Summary.

A spectrum of bands attributed to nitrogen sulphide (NS) has been obtained by passing uncondensed discharges through tubes containing nitrogen and sulphur vapour. The spectrum is strikingly similar to that of nitric oxide (NO) and appears in nearly the same (ultra-violet) region. Two systems of bands, corresponding with the well-known γ and β bands of NO, are clearly shown. In NS the doublet separations are larger, and the separations of successive bands smaller, than in NO.

DESCRIPTION OF PLATE.

a.—Nitrogen afterglow, showing γ and β bands of NO strongly. (For 0·2 ... *read* 0, 2 ...)
b.—Air vacuum tube; γ bands strong, β bands weak.
c.—Vacuum tube containing nitrogen and sulphur vapour. (The bands at λ 2576 and λ 2662 are due to an impurity of CS.) Copper arc comparison.

Notes on the Theory of Radiation.

By C. G. Darwin, F.R.S.

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It will probably be agreed that among all the recent developments of the quantum theory, one of the least satisfactory is the theory of radiation. The present paper is intended as a preliminary to a new line of attack on the subject. It was begun some time ago, but owing to lack of success in carrying it to a conclusion, its publication has been much delayed. In the meantime other papers have appeared,* which in some respects follow the same train of thought. The authors of these works have carried their methods further in some directions than I have attempted, but there is still perhaps room for the discussion of a number of questions from the rather different point of view adopted here.

1. The main principle of the present work is the idea that, since matter and light both possess the dual characters of particle and wave, a similar mathematical treatment ought to be applied to both, and that this has not yet been

* Landau and Peierls, 'Z. Physik,' vol. 62, p. 188 (1930); Oppenheimer, 'Phys. Rev.,' vol. 38, p. 725 (1931).

done as fully as should be possible. Whereas we have a fairly complete calculus for dealing with the behaviour of any number of electrons or atoms, for photons the existing processes are much less satisfactory. The central difficulty, which makes it hard to apply the ordinary methods of wave mechanics to light, is the fact that (at least according to our present ideas) photons can be created and annihilated, and to represent this in a wave system we have to be able to think of a medium suddenly coming into existence and then going out again, when the light that it was carrying is absorbed. Such behaviour is a grave difficulty in the way of allowing us to think of the photon as a wave, and tends to make us think with more favour of its particle aspect, until we recall that after all it is quite unlike any known particle to come into existence and later to disappear without trace. The theories at present current, such as that of Heisenberg and Pauli,* avoid these difficulties because they are mainly formal generalisations of the classical theory ; this frees them from the above difficulties, but they pay for it in being highly abstract, and, as it has turned out, rather unsuccessful.

The guiding idea adopted here was that for the present one should set aside difficulties of creation and annihilation, and should see to it that in cases where the photon can be endowed with some measure of individuality, its general description should follow the lines which have been so successful for the electron. In particular the Compton effect, at its discovery, was regarded as a simple collision of two bodies, and yet the detailed discussion at the present time involves the idea of the annihilation of one photon and the simultaneous creation of one among an infinity of other possible ones. We would like to be able to treat the effect as a two-body problem, with the scattered photon regarded as the same individal as the incident, in just the way we treat of the collisions of electrons. Also we must include in our account of it all the associated phenomena of the effect ; in a good theory the change of wave-length of the light, the velocity of recoil of the electron, the association of the directions of the two and their simultaneity will all be represented at once. It was with a view to finding ways of describing all these features of the Compton effect that I discussed, now nearly three years ago, the collision of two material particles in the wave mechanics.† That work contributed little new to the wave theory of matter, but it was the result of an investigation as

* 'Z. Physik,' vol. 56, p. 1 (1929), and vol. 59, p. 168 (1930).

† 'Proc. Roy. Soc.,' A, vol. 124, p. 375 (1929). Among other things a rather complicated problem was discussed in which the particles were made to collide twice over. In relativistic processes it is dangerous to think of simultaneity in different places, and the point of this problem was that it avoided this danger and yet implied the simultaneity.

to how all the phenomena of the Compton effect could be represented together, and this investigation clearly showed that the configuration space of the two bodies would yield the description so perfectly that it could hardly be right to look further. It seemed therefore right to apply a similar method to the Compton effect, treating the collision of photon and electron by the same method as is used in the wave mechanics for two material particles. It will be seen that this is very like the work of Landau and Peierls (*loc. cit.*), and it is immediately clear that the method will yield qualitatively all the features of the Compton effect. The trouble begins when we try to make the theory exact, for this demands a relativistically invariant theory of two bodies, which hardly exists yet. In view of this difficulty, and also of the complicated calculations necessary to relate the work with the formulæ of Klein and Nishina,* I shall not in the present note attempt to discuss the exact form of the interaction of a photon and an electron. Simpler questions in this connection are dealt with in the present work, and I hope to return to the question of the interaction in a later communication.

We must recognise that the present idea involves rather a radical change in our conception of electromagnetic waves. In the theory of Heisenberg and Pauli (*loc. cit.*) the electric force of such a wave is regarded as an observable quantity, but here we treat it as of the nature of the unobservable ψ of the wave mechanics. Consequently we abandon the idea of identifying the electric force of a light wave with a static electric force, or even with the force in wireless waves. This is unavoidable, but it is made plausible by the consideration that observable electric forces always arise from millions of electrons, and wireless waves always involve millions of photons, and that it is reasonable to demand a mastery of the behaviour of a single photon before proceeding to discuss that of a large number. Moreover it is certainly in accordance with fact that for light waves, just as for electron waves, it is the intensity that is observed, while the amplitude of the light wave is just as unobservable as the ψ of an electron. We shall therefore abandon entirely the quasi-classical view that a light wave shakes an electron to and fro, and so causes it to emit light by virtue of its acceleration ; with the appropriate modifications this is the method of consideration prevalent at present. Instead we shall consider electron and photon as two bodies, and shall find their mutual potential, though only in incomplete form in the present paper.

2. Our first task is to throw the description of the unperturbed photon into a form acceptable to the wave mechanics. In doing this we have to accom-

* 'Z. Physik,' vol. 52, p. 853 (1929).

modate the facts of polarisation, and so must start with some form of the electromagnetic equations, but we have to consider which of them is suitable. Thus we might choose for our variables either the electric or magnetic forces or both, or the potentials, or even such other types as the components of the Hertzian vector, but with the aims of the present work there could be no doubt as to which should be chosen. The equations of the spinning electron involve differentials of the first order in the time, so if we wish to combine an electron with a photon in a single system of equations we must have the photon's equations also of the first order in the time. This means that we do not want the potentials, but must use some combinations of the electric and magnetic forces. If we only needed to consider the unperturbed photon it would be possible to combine the six quantities, E, H into four independent ones $E_x + iE_y$, E_z, etc., which exactly satisfy Dirac's equations with m zero; but it is inadvisable to do this, because it conceals the actual vector character of the electric force of a light wave, as exhibited by the polarisation of scattered light. It is simplest to face a certain amount of redundancy and to make use of the six components of E, H, as our wave functions.

We submit the ordinary electromagnetic equations to the process which may be regarded as the fundamental process of all wave motions, the problem of finding the condition of the medium at any time in terms of arbitrary initial conditions. This is a well-known process and it will suffice merely to quote the result. Consider a radiation field in free space, supposing that at $t = 0$ the values of E and H are given everywhere. We shall denote vectors by a subscript, and shall make use of the ordinary summation convention (over the three space dimensions) for duplicate subscripts. The initial field is then E_a^0, H_a^0, but the six quantities are not independent on account of the divergence relations

$$\frac{\partial E_a^0}{\partial x_a} = 0, \quad \frac{\partial H_a^0}{\partial x_a} = 0, \tag{2.1}$$

and so they really only involve four arbitrary functions. In introducing Fourier integrals we shall increase the resemblance to the wave of an electron by writing the quantum in (denoting by h the quantum divided by 2π) even though in fact it plays no part in the work. We derive six functions $A_a^0(p)$, $B_a^0(p)$ of the three variables p_β given by

$$\left.\begin{aligned} E_a^0(x) &= \int A_a^0(p)\, e^{ip_\beta x_\beta/h}\, dp^{(3)} \\ H_a^0(x) &= \int B_a^0(p)\, e^{ip_\beta x_\beta/h}\, dp^{(3)} \end{aligned}\right\} \tag{2.2}$$

where $dp^{(3)}$ is short for $dp_x\, dp_y\, dp_z$; and A^0, B^0 can be explicitly determined by inversion of the Fourier integrals. Now take

$$\left.\begin{aligned} A_\alpha(p) &= \tfrac{1}{2}\{A_\alpha^0(p) - [p, B^0]_\alpha/w\} \\ B_\alpha(p) &= \tfrac{1}{2}\{B_\alpha^0(p) + [p, A^0]_\alpha/w\} = [p, A]_\alpha/w, \end{aligned}\right\} \quad (2.3)$$

where w is the positive square root of p_β^2. On account of (2.1) we have

$$p_\alpha A_\alpha = 0. \quad (2.4)$$

Then the solution of our problem is

$$\left.\begin{aligned} E_\alpha(x, t) &= \int A_\alpha(p)\, e^{i(p_\beta x_\beta - wct)/h}\, dp^{(3)} + \int A_\alpha^*(p)\, e^{-i(p_\beta x_\beta - wct)/h}\, dp^{(3)} \\ H_\alpha(x, t) &= \int B_\alpha(p)\, e^{i(p_\beta x_\beta - wct)/h}\, dp^{(3)} + \int B_\alpha^*(p)\, e^{-i(p_\beta x_\beta - wct)/h}\, dp^{(3)}, \end{aligned}\right\} \quad (2.5)$$

where as usual A_α^* is conjugate to A_α.

Now compare this with the corresponding process for the spinning electron. If the four components of ψ are chosen initially quite arbitrarily, so that their real and imaginary parts constitute eight real arbitrary functions, then the values at other times will depend on two Fourier integrals involving respectively

$$\exp i\,[p_\beta x_\beta \mp wct]/h,$$

where $w = \sqrt{(m^2c^2 + p_\beta^2)}$. This is, of course, the well-known fact that negative energy cannot be avoided in the description of the spinning electron, if complete generality is to be attained. If we want to have only positive energy we must leave four of the eight real functions adjustable; it is a matter of indifference whether we choose two of the four ψ's as arbitrary complex functions and determine the other two from them, or whether we regard the real parts of all four as given and deduce the imaginary parts from them.

Returning to the photon, if we admit the existence of imaginary solutions, then we have in fact assigned eight functions, the six real functions for E^0 and H^0, reduced to four by (2.1), and the corresponding imaginary parts defined to be zero. To represent the propagation of the disturbance we must again use two sets of terms with exponents of both signs. The fact that the electromagnetic equations can be put in a real form implies that a completely real solution can be found, but if we regard the exponential type of solution as the right primitive solution, then we must regard the real solution as a superposition of two primitive ones of equal amplitude, one with positive frequency and the other with negative.

Now the energy of material particles of all kinds is described in the quantum theory by means of the frequency of their expansion in exponentials, and we would like to be able to regard the energy of a photon as on the same footing. It would seem natural then to regard an electromagnetic wave, written in the usual real form, as the superposition of two waves, one with positive energy and momentum and the other with both negative. For an actual light wave it would seem natural to exclude the wave of negative energy, and to take as our wave the first term in (2.5). In the case of the electron Dirac* has made an attempt to justify the exclusion of negative energy with the help of the exclusion principle and of an infinite number of electrons of negative energy. We cannot, of course, invoke the same idea here because of the different statistics of photons, nevertheless Dirac's theory does not seem to have turned out very well. So we must, I think, exclude the negative energy in both cases for the same reason —that is to say, nobody knows how or why. It might be thought that we are increasing the troubles of the quantum theory by introducing a new case of negative energy, but it is usually found that the best hope of resolving a deep difficulty is to extend its application as widely as possible.

It should be noticed that the suppression of the second term in (2.5) leads to one significant difference. When a harmonic wave is represented by real quantities, it involves a factor $\cos\phi$ where the phase $\phi = (px - wct)/h$. For the intensity this is squared, and then averaged, yielding $\frac{1}{2}$, but there is in addition a term in $\frac{1}{2}\cos 2\phi$, and in all ordinary waves, such as water waves or sound waves, there is a real pulsation in the pressure exerted by the wave, of frequency twice that of the wave. Classical theory indicates that the same should be true for the momentum of light waves, but the general trend of the quantum theory points the other way. Imagine an experiment in which a mirror is mounted with a spring, so that it can vibrate along the normal to its face. When the mirror is reflecting light of frequency ν, adjust the spring so that it will vibrate with frequency 2ν; then if the term in $\cos 2\phi$ is present the mirror will be thrown into resonance. Now when we consider the actual mechanism of reflection, we may think this experiment conceived in too macroscopic a manner, but it is easy to see what modification will fit it into the quantum theory. For interaction between two waves of phases ϕ and ϕ' we shall have a chance depending on $e^{i(\phi-\phi')}$, and the analogue of the term in $\cos 2\phi$ will be a term depending on $e^{i(\phi+\phi')}$. To make this term display its effect we should need to make the system interact with a further system possessing phase nearly equal to $\phi + \phi'$. It seems very improbable that such

* 'Proc. Roy. Soc.,' A, vol. 126, p. 360 (1930).

interaction would occur, but if it does it would be explained in the language of the quantum theory by saying that our first system has jumped to negative energy. So once again it seems that we have reason to reject the second term in (2.5).

3. We will now consider certain dynamical properties of the photon, especially those connected with its polarisation, and compare them with the corresponding quantities for the electron. With a view to this comparison we first give certain properties of a wave packet of light, treated classically, and afterwards we shall develop the same results by methods analogous to those used for the electron.

Suppose that we have a field of radiation given by the real parts of

$$\left.\begin{aligned} \mathrm{E}_\alpha &= \int \mathrm{A}_\alpha(p)\, e^{i(p_\beta x_\beta - wct)/h}\, dp^{(3)} \\ \mathrm{H}_\alpha &= \int [(p/w), \mathrm{A}]_\alpha\, e^{i(p_\beta x_\beta - wct)/h}\, dp^{(3)} \end{aligned}\right\} \quad (3.1)$$

The divergence relation requires that A should satisfy

$$p_\alpha \mathrm{A}_\alpha = 0. \quad (3.2)$$

The ensuing calculations depend only on Fourier integrations and need not be shown in detail; in the course of them the pulsating terms of octave frequency are rejected as usual. The following results are obtained :—

(i) The energy

$$\left.\begin{aligned} \mathrm{W} &= \int \frac{1}{8\pi}(\mathrm{E}_\alpha{}^2 + \mathrm{H}_\alpha{}^2)\, dx^{(3)} \\ &= \pi^2 h^3 \int |\mathrm{A}_\alpha|^2 dp^{(3)} \end{aligned}\right\} \quad (3.3)$$

(ii) The momentum

$$\left.\begin{aligned} \mathrm{P}_\alpha &= \int \frac{1}{4\pi c}\, [\mathrm{E}, \mathrm{H}]_\alpha\, dx^{(3)} \\ &= \frac{\pi^2 h^3}{c} \int \frac{p_\alpha}{w}\, |\mathrm{A}_\beta|^2 dp^{(3)}. \end{aligned}\right\} \quad (3.4)$$

This result is general; for a wave packet $\mathrm{A}_\alpha = 0$ except for p near some definite value and hence $\mathrm{P}_\alpha = \mathrm{W}p_\alpha/wc$.

(iii) The motion of the packet can be studied by finding how its centre of gravity moves. Then

$$\overline{x}_\beta \mathrm{W} = \int x_\beta \frac{1}{8\pi}(\mathrm{E}_\alpha{}^2 + \mathrm{H}_\alpha{}^2)\, dx^{(3)}.$$

The reduction is a little more complicated, but is done by writing $x_\beta\, e^{ip_\gamma x_\gamma/h}$ as $\frac{h}{i}\frac{\partial}{\partial p_\beta}\, e^{ip_\gamma x_\gamma/h}$ and integrating by parts. The result is

$$\overline{x}_\beta\, \mathrm{W} = \pi^2 h^3 \int dp^{(3)} \left\{ ct\, \frac{p_\beta}{w}\, |\mathrm{A}_\alpha|^2 + ih\mathrm{A}_\alpha{}^* \frac{\partial \mathrm{A}_\alpha}{\partial p_\beta} \right\}. \tag{3.5}$$

The centre of gravity therefore travels with velocity c along the direction given by the momentum P; the second term gives its initial position.

(iv) The angular momentum about an axis through the origin is given by

$$\mathrm{M}_\beta = \int \left[x,\, \frac{1}{4\pi c}\, [\mathrm{E},\, \mathrm{H}] \right]_\beta dx^{(3)}.$$

This can be reduced to

$$\frac{\pi^2 h^3}{c} \int dp^{(3)} \left\{ ih \left[\mathrm{A}_\alpha{}^* \frac{\partial \mathrm{A}_\alpha}{\partial p},\, \frac{p}{w} \right]_\beta + \frac{ih}{w}\, [\mathrm{A},\, \mathrm{A}^*]_\beta \right\}. \tag{3.6}$$

If we take the special case of a wave packet, we see that the first term represents $[\overline{x},\, \mathrm{P}]_\beta$ which is the angular momentum of a particle of momentum P, moving with the centre of gravity of the packet. The second term is intrinsic momentum and may be written as

$$\frac{\mathrm{W}}{cw} \frac{\int ih\, [\mathrm{A},\, \mathrm{A}^*]\, dp^{(3)}}{\int (\mathrm{A},\, \mathrm{A}^*)\, dp^{(3)}}. \tag{3.7}$$

If we quantise we write $\mathrm{W} = wc$. We then see that a photon can have intrinsic momentum about its direction of motion ranging between $\pm h$ the extreme values corresponding to circular polarisation.

This angular momentum is analogous to the spin of the electron, but we should observe that the analogy is imperfect, because we have only got one quantity instead of two, since the momentum of the photon is always about the line of its motion. The polarisation of the light is incompletely described by the angular momentum, whereas the axis of spin of an electron completely describes its polarisation. It is possible, of course, for the case of the photon, to establish a geometrical correspondence so that the specification of a direction in space should yield not only the ellipticity of the light, but also the axes of the ellipse, but such a correspondence is purely conventional and without dynamical significance. This is natural when we consider that the electromagnetic equations for free space are symmetrical between the electric and magnetic forces, so that we can never hope to get a discrimination between them, until we introduce a perturbation (say an electron) which is unsymmetrical in its reaction to electric and magnetic forces.

We now apply the methods of quantum theory to the same problem. The wave function ψ has six components E_x, E_y, E_z; H_x, H_y, H_z, which we denote by $\psi_1, \ldots, \psi_6$, taken in this order. Here there is one peculiar feature which does not usually occur in quantum processes, and this is that the six components are restricted by the two divergence relations. If we keep in mind the differential equations to which the quantum processes are equivalent, we shall run no danger from this. The six curl equations of the electromagnetic field can be derived from a Hamiltonian

$$\mathfrak{H} = c\,(\beta_x p_x + \beta_y p_y + \beta_z p_z), \tag{3.8}$$

where $p_x = -ih\partial/\partial x$, etc., as usual, and the β's are the matrices

$$\beta_x = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \beta_y = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \beta_z = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{3.9}$$

The equations are then

$$\mathfrak{H}\psi = ih\,\frac{\partial\psi}{\partial t}. \tag{3.10}$$

The stationary states are given by taking all components proportional to $\exp i\,(p_x x + p_y y + p_z z - wct)/h$. The result is a six-rowed determinant which reduces to

$$w^2\,(w^2 - p_x^2 - p_y^2 - p_z^2)^2 = 0. \tag{3.11}$$

The solutions $w = 0$ represent static electric and magnetic fields, derivable from potentials; they are to be excluded as having nothing to do with a photon. Actually they are excluded by the relations $div\,\mathrm{E} = 0$, $div\,\mathrm{H} = 0$, which cannot be derived from the Hamiltonian though their time-differentials can. We shall also exclude the negative values of w yielded by the second factor of (3.11), as discussed earlier.

We can now derive many results like those for the electron, but in general they are more troublesome, because the β's have not the simple commutation rules of the α's of Dirac; in particular they have no reciprocals. In fact, the required relations are only easily found by using ordinary vector methods and then translating them. In this way we can show that, provided $w \neq 0$,

$$(\beta_x p_x + \beta_y p_y + \beta_z p_z)^2 = p^2, \tag{3.12}$$

so that each component of ψ satisfies $\Box^2\psi = 0$. We may also mention another relation which will be used later. If we form the six-rowed matrix

$$\gamma_z = \beta_x\beta_y - \beta_y\beta_x \tag{3.13}$$

we can verify by matrix multiplication that

$$\beta_x\gamma_z - \gamma_z\beta_x = \beta_y, \quad \beta_y\gamma_z - \gamma_z\beta_y = -\beta_x, \quad \beta_z\gamma_z - \gamma_z\beta_z = 0. \tag{3.14}$$

As in the case of the electron we have a current function derived from the equation

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x}(c\psi^*\beta_x\psi) + \frac{\partial}{\partial y}(c\psi^*\beta_y\psi) + \frac{\partial}{\partial z}(c\psi^*\beta_z\psi) = 0. \tag{3.15}$$

By substituting for ψ in terms of E and H we see that the components $c\psi^*\beta\psi$ are simply the Poynting vector associated with the electromagnetic energy $\psi^*\psi$. So, too, we can work out the momentum $P_x = \int \psi^* p_x \psi \, dx^{(3)}$, but it will be observed that though the integral comes out the same as (3.4), the integrand is not the same; this will make an important difference for the angular momentum. We can also work out the centre of gravity of a packet, and show that it travels with the speed of light.

The system of either an electron or a photon in free space is unduly simple in that there are two separate theorems of conservation of angular momentum. There is first the angular momentum of a particle moving with the centre of gravity, and there is also the intrinsic momentum, and each is conserved separately. In Dirac's theory of the electron the angular momentum, as formally defined, is of the first type, and in order to get the general conservation and not merely the uninteresting first type of it, he had to impose a spherically symmetrical perturbing force. We have worked out the angular momentum on classical principles and have got both types together, and it should be noticed that this was because in effect we did have a perturbing force, since the classical formula for the momentum of radiation is derived by considering the interaction with matter. We will now attack the same question in the manner of Dirac, and obtain the same result as before.

We suppose that a spherically symmetrical perturbation Q acts on the photon, so that

$$\mathfrak{H} = c(\beta_x p_x + \beta_y p_y + \beta_z p_z) + Q. \tag{3.16}$$

Using the quantum definition of angular momentum we have

$$\mathrm{M}'_z = \int \psi^* (xp_y - yp_x) \psi \, dx^{(3)}. \tag{3.17}$$

The time differential of this is reduced with the help of (3.10) and its conjugate to

$$\frac{d\mathrm{M}'_z}{dt} = \int \psi^* \left[c\beta_x p_y - c\beta_y p_x - x \frac{\partial \mathrm{Q}}{\partial y} + y \frac{\partial \mathrm{Q}}{\partial x} \right] \psi \, dx^{(3)}.$$

The last two terms vanish on account of the symmetry of Q; to deal with the first two, consider

$$\mathrm{M}''_z = -\int \psi^* ih\gamma_z \psi \, dx^{(3)}, \tag{3.18}$$

where γ_z is given by (3.13). Then with the help of (3.14), we obtain

$$\frac{d\mathrm{M}''_z}{dt} = \int \psi^* [c\beta_y p_x - c\beta_x p_y] \psi \, dx^{(3)}.$$

Thus we have

$$\frac{d}{dt}(\mathrm{M}'_z + \mathrm{M}''_z) = 0,$$

and the quantity conserved is

$$\mathrm{M}_z = \int \psi^* [xp_y - yp_x - ih\gamma_z] \psi \, dx^{(3)}. \tag{3.19}$$

If we write out γ_z and replace ψ by E, H, we find that this is exactly the same as (3.6).

It should perhaps be remarked that though we have spoken of the angular momentum as having any values between $\pm h$, this is not contrary to the statement of Dirac† that plane polarised light is to be regarded as having equal probabilities of being light with eigenvalues $\pm h$ and not as having angular momentum zero. To reconcile the two statements it is only necessary to think of the analogy of the spinning electron; in one sense it may only have angular momentum either $\frac{1}{2}h$ or $-\frac{1}{2}h$ along the z axis, but in another we may say that it has no angular momentum along z if it is pointing along x. It is merely a question of whether we think of the particle as simultaneously in two stationary states, or insist that it shall be in only one.

4. We have discussed the angular momentum of the photon and have obtained an expression for it of broadly the same character as that of the electron. It is now proper to point out that this angular momentum suffers

† "Quantum Mechanics," p. 131.

from just the same disability as does that of the electron. In both cases the momentum is divisible into two parts which we may call respectively the external, due to the motion of the centre of gravity, and the intrinsic, due to the polarisation, but the division is really quite artificial and dynamically meaningless. This fact was shown by Bohr,* as far as concerns the magnetic moment of the electron, and it is only a trifling extension to apply it to the angular momentum. The ensuing argument deals with angular momentum in general; it is nothing but a simple application of the principle that no experiment interpreted according to classical ideas can yield the quantum, and it must, I think, be familiar to many, though I can give no citation.

By the type of argument with which the uncertainty principle has made us familiar, it is easy to show that no experiment can ever reveal the intrinsic momentum of a photon or free electron. In order to know it we should have to know what allowance to make for the external momentum, and this requires a knowledge of the line of motion. Suppose that we want the intrinsic momentum about the axis of z. Then we try to reduce as low as possible the external momentum about this axis, and this we may do by sending the particle along the z-direction through a small hole at the origin. Then if b is the radius of the hole, diffraction will introduce a transverse linear momentum of order h/b, and as the particle may pass through anywhere at distance less than b from the axis, it clearly may have external momentum anything up to h in amount. This is sufficient to mask the intrinsic momentum. Moreover, the same result is true for atoms or molecules, when allowance is made for the possibility that they may carry several quanta of momentum; in this case our measure will refuse to tell us in exactly which quantised state the atom is.

It may be well to elaborate the argument a little further. In conformity with the general principles of resolving power we know that no system of lenses, etc., can be better than the small hole considered, but there is another device that needs closer examination. Suppose that we intend to measure the momentum by absorbing the particle in a body that is free to turn. We might take this body as of very small size and at a large distance from the hole defining the incident particle's path. Consider the momentum about the axis joining the centre of the hole to that of the absorber. Then, though the particle may emerge from the hole with any external momentum up to h, yet it will only strike the absorber if its external momentum about this axis is very small, and we might think that we could isolate the intrinsic momentum in this way. The fallacy, however, is the same as that which asserts that we

* 'Proc. Roy. Soc.,' A, vol. 124, p. 440 (1929).

can know the path of a particle and its momentum as accurately as we like by choosing a particle that has gone through two small holes one after the other; we can observe when this happens, but it is useless for knowing what will happen to the particle later. In the present case we must suppose that we can measure the change in angular momentum of the absorber, and to observe the intrinsic momentum given to it by our particle we must measure the momentum to accuracy better than h; but this we can never do, since the absorber will itself have uncertain external momentum of order h, and that no matter how massive it may be. It is a rather striking example of the duality of the quantum theory, that one aspect insists that every system always has angular momentum an exact multiple of the quantum, while the other insists that it can never be possible to measure the angular momentum of any system to the nearest quantum; indeed it is really this second fact that allows us to make the first assertion without fear of contradiction.

Although the angular momentum of the free electron or photon is a single quantity not to be separated into two parts, yet it is, of course, possible to do statistical experiments from which the intrinsic momentum can be inferred. Thus if a beam of circularly polarised light is collimated as accurately as possible, each of its photons will have intrinsic momentum h in the same sense, though their external momenta will range between $\pm h$. If N of these photons are absorbed at a surface, the surface should acquire angular momentum $(N \pm \sqrt{N})\,h$ and the uncertainty becomes insignificant when N is large. It is not, of course, to be expected that the angular momentum should be practically observed, for it will only give rise to the very small couple produced by multiplying the force due to the radiation pressure by an arm equal to a wave-length of the light.

There are certain points about the angular momentum of radiation that should be noticed, though they are not so fundamental as the preceding. We may recall that long ago Rubinowitz* discussed the question in connection with the change of an atom's azimuthal quantum number. If we take the classical problem of an electron describing a small circle, we find that in addition to the terms in $1/r$ the electric force at a distance has others in λ/r^2, and this means that the wave front of the emitted radiation faces not exactly away from the origin, but from a point about a wave-length away from it. The same is true for the Poynting vector. This, of course, does not matter, since the origin of a source of light is always indefinite to a wave-length, but it is contrary to what we expect at first sight of a particle. It implies that the photon which

* 'Phys. Z.,' vol. 19, p. 441 (1918).

is to carry away the energy and angular momentum from an atom of radius 10^{-8} cm. starts its life outside the atom at a distance 10^{-5} cm. away.

It is also interesting to consider whether the phenomenon of emission can be described regarding the photon as a pure particle. The atom is to lose angular momentum h about the z axis by the emission of a single photon of linear momentum $2\pi h/\lambda$ in an arbitrary direction. The intrinsic momentum of a photon is always directed along its direction of motion; let m be its magnitude. Let θ, ϕ, give the direction of motion of the photon and let ξ, η, 0, be its birth-place in the equator. Then we easily see that the three conditions of angular momentum can only be satisfied if

$$\left.\begin{aligned} m &= h\cos\theta \\ \xi &= \lambda\sin\theta\sin\phi/2\pi \\ \eta &= -\lambda\sin\theta\cos\phi/2\pi \end{aligned}\right\}, \tag{4.1}$$

The polarisation of the photon if it goes into the direction θ is just such as is indicated in the classical theory and confirmed by observation in the Zeeman effect. As far as it goes this is satisfactory, but its scope is limited, because we must also consider the quadrupole emission where $2h$ of momentum is lost in a single photon. The same argument now would give $m = 2h\cos\theta$ and even though for this type of quadrupole there is not much emission in high latitudes, still there is some, which would give to m inadmissible values greater than h. For quadrupole emission the pure particle concept is a failure.

5. We now consider the perturbation of a photon. This may be due either to the refraction of a medium (of any type of anisotropy) or, as in the case we shall study here, it may be an electron. We only consider the case where the momentum of the photon is insufficient to move the electron perceptibly, and shall only consider the first approximation. The photon is perturbed by a potential Q, a function of position; Q is in general a matrix of 36 components, but we may take all but 9 of them as zero, as the following considerations show. When the momentum and energy of a photon are given, the magnetic components can be expressed in terms of the electric. Thus suppose that $u\,(p)$ represents the six components of the solution

$$u\,(p)\,e^{i(p_\beta x_\beta - wct)/h}, \tag{5.1}$$

then u_4 can be expressed in terms of u_1, u_2, u_3 with the help of p and w. The amplitude of the scattered wave will in the usual way depend on

$$\int u^*(p')\,Qu\,(p)\,e^{i(p_\beta - p'_\beta)x_\beta/h}\,dx^{(3)}, \tag{5.2}$$

and in this it is possible to replace u_4, etc., on both sides of Q by u_1, etc., that is to say, to transfer all the other components of Q to the first nine. In doing so we, of course, have to introduce p and p' into Q and so we must be ready to regard it as possibly involving differential operators. It should be noted that the possibility of reducing Q to nine members depends on the exclusion of negative energy; with given p, the magnetic components are only uniquely expressed in terms of the electric, provided that w is also given. In our present problem there is no danger of the energy changing sign, and so we need only regard Q as consisting of the nine electric-electric members.

The best method of solving the perturbation problem is by means of the Green function. It is easily verified in the present case that for a ψ proportional to $e^{-iwct/h}$ the wave equation can be written as

$$\left.\begin{aligned} E_\alpha(x) &= E_\alpha^0(x) - \frac{1}{4\pi wc}\int dx'^{(3)}\frac{e^{iwr/h}}{r}\left\{\frac{\partial^2}{\partial x'_\alpha\,\partial x'_\beta}Q'_{\beta\gamma}E'_\gamma + \frac{w^2}{h^2}Q'_{\alpha\gamma}E'_\gamma\right\} \\ H_\alpha(x) &= H_\alpha^0(x) + \frac{i}{4\pi hc}\int dx'^{(3)}\frac{e^{iwr/h}}{r}\left[\operatorname{curl}(Q'E')\right]_\alpha \end{aligned}\right\}. \tag{5.3}$$

Here E^0, H^0, is any solution of the unperturbed equation, $r = \sqrt{(x_\alpha - x'_\alpha)^2}$ and Q', E' represent the values of the respective quantities at the point x'. This is exact, but we can approximate by substituting E^0 for E under the integral sign and expanding r. Then incident wave

$$E_\alpha = A_\alpha\, e^{ip_\beta x_\beta/h} \tag{5.4}$$

yields scattered wave in direction p'

$$E'_\alpha = \frac{1}{4\pi}\frac{e^{iwr_0/h}}{r_0 wc}\int dx'^{(3)}\, e^{-ip'_\gamma x'_\gamma/h}\{p'_\alpha p'_\beta Q'_{\beta\delta}A_\delta - w^2 Q'_{\alpha\delta}A_\delta\}. \tag{5.5}$$

This is the form that we shall actually need, but it is of course, quite easy to go on in the ordinary way and show that the scattered wave has amplitude proportional to (5.2).

We want to find what form of Q will give the scattering which is actually caused by an electron, in fact, the Thomson scattering. According to the classical formulæ the incident wave

$$E_\alpha = A_\alpha\, e^{i(p_\beta x_\beta - wct)/h} \tag{5.6}$$

gives rise to a scattered wave

$$E'_\alpha = \frac{e^2}{mc^2}\frac{1}{w^2}\{p'_\alpha p'_\beta A_\beta - w^2 A_\alpha\}\frac{e^{iw(r-ct)}}{r}. \tag{5.7}$$

If we compare this with (5.5) we see that the solutions can be identified if

$$Q_{11} = Q_{22} = Q_{33} = 4\pi \frac{e^2}{mc^2} \frac{h^2 c}{w} \delta(x_1)\, \delta(x_2)\, \delta(x_3)\,, \tag{5.8}$$

while the other components all vanish. Here δ is the singular function of Dirac. This form of Q is much the simplest, but is by no means unique. We could, for example, make the first three components of ψ mean dielectric displacement instead of electric force; this would annul Q_{11}, etc., and introduce Q_{41}, etc., in place, thus undoing again the simplification made above.

It should be noticed that Q is an "improper" expression, since we cannot replace w in it by a time differentiation. This, of course, means that there is no first order differential equation which can yield the observed scattered wave. That does not matter, for in a fuller theory the electron would be free, and the mc^2 in the coefficient would then be replaced by the energy of the electron; so even if the w had not occurred in the denominator, the fuller theory would involve an improper expression. As a matter of fact, the product of w and mc^2 is admirably suited to turn into a relativistic invariant. It should also be observed that the infinity of δ in (5.8) prevents the carrying out of any higher approximations. As far as the present work goes the photon must not have wave-length much less than 10^{-8} cm., or it would set the electron in motion; and so we may imagine the singular function replaced by any function with a single peak much narrower than this, and we could carry the approximation further with such a function, though it would not be significant to do so. If the theory is extended to cover the Compton effect, the restriction on the singular function becomes much more severe, but even in that case we cannot claim to have any data for distances less than, say, 10^{-12} cm. Thus the singularity at the origin is only an approximation to a state of affairs about which we have no experimental evidence. From the other side of the question, it should be noticed that the classical calculation of (5.7) is itself an approximation, worked by supposing that the electron's radiation is negligible during the calculation of its own motion. The higher approximations of the classical theory are merely constructed out of general dynamical principles, and it will be easier to apply such general principles (with the appropriate modifications) through the quantum theory direct, rather than by means of a laborious translation of the classical forms.

Summary.

The general aim of the work was to apply the methods of wave mechanics to the single photon, following out as far as possible the process which has succeeded for the electron.

The usual real form of electromagnetic waves implies the presence of two waves of equal amplitude, but respectively of positive and negative energy and momentum, and it is suggested that the latter should be rejected.

The polarisation of light is connected with its angular momentum in a way resembling the spin of the electron. The separation of angular momentum into two parts, one external, due to the motion of the centre of gravity, and the other intrinsic, is shown to be conventional; no experiment can divide it in this way, and since the external momentum is always uncertain to h, the intrinsic momentum of a photon is unobservable. Certain features of the radiation of an atom are considered.

The perturbation of the photon by an electron is considered and the energy of perturbation is given for the case of Thomson scattering.

The Phenomena of Superconductivity with Alternating Currents of High Frequency.

By J. C. McLennan, F.R.S., A. C. Burton, A. Pitt, and J. O. Wilhelm.

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[Plate 2.]

In a previous paper* the authors described experiments on the resistance of lead to currents having the high frequency of 10^7 per second. In these experiments no evidence was found of an abrupt change of resistance corresponding to the phenomenon of superconductivity that appears with direct currents when the critical temperature of 7·2° K. is passed, and it was pointed out that the partial silvering of the vacuum flasks that contained the lead coil was a source of error.

In a set of new experiments† to be described below unsilvered flasks were used, but the same technique of measurement was adopted. It was found that there was an abrupt decrease of the high frequency resistance at a temperature which appeared to be slightly lower than that characteristic of the transition to superconductivity with direct currents. Experiments with tin instead

* 'Phil. Mag.,' vol. 12, p. 707 (1931).

† Preliminary notes on these experiments were published in 'Nature,' vol. 128, p. 1004 (1931), and 'Trans. Roy. Soc. Canada,' section III, p. 191 (1931).