots s_2 s_1$ equals 1_G when regarded as a product in G.

Proof. First suppose Γ is a Schreier graph $\Gamma_{\mathrm{Sch}}(G, S; A)$. Consider an arbitrary directed path in the graph Γ. Let v be the beginning of the path and $s_1 s_2 \dots s_k$ be its code word. Then the consecutive vertices of the path are $v_0 = v, v_1, \dots, v_k$, where $v_i = A_{s_i}(v_{i-1})$ for $1 \le i \le k$. Hence the end of the path is $A_{s_k} \dots A_{s_2} A_{s_1}(v) = A_g(v)$, where g denotes $s_k \dots s_2 s_1$ regarded as a product in G. Clearly, the path is closed whenever $g = 1_G$. Thus any Schreier graph of the group G satisfies the condition (iii). The conditions (i) and (ii) are trivially satisfied as well. It is easy to see that the conditions (i), (ii), and (iii) are preserved under isomorphisms of graphs. In particular, they hold for any graph that coincides with a Schreier graph up to renaming edges.

Now suppose Γ is a directed graph with labeled edges that satisfies the conditions (i), (ii), and (iii). Let M denote the vertex set of Γ. Given a word $w = s_1 s_2 \dots s_k$ over the alphabet S, we define a transformation $B_w : M \to M$ as follows. The condition (ii) implies that for any vertex $v \in M$ there is a unique directed path in Γ with beginning v and code word $s_k \dots s_2 s_1$ (the word w reversed). We set $B_w(v)$ to be the end of that path. For any words $w = s_1 s_2 \dots s_k$ and $w' = s'_1 s'_2 \dots s'_m$ over the alphabet S let ww' denote the concatenated word $s_1 s_2 \dots s_k s'_1 s'_2 \dots s'_m$. Then $B_{ww'}(v) = B_w(B_{w'}(v))$ for all $v \in M$. Any word over the alphabet S can be regarded as a product in the group G thus representing an element $g \in G$. Clearly, the concatenation of words corresponds to the multiplication in the group. The condition (iii) means that B_w is the identity transformation whenever the word w represents the unity 1_G. This implies that transformations B_w and $B_{w'}$ are the same if the words w and w' represent the same element $g \in G$. Indeed, let $w = s_1 s_2 \dots s_k$, $w' = s'_1 s'_2 \dots s'_m$ and consider the third word $z = s_k^{-1} \dots s_2^{-1} s_1^{-1}$. The word z represents the inverse g^{-1}. Therefore the words wz and zw' both represent the unity. Then $B_w = B_w B_{zw'} = B_{wzw'} = B_{wz} B_{w'} = B_{w'}$. Now for any $g \in G$ we let $A_g = B_w$, where w is an arbitrary word over the alphabet S representing g. By the above A_g is a well-defined transformation of M, A_{1_G} is the identity transformation, and $A_{gg'} = A_g A_{g'}$ for all $g, g' \in G$. Hence the transformations A_g, $g \in G$ constitute an action A of the group G on the vertex set M. By construction, for any $v \in M$ and $s \in S$ the vertex $A_s(v)$ is the end of the edge with beginning v and label s. In view of the conditions (i) and (ii), this means that the graph Γ coincides with the Schreier graph $\Gamma_{\text{Sch}}(G, S; A)$ up to renaming edges. ∎

For any $x \in M$ let $\Gamma_{\text{Sch}}(G, S; A, x)$ denote the Schreier graph of the restriction of the action A to the orbit of x. We refer to $\Gamma_{\text{Sch}}(G, S; A, x)$ as the Schreier graph of the orbit of x. It is easy to observe that $\Gamma_{\text{Sch}}(G, S; A, x)$ is the connected component of the graph $\Gamma_{\text{Sch}}(G, S; A)$ containing the vertex x. In particular, the Schreier graph of the action A is connected if and only if the action is transitive, in which case $\Gamma_{\text{Sch}}(G, S; A, x) = \Gamma_{\text{Sch}}(G, S; A)$ for all $x \in M$. Let $\Gamma^*_{\text{Sch}}(G, S; A, x)$ denote a marked graph obtained from $\Gamma_{\text{Sch}}(G, S; A, x)$ by marking the vertex x. We refer to it as the *marked Schreier graph* of the point x (under the action A). Notice that the point x and the restriction of the action A to its orbit are uniquely recovered from the graph $\Gamma^*_{\text{Sch}}(G, S; A, x)$. Any graph of the form $\Gamma^*_{\text{Sch}}(G, S; A, x)$ is called a marked Schreier graph of the group G (relative to the generating set S).

Let $\text{Sch}(G, S)$ denote the set of isomorphism classes of all marked Schreier graphs of the group G relative to the generating set S. A graph $\Gamma \in \mathcal{MG}$ belongs to $\text{Sch}(G, S)$ if it is a marked directed graph that is connected and satisfies conditions (i), (ii), (iii) of Proposition 4.1.

The group G acts naturally on the set of the marked Schreier graphs of G by changing the marked vertex. The action $\mathcal{A}$ is given by $\mathcal{A}_g\big(\Gamma^*_{\text{Sch}}(G, S; A, x)\big) = \Gamma^*_{\text{Sch}}(G, S; A, A_g(x))$, $g \in G$. It turns out that $\mathcal{A}$ is well defined as an action on $\text{Sch}(G, S)$. Indeed, let $\Gamma^*_{\text{Sch}}(G, S; B, y)$ be a marked Schreier graph isomorphic to $\Gamma^*_{\text{Sch}}(G, S; A, x)$. Then any isomorphism f of the latter graph with the former one is simultaneously a conjugacy of the restriction of the action A to the orbit of x with the restriction of the action B to the orbit of y. Since $f(x) = y$, it follows that $f(A_g(x)) = B_g(y)$ for all $g \in G$. Hence for any $g \in G$ the map f is also an isomorphism of the graph $\Gamma^*_{\text{Sch}}(G, S; A, A_g(x))$ with $\Gamma^*_{\text{Sch}}(G, S; B, B_g(y))$.

Proposition 4.2 $\text{Sch}(G, S)$ *is a compact subset of the metric space* $\mathcal{MG}_0$. *The action of the group G (regarded as a discrete group) on* $\text{Sch}(G, S)$ *is continuous.*

Proof. Let N be the number of elements in the generating set S. Then every vertex v of a graph $\Gamma \in \text{Sch}(G, S)$ is the beginning of exactly N edges. Furthermore, v is the end of an edge with beginning v' and label s if and only if v' is the end of the edge with beginning v and label s^{-1}. It follows that v is also the end of exactly N edges. Hence any vertex of Γ is an endpoint for at most $2N$ edges. Therefore $\text{Sch}(G, S) \subset \mathcal{MG}(2N, S)$. Since all marked Schreier graphs are connected, we have $\text{Sch}(G, S) \subset \mathcal{MG}_0(2N, S) \subset \mathcal{MG}_0$.

Now let us show that the set $\text{Sch}(G, S)$ is closed in the topological space $\mathcal{MG}_0$. Take any graph $\Gamma \in \mathcal{MG}_0$ not in that set. Then Γ does not satisfy at least one of the conditions (i), (ii), and (iii) in Proposition 4.1. First consider the case when the condition (i) or (iii) does not hold. Since the graph Γ is locally finite, it has a finite subgraph Γ_0 for which the same condition does not hold. Since Γ is connected, we can choose the subgraph Γ_0 to be marked and connected so that $\Gamma_0 \in \mathcal{MG}_0$. Clearly, the same condition does not hold for any graph Γ' such that Γ_0 is a subgraph of Γ'. It follows that the neighborhood $\mathcal{U}(\Gamma_0, \emptyset)$ of the graph Γ is disjoint from $\mathrm{Sch}(G, S)$. Next consider the case when Γ does not satisfy the condition (ii). Let v be the vertex of Γ such that for some generator $s \in S$ there are either several edges with beginning v and label s or no such edges at all. Since $\Gamma \in \mathcal{MG}_0$, there exists a finite connected subgraph Γ_0 of Γ that contains the marked vertex, the vertex v, and all edges for which v is an endpoint. Then $\Gamma_0 \in \mathcal{MG}_0$ and the open set $\mathcal{U}(\Gamma_0, \{v\})$ is a neighborhood of Γ. By construction, the condition (ii) fails in the entire neighborhood so that $\mathcal{U}(\Gamma_0, \{v\})$ is disjoint from $\mathrm{Sch}(G, S)$. Thus the set $\mathcal{MG}_0 \setminus \mathrm{Sch}(G, S)$ is open in $\mathcal{MG}_0$. Therefore the set $\mathrm{Sch}(G, S)$ is closed.

Since the closed set $\mathrm{Sch}(G, S)$ is contained in $\mathcal{MG}_0(2N, S)$, which is a compact set due to Proposition 2.4, the set $\mathrm{Sch}(G, S)$ is compact as well.

An action of the group G is continuous whenever the generators act continuously. To prove that the transformations $\mathcal{A}_s$, $s \in S$ are continuous, we are going to show that $\delta(\mathcal{A}_s(\Gamma), \mathcal{A}_s(\Gamma')) \leq 2\delta(\Gamma, \Gamma')$ for any graphs $\Gamma, \Gamma' \in \mathrm{Sch}(G, S)$ and any generator $s \in S$. If the graphs Γ and Γ' are isomorphic, then the graphs $\mathcal{A}_s(\Gamma)$ and $\mathcal{A}_s(\Gamma')$ are also isomorphic so that $\delta(\mathcal{A}_s(\Gamma), \mathcal{A}_s(\Gamma')) = \delta(\Gamma, \Gamma') = 0$. Otherwise $\delta(\Gamma, \Gamma') = 2^{-n}$ for some nonnegative integer n. Since the distance between any graphs in $\mathcal{MG}_0$ never exceeds 1, it is enough to consider the case $n \geq 2$. Let v denote the marked vertex of Γ and v' denote the marked vertex of Γ'. By definition of the distance function, the closed balls $\overline{B}_\Gamma(v, n-1)$ and $\overline{B}_{\Gamma'}(v', n-1)$ are isomorphic. Consider an isomorphism f of these graphs. Clearly, $f(v) = v'$. Let v_1 denote the marked vertex of the graph $\mathcal{A}_s(\Gamma)$ and v'_1 denote the marked vertex of $\mathcal{A}_s(\Gamma')$. Then v_1 is the end of the edge with beginning v and label s in the graph Γ. Similarly, v'_1 is the end of the edge with beginning v' and label s in Γ'. It follows that $f(v_1) = v'_1$. Since the vertex v_1 is joined to v by an edge, the closed ball $\overline{B}_\Gamma(v_1, n-2)$ is a subgraph of $\overline{B}_\Gamma(v, n-1)$. Note that $\overline{B}_\Gamma(v_1, n-2)$ remains the closed ball with the same center and radius in the graph $\overline{B}_\Gamma(v, n-1)$. Similarly, $\overline{B}_{\Gamma'}(v'_1, n-2)$ is a subgraph of $\overline{B}_{\Gamma'}(v, n-1)$ and it is also the closed ball of radius $n-2$ centered at v'_1 in the graph $\overline{B}_{\Gamma'}(v, n-1)$. Since $f(v_1) = v'_1$ and any isomorphism of graphs preserves distance between vertices, the restriction f_0 of f to the vertex set of $\overline{B}_\Gamma(v_1, n-2)$ is an isomorphisms of the graphs $\overline{B}_\Gamma(v_1, n-2)$ and $\overline{B}_{\Gamma'}(v'_1, n-2)$. It remains to notice that the closed ball $\overline{B}_{\mathcal{A}_s(\Gamma)}(v_1, n-2)$ differs from $\overline{B}_\Gamma(v_1, n-2)$ in that the marked vertex is v_1 and, similarly, $\overline{B}_{\mathcal{A}_s(\Gamma')}(v'_1, n-2)$ differs from $\overline{B}_{\Gamma'}(v'_1, n-2)$ in that the marked vertex is v'_1. Therefore f_0 is also an isomorphism of $\overline{B}_{\mathcal{A}_s(\Gamma)}(v_1, n-2)$ and $\overline{B}_{\mathcal{A}_s(\Gamma')}(v'_1, n-2)$. By definition of the distance function, $\delta(\mathcal{A}_s(\Gamma), \mathcal{A}_s(\Gamma')) \leq 2^{-(n-1)} = 2\delta(\Gamma, \Gamma')$. ∎

Let A be an action of the group G on a set M. To any point $x \in M$ we associate three subgroups of G: the stabilizer $\mathrm{St}_A(x)$ of x, the stabilizer $\mathrm{St}_{\mathcal{A}}(\Gamma^*_x)$ of the marked Schreier graph $\Gamma^*_x = \Gamma^*_{\rm Sch}(G,S;A,x)$, and the neighborhood stabilizer $\mathrm{St}^o_{\mathcal{A}}(\Gamma^*_x)$ (if the action A is continuous then there is the fourth subgroup, the neighborhood stabilizer of x). Clearly, the graph $\Gamma^*_{A_g(x)}$ coincides with Γ^*_x if and only if $A_g(x) = x$. However this does not imply that the stabilizer of the graph is the same as the stabilizer of x. Since $\mathcal{A}$ is an action on isomorphism classes of graphs, we have $g \in \mathrm{St}_{\mathcal{A}}(\Gamma^*_x)$ if and only if the graph $\Gamma^*_{A_g(x)}$ is isomorphic to Γ^*_x.

Lemma 4.3 *(i)* $\mathrm{St}_A(x)$ *is a normal subgroup of* $\mathrm{St}_{\mathcal{A}}\big(\Gamma^*_{\rm Sch}(G,S;A,x)\big)$.

(ii) The quotient of $\mathrm{St}_{\mathcal{A}}\big(\Gamma^*_{\rm Sch}(G,S;A,x)\big)$ *by* $\mathrm{St}_A(x)$ *is isomorphic to the group of all automorphisms of the unmarked graph* $\Gamma_{\rm Sch}(G,S;A,x)$.

(iii) $\mathrm{St}_A(x)$ *is a subgroup of* $\mathrm{St}^o_{\mathcal{A}}\big(\Gamma^*_{\rm Sch}(G,S;A,x)\big)$.

Proof. Without loss of generality we can assume that the action A is transitive. For brevity, let Γ^* denote the marked graph $\Gamma^*_{\rm Sch}(G,S;A,x)$, Γ denote the unmarked graph $\Gamma_{\rm Sch}(G,S;A,x)$, and R denote the group of all automorphisms of Γ. Consider an arbitrary $f \in R$. For any vertex $y \in O_A(x)$ and any label $s \in S$ the unique edge of Γ with beginning y and label s has end $A_s(y)$. It follows that $f(A_s(y)) = A_s(f(y))$. Since the action A is transitive, the automorphism f commutes with transformations A_s, $s \in S$. Then f commutes with A_g for all $g \in G$. Notice that the automorphism f is uniquely determined by the vertex $f(x)$. Indeed, any vertex y of Γ is represented as $A_g(x)$ for some $g \in G$, then $f(y) = f(A_g(x)) = A_g(f(x))$. In particular, f is the identity if $f(x) = x$.

To prove the statements (i) and (ii), we are going to construct a homomorphism Ψ of the stabilizer $\mathrm{St}_{\mathcal{A}}(\Gamma^*)$ onto the group R with kernel $\mathrm{St}_A(x)$. An element $g \in G$ belongs to $\mathrm{St}_{\mathcal{A}}(\Gamma^*)$ if the graph Γ^* is isomorphic to $\Gamma^*_{\rm Sch}(G,S;A,A_g(x))$. An isomorphism of these marked graphs is an automorphism of the unmarked graph Γ that sends x to $A_g(x)$. Hence $g \in \mathrm{St}_{\mathcal{A}}(\Gamma^*)$ if and only if $A_g(x) = \psi_g(x)$ for some $\psi_g \in R$. By the above the automorphism ψ_g is uniquely determined by $A_g(x)$. Now we define a mapping $\Psi : \mathrm{St}_{\mathcal{A}}(\Gamma^*) \to R$ by $\Psi(g) = \psi_{g^{-1}}$. It is easy to observe that Ψ maps $\mathrm{St}_{\mathcal{A}}(\Gamma^*)$ onto R and the preimage of the identity under Ψ is $\mathrm{St}_A(x)$. Further, for any $g,h \in \mathrm{St}_{\mathcal{A}}(\Gamma^*)$ we have $\psi_{(gh)^{-1}}(x) = A^{-1}_{gh}(x) = A^{-1}_h(A^{-1}_g(x)) = A^{-1}_h(\psi_{g^{-1}}(x))$. Recall that the automorphism $\psi_{g^{-1}}$ commutes with the action A, in particular, $A^{-1}_h\psi_{g^{-1}} = \psi_{g^{-1}}A^{-1}_h$. Then $\psi_{(gh)^{-1}}(x) = \psi_{g^{-1}}(A^{-1}_h(x)) = \psi_{g^{-1}}(\psi_{h^{-1}}(x))$, which implies that $\Psi(gh) = \Psi(g)\Psi(h)$. Thus Ψ is a homomorphism.

We proceed to the statement (iii). Take any element $g \in \mathrm{St}_A(x)$. It can be represented as a product $s_1s_2\dots s_k$, where each s_i is in S. Let γ denote the unique directed path in Γ^* with beginning x and code word $s_k\dots s_2s_1$. By construction, the end of the path γ is $A_g(x)$ so that the path is closed. Let Γ^*_0 denote the subgraph of Γ^* whose vertex set consists of all vertices of the path γ. Clearly, Γ^*_0 is a marked graph, finite and connected. Hence $\Gamma^*_0 \in \mathcal{MG}_0$. Any graph $\Gamma^*_1 \in \mathcal{U}(\Gamma^*_0,\emptyset)$ admits a closed directed path with beginning at the marked point and code word $s_k\dots s_2s_1$. If $\Gamma^*_1 = \Gamma^*_{\rm Sch}(G,S;B,y)$, this implies that $B_g(y) = y$.

Hence $g \in \mathrm{St}_B(y) \subset \mathrm{St}_{\mathcal{A}}\big(\Gamma^*_{\mathrm{Sch}}(G,S;B,y)\big)$. Thus the transformation $\mathcal{A}_g$ fixes the set $\mathcal{U}(\Gamma^*_0,\emptyset) \cap \mathrm{Sch}(G,S)$, which is an open neighborhood of the graph Γ^* in $\mathrm{Sch}(G,S)$. ■

Any group G acts naturally on itself by left multiplication. The action $\mathrm{adj}_G : G \times G \to G$, called *adjoint*, is given by $\mathrm{adj}_G(g_0,g) = g_0 g$. The Schreier graph of this action relative to any generating set S is the *Cayley graph* of the group G relative to S. Given a subgroup H of G, the adjoint action of the group G descends to an action on G/H. The action $\mathrm{adj}_{G,H} : G \times G/H \to G/H$ is given by $\mathrm{adj}_{G,H}(g_0,gH) = (g_0 g)H$. The Schreier graph of the latter action relative to a generating set S is denoted $\Gamma_{\mathrm{coset}}(G,S;H)$. It is called a *Schreier coset graph*. The *marked Schreier coset graph* $\Gamma^*_{\mathrm{coset}}(G,S;H)$ is the marked Schreier graph of the coset H under the action $\mathrm{adj}_{G,H}$. It is obtained from $\Gamma_{\mathrm{coset}}(G,S;H)$ by marking the vertex H.

Proposition 4.4 *A marked Schreier graph $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ is isomorphic to a marked Schreier coset graph $\Gamma^*_{\mathrm{coset}}(G,S;H)$ if and only if $H = \mathrm{St}_A(x)$.*

Proof. Let H_0 denote the stabilizer $\mathrm{St}_A(x)$. Suppose $A_{g_1}(x) = A_{g_2}(x)$ for some $g_1, g_2 \in G$. Then $A_{g_2^{-1}g_1}(x) = A_{g_2}^{-1}(A_{g_1}(x)) = x$ so that $g_2^{-1}g_1 \in H_0$. Hence $g_2^{-1}g_1 H_0 = H_0$ and $g_1 H_0 = g_2 H_0$. Conversely, if $g_1 H_0 = g_2 H_0$ then $g_1 = g_2 h$ for some $h \in H_0$. It follows that $A_{g_1}(x) = A_{g_2}(A_h(x)) = A_{g_2}(x)$.

Let us define a mapping $f : G/H_0 \to O_A(x)$ by $f(gH_0) = A_g(x)$. By the above f is well defined and one-to-one. Clearly, it maps G/H_0 onto the entire orbit $O_A(x)$. For any $g_0, g \in G$ we have $f(g_0 g H_0) = A_{g_0 g}(x) = A_{g_0}(A_g(x)) = A_{g_0}(f(gH_0))$. Therefore f is a conjugacy of the action adj_{G,H_0} with the restriction of the action A to the orbit $O_A(x)$. It follows that f is also an isomorphism of the unmarked graphs $\Gamma_{\mathrm{coset}}(G,S;H_0)$ and $\Gamma_{\mathrm{Sch}}(G,S;A,x)$. As $f(H_0) = x$, the mapping f is an isomorphism of the marked graphs $\Gamma^*_{\mathrm{coset}}(G,S;H_0)$ and $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ as well.

Since any isomorphism of Schreier graphs of the group G is also a conjugacy of the corresponding actions, it preserves stabilizers of vertices. In particular, marked Schreier graphs cannot be isomorphic if the stabilizers of their marked vertices do not coincide. For any subgroup H of G the stabilizer of the coset H under the action $\mathrm{adj}_{G,H}$ is H itself. Therefore the graph $\Gamma^*_{\mathrm{coset}}(G,S;H)$ is not isomorphic to $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ if $H \neq \mathrm{St}_A(x)$. ■

5 Space of subgroups

Let G be a discrete countable group. Denote by $\mathrm{Sub}(G)$ the set of all subgroups of G. We endow the set $\mathrm{Sub}(G)$ with a topology as follows. First we consider the product topology on $\{0,1\}^G$. The set $\{0,1\}^G$ is in a one-to-one correspondence with the set of all functions $f : G \to \{0,1\}$. Also, any subset $H \subset G$ (in particular, any subgroup) is assigned the indicator function $\chi_H : G \to \{0,1\}$ defined by

$$\chi_H(g) = \begin{cases} 1 \text{ if } g \in H, \\ 0 \text{ if } g \notin H. \end{cases}$$

This gives rise to a mapping $j : \mathrm{Sub}(G) \to \{0,1\}^G$, which is an embedding. Now the topology on $\mathrm{Sub}(G)$ is the smallest topology such that the embedding j is continuous. By definition, the base of this topology consists of sets of the form

$$U_G(S^+, S^-) = \{H \in \mathrm{Sub}(G) \mid S^+ \subset H \text{ and } S^- \cap H = \emptyset\},$$

where S^+ and S^- run independently over all finite subsets of G. Notice that $U_G(S_1^+, S_1^-) \cap U_G(S_2^+, S_2^-) = U_G(S_1^+ \cup S_2^+, S_1^- \cup S_2^-)$.

The topological space $\mathrm{Sub}(G)$ is ultrametric and compact (since $\{0,1\}^G$ is ultrametric and compact, and $j(\mathrm{Sub}(G))$ is closed in $\{0,1\}^G$). Suppose $g_1, g_2, g_3, \dots$ is a complete list of elements of the group G. For any subgroups $H_1, H_2 \subset G$ let $d(H_1, H_2) = 0$ if $H_1 = H_2$; otherwise let $d(H_1, H_2) = 2^{-n}$, where n is the smallest index such that g_n belongs to the symmetric difference of H_1 and H_2. Then d is a distance function on $\mathrm{Sub}(G)$ compatible with the topology.

Note that the above construction also applies to a finite group G, in which case $\mathrm{Sub}(G)$ is a finite set with the discrete topology.

The following three lemmas explore properties of the topological space $\mathrm{Sub}(G)$.

Lemma 5.1 *The intersection of subgroups is a continuous operation on the space* $\mathrm{Sub}(G)$*.*

Proof. We have to show that the mapping $I : \mathrm{Sub}(G) \times \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ defined by $I(H_1, H_2) = H_1 \cap H_2$ is continuous. Take any finite sets $S^+, S^- \subset G$. Given subgroups $H_1, H_2 \subset G$, the intersection $H_1 \cap H_2$ is an element of the set $U_G(S^+, S^-)$ if and only if $H_1 \in U_G(S^+, S_1)$ and $H_2 \in U_G(S^+, S_2)$ for some sets S_1 and S_2 such that $S_1 \cup S_2 = S^-$. Clearly, the sets S_1 and S_2 are finite. It follows that

$$I^{-1}\big(U_G(S^+, S^-)\big) = \bigcup_{S_1, S_2 \,:\, S_1 \cup S_2 = S^-} U_G(S^+, S_1) \times U_G(S^+, S_2).$$

It remains to notice that any open subset of $\mathrm{Sub}(G)$ is a union of sets of the form $U_G(S^+, S^-)$ while any set of the form $U_G(S^+, S_1) \times U_G(S^+, S_2)$ is open in $\mathrm{Sub}(G) \times \mathrm{Sub}(G)$. ∎

Lemma 5.2 *For any subgroups H_1 and H_2 of the group G, let $H_1 \vee H_2$ denote the subgroup generated by all elements of H_1 and H_2. Then $\vee$ is a Borel measurable operation on* $\mathrm{Sub}(G)$*.*

Proof. We have to show that the mapping $J : \mathrm{Sub}(G) \times \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ defined by $J(H_1, H_2) = H_1 \vee H_2$ is Borel measurable. Take any $g \in G$ and consider arbitrary subgroups $H_1, H_2 \in \mathrm{Sub}(G)$ such that $J(H_1, H_2) \in U_G(\{g\}, \emptyset)$, i.e., $H_1 \vee H_2$ contains g. The element g can be represented as a product $g = h_1 h_2 \dots h_k$, where each h_i belongs to H_1 or H_2. Let S_1 denote the set of all elements of H_1 in the sequence $h_1, h_2, \dots, h_k$ and S_2 denote the set of all elements of H_2 in the same sequence. Then the element g belongs to $K_1 \vee K_2$ for any subgroups $K_1 \in U_G(S_1, \emptyset)$ and $K_2 \in U_G(S_2, \emptyset)$. Hence the pair (H_1, H_2) is contained in the preimage of $U_G(\{g\}, \emptyset)$ under the mapping J along with its open neighborhood $U_G(S_1, \emptyset) \times U_G(S_2, \emptyset)$. Thus the preimage $J^{-1}\big(U_G(\{g\}, \emptyset)\big)$ is an open set. Since the set $U_G(\emptyset, \{g\})$ is the complement of $U_G(\{g\}, \emptyset)$, its preimage under J is closed.

Given finite sets $S^+, S^- \subset G$, the set $U_G(S^+, S^-)$ is the intersection of sets $U_G(\{g\}, \emptyset)$, $g \in S^+$ and $U_G(\emptyset, \{h\})$, $h \in S^-$. By the above $J^{-1}\big(U_G(S^+, S^-)\big)$ is a Borel set, the intersection of an open set with a closed one. Finally, any open subset of $\mathrm{Sub}(G)$ is the union of some sets $U_G(S^+, S^-)$. Moreover, it is a finite or countable union since there are only countably many sets of the form $U_G(S^+, S^-)$. It follows that the preimage under J of any open set is a Borel set. ∎

Lemma 5.3 *Suppose H is a subgroup of G. Then* $\mathrm{Sub}(H)$ *is a closed subset of* $\mathrm{Sub}(G)$. *Moreover, the intrinsic topology on* $\mathrm{Sub}(H)$ *coincides with the topology induced by* $\mathrm{Sub}(G)$.

Proof. The intrinsic topology on $\mathrm{Sub}(H)$ is generated by all sets of the form $U_H(P^+, P^-)$, where P^+ and P^- are finite subsets of H. The topology induced by $\mathrm{Sub}(G)$ is generated by all sets of the form $U_G(S^+, S^-) \cap \mathrm{Sub}(H)$, where S^+ and S^- are finite subsets of G. Clearly, $U_G(S^+, S^-) \cap \mathrm{Sub}(H) = U_H(S^+, S^- \cap H)$ if $S^+ \subset H$ and $U_G(S^+, S^-) \cap \mathrm{Sub}(H) = \emptyset$ otherwise. It follows that the two topologies coincide.

For any $g \in G$ the open set $U_G(\emptyset, \{g\})$ is also closed in $\mathrm{Sub}(G)$ as it is the complement of another open set $U_G(\{g\}, \emptyset)$. Then the set $\mathrm{Sub}(H)$ is closed in $\mathrm{Sub}(G)$ since it is the intersection of closed sets $U_G(\emptyset, \{g\})$ over all $g \in G \setminus H$. ∎

Let A be an action of the group G on a set M. Let us consider the stabilizer $\mathrm{St}_A(x)$ of a point $x \in M$ under the action (see Section 3) as the value of a mapping $\mathrm{St}_A : M \to \mathrm{Sub}(G)$.

Lemma 5.4 *Suppose A is a continuous action of the group G on a Hausdorff topological space M. Then*

(i) the mapping St_A *is Borel measurable;*

(ii) St_A *is continuous at a point $x \in M$ if and only if the stabilizer of x under the action coincides with its neighborhood stabilizer:* $\mathrm{St}^o_A(x) = \mathrm{St}_A(x)$;

(iii) if a sequence of points in M converges to the point x and the sequence of their stabilizers converges to a subgroup H, then $\mathrm{St}^o_A(x) \subset H \subset \mathrm{St}_A(x)$.

Proof. For any $g \in G$ let $\mathrm{Fix}_A(g)$ denote the set of all points in M fixed by the transformation A_g. Let us show that $\mathrm{Fix}_A(g)$ is a closed set. Take any point $x \in M$ not in $\mathrm{Fix}_A(g)$. Since the points x and $A_g(x)$ are distinct, they have disjoint open neighborhoods X and Y, respectively. Since A_g is continuous, there exists an open neighborhood Z of x such that $A_g(Z) \subset Y$. Then $X \cap Z$ is an open neighborhood of x and $A_g(X \cap Z)$ is disjoint from $X \cap Z$. In particular, $X \cap Z$ is disjoint from $\mathrm{Fix}_A(g)$.

For any finite sets $S^+, S^- \subset G$ the preimage of the open set $U_G(S^+, S^-)$ under the mapping St_A is

$$\bigcap_{g \in S^+} \mathrm{Fix}_A(g) \setminus \bigcup_{h \in S^-} \mathrm{Fix}_A(h).$$

This is a Borel set as $\mathrm{Fix}_A(g)$ is closed for any $g \in G$. Since sets of the form $U_g(S^+, S^-)$ constitute a base of the topology on $\mathrm{Sub}(G)$, the mapping St_A is Borel measurable.

The mapping St_A is continuous at a point $x \in M$ if and only if x is an interior point in the preimage under St_A of any set $U_G(S^+, S^-)$ containing $\mathrm{St}_A(x)$. The latter holds true if and only if x is an interior point in any set $\mathrm{Fix}_A(g)$ containing this point. Clearly, x is an interior point of $\mathrm{Fix}_A(g)$ if and only if g belongs to the neighborhood stabilizer $\mathrm{St}^o_A(x)$. Thus St_A is continuous at x if and only if any element of $\mathrm{St}_A(x)$ belongs to $\mathrm{St}^o_A(x)$ as well.

Now suppose that a sequence $x_1, x_2, \dots$ of points in M converges to the point x and, moreover, the stabilizers $\mathrm{St}_A(x_1), \mathrm{St}_A(x_2), \dots$ converge to a subgroup H. Consider an arbitrary $g \in G$. In the case $g \in H$, the subgroup H belongs to the open set $U_G(\{g\}, \emptyset)$. Since $\mathrm{St}_A(x_n) \to H$ as $n \to \infty$, we have $\mathrm{St}_A(x_n) \in U_G(\{g\}, \emptyset)$ for large n. In other words, $x_n \in \mathrm{Fix}_A(g)$ for large n. Since the set $\mathrm{Fix}_A(g)$ is closed, it contains the limit point x as well. That is, $g \in \mathrm{St}_A(x)$. In the case $g \notin H$, the subgroup H belongs to the open set $U_G(\emptyset, \{g\})$. Then $\mathrm{St}_A(x_n) \in U_G(\emptyset, \{g\})$ for large n. In other words, $x_n \notin \mathrm{Fix}_A(g)$ for large n. Since $x_n \to x$ as $n \to \infty$, the action of g fixes no neighborhood of x. That is, $g \notin \mathrm{St}^o_A(x)$. ■

The group G acts naturally on the set $\mathrm{Sub}(G)$ by conjugation. The action $\mathcal{C} : G \times \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ is given by $\mathcal{C}(g, H) = gHg^{-1}$. This action is continuous. Indeed, one easily observes that $\mathcal{C}_g^{-1}\big(U_G(S_1, S_2)\big) = U_G(g^{-1}S_1 g, g^{-1}S_2 g)$ for all $g \in G$ and finite sets $S_1, S_2 \subset G$.

Proposition 5.5 *The action $\mathcal{C}$ of the group G on $\mathrm{Sub}(G)$ is continuously conjugated to the action $\mathcal{A}$ on the space $\mathrm{Sch}(G, S)$ of the marked Schreier graphs of G relative to a generating set S. Moreover, the mapping $f : \mathrm{Sub}(G) \to \mathrm{Sch}(G, S)$ given by $f(H) = \Gamma^*_{\mathrm{coset}}(G, S; H)$ is a continuous conjugacy.*

Proof. Proposition 4.4 implies that the mapping f is bijective.

Consider arbitrary element g and subgroup H of the group G. The stabilizer of the coset gH under the action $\mathrm{adj}_{G,H}$ consists of those $g_0 \in G$ for which $g_0gH = gH$. The latter condition is equivalent to $g^{-1}g_0g \in H$. Therefore the stabilizer is $gHg^{-1} = \mathcal{C}_g(H)$. As $\mathcal{A}_g\big(\Gamma^*_{\mathrm{coset}}(G,S;H)\big) = \Gamma^*_{\mathrm{Sch}}(G,S;\mathrm{adj}_{G,H},gH)$, it follows from Proposition 4.4 that $\mathcal{A}_g(f(H)) = f(\mathcal{C}_g(H))$. Thus f conjugates the action $\mathcal{C}$ with $\mathcal{A}$.

Now we are going to show that for any finite sets $S^+, S^- \subset G$ the image of the open set $U_G(S^+,S^-)$ under the mapping f is open in $\mathrm{Sch}(G,S)$. Let $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ be an arbitrary graph in that image. Any element $g \in G$ can be represented as a product $s_1s_2\dots s_k$, where $s_i \in S$. Let us fix such a representation and denote by γ_g the unique directed path in $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ with beginning x and code word $s_k\dots s_2s_1$. Then the end of the path γ_g is $A_g(x)$. In particular, the path γ_g is closed if and only if $g \in \mathrm{St}_A(x)$. By Proposition 4.4, the preimage of the graph $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ under f is $\mathrm{St}_A(x)$. Since $\mathrm{St}_A(x) \in U_G(S^+,S^-)$, the path γ_g is closed for $g \in S^+$ and not closed for $g \in S^-$. Let Γ_0 denote the smallest subgraph of $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ containing all paths γ_g, $g \in S^+ \cup S^-$. Clearly, Γ_0 is a marked graph, finite and connected. Hence $\Gamma_0 \in \mathcal{MG}_0$. For any marked Schreier graph $\Gamma^*_{\mathrm{Sch}}(G,S;B,y)$ in $\mathcal{U}(\Gamma_0,\emptyset)$, the directed path with beginning y and the same code word as in γ_g is closed for all $g \in S^+$ and not closed for all $g \in S^-$. It follows that $\mathrm{St}_B(y) \in U_G(S^+,S^-)$. Therefore the graph $\Gamma^*_{\mathrm{Sch}}(G,S;A,x)$ is contained in $f(U_G(S^+,S^-))$ along with its neighborhood $\mathcal{U}(\Gamma_0,\emptyset) \cap \mathrm{Sch}(G,S)$.

Any open set in $\mathrm{Sub}(G)$ is the union of some sets $U_G(S^+,S^-)$. Hence it follows from the above that the mapping f maps open sets onto open sets. In other words, the inverse mapping f^{-1} is continuous. Since the topological spaces $\mathrm{Sub}(G)$ and $\mathrm{Sch}(G,S)$ are compact, f is continuous as well. ∎

Proposition 5.5 allows for a short (although not constructive) proof of the following statement.

Proposition 5.6 *Any subgroup of finite index of a finitely generated group is also finitely generated.*

Proof. Suppose G is a finitely generated group and H is a subgroup of G of finite index. Let S be a finite symmetric generating set for G. By Proposition 5.5, the space $\mathrm{Sub}(G)$ of subgroups of G is homeomorphic to the space $\mathrm{Sch}(G,S)$ of marked Schreier graphs of G relative to the generating set S. Moreover, there is a homeomorphism that maps the subgroup H to the marked Schreier coset graph $\Gamma^*_{\mathrm{coset}}(G,S;H)$. The vertices of the graph are cosets of H in G. Since H has finite index in G, the graph $\Gamma^*_{\mathrm{coset}}(G,S;H)$ is finite. Notice that any finite graph in the topological space $\mathcal{MG}_0$, which contains $\mathrm{Sch}(G,S)$, is an isolated point. It follows that H is an isolated point in $\mathrm{Sub}(G)$. Then there exist finite sets $S^+, S^- \subset G$ such that H is the only element of the open set $U_G(S^+,S^-)$. Let H_0 be the subgroup of G generated by the finite set S^+. Since $S^+ \subset H$ and $S^- \cap H = \emptyset$, the subgroup H_0 is disjoint from S^-. Thus $H_0 \in U_G(S^+,S^-)$ so that $H_0 = H$. ∎

6 Automorphisms of regular rooted trees

Consider an arbitrary graph Γ. Let γ be a path in this graph, $v_0, v_1, \dots, v_m$ be consecutive vertices of γ, and $e_1, \dots, e_m$ be consecutive edges. A *backtracking* in the path γ occurs if $e_{i+1} = e_i$ for some i (then $v_{i+1} = v_{i-1}$). The graph Γ is called a *tree* if it is connected and admits no closed path of positive length without backtracking. In particular, this means no loops and no multiple edges. A *rooted tree* is a tree with a distinguished vertex called the *root*. Clearly, the root is a synonym for the marked vertex. For any integer $n \geq 0$ the *level* n (or the nth level) of the tree is defined as the set of vertices at distance n from the root. If $n \geq 1$ then any vertex on the nth level is joined to exactly one vertex on the level $n-1$ and, optionally, to some vertices on the level $n+1$. The rooted tree is called *k-regular* if every vertex on any level n is joined to exactly k vertices on level $n+1$. The 2-regular rooted tree is also called *binary*.

All k-regular rooted trees are isomorphic to each other. A standard model of such a tree is built as follows. Let X be a set of cardinality k referred to as the *alphabet* (usually $X = \{0, 1, \dots, k-1\}$). A *word* (or *finite word*) in the alphabet X is a finite string of elements from X (referred to as *letters*). The set of all words in the alphabet X is denoted X^*. X^* is a monoid with respect to the concatenation (the unit element is the empty word, denoted $\varnothing$). Moreover, it is the free monoid generated by elements of X. Now we define a plain graph $\mathcal{T}$ with the vertex set X^* in which two vertices w_1 and w_2 are joined by an edge if $w_1 = w_2 x$ or $w_2 = w_1 x$ for some $x \in X$. Then $\mathcal{T}$ is a k-regular rooted tree with the root $\varnothing$. The nth level of the tree $\mathcal{T}$ consists of all words of length n.

A bijection $f : X^* \to X^*$ is an automorphism of the rooted tree $\mathcal{T}$ if and only if it preserves the length of any word and the length of the common beginning of any two words. Given an automorphism f and a word $u \in X^*$, there exists a unique transformation $h : X^* \to X^*$ such that $f(uw) = f(u)h(w)$ for all $w \in X^*$. It is easy to see that h is also an automorphism of the tree $\mathcal{T}$. This automorphism is called the *section* of f at the word u and denoted $f|_u$. A set of automorphisms of the tree $\mathcal{T}$ is called *self-similar* if it is closed under taking sections. For any automorphisms f and h and any word $u \in X^*$ one has $(fh)|_u = f|_{h(u)} h|_u$ and $f^{-1}|_u = \big(f|_{f^{-1}(u)}\big)^{-1}$. It follows that any group of automorphisms generated by a self-similar set is itself self-similar.

Suppose G is a group of automorphisms of the tree $\mathcal{T}$. Let α denote the natural action of G on the vertex set X^*. Given a word $u \in X^*$, the section mapping $g \mapsto g|_u$ is a homomorphism when restricted to the stabilizer $\mathrm{St}_\alpha(u)$. If G is self-similar then this is a homomorphism to G. The self-similar group G is called *self-replicating* if for any $u \in X^*$ the mapping $g \mapsto g|_u$ maps the subgroup $\mathrm{St}_\alpha(u)$ onto the entire group G.

Suppose that letters of the alphabet X are canonically ordered: $x_1, x_2, \dots, x_k$. For any permutation π on X and automorphisms $h_1, h_2, \dots, h_k$ of the tree $\mathcal{T}$ we denote by $\pi(h_1, h_2, \dots, h_k)$ a transformation $f : X^* \to X^*$ given by $f(x_i w) =$

$\pi(x_i)h_i(w)$ for all $w \in X^*$ and $1 \le i \le k$. It is easy to observe that f is also an automorphism of $\mathcal{T}$ and $h_i = f|_{x_i}$ for $1 \le i \le k$. The expression $\pi(h_1, h_2, \dots, h_k)$ is called the *wreath recursion* for f. Any self-similar set of automorphisms f_j, $j \in J$ satisfies a system of "self-similar" wreath recursions

$$f_j = \pi_j(f_{m(j,x_1)}, f_{m(j,x_2)}, \dots, f_{m(j,x_k)}),\ j \in J,$$

where π_j, $j \in J$ are permutations on X and m maps $J \times X$ to J.

Lemma 6.1 *Any system of self-similar wreath recursions over the alphabet X is satisfied by a unique self-similar set of automorphisms of the regular rooted tree $\mathcal{T}$.*

Proof. Consider a system of wreath recursions $f_j = \pi_j(f_{m(j,x_1)}, \dots, f_{m(j,x_k)})$, $j \in J$, where π_j, $j \in J$ are permutations on X and m is a mapping of $J \times X$ to J. We define transformations F_j, $j \in J$ of the set X^* inductively as follows. First $F_j(\varnothing) = \varnothing$ for all $j \in J$. Then, once the transformations are defined on words of a particular length $n \ge 0$, we let $F_j(x_i w) = \pi_j(x_i)F_{m(j,x_i)}(w)$ for all $j \in J$, $1 \le i \le k$, and words w of length n. By definition, each F_j preserves the length of words. Besides, it follows by induction on n that F_j is bijective when restricted to words of length n and that F_j preserves having a common beginning of length n for any two words. Therefore each F_j is an automorphism of the tree $\mathcal{T}$. By construction, the automorphisms F_j, $j \in J$ form a self-similar set satisfying the above system of wreath recursions. Moreover, they provide the only solution to that system. ∎

An *infinite path* in the tree $\mathcal{T}$ is an infinite sequence of vertices $v_0, v_1, v_2, \dots$ together with a sequence of edges $e_1, e_2 \dots$ such that the endpoints of any e_i are v_{i-1} and v_i. The vertex v_0 is the beginning of the path. Clearly, the path is uniquely determined by the sequence of vertices alone. The *boundary* of the rooted tree $\mathcal{T}$, denoted $\partial\mathcal{T}$, is the set of all infinite paths without backtracking that begin at the root. There is a natural one-to-one correspondence between $\partial\mathcal{T}$ and the set $X^{\mathbb{N}}$ of infinite words over the alphabet X. Namely, an infinite word $x_1x_2x_3\dots$ corresponds to the path going through the vertices $\varnothing, x_1, x_1x_2, x_1x_2x_3, \dots$. The set $X^{\mathbb{N}}$ is equipped with the product topology and the uniform Bernoulli measure. This allows us to regard the tree boundary $\partial\mathcal{T}$ as a compact topological space with a Borel probability measure (called *uniform*).

Suppose G is a group of automorphisms of the regular rooted tree $\mathcal{T}$. The natural action of G on the vertex set X^* gives rise to an action on the boundary $\partial\mathcal{T}$. The latter is continuous and preserves the uniform measure on $\partial\mathcal{T}$.

Proposition 6.2 ([3]) *Let G be a countable group of automorphisms of a regular rooted tree $\mathcal{T}$. Then the following conditions are equivalent:*

(i) the group G acts transitively on each level of the tree;

(ii) *the action of G on the boundary $\partial\mathcal{T}$ of the tree is topologically transitive;*

(iii) *the action of G on $\partial\mathcal{T}$ is minimal;*

(iv) *the action of G on $\partial\mathcal{T}$ is ergodic with respect to the uniform measure;*

(v) *the action of G on $\partial\mathcal{T}$ is uniquely ergodic.*

Let G be a countable group of automorphisms of a regular rooted tree $\mathcal{T}$. Let α denote the natural action of G on the vertex set of the tree $\mathcal{T}$ and β denote the induced action of G on the boundary $\partial\mathcal{T}$ of the tree.

Proposition 6.3 *The mapping St_β is continuous on a residual (dense G_δ) set.*

Proof. For any $g \in G$ let $\mathrm{Fix}_\beta(g)$ denote the set of all points in $\partial\mathcal{T}$ fixed by the transformation β_g. If $g \in \mathrm{St}_\beta(\xi)$ but $g \notin \mathrm{St}^o_\beta(\xi)$, then ξ is a boundary point of the set $\mathrm{Fix}_\beta(g)$, and vice versa. Since $\mathrm{Fix}_\beta(g)$ is a closed set, its boundary is a closed, nowhere dense set. It follows that the set of points $\xi \in \partial\mathcal{T}$ such that $\mathrm{St}^o_\beta(\xi) = \mathrm{St}_\beta(\xi)$ is the intersection of countably many dense open sets (it is dense since $\partial\mathcal{T}$ is a complete metric space). By Lemma 5.4, the latter set consists of points at which the mapping St_β is continuous. ∎

The mapping St_β is Borel due to Lemma 5.4. If St_β is injective then, according to the descriptive set theory, it also maps Borel sets onto Borel sets (see, e.g., [4]). The following two lemmas show the same can hold under a little weaker condition.

Lemma 6.4 *Assume that for any points $\xi, \eta \in \partial\mathcal{T}$ either $\mathrm{St}_\beta(\xi) = \mathrm{St}_\beta(\eta)$ or $\mathrm{St}_\beta(\xi)$ is not contained in $\mathrm{St}_\beta(\eta)$. Then the mapping St_β maps any open set, any closed set, and any intersection of an open set with a closed one onto Borel sets.*

Proof. First let us show that St_β maps any closed subset C of the boundary $\partial\mathcal{T}$ onto a Borel subset of $\mathrm{Sub}(G)$. For any positive integer n let C_n denote the set of all words of length n in the alphabet X that are beginnings of infinite words in C. Further, let W_n be the union of sets $\mathrm{Sub}(\mathrm{St}_\alpha(w))$ over all words $w \in C_n$. Finally, let W be the intersection of the sets W_n over all $n \geq 1$. By Lemma 5.3, the set $\mathrm{Sub}(H)$ is closed in $\mathrm{Sub}(G)$ for any subgroup $H \in \mathrm{Sub}(G)$. Hence each W_n is closed as the union of finitely many closed sets. Then the intersection W is closed as well.

The stabilizer $\mathrm{St}_\beta(\xi)$ of an infinite word $\xi \in \partial\mathcal{T}$ is a subgroup of the stabilizer $\mathrm{St}_\alpha(w)$ of a finite word $w \in X^*$ whenever w is a beginning of ξ. It follows that $\mathrm{St}_\beta(\xi) \in W$ for all $\xi \in C$. By construction of the set W, any subgroup of an element of W is also an element of W. Hence W contains all subgroups of the groups $\mathrm{St}_\beta(\xi)$, $\xi \in C$.

Conversely, for any subgroup $H \in W$ there is a sequence of words $w_1, w_2, \dots$ such that $w_n \in C_n$ and $H \subset \mathrm{St}_\alpha(w_n)$ for $n = 1, 2, \dots$. Since the number of words of a fixed length is finite, one can find nested infinite sets of indices $I_1 \supset I_2 \supset \dots$ such that the beginning of length k of the word w_n is the same for all $n \in I_k$. Choose an increasing sequence of indices $n_1, n_2, \dots$ such that $n_k \in I_k$ for all k, and let w'_k be the beginning of length k of the word w_{n_k}. Then $w'_k \in C_k$ as $w_{n_k} \in C_{n_k}$. Besides, $\mathrm{St}_\alpha(w_{n_k}) \subset \mathrm{St}_\alpha(w'_k)$, in particular, the group H is a subgroup of $\mathrm{St}_\alpha(w'_k)$. By construction, the word w'_k is a beginning of w'_m whenever $k < m$. Therefore all w'_k are beginnings of the same infinite word $\xi' \in \partial\mathcal{T}$. Since every beginning of ξ' coincides with a beginning of some infinite word in C and the set C is closed, it follows that $\xi' \in C$. The stabilizer $\mathrm{St}_\beta(\xi')$ is the intersection of stabilizers $\mathrm{St}_\alpha(w'_k)$ over all $k \geq 1$. Hence H is a subgroup of $\mathrm{St}_\beta(\xi')$.

By the above a subgroup H of G belongs to the set W if and only if it is a subgroup of the stabilizer $\mathrm{St}_\beta(\xi)$ for some $\xi \in C$. The assumption of the lemma implies that stabilizers $\mathrm{St}_\beta(\xi)$, $\xi \in C$ can be distinguished as the maximal subgroups in the set W. That is, such a stabilizer is an element of W which is not a proper subgroup of another element of W. For any $g \in G$ we define a transformation ψ_g of $\mathrm{Sub}(G)$ by $\psi_g(H) = \langle g \rangle \vee H$, where $\langle g \rangle$ is a cyclic subgroup of G generated by g. The group $\psi_g(H)$ is generated by g and all elements of the group H. Clearly, a subgroup $H \in W$ is not maximal in W if and only if $\psi_g(H) \in W$ for some $g \notin H$. An equivalent condition is that H belongs to the set $W'_g = W \cap \psi_g^{-1}(W) \cap U_G(\emptyset, \{g\})$. It follows from Lemma 5.2 that the mapping ψ_g is Borel measurable. Therefore W'_g is a Borel set. Now the image of the set C under the mapping St_β is the difference of the closed set W and the union of Borel sets W'_g, $g \in G$. Hence this image is a Borel set.

Any open set $D \subset \partial\mathcal{T}$ is the union of a finite or countable collection of cylinders $Z_1, Z_2, \dots$, which are both open and closed sets. By the above each cylinder is mapped by St_β onto a Borel set in $\mathrm{Sub}(G)$. Then the union D is mapped onto the union of images of the cylinders, which is a Borel set as well. Further, for any closed set $C \subset \partial\mathcal{T}$ the intersection $C \cap D$ is the union of closed sets $C \cap Z_1, C \cap Z_2, \dots$. Hence it is also mapped by St_β onto a Borel set. ∎

Lemma 6.5 *Under the assumption of Lemma 6.4, if the mapping* St_β *is finite-to-one, i.e., the preimage of any subgroup in* $\mathrm{Sub}(G)$ *is finite, then it maps Borel sets onto Borel sets.*

Proof. Recall that the class $\mathfrak{B}$ of the Borel sets in $\partial\mathcal{T}$ is the smallest collection of subsets of $\partial\mathcal{T}$ that contains all closed sets and is closed under taking countable intersections, countable unions, and complements. Let $\mathfrak{U}$ denote the smallest collection of subsets of $\partial\mathcal{T}$ that contains all closed sets and is closed under taking countable intersections of nested sets and countable unions of any sets. Note that $\mathfrak{U}$ is well defined; it is the intersection of all collections satisfying these conditions. In particular, $\mathfrak{U} \subset \mathfrak{B}$. Further, let $\mathfrak{W}$ denote the collection of all Borel sets in $\partial\mathcal{T}$ mapped onto Borel sets in $\mathrm{Sub}(G)$ by the mapping St_β.

For any mapping $f : \partial\mathcal{T} \to \mathrm{Sub}(G)$ and any sequence $U_1, U_2, \dots$ of subsets of $\partial\mathcal{T}$ the image of the union $U_1 \cup U_2 \cup \dots$ under f is the union of images $f(U_1), f(U_2), \dots$. On the other hand, the image of the intersection $U_1 \cap U_2 \cap \dots$ under f is contained in $f(U_1) \cap f(U_2) \cap \dots$ but need not coincide with the latter when the mapping f is not one-to-one. The two sets do coincide if f is finite-to-one and $U_1 \supset U_2 \supset \dots$. Since the mapping St_β is assumed to be finite-to-one, it follows that the collection $\mathfrak{W}$ is closed under taking countable intersections of nested sets and countable unions of any sets. By Lemma 6.4, $\mathfrak{W}$ contains all closed sets. Therefore $\mathfrak{U} \subset \mathfrak{W}$.

To complete the proof, we are going to show that $\mathfrak{U} = \mathfrak{B}$, which will imply that $\mathfrak{W} = \mathfrak{B}$. Given a set $Y \in \mathfrak{U}$, let $\mathfrak{U}_Y$ denote the collection of all sets $U \in \mathfrak{U}$ such that the intersection $U \cap Y$ also belongs to $\mathfrak{U}$. For any sequence $U_1, U_2, \dots$ of elements of $\mathfrak{U}_Y$ we have

$$\Bigl(\bigcup\nolimits_{n\ge1} U_n\Bigr) \cap Y = \bigcup\nolimits_{n\ge1} (U_n \cap Y), \qquad \Bigl(\bigcap\nolimits_{n\ge1} U_n\Bigr) \cap Y = \bigcap\nolimits_{n\ge1} (U_n \cap Y).$$

Besides, the sets $U_1 \cap Y, U_2 \cap Y, \dots$ are nested whenever the sets $U_1, U_2, \dots$ are nested. It follows that the class $\mathfrak{U}_Y$ is closed under taking countable intersections of nested sets and countable unions of any sets. Consequently, $\mathfrak{U}_Y = \mathfrak{U}$ whenever $\mathfrak{U}_Y$ contains all closed sets. The latter condition obviously holds if the set Y is itself closed. Notice that for any sets $Y, Z \in \mathfrak{U}$ we have $Z \in \mathfrak{U}_Y$ if and only if $Y \in \mathfrak{U}_Z$. Since $\mathfrak{U}_Y = \mathfrak{U}$ for any closed set Y, it follows that $\mathfrak{U}_Z$ contains all closed sets for any $Z \in \mathfrak{U}$. Then $\mathfrak{U}_Z = \mathfrak{U}$ for any $Z \in \mathfrak{U}$. In other words, the class $\mathfrak{U}$ is closed under taking finite intersections. Combining finite intersections with countable intersections of nested sets, we can obtain any countable intersection of sets from $\mathfrak{U}$. Namely, if $U_1, U_2, \dots$ are arbitrary elements of $\mathfrak{U}$, then their intersection coincides with the intersection of sets $Y_n = U_1 \cap U_2 \cap \dots \cap U_n$, $n = 1, 2, \dots$, which are nested: $Y_1 \supset Y_2 \supset \dots$. Therefore $\mathfrak{U}$ is closed under taking any countable intersections.

Let $\mathfrak{U}'$ be the collection of complements in $\partial\mathcal{T}$ of all sets from $\mathfrak{U}$. For any subsets $U_1, U_2, \dots$ of $\partial\mathcal{T}$ the complement of their union is the intersection of their complements $\partial\mathcal{T} \setminus U_1, \partial\mathcal{T} \setminus U_2, \dots$ while the complement of their intersection is the union of their complements. Since the class $\mathfrak{U}$ is closed under taking countable intersections and countable unions, so is $\mathfrak{U}'$. Further, any open subset of $\partial\mathcal{T}$ is the union of at most countably many cylinders, which are closed (as well as open) sets. Therefore $\mathfrak{U}$ contains all open sets. Then $\mathfrak{U}'$ contains all closed sets. Now it follows that $\mathfrak{U} \subset \mathfrak{U}'$. In other words, the class $\mathfrak{U}$ is closed under taking complements.

Thus the collection $\mathfrak{U}$ is closed under taking any countable intersections and complements. This implies that $\mathfrak{U} = \mathfrak{B}$. ∎

Let A be a continuous action of a countable group G on a compact metric space M. Let Ω denote the image of M under the mapping St_A.

Lemma 6.6 *Assume that for any distinct points $x, y \in M$ the neighborhood stabilizer $\mathrm{St}^o_A(x)$ is not contained in the stabilizer $\mathrm{St}_A(y)$. Then the inverse of St_A, defined on the set Ω, can be extended to a continuous mapping of the closure of Ω onto M.*

Proof. Since $\mathrm{St}^o_A(x)$ is a subgroup of $\mathrm{St}_A(x)$ for any $x \in M$, the assumption of the lemma implies that the mapping St_A is one-to-one so that the inverse is well defined on Ω. To prove that the inverse can be extended to a continuous mapping of the closure of Ω onto M, it is enough to show that any sequence $x_1, x_2, \dots$ of points in M is convergent whenever the sequence of stabilizers $\mathrm{St}_A(x_1), \mathrm{St}_A(x_2), \dots$ converges in $\mathrm{Sub}(G)$. Suppose that $\mathrm{St}_A(x_n) \to H$ as $n \to \infty$. Since M is a compact metric space, the sequence $x_1, x_2, \dots$ has at least one limit point. By Lemma 5.4(iii), any limit point x satisfies $\mathrm{St}^o_A(x) \subset H \subset \mathrm{St}_A(x)$. In particular, $\mathrm{St}^o_A(x) \subset \mathrm{St}_A(y)$ for any limit points x and y. Then $x = y$ due to the assumption of the lemma. It follows that the sequence $x_1, x_2, \dots$ is convergent. ∎

7 The Grigorchuk group

Let $X = \{0, 1\}$ be the binary alphabet, X^* be the set of finite words over X regarded as the vertex set of a binary rooted tree $\mathcal{T}$, and $X^{\mathbb{N}}$ be the set of infinite words over X regarded as the boundary $\partial\mathcal{T}$ of the tree $\mathcal{T}$.

We define the Grigorchuk group $\mathcal{G}$ as a self-similar group of automorphisms of the tree $\mathcal{T}$ (for alternative definitions, see [2]). The group is generated by four automorphisms a, b, c, d that, together with the trivial automorphism, form a self-similar set. Consider the following system of wreath recursions:

$$\begin{cases} a = (0\,1)(e, e), \\ b = (a, c), \\ c = (a, d), \\ d = (e, b), \\ e = (e, e). \end{cases}$$

By Lemma 6.1, this system uniquely defines a self-similar set of automorphisms of the tree $\mathcal{T}$. The automorphism e is clearly the identity (e.g., by Lemma 6.1). It is the unity of the group $\mathcal{G}$. We shall denote the unity by $1_{\mathcal{G}}$ to avoid confusion with a letter of the alphabet X. The set $S = \{a, b, c, d\}$ shall be considered the standard set of generators for the group $\mathcal{G}$.

All 4 generators of the Grigorchuk group are involutions. Indeed, the transformations $a^2, b^2, c^2, d^2, 1_{\mathcal{G}}$ form a self-similar set satisfying wreath recursions $a^2 = (1_{\mathcal{G}}, 1_{\mathcal{G}})$, $b^2 = (a^2, c^2)$, $c^2 = (a^2, d^2)$, $d^2 = (1_{\mathcal{G}}, b^2)$, and $1_{\mathcal{G}} = (1_{\mathcal{G}}, 1_{\mathcal{G}})$. Then Lemma 6.1 implies that $a^2 = b^2 = c^2 = d^2 = 1_{\mathcal{G}}$. This fact allows us to regard the Schreier graphs of the group $\mathcal{G}$ relative to the generating set S as graphs with undirected edges (as explained in Section 2).

Since $a^2 = 1_{\mathcal{G}}$, the automorphisms bcd, cdb, dbc, and $1_{\mathcal{G}}$ form a self-similar set satisfying wreath recursions $bcd = (1_{\mathcal{G}}, cdb)$, $cdb = (1_{\mathcal{G}}, dbc)$, $dbc = (1_{\mathcal{G}}, bcd)$, and $1_{\mathcal{G}} = (1_{\mathcal{G}}, 1_{\mathcal{G}})$. Lemma 6.1 implies that $bcd = cdb = dbc = 1_{\mathcal{G}}$. Then $bc = bcd^2 = d = d^2bc = bc$. It follows that $\{1_{\mathcal{G}}, b, c, d\}$ is a subgroup of $\mathcal{G}$ isomorphic to the Klein 4-group.

We denote by α the generic action of the group $\mathcal{G}$ on vertices of the binary rooted tree $\mathcal{T}$. The induced action on the boundary $\partial\mathcal{T}$ of the tree is denoted β. For brevity, we write $g(\xi)$ instead of $\beta_g(\xi)$. The action of the generator a is very simple: it changes the first letter in every finite or infinite word while keeping the other letters intact. In particular, the empty word is the only word fixed by a. To describe the action of the other generators, we need three observations. First of all, b, c, and d fix one-letter words. Secondly, any word beginning with 0 is fixed by d while b and c change only the second letter in such a word. Thirdly, the section mapping $g \mapsto g|_1$ induces a cyclic permutation on the set $\{b, c, d\}$. It follows that a finite or infinite word w is simultaneouly fixed by b, c, and d if it contains no zeros or the only zero is the last letter. Otherwise two of the three generators change the letter following the first zero in w (keeping the other letters intact) while the third generator fixes w. In the latter case, it is the position k of the first zero in w that determines the generator fixing w. Namely, $b(w) = w$ if $k \equiv 0 \bmod 3$, $c(w) = w$ if $k \equiv 2 \bmod 3$, and $d(w) = w$ if $k \equiv 1 \bmod 3$.

Lemma 7.1 *The group $\mathcal{G}$ is self-replicating.*

Proof. We have to show that for any word $w \in X^*$ the section mapping $g \mapsto g|_w$ maps the stabilizer $\mathrm{St}_\alpha(w)$ onto the entire group $\mathcal{G}$. Let W be the set of all words with this property. Clearly, $\varnothing \in W$ as $\mathrm{St}_\alpha(\varnothing) = \mathcal{G}$ and $g|_\varnothing = g$ for all $g \in \mathcal{G}$. Suppose $w_1, w_2 \in W$. Given an arbitrary $g \in \mathcal{G}$, there exists $g' \in \mathcal{G}$ such that $g'(w_2) = w_2$ and $g'|_{w_2} = g$. Further, there exists $g'' \in \mathcal{G}$ such that $g''(w_1) = w_1$ and $g''|_{w_1} = g'$. Then $g''(w_1w_2) = g''(w_1)g''|_{w_1}(w_2) = w_1w_2$ and $g''|_{w_1w_2} = (g''|_{w_1})|_{w_2} = g$. Since g is arbitrary, $w_1w_2 \in W$. That is, the set W is closed under concatenation.

Any automorphism of the tree $\mathcal{T}$ either interchanges the vertices 0 and 1 or fixes them both. Hence the stabilizer $\mathrm{St}_\alpha(0)$ coincides with $\mathrm{St}_\alpha(1)$. This stabilizer contains the elements b, c, d, aba, aca, ada. The wreath recursions for these elements are $b = (a, c)$, $c = (a, d)$, $d = (1_{\mathcal{G}}, b)$, $aba = (c, a)$, $aca = (d, a)$, $ada = (b, 1_{\mathcal{G}})$. It follows that the images of the group $\mathrm{St}_\alpha(0)$ under the section mappings $g \mapsto g|_0$ and $g \mapsto g|_1$ contain the generating set S. As the restrictions of these mappings to $\mathrm{St}_\alpha(0)$ are homomorphisms, both images coincide with $\mathcal{G}$. Therefore the words 0 and 1 are in the set W. By the above W is closed under concatenation and contains the empty word. This implies $W = X^*$. ■

The orbits of the actions α and β are very easy to describe.

Lemma 7.2 *The group $\mathcal{G}$ acts transitively on each level of the binary rooted tree $\mathcal{T}$. Any two infinite words in $\partial\mathcal{T}$ are in the same orbit of the action β if and only if they differ in only finitely many letters.*

Proof. For any infinite word $\xi \in \partial\mathcal{T}$ and any generator $h \in \{a, b, c, d\}$ the infinite word $h(\xi)$ differs from ξ in at most one letter. Any $g \in \mathcal{G}$ can be represented as a product $g = h_1 h_2 \dots h_k$, where each h_i is in $\{a, b, c, d\}$. It follows that for any $\xi \in \partial\mathcal{T}$ the infinite words $g(\xi)$ and ξ differ in at most k letters. Thus any two infinite words in the same orbit of the action β differ in only finitely many letters.

Now we are going to show that for any finite words $w_1, w_2 \in X^*$ of the same length there exists $g \in \mathcal{G}$ such that $g(w_1) = w_2$ and $g|_{w_1} = 1_{\mathcal{G}}$. Equivalently, $g(w_1\xi) = w_2\xi$ for all $\xi \in \partial\mathcal{T}$. This will complete the proof of the lemma. Indeed, the claim contains the statement that the group $\mathcal{G}$ acts transitively on each level of the tree $\mathcal{T}$. Moreover, it implies that two infinite words in $\partial\mathcal{T}$ are in the same orbit of the action β whenever they differ in a finite number of letters.

We prove the claim by induction on the length n of the words w_1 and w_2. The case $n = 0$ is trivial. Here w_1 and w_2 are the empty words so that we take $g = 1_{\mathcal{G}}$. Now assume that the claim is true for all pairs of words of specific length $n \geq 0$ and consider words w_1 and w_2 of length $n + 1$. Let x_1 be the first letter of w_1 and x_2 be the first letter of w_2. Then $w_1 = x_1 u_1$ and $w_2 = x_2 u_2$, where u_1 and u_2 are words of length n. By the inductive assumption, there exists $h \in \mathcal{G}$ such that $h(u_1\xi) = u_2\xi$ for all $\xi \in \partial\mathcal{T}$. Since the group $\mathcal{G}$ is self-replicating, there exists $g_0 \in \mathcal{G}$ such that $g_0(x_1\eta) = x_1 h(\eta)$ for all $\eta \in \partial\mathcal{T}$. In particular, $g_0(x_1 u_1 \xi) = x_1 u_2 \xi$ for all $\xi \in \partial\mathcal{T}$. It remains to take $g = g_0$ if $x_2 = x_1$ and $g = a g_0$ otherwise. Then $g(x_1 u_1 \xi) = x_2 u_2 \xi$ for all $\xi \in \partial\mathcal{T}$. ∎

Lemma 7.3 *Suppose w_1 and w_2 are words in the alphabet $\{0, 1\}$ such that w_1 is not a beginning of w_2 while w_2, even with the last two letters deleted, is not a beginning of w_1. Then there exists $g \in \mathcal{G}$ that does not fix w_2 while fixing all words with beginning w_1.*

Proof. First we consider a special case when $w_2 = 100$. To satisfy the assumption of the lemma, the word w_1 has to begin with 0. Then we can take $g = d$. Indeed, the transformation d fixes all words that begin with 0, which includes all words with beginning w_1. At the same time, $d(100) = 1b(00) = 10a(0) = 101 \neq 100$.

Next we consider a slightly more general case when w_2 is an arbitrary word of length 3. By Lemma 7.2, the group $\mathcal{G}$ acts transitively on the third level of the tree $\mathcal{T}$. Therefore $h(w_2) = 100$ for some $h \in \mathcal{G}$. The words $h(w_1)$ and $h(w_2)$ satisfy the assumption of the lemma since the words w_1 and w_2 do. By the above, $dh(w_2) \neq h(w_2)$ while $d(h(w_1)u) = h(w_1)u$ for all $u \in X^*$. Let $g = h^{-1}dh$. Then $g(w_2) \neq w_2$ while $g(w_1 w) = w_1 w$ for all $w \in X^*$.

Finally, consider the general case. Let w_0 be the longest common beginning of the words w_1 and w_2. Then $w_1 = w_0 u_1$ and $w_2 = w_0 u_2$, where the words u_1

and u_2 also satisfy the assumption of the lemma. In particular, u_1 is nonempty and the length of u_2 is at least 3. We have $u_2 = u_2' u_2''$, where $u_2', u_2'' \in X^*$ and the length of u_2' is 3. Since the first letters of the words u_1 and u_2' are distinct, these words satisfy the assumption of the lemma. By the above there exists $g_0 \in \mathcal{G}$ such that $g_0(u_2') \neq u_2'$ and $g_0(u_1 u) = u_1 u$ for all $u \in X^*$. Since the group $\mathcal{G}$ is self-replicating, there exists $g \in \mathcal{G}$ such that $g(w_0 w) = w_0 g_0(w)$ for all $w \in X^*$. Then g does not fix the word $w_0 u_2'$ while fixing all words with beginning w_1. Since $w_0 u_2'$ is a beginning of w_2, the transformation g does not fix w_2 as well. ∎

Lemma 7.4 *For any distinct points $\xi, \eta \in \partial\mathcal{T}$ the neighborhood stabilizer $\mathrm{St}^o_\beta(\xi)$ is not contained in $\mathrm{St}_\beta(\eta)$.*

Proof. Let n denote the length of the longest common beginning of the distinct infinite words ξ and η. Let w_1 be the beginning of ξ of length $n+1$ and w_2 be the beginning of η of length $n+3$. It is easy to see that the words w_1 and w_2 satisfy the assumption of Lemma 7.3. Therefore there exists a transformation $g \in \mathcal{G}$ that does not fix w_2 while fixing all finite words with beginning w_1. Clearly, the action of g on $\partial\mathcal{T}$ fixes all infinite words with beginning w_1. As such infinite words form an open neighborhood of the point ξ, we have $g \in \mathrm{St}^o_\beta(\xi)$. At the same time, g does not fix the infinite word η since it does not fix its beginning w_2. Hence $g \notin \mathrm{St}_\beta(\eta)$ so that $\mathrm{St}^o_\beta(\xi) \not\subset \mathrm{St}_\beta(\eta)$. ∎

Lemma 7.5 $\mathrm{St}^o_\beta(\xi) = \mathrm{St}_\beta(\xi)$ *for any infinite word $\xi \in \partial\mathcal{T}$ containing infinitely many zeros.*

Proof. We are going to show that, given an automorphism $g \in \mathcal{G}$ and an infinite word $\xi \in \partial\mathcal{T}$ with infinitely many zeros, one has $g|_w = 1_{\mathcal{G}}$ for a sufficiently long beginning w of ξ. This claim implies the lemma. Indeed, in the case $g(\xi) = \xi$ the action of g fixes all infinite words with beginning w, which form an open neighborhood of ξ.

Let R be the set of all $g \in \mathcal{G}$ such that the claim holds true for g and any $\xi \in \partial\mathcal{T}$ with infinitely many zeros. The set R contains the generating set S. Indeed, $a|_w = 1_{\mathcal{G}}$ for any nonempty word $w \in X^*$ and $b|_w = c|_w = d|_w = 1_{\mathcal{G}}$ for any word w that contains a zero which is not the last letter of w. Now suppose $g, h \in R$ and consider an arbitrary $\xi \in \partial\mathcal{T}$ with infinitely many zeros. Then $h|_w = 1_{\mathcal{G}}$ for a sufficiently long beginning w of ξ. Lemma 7.2 implies that the infinite word $h(\xi)$ also has infinitely many zeros. Since $h(w)$ is a beginning of $h(\xi)$ and $g \in R$, we have $g|_{h(w)} = 1_{\mathcal{G}}$ provided w is long enough. Since $(gh)|_w = g|_{h(w)} h|_w$, we have $(gh)|_w = 1_{\mathcal{G}}$ provided w is long enough. Thus $gh \in R$. That is, the set R is closed under multiplication. Since $S \subset R$ and all generators are involutions, it follows that $R = \mathcal{G}$. ∎

The infinite word $\xi_0 = 111\dots$ (also denoted 1^∞) is an exceptional point for the action β.

Lemma 7.6 *The quotient of* $\mathrm{St}_\beta(\xi_0)$ *by* $\mathrm{St}^o_\beta(\xi_0)$ *is the Klein* 4*-group. The coset representatives are* $1_\mathcal{G}, b, c, d$.

Proof. Recall that $H = \{1_\mathcal{G}, b, c, d\}$ is a subgroup of $\mathcal{G}$ isomorphic to the Klein 4-group. Clearly, $H \subset \mathrm{St}_\beta(\xi_0)$. We are going to show that $H \cap \mathrm{St}^o_\beta(\xi_0) = \{1_\mathcal{G}\}$ and $\mathrm{St}_\beta(\xi_0) = \mathrm{St}^o_\beta(\xi_0)H$, which implies the lemma.

For any positive integer n let η_n denote the infinite word over the alphabet X that has a single zero in the position n. The sequence $\eta_1, \eta_2, \dots$ converges to ξ_0. One observes that any of the generators b, c, and d fixes η_n only if n leaves a specific remainder under division by 3 (0 for b, 2 for c, and 1 for d). It follows that $H \cap \mathrm{St}^o_\beta(\xi_0) = \{1_\mathcal{G}\}$.

Now let us show that any $g \in \mathrm{St}_\beta(\xi_0)$ is contained in the set $\mathrm{St}^o_\beta(\xi_0)H$. The proof is by strong induction on the length n of g, which is the smallest possible number of factors in an expansion $g = s_m \dots s_2 s_1$ such that each $s_i \in S$. The case $n = 0$ is trivial as $1_\mathcal{G}$ is the only element of length 0. Assume that the claim is true for all elements of length less than some $n > 0$ and consider an arbitrary element $g \in \mathrm{St}_\beta(\xi_0)$ of length n. We have $g = s_n \dots s_2 s_1$, where each s_i is a generator from S. Let $\xi_k = (s_k \dots s_2 s_1)(\xi_0)$, $k = 1, 2, \dots, n$. If $\xi_k = \xi_0$ for some $0 < k < n$, then $g_1 = s_n \dots s_{k+1}$ and $g_2 = s_k \dots s_2 s_1$ both fix ξ_0. Since the length of g_1 and g_2 is less than n, they belong to $\mathrm{St}^o_\beta(\xi_0)H$ by the inductive assumption. As $\mathrm{St}^o_\beta(\xi_0)H$ is a group, so does $g = g_1 g_2$. If $\xi_k \neq \xi_0$ for all $0 < k < n$, then $s_{i+1}|_{w_i} = 1_\mathcal{G}$ for any $0 \leq i < n$ and sufficiently long beginning w_i of the infinite word ξ_i. It follows that $g|_w = 1_\mathcal{G}$ for a sufficiently long beginning w of ξ_0. Thus $g \in \mathrm{St}^o_\beta(\xi_0)$. ∎

Recall that we consider the Schreier graphs of the group $\mathcal{G}$ relative to the generating set $S = \{a, b, c, d\}$ as graphs with undirected edges. The Schreier graphs of all orbits of the action β except $O_\beta(\xi_0)$ are similar. Any vertex is joined to two other vertices. Moreover, it is joined to one of the neighbors by a single edge labeled a and to the other neighbor by two edges. Also, there is one loop at each vertex. Hence the Schreier graph has a linear structure (see Figure 1) and all such graphs are isomorphic as graphs with unlabeled edges. The Schreier graph of the orbit of $\xi_0 = 1^\infty$ is different in that there are three loops labeled b, c, and d at the vertex ξ_0 (see Figure 2).

Let $F : \partial\mathcal{T} \to \mathrm{Sch}(\mathcal{G}, S)$ be the mapping that assigns to any point on the boundary of the binary rooted tree $\mathcal{T}$ its marked Schreier graph under the action β. Using notation of Section 4, $F(\xi) = \Gamma^*_{\mathrm{Sch}}(\mathcal{G}, S; \beta, \xi)$ for all $\xi \in \partial\mathcal{T}$.

Lemma 7.7 *The graph* $F(\xi_0)$ *is an isolated point in the image* $F(\partial\mathcal{T})$.

Proof. Let Γ_0 denote the marked graph with a single vertex and three loops labeled b, c, and d. Recall that $\mathcal{U}(\Gamma_0, \emptyset)$ is an open subset of $\mathcal{MG}_0$ consisting of all

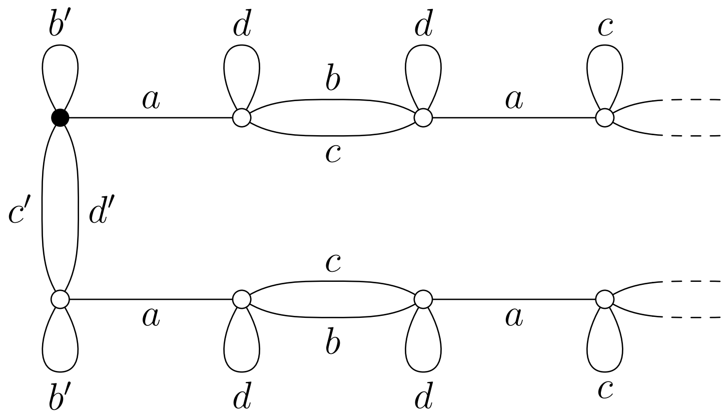

Figure 3: Limit graphs $\Delta_0^*, \Delta_1^*, \Delta_2^*$.

graphs in $\mathcal{MG}_0$ that have a subgraph isomorphic to Γ_0. Hence $\mathcal{U}(\Gamma_0, \emptyset) \cap \mathrm{Sch}(\mathcal{G}, S)$ is an open subset of $\mathrm{Sch}(\mathcal{G}, S)$. Given $\xi \in \partial\mathcal{T}$, the graph $F(\xi)$ belongs to that open subset if and only if $a(\xi) \neq \xi$ and $b(\xi) = c(\xi) = d(\xi) = \xi$. The latter conditions are satisfied only for $\xi = \xi_0$. The lemma follows. ■

It turns out that the image $F(\partial\mathcal{T})$ is not closed in $\mathrm{Sch}(\mathcal{G}, S)$. The following construction will help to describe the closure of $F(\partial\mathcal{T})$. Let us take two copies of the Schreier graph $\Gamma_{\mathrm{Sch}}(\mathcal{G}, S; \beta, \xi_0)$. We remove two out of three loops at the vertex ξ_0 (loops with the same labels in both copies) and replace them with two edges joining the two copies. Let c' and d' denote labels of the removed loops and b' denote the label of the retained loop. Then b', c', d' is a permutation of b, c, d. To be rigorous, the new graph has the vertex set $O_\beta(\xi_0) \times \{0, 1\}$, the set of edges $O_\beta(\xi_0) \times \{0, 1\} \times S$, and the set of labels S. An arbitrary edge (ξ, i, s) has beginning (ξ, i) and label s. The end of this edge is $(s(\xi), i)$ unless $\xi = \xi_0$ and $s = c'$ or $s = d'$, in which case the end is $(s(\xi), 1 - i) = (\xi_0, 1 - i)$. There are three ways to perform the above construction depending on the choice of b'. We denote by Δ_0, Δ_1, and Δ_2 the graphs obtained when $b' = b$, $b' = d$, and $b' = c$, respectively. Further, for any $i \in \{0, 1, 2\}$ we denote by Δ_i^* a marked graph obtained from Δ_i by marking the vertex $(\xi_0, 0)$ (see Figure 3).

Consider an arbitrary sequence of points $\eta_1, \eta_2, \dots$ in $\partial\mathcal{T}$ such that $\eta_n \to \xi_0$ as $n \to \infty$, but $\eta_n \neq \xi_0$. Let z_n denote the position of the first zero in the infinite word η_n.

Lemma 7.8 *The marked Schreier graphs $F(\eta_n)$ converge to Δ_i^*, $0 \leq i \leq 2$, as $n \to \infty$ if $z_n \equiv i \bmod 3$ for large n.*

Proof. For any $n \geq 1$ we define a map $f_n : O_\beta(\xi_0) \times \{0, 1\} \to O_\beta(\eta_n)$ as follows. Given $\xi \in O_\beta(\xi_0)$ and $x \in \{0, 1\}$, let $f_n(\xi, x)$ be an infinite word obtained from η_n after replacing the first $z_n - 1$ letters by the first $z_n - 1$ letters of ξ and adding $x \bmod 2$ to the $(z_n + 1)$-th letter. Clearly, $f_n(\xi_0, 0) = \eta_n$. Let i_n be the remainder

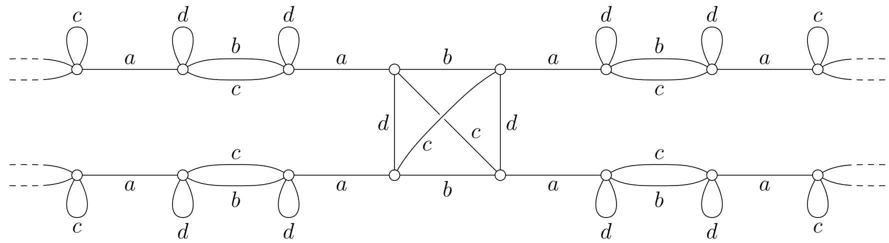

Figure 4: The Schreier coset graph of $\mathrm{St}^o_\beta(\xi_0)$.

of z_n under division by 3. One can check that the restriction of f_n to the vertex set of the closed ball $\overline{B}_{\Delta^*_{i_n}}((\xi_0,0),N)$ is an isomorphism of this ball with the closed ball $\overline{B}_{F(\eta_n)}(\eta_n,N)$ whenever $N \leq 2^{z_n-2}$. Therefore $\delta(F(\eta_n),\Delta^*_{i_n}) \to 0$ as $n \to \infty$. ■

One consequence of Lemma 7.8 is that the graphs Δ_0, Δ_1, and Δ_2 are Schreier graphs of the group $\mathcal{G}$. By construction, each of these graphs admits a nontrivial automorphism, which interchanges vertices corresponding to the same vertex of $\Gamma_{\mathrm{Sch}}(\mathcal{G},S;\beta,\xi_0)$. This property distinguishes Δ_0, Δ_1, and Δ_2 from the Schreier graphs of orbits of the action β.

Lemma 7.9 *The Schreier graphs $\Gamma_{\mathrm{Sch}}(\mathcal{G},S;\beta,\xi)$, $\xi \in \partial\mathcal{T}$ do not admit nontrivial automorphisms. The graphs Δ_0, Δ_1, and Δ_2 admit only one nontrivial automorphism.*

Proof. It follows from Proposition 4.4 and Lemma 7.4 that marked Schreier graphs $F(\xi)$ and $F(\eta)$ are isomorphic only if $\xi = \eta$. Therefore the Schreier graphs $\Gamma_{\mathrm{Sch}}(\mathcal{G},S;\beta,\xi)$, $\xi \in \partial\mathcal{T}$ admit no nontrivial automorphisms.

The graphs Δ_0, Δ_1, and Δ_2 have linear structure. Namely, one can label their vertices by v_j, $j \in \mathbb{Z}$ so that each v_j is adjacent only to v_{j-1} and v_{j+1}. If f is an automorphism of such a graph, then either $f(v_j) = v_{n-j}$ for some $n \in \mathbb{Z}$ and all $j \in \mathbb{Z}$ or $f(v_j) = v_{n+j}$ for some $n \in \mathbb{Z}$ and all $j \in \mathbb{Z}$. Assume that some Δ_i has more than one nontrivial automorphism. Then we can choose f above so that the latter option holds with $n \neq 0$. Take any path in Δ_i that begins at v_0 and ends at v_n and let w be the code word of that path. Since $f^m(v_0) = v_{mn}$ and $f^m(v_n) = v_{(m+1)n}$ for any integer m, the path in Δ_i with beginning v_{mn} and code word w ends at $v_{(m+1)n}$. It follows that for any integer $m > 0$ the path with beginning v_0 and code word w^m ends at v_{mn}. In particular, this path is not closed. However every element of the Grigorchuk group $\mathcal{G}$ is of finite order (see [2]) so that for some $m > 0$ the reversed word w^m equals $1_{\mathcal{G}}$ when regarded as a product in $\mathcal{G}$. This conradicts with Proposition 4.1. Thus the graph Δ_i admits only one nontrivial automorphism. ■

Lemma 7.10 *The Schreier graph $\Gamma_{\mathrm{Sch}}(\mathcal{G}, S; \beta, \xi_0)$ is a double quotient of each of the graphs Δ_0, Δ_1, and Δ_2. On the other hand, each of the graphs Δ_0, Δ_1, and Δ_2 is a double quotient of the Schreier coset graph $\Gamma_{\mathrm{coset}}(\mathcal{G}, S; \mathrm{St}^o_\beta(\xi_0))$.*

Proof. The Schreier coset graph of the subgroup $\mathrm{St}^o_\beta(\xi_0)$ is shown in Figure 4. In view of Lemmas 7.6 and 7.9, the automorphism group of this graph is the Klein 4-group. The quotient of the graph by the entire automorphism group is the Schreier graph of the orbit of ξ_0. The quotients by subgroups of order 2 are the graphs Δ_0, Δ_1, and Δ_2. ∎

Now it remains to collect all parts in Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We are concerned with the mapping $F : \partial\mathcal{T} \to \mathrm{Sch}(\mathcal{G}, S)$ given by $F(\xi) = \Gamma^*_{\mathrm{Sch}}(\mathcal{G}, S; \beta, \xi)$. Let us also consider a mapping $\psi : \partial\mathcal{T} \to \mathrm{Sub}(\mathcal{G})$ given by $\psi(\xi) = \mathrm{St}_\beta(\xi)$ and a mapping $f : \mathrm{Sub}(\mathcal{G}) \to \mathrm{Sch}(\mathcal{G}, S)$ given by $f(H) = \Gamma^*_{\mathrm{coset}}(\mathcal{G}, S; H)$. By Proposition 4.4, $F(\xi) = f(\psi(\xi))$ for all $\xi \in \partial\mathcal{T}$. By Proposition 5.5, f is a homeomorphism. Lemma 7.4 implies that the mapping ψ is injective. It is Borel measurable due to Lemma 5.4. Also, ψ is continuous at a point $\xi \in \partial\mathcal{T}$ if and only if $\mathrm{St}^o_\beta(\xi) = \mathrm{St}_\beta(\xi)$. Lemmas 7.5 and 7.6 imply that the latter condition fails only if the infinite word ξ contains only finitely many zeros. According to Lemma 7.2, an equivalent condition is that ξ is in the orbit of $\xi_0 = 1^\infty$ under the action β. Since the mapping F is f postcomposed with a homeomorphism, it is also injective, Borel measurable, and continuous everywhere except the orbit of ξ_0.

By Lemma 7.7, the graph $F(\xi_0)$ is an isolated point of the image $F(\partial\mathcal{T})$. Since $F(g(\xi)) = \mathcal{A}_g(F(\xi))$ for any $\xi \in \partial\mathcal{T}$ and $g \in \mathcal{G}$ and since the action $\mathcal{A}$ is continuous (see Proposition 4.2), the graph $F(g(\xi_0))$ is an isolated point of $F(\partial\mathcal{T})$ for all $g \in \mathcal{G}$. On the other hand, if $\xi \in \partial\mathcal{T}$ is not in the orbit of ξ_0, then the graph $F(\xi)$ is not an isolated point of $F(\partial\mathcal{T})$ as the mapping F is injective and continuous at ξ.

It follows from Lemma 7.9 that the image $F(\partial\mathcal{T})$ and the orbits $O_{\mathcal{A}}(\Delta^*_i)$, $i \in \{0, 1, 2\}$ are disjoint sets. Note that the orbit $O_{\mathcal{A}}(\Delta^*_i)$ consists of marked graphs obtained from the graph Δ_i by marking an arbitrary vertex. Lemma 7.8 implies the union of those 4 sets is the closure of $F(\partial\mathcal{T})$.

Finally, the statement (v) of Theorem 1.1 follows from Lemma 7.10. ∎

Proof of Theorem 1.2. Lemma 6.6 combined with Lemma 7.4 implies that the action of $\mathcal{G}$ on the closure of $F(\partial\mathcal{T})$ is a continuous extension of the action β. The extension is one-to-one everywhere except for the orbit $O_\beta(\xi_0)$ where it is four-to-one. Namely, for any $g \in \mathcal{G}$ the point $g(\xi_0)$ is covered by 4 graphs $F(g(\xi_0))$, $\mathcal{A}_g(\Delta^*_0)$, $\mathcal{A}_g(\Delta^*_1)$, and $\mathcal{A}_g(\Delta^*_0)$. According to Theorem 1.1, the graph $F(g(\xi_0))$ is an isolated point of the closure of $F(\partial\mathcal{T})$. When we restrict our attention to the set Ω of non-isolated points of the closure, we still have a continuous extension of the action β, but it is three-to-one on the orbit $O_\beta(\xi_0)$.

By Lemma 7.2, the group $\mathcal{G}$ acts transitively on each level of the binary rooted tree $\mathcal{T}$. Then Proposition 6.2 implies that the action β is minimal and uniquely ergodic, the only invariant Borel probability measure being the uniform measure on $\partial\mathcal{T}$. Since the action of $\mathcal{G}$ on the set Ω is a continuous extension of the action β that is one-to-one except for a countable set and since this action has no finite orbits, it follows that the action is minimal, uniquely ergodic, and isomorphic to β as the action with an invariant measure. ∎

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