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Pantelis E. Eleftheriou, Costas D. Koutras, Christos Nomikos

Institutions: University of Lisbon, University of Peloponnese, University of Ioannina

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Notions of Bisimulation for Heyting-Valued Modal Languages

Pantelis E. Eleftheriou*
CMAF, Universidade de Lisboa,
Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
pelefthe@gmail.com

Costas D. Koutras
Department of Computer Science and Technology,
University of Peloponnese,
end of Karaiskaki Street, 22100 Tripolis, Greece
ckoutras@uop.gr

Christos Nomikos
Department of Computer Science, University of Ioannina,
P.O. Box 1186, 45110 Ioannina, Greece
cnomikos@cs.uoi.gr

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Abstract

We examine the notion of bisimulation and its ramifications, in the context of the family of Heyting-valued modal languages introduced by M. Fitting. Each modal language in this family is built on an underlying space of truth values, a Heyting algebra $\mathcal{H}$. All the truth values are directly represented in the language, which is interpreted on relational frames with an $\mathcal{H}$-valued accessibility relation. We define two notions of bisimulation that allow us to obtain truth invariance results. We provide game semantics and, for the more interesting and complicated notion, we are able to provide characteristic formulae and prove a Hennessy-Milner type theorem. If the underlying algebra $\mathcal{H}$ is finite, Heyting-valued modal models can be equivalently reformulated to a form relevant to epistemic situations with many interrelated experts. Our definitions and results draw inspiration from this formulation, which is of independent interest to Knowledge Representation applications.

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1 Introduction

The concept of bisimulation is a very rich one. It has played an important role in many areas of Computer Science, Logic and Set Theory [San07] and has been a very fruitful idea in the model theory of modal and temporal logics [KdR97] and the logics of computation[1]. It is not an overstatement that, nowadays, some of the most interesting and elegant results in the expressibility of modal languages are based on the notion of bisimulation equivalence. Its origins can be found in the analysis of Modal Logic and, independently, in the discoveries made by computer scientists in their efforts to understand concurrency.

In the area of Modal Logic, *bisimulation equivalence* is the correct algebraic counterpart to modal model equivalence, or, equivalently, the correct notion of similarity between two modal models: modal formulas are unable to distinguish bisimilar points of the two models. Following Segerberg's *p-morphisms*, bisimulations were introduced by van Benthem, under the name of *p-relations* or *zig-zag relations*, in the course of his work on the correspondence theory of Modal Logic [vB83, vB84]. The celebrated *van Benthem characterization theorem*, published in the mid-'70s, states that invariance for bisimulation captures the essential property of the 'modal fragment' of first-order logic: *a first-order formula is invariant for bisimulation iff it is (equivalent to) the syntactical translation of a modal formula* (the so-called *standard translation*) [BdRV01]. This characterization theorem has generated an important stream of research in the analysis of logical languages. In particular, the bisimulation-based analysis of modal languages has been extensively studied in the Amsterdam school of modal logic [dR93, Ger99, MV03], and it has even been suggested that this notion is as important for modal logic as the notion of partial isomorphism has been for the model theory of classical logic [dR93]. It is worth mentioning that bisimulations have been used beyond the realm of classical modal logic: see for instance the variant used to analyze *since-until temporal languages* [KdR97], M. Otto's work related to *Finite Model Theory* [Ott99] and J. Gerbrandy's dissertation [Ger99]. The basic theory of bisimulation can be found in the recent excellent handbook chapter [GO07].

Many variants of this notion have been employed in the effort to characterize the *behavioural equivalence* between transition systems which appear in many areas of Computer Science. In this context, bisimulations represent a fundamental notion of identity between process states and every language designed to capture the essential properties of processes should be blind for bisimilar states. The origins of bisimulation, in this stream, can be traced back to the research on the algebraic theory of automata. The decisive progress towards bisimulations has been made by R. Milner in the '70s [Mil71], with important contributions by D. Park [Par81] and M. Hennessy [HM85]; the reader can consult [San07] for a nice exposition of this story and [Sti01] for an introductory text on modal and temporal treatment of computational processes.

[1] The reader can consult the textbook [Sti01, Section 5.4] for a short exposition (and further references) of *preservation of bisimulation equivalence* in the modal μ-calculus, and the classical text of Yiannis N. Moschovakis [Mos74] for one of the earlier applications of bisimulation, in the field of inductive definability.

Also, bisimulations have been used as a fundamental tool in the area of non-well founded set theory ([Acz88, BM96], see [BdRV01, San07] for a few details and further references).

In this paper, we address the question of what constitutes a suitable notion of bisimulation for the family of many-valued modal languages introduced by M. Fitting in the early '90s [Fit92, Fit91]. Each language of this family is built on an underlying space of truth values, a Heyting algebra $\mathcal{H}$. There exist three features that give these logics their distinctive character. The first one is *syntactic*: the elements of $\mathcal{H}$ are directly encoded in the language as special constants and this permits the formation of 'weak', uncertainty-oriented versions of the classical modal epistemic actions [Kou03, KNP02, KP02]. The second is *semantic*: the languages we discuss are interpreted on $\mathcal{H}$-labelled directed graphs which provide us a form of many-valued accessibility relation. Finally, the third one concerns the potential applicability of these logics in epistemic situations with multiple intelligent agents. More specifically, assuming that $\mathcal{H}$ is a finite Heyting algebra, these logics can be formulated in a way that expresses the epistemic consensus of many experts, interrelated through a binary 'dominance' relation [Fit92]. It is worth mentioning that, model-theoretically, every complete Heyting algebra can serve as the space of truth values. However, apart from the equivalent multiple-expert formulation of the logics, the finiteness assumption for $\mathcal{H}$ is essential for the elegant canonical model construction of [Fit92] which leads to a completeness theorem; note that this finiteness restriction seems to be also necessary for obtaining a many-valued analog of the ultrafilter extension construction [EK05].

In Section 2, we provide a few basic definitions and facts on Heyting algebras, that will be used in the sequel. Then, syntax and semantics for the family of Heyting-valued modal languages are given. Section 3 contains the results of the paper. We provide two notions of bisimulation: a rather strong and restrictive one in Section 3.1, and a more fine-grained in Section 3.2; for both, we derive modal equivalence results and simple appropriate Ehrenfeucht-Fraissé type combinatorial games. For the 'weak' bisimulation notion of Section 3.2, we are able to obtain an interesting notion of a Hennessy-Milner class of Heyting-valued modal models in Section 3.2.3 and, in Section 3.3 we define characteristic formulae, which capture syntactically the bisimulation game in a precise sense. Both notions of bisimulations draw inspiration from the equivalent multiple-expert formulation of these logics, which is actually a mixture of Kripke modal and Kripke intuitionistic semantics. In Section 4, we review this alternative formulation and interpret our results in that context.

The interest in our results stems from the fact that model theory places special emphasis on the characterization of logical equivalence, in terms of other forms of structural equivalences between models. In particular, obtaining truth preservation - or invariance - results, provides useful insight in the semantics of a logic and furnishes tools for studying its model theory. We show in this paper that, also in the Heyting-valued context, *bisimulation equivalence* is a valuable tool for examining the relationship between Heyting-valued *possible-worlds semantics* and *structural/combinatorial equivalences* between Heyting-valued modal models.

2 Many-Valued Modal Languages

In this section we provide the syntax and semantics of many-valued modal languages, as introduced in [Fit92], with only minor changes in the notation. To construct a modal language of this family, we first fix a Heyting algebra $\mathcal{H}$, which will serve as the space of truth values. Thus, we first briefly expose the basic definitions and properties of Heyting algebras, fixing also notation and terminology. We assume that the reader already has some familiarity with the elements of lattice theory; for more details the classical texts [RS70, DP90, BD74] can be consulted.

Heyting Algebras A *lattice* $\mathcal{L}$ is a pair $\langle L, \leq \rangle$ consisting of a non-empty set L equipped with a partial-order relation $\leq$, such that every two-element subset $\{a, b\}$ of L has a *least upper bound* or *join*, denoted by $a \vee b$, and a *greatest lower bound* or *meet*, denoted by $a \wedge b$. A lattice $\mathcal{L}$ is *complete* if a join and a meet exist for *every* subset of $\mathcal{L}$. A *least* (or *bottom*) element of a lattice is denoted by $\bot$ and a *greatest* (or *top*) one by $\top$. An element $x \in L$ is *join-irreducible* if $x \neq \bot$ (in case L has a bottom element) and $x = a \vee b$ implies $x = a$ or $x = b$. If every nonempty subset of a partially ordered set $\mathcal{L}$ contains a minimal element, then $\mathcal{L}$ satisfies the *descending chain condition* (DCC). For the class of intended applications of the logics we examine, it is natural to consider truth spaces which do not have infinite descending chains, and thus satisfy DCC. The following theorem states a fact that will be useful in Section 3.2; see [BD74, Theorem III.2.2] for the proof.

Theorem 2.1 *If a lattice L satisfies DCC, then each element of L is a sum of a finite non-empty set of mutually incomparable join-irreducible elements. If, in addition, L is distributive, this representation is unique.*

It can be verified that every element of a lattice L satisfying DCC can be obtained as the sum of the join-irreducible elements below it in the lattice ordering [DP90].

We frequently use indexed sets and denote (possibly infinite) meets and joins by $\bigwedge_{t \in T} a_t$ and $\bigvee_{t \in T} a_t$. Some fairly obvious properties of infinite joins and meets, such as $a \wedge \bigwedge_{t \in T} a_t = a \wedge \bigwedge_{t \in T} (a \wedge a_t)$ will be used, generally without comment.

A lattice $\mathcal{H} = \langle H, \leq \rangle$ with the additional property that, for every pair of elements $\langle a, b \rangle$, the set $\{x \mid a \wedge x \leq b\}$ has a greatest element, is called a *relatively pseudo-complemented lattice*. This element is denoted by $a \Rightarrow b$ and is called the *pseudo-complement of a relative to b*. A relatively pseudo-complemented lattice is always a topped ordered set [RS70]. It is not always the case that a relatively pseudo-complemented lattice has a least element. A relatively pseudo-complemented lattice $\mathcal{H}$ with a least element is called a **Heyting algebra** (HA) or a **pseudo-Boolean algebra**. It is known that the class of HAs includes the class of Boolean algebras and is included in the class of distributive lattices; both inclusions are proper. For finite lattices, the second inclusion becomes an equality: the class of finite HAs coincides with the class of finite distributive lattices [RS70]. The next lemma gathers some useful properties of relatively pseudocomplemented lattices that will be used in the rest of the paper.

Lemma 2.2 *Let a, b, c be elements of a Heyting algebra $\mathcal{H}$. Then the following statements hold:*

1. $c \leq a \Rightarrow b$ *iff* $c \wedge a \leq b$
2. $a \Rightarrow b = \top$ *iff* $a \leq b$
3. *If* $a \leq b$ *then* $b \Rightarrow c \leq a \Rightarrow c$
4. $a \Rightarrow b = a \Rightarrow (a \wedge b)$
5. $(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c)$
6. $c \wedge (a \Rightarrow b) = c \wedge \big((c \wedge a) \Rightarrow (c \wedge b)\big)$
7. $c \wedge (a \Rightarrow b) = c \wedge \big((c \wedge a) \Rightarrow b\big)$
8. *If c is join-irreducible and $c \leq a \vee b$ then $c \leq a$ or $c \leq b$*

Proof. The proof of items (1)-(3), (5), (6) can be found in [RS70, pp. 58-60]. Item (4) follows immediately from the following simple property of Heyting algebras recorded in [BD74, Thm. 3.(x), p. 174]: $(a \Rightarrow [b \wedge c]) = (a \Rightarrow c) \wedge (b \Rightarrow c)$. Item (8) is known to hold in any distributive lattice [BD74, p. 65]. Item (7) can be proved as follows:

$$c \wedge (a \Rightarrow b) = c \wedge \big((c \wedge a) \Rightarrow (c \wedge b)\big) = c \wedge \big((c \wedge c \wedge a) \Rightarrow (c \wedge b)\big) = c \wedge \big((c \wedge a) \Rightarrow b\big)$$

where the first and third equalities are by item (6). ■

Syntax of Many-Valued Modal Languages Having fixed a complete Heyting algebra $\mathcal{H}$ we proceed to define the syntax of the modal language. The elements of $\mathcal{H}$ are directly represented in the language by special constants, called *propositional constants*, and we reserve lowercase letters (along with $\bot, \top$) to denote them. To facilitate notation, we use the same letter for the element of $\mathcal{H}$ and the constant which represents it in the language; context will clarify what is meant. Assuming also a set Φ of *propositional variables* (also called propositional letters) we define the many-valued modal language $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ with the following BNF specification, where t ranges over elements of $\mathcal{H}$, P ranges over elements of Φ and A is a formula of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$.

$$A \ ::= \ t \ | \ P \ | \ A_1 \vee A_2 \ | \ A_1 \wedge A_2 \ | \ A_1 \supset A_2 \ | \ \Box A \ | \ \Diamond A$$

As we will see below, the modal logics defined are in general bimodal, thus we need both modal operators. Note also that $\vee$ and $\wedge$ serve both as logical connectives, as well as lattice operation symbols but it should be clear by context what is meant. A (non-classical) negation $\neg X$ can be defined as $(X \supset \bot)$.

Semantics of Many-Valued Modal Languages $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ is interpreted on an interesting variant of a relational frame, which possesses a kind of Heyting-valued accessibility relation. Note that there have been other approaches in the literature for defining many-valued modal logics, but all of them have kept the essence of classical relational semantics intact (see [Fit92] for references). Given $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, we define $\mathcal{H}$-modal frames and $\mathcal{H}$-modal models as follows:

Definition 2.3 An $\mathcal{H}$-*modal frame* for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ is a pair $\mathfrak{F} = \langle \mathfrak{S}, \mathfrak{g} \rangle$, where $\mathfrak{S}$ is a non-empty set of *states* and $\mathfrak{g} : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathcal{H}$ is a total function mapping pairs of states to elements of $\mathcal{H}$.
An $\mathcal{H}$-*modal model* $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ is built on $\mathfrak{F}$ by providing a *valuation* v, that is a function $v : \mathfrak{S} \times (\mathcal{H} \cup \Phi) \rightarrow \mathcal{H}$ which assigns a $\mathcal{H}$-truth value to atomic formulae in each state, such that $v(\mathfrak{s}, t) = t$, for every $\mathfrak{s} \in \mathfrak{S}$ and $t \in \mathcal{H}$.

In other words, the propositional constants are always mapped to 'themselves'. In the sequel, we shall often omit the adjective 'modal' and talk simply of $\mathcal{H}$-frames and $\mathcal{H}$-models.

The valuation v extends to all the formulae of $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ in a standard recursive fashion:

Definition 2.4 Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ be an $\mathcal{H}$-*model* and $\mathfrak{s}$ a state of $\mathfrak{S}$. The extension of the valuation v to the whole language $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ is given by the following clauses:

- $v(\mathfrak{s}, A \wedge B) = v(\mathfrak{s}, A) \wedge v(\mathfrak{s}, B)$
- $v(\mathfrak{s}, A \vee B) = v(\mathfrak{s}, A) \vee v(\mathfrak{s}, B)$
- $v(\mathfrak{s}, A \supset B) = v(\mathfrak{s}, A) \Rightarrow v(\mathfrak{s}, B)$
- $v(\mathfrak{s}, \Box A) = \bigwedge_{\mathfrak{r} \in \mathfrak{S}} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \Rightarrow v(\mathfrak{r}, A)\big)$
- $v(\mathfrak{s}, \Diamond A) = \bigvee_{\mathfrak{r} \in \mathfrak{S}} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge v(\mathfrak{r}, A)\big)$

The operators of *necessity* ($\Box$) and *possibility* ($\Diamond$) are not each other's dual, unless $\mathcal{H}$ is a Boolean algebra [Fit92]. Note also that all the definitions above collapse to the familiar ones from the classical case, in the case of the classical language $L^{\mathbf{2}}_{\Box\Diamond}$, where **2** is the lattice of two-valued classical logic.

3 Bisimulations for Heyting-Valued Modal Languages

In this section, we define two suitable general notions of bisimulation for a language $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ of the family defined in the previous section. Before proceeding, we have to define a refined notion of modal truth invariance, which fits our aims and which also has an interesting interpretation in the multiple-expert context. Note that the following notion is trivial for $t = \bot$.

Definition 3.1 (t-invariance) Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ two states and $t \in \mathcal{H}$ a truth value ($t \neq \bot$). We say that modal truth is *t-invariant for the transition between $\mathfrak{s}$ and $\mathfrak{s}'$* if for every $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$

$$t \wedge v(\mathfrak{s}, X) \;=\; t \wedge v'(\mathfrak{s}', X)$$

3.1 Strong Bisimulation for Many-Valued Modal Languages

The following definition captures the idea of moving 'back and forth' between two $\mathcal{H}$-models by matching steps ('modulo' t) in both directions.

Definition 3.2 (t-bisimulations) Given two $\mathcal{H}$-models $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ and a truth value $t \in \mathcal{H}$ ($t \neq \bot$), a non-empty relation $B \subseteq \mathfrak{S} \times \mathfrak{S}'$ is a *t-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$* if for any pair $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in B$

(*base*) $t \wedge v(\mathfrak{s}, P) \;=\; t \wedge v'(\mathfrak{s}', P)$ for every $P \in \Phi$

(*forth*) for every $\mathfrak{r} \in \mathfrak{S}$ such that $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$,
there exists an $\mathfrak{r}' \in \mathfrak{S}'$ such that $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \;=\; t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in B$

(*back*) for every $\mathfrak{r}' \in \mathfrak{S}'$ such that $t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \neq \bot$,
there exists an $\mathfrak{r} \in \mathfrak{S}$ such that $t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \;=\; t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in B$

Two states $\mathfrak{s}$ and $\mathfrak{s}'$ are called *t-bisimilar* (notation $\mathfrak{s} \leftrightarroweq_t \mathfrak{s}'$ or $\mathfrak{M}, \mathfrak{s} \leftrightarroweq_t \mathfrak{M}', \mathfrak{s}'$) if there is a t-bisimulation B between $\mathfrak{M}$ and $\mathfrak{M}'$ such that $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in B$.

We can now state the basic theorem, which asserts that t-bisimulation implies t-invariance.

Theorem 3.3 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ two states and $t \in \mathcal{H}$ a truth value ($t \neq \bot$). If $\mathfrak{M}, \mathfrak{s} \leftrightarroweq_t \mathfrak{M}', \mathfrak{s}'$, then $t \wedge v(\mathfrak{s}, X) \;=\; t \wedge v'(\mathfrak{s}', X)$ for every $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$.*

Proof. The theorem follows immediately from Lemma 3.9 and Theorem 3.11, which are presented in the next subsection. A direct proof can be obtained using induction on the formation of X, as in the proof of Theorem 3.11. ∎

EF-type games for t-bisimulation Bisimulation games are very natural variants of the Ehrenfeucht-Fraissé (EF) game played in First-Order Logic. Actually, they are so close to it that, looking back, one can say that the bisimulation technique of characterizing modal properties, already existed in EF games, awaiting its 'discovery'. Indeed, bisimulation relations can be naturally viewed as descriptions of non-deterministic winning strategies for one player in the corresponding model comparison games [GO07].

The t-bisimulation game is a simple variant of the classical EF game. For the purposes of the rest of this section call a state $\mathfrak{r}$ a *t-compatible successor* state of $\mathfrak{s}$ if $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$. Two elements a, b of $\mathcal{H}$ are called *t-equivalent* if $t \wedge a = t \wedge b$. We call *labels* the $\mathcal{H}$-truth values attached to the graph's edges and to the propositional letters of the language in each possible world. The t-bisimulation game is played on two *pointed*

$\mathcal{H}$-models (models with a single distinguished state) $\mathfrak{M}, \mathfrak{s}_0$ and $\mathfrak{M}', \mathfrak{s}'_0$. There exists one marked element in each $\mathcal{H}$-model; initially, the marked elements are the distinguished nodes $\mathfrak{s}_0$ and $\mathfrak{s}'_0$. In each *round* of the game

- **Player I** selects one of the $\mathcal{H}$-models, chooses a t-compatible successor of the marked element and moves the marker along the edge (labelled with a_I) to its target.
- **Player II** responds with a move of the marker in the other $\mathcal{H}$-model in a corresponding t-compatible transition (labelled with a_{II}) such that a_I and a_{II} are t-equivalent and the labels of the propositional letters in the marked elements (states) of the models are also t-equivalent.

The *length* of the game is the (finite or infinite) number of rounds and Player II loses the *match* if at a certain round cannot respond with an appropriate move or if the initial states $\mathfrak{s}_0$ and $\mathfrak{s}'_0$ are not t-equivalent. It is obvious that Player I is trying to spoil a t-bisimulation while Player II is trying to reveal one. Player II has a *winning strategy* in a game of n rounds if she can win every n-round game played on $\mathfrak{M}, \mathfrak{s}_0$ and $\mathfrak{M}', \mathfrak{s}'_0$. In a classical fashion, we can proceed to a finer analysis of t-bisimulations using the inductively defined notion of the modal depth of a formula (the maximum depth of nesting of modal operators, [BdRV01, MV03]). The notion of a *t-bisimulation bounded by a positive integer n*, or any ordinal actually, can easily be defined (see [Ger99, Chapter 2.1] for the classical case), but we will not give further details here, since the whole construction is identical to the classical one. We only provide the following proposition, which generalizes the known classical results from two-valued modal logic:

Proposition 3.4 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be $\mathcal{H}$-models and $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ be two states. The following statements hold:*

1. *Player II has a winning strategy for the infinite game played on $\mathfrak{M}, \mathfrak{s}_0$ and $\mathfrak{M}', \mathfrak{s}'_0$ iff $\mathfrak{s}_0 \leftrightarrows_t \mathfrak{s}'_0$.*
2. *If Player II has a winning strategy in the n-round game played on $\mathfrak{M}, \mathfrak{s}_0$ and $\mathfrak{M}', \mathfrak{s}'_0$ then modal truth is t-invariant for the transition between $\mathfrak{s}_0$ and $\mathfrak{s}'_0$ for every formula up to modal depth n.*

Proof. The first item follows from the definitions of t-bisimulation and the corresponding game. The proof of the second item runs by induction on n; it also follows from Proposition 3.12 and Lemma 3.9, using the first item. ∎

t-unravellings and the tree-model property The idea of unravelling a model into a modally-equivalent tree model is known both from modal logic and the theory of processes. In the latter field, the states of the unravelled model represent *traces* (*histories*) of processes, starting from a state $\mathfrak{s}$. The following definition provides the many-valued analog of this notion.

Definition 3.5 Given a pointed model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle, \mathfrak{s}_1$, its *t-unravelling* is the model $\mathfrak{M}^t_{\mathfrak{s}_1} = \langle \mathfrak{S}^t_{\mathfrak{s}_1}, \mathfrak{g}^t_{\mathfrak{s}_1}, v^t_{\mathfrak{s}_1} \rangle$, where

1. $\mathfrak{S}^t_{\mathfrak{s}_1}$ consists of all tuples $\langle \mathfrak{s}_1, \cdots, \mathfrak{s}_k \rangle$ where $\mathfrak{s}_{i+1}$ is a t-compatible successor of $\mathfrak{s}_i$
2. $\mathfrak{g}^t_{\mathfrak{s}_1}(\langle \mathfrak{s}_1, \cdots, \mathfrak{s}_k \rangle, \langle \mathfrak{s}_1, \cdots, \mathfrak{s}_{k+1} \rangle) = t \wedge \mathfrak{g}(\mathfrak{s}_k, \mathfrak{s}_{k+1})$, and $\mathfrak{g}^t_{\mathfrak{s}_1}(x, y) = \bot$ for any other pair of tuples $x, y \in \mathfrak{S}^t_{\mathfrak{s}_1}$.
3. $v^t_{\mathfrak{s}_1}(\langle \mathfrak{s}_1, \cdots, \mathfrak{s}_k \rangle, P) = t \wedge v(\mathfrak{s}_k, P)$, $(P \in \Phi)$

Obviously $\mathfrak{M}^t_{\mathfrak{s}_1}$ is a tree model and the following proposition can be proved by a careful inspection on the definition of a t-bisimulation.

Proposition 3.6 The graph of the function from $\mathfrak{S}^t_{\mathfrak{s}_1}$ to $\mathfrak{S}$, which maps every tuple to its last component (and $\langle \mathfrak{s}_1 \rangle$ to $\mathfrak{s}_1$) is a t-bisimulation.

Thus, modal truth is t-invariant for the transition from $\mathfrak{s}_1$ to the root $\langle \mathfrak{s}_1 \rangle$ of the tree and this is a generalized version of the *tree model property* [Var97].

Satisfiability in Many-Valued Modal Languages The general satisfiability problem in this context can be phrased as follows: given $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ and $t \in \mathcal{H}$, is there a state $\mathfrak{s}$ of an $\mathcal{H}$-model $\mathfrak{M}$ in which $v(\mathfrak{s}, X) \geq t$? This is equivalent (by Lemma 2.2(2)) to $t \Rightarrow v(\mathfrak{s}, X) = \top$, which is equivalent (by Def. 2.4) to $v(\mathfrak{s}, t \supset X) = \top$. Thus, the general satisfiability problem is subsumed by the question of finding a state in which a formula is mapped to the top element of the lattice. By the previous paragraph, if such a state/model exists, then this formula can be also satisfied at the root of a ($\top$-unravelled) tree. Imitating the classical arguments ([MV03]), it is easy to prove that, if $\mathcal{H}$ is finite, every formula can be satisfied in a finite tree whose size is bounded: its depth is bounded by the modal depth of X and its branching degree is bounded by the number of box and diamond subformulas of X multiplied by the number of elements in $\mathcal{H}$. This leads to a simple way of proving the fact that the many-valued analog of the system **K** (which is determined by the class of all $\mathcal{H}$-models [Fit92]) has a decidable general satisfiability problem.

3.2 Weak Bisimulations for Many-Valued Modal Languages

We proceed now to define, a weaker, more fine-grained notion of bisimulation that is directly inspired from (and can be better explained in the context of) the multiple-expert semantics of these languages.

3.2.1 Defining Weak Bisimulations

We first fix some notation. Let $\mathcal{H}$ be a complete Heyting algebra and let $I_{\mathcal{H}}$ denote the set of join-irreducible elements of $\mathcal{H}$. Define the function $D_{\mathcal{H}}$ from $\mathcal{H} - \{\bot\}$ to $2^{I_{\mathcal{H}}}$, such that $D_{\mathcal{H}}(t) = \{c \in I_{\mathcal{H}} \mid c \leq t\}$. The following lemma states an useful property of $D_{\mathcal{H}}(t)$.

Lemma 3.7 *Let $\mathcal{H}$ be a Heyting algebra. Then, for every $t \in \mathcal{H}$*

$$t = \bigvee_{c \in D_{\mathcal{H}}(t)} c.$$

PROOF. Since every Heyting algebra is a distributive lattice, by Theorem 2.1, for every $t \in \mathcal{H}$ there exists a set of join-irreducible elements $S_t \subseteq I_{\mathcal{H}}$ such that $t = \bigvee_{c \in S_t} c$. Obviously, for every $c \in S_t$ it is $c \leq t$, which implies that $S_t \subseteq D_{\mathcal{H}}(t)$. Therefore

$$\bigvee_{c \in D_{\mathcal{H}}(t)} c = \bigvee_{c \in S_t} c \vee \bigvee_{c \in D_{\mathcal{H}}(t) - S_t} c = t \vee \bigvee_{c \in D_{\mathcal{H}}(t) - S_t} c = t$$

∎

In the next definition, a bisimulation relation is defined for every truth value, but in a way that it is 'upwards consistent' (with respect to the lattice of truth values). The intuition behind the consistency property and the role of join-irreducible elements can be better understood using the multiple-expert interpretation of the truth values in a $\mathcal{H}$. We will discuss this issue in section 4.

Definition 3.8 (Weak bisimulation) Given two $\mathcal{H}$-models $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$, a function Z from $\mathcal{H} - \{\bot\}$ to $2^{\mathfrak{S} \times \mathfrak{S}'}$ is a *weak bisimulation between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if it satisfies the following properties:

- for every $t_1, t_2 \in \mathcal{H} - \{\bot\}$

  (*consistency*) $Z(t_1 \vee t_2) = Z(t_1) \cap Z(t_2)$

- for every join-irreducible value $t \in I_{\mathcal{H}}$ and any pair $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t)$

  (*base*) $t \wedge v(\mathfrak{s}, P) = t \wedge v'(\mathfrak{s}', P)$ for every $P \in \Phi$

  (*forth*) for every $\mathfrak{r} \in \mathfrak{S}$ such that $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$
  and for every $c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))$,
  there exists an $\mathfrak{r}' \in \mathfrak{S}'$ such that $c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in Z(c)$

  (*back*) for every $\mathfrak{r}' \in \mathfrak{S}'$ such that $t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \neq \bot$
  and for every $c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}'))$,
  there exists an $\mathfrak{r} \in \mathfrak{S}$ such that $c \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r})$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in Z(c)$

Two states $\mathfrak{s}$ and $\mathfrak{s}'$ are called *weakly t-bisimilar* (notation $\mathfrak{s} \rightsquigarrow_t \mathfrak{s}'$ or $\mathfrak{M}, \mathfrak{s} \rightsquigarrow_t \mathfrak{M}', \mathfrak{s}'$) if there is a weak bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that $\langle \mathfrak{s}, \mathfrak{s}' \rangle$ belongs to $Z(t)$.

We next prove that we have indeed defined a weaker notion than that of a t-bisimulation:

Lemma 3.9 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ two states and $t \in \mathcal{H}$ a truth value ($t \neq \bot$). Then $\mathfrak{M}, \mathfrak{s} \leftrightarrows_t \mathfrak{M}', \mathfrak{s}'$ implies $\mathfrak{M}, \mathfrak{s} \rightsquigarrow_t \mathfrak{M}', \mathfrak{s}'$.*

Proof. Suppose that B is a t-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ such that $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in B$. Define Z as follows:

$$Z(a) = \begin{cases} B & \text{if } a \leq t \\ \emptyset & \text{otherwise} \end{cases}$$

We will prove that Z is a weak bisimulation. For the consistency condition it suffices to prove that $Z(a \vee b) = B$ iff $Z(a) \cap Z(b) = B$:

$$\begin{array}{rll} Z(a \vee b) = B & \text{iff} & a \vee b \leq t \\ & \text{iff} & a \leq t \text{ and } b \leq t \\ & \text{iff} & Z(a) = B \text{ and } Z(b) = B \\ & \text{iff} & Z(a) \cap Z(b) = B \end{array}$$

For the remaining properties, suppose that $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(a)$, for some join irreducible element $a \in \mathcal{H} - \{\bot\}$. From the definition of Z it follows that $a \leq t$, which implies $a \wedge t = a$.

We now prove that Z satisfies the base condition. From the definition of Z, it follows that $\langle \mathfrak{s}, \mathfrak{s}' \rangle$ is also in B; using the base condition for B we obtain that for every $P \in \Phi$ it is $t \wedge v(\mathfrak{s}, P) = t \wedge v'(\mathfrak{s}', P)$. By taking the join of both sides of this equation with a and using the fact that $a \wedge t = a$, we get the base condition for Z.

We finally prove that Z satisfies the forth condition (the back condition can be proved using completely symmetric arguments). Suppose that $a \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$ for some $\mathfrak{r} \in \mathfrak{S}$ and consider any $c \in D_{\mathcal{H}}(a \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))$. Since $a \leq t$, it is also $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$ and by the forth condition for B we conclude that there exists an $\mathfrak{r}' \in \mathfrak{S}'$ such that $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) = t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in B$. Therefore $c \leq a \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \leq t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) = t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$. Moreover $c \leq t$, which implies that $\langle \mathfrak{r}, \mathfrak{r}' \rangle$ is also in $Z(c)$. Therefore, Z satisfies the forth condition.

From the definition of Z, $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t)$, which implies that $\mathfrak{M}, \mathfrak{s} \simeq_t \mathfrak{M}', \mathfrak{s}'$. ∎

The converse of the above lemma does not hold, unless the truth space $\mathcal{H}$ has only two elements; in this case, the $\mathcal{H}$-model is a modal model in the classical sense and it is easy to verify that the two notions of bisimulation coincide with the standard notion of bisimulation in modal logic. The following example demonstrates that **if $\mathcal{H}$ contains at least three elements, the two notions of bisimulation do not coincide**, even in the case that Φ contains a single propositional variable and the $\mathcal{H}$-models have only two states.

Example 3.10 Let $\Phi = \{P\}$, $\mathcal{H}$ be a Heyting algebra with at least three elements and let c be an element in $\mathcal{H}$ different from $\bot$ and $\top$. Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$, where $\mathfrak{S} = \{\mathfrak{s}, \mathfrak{r}\}$, $\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) = c$, $\mathfrak{g}(\mathfrak{r}, \mathfrak{s}) = \mathfrak{g}(\mathfrak{s}, \mathfrak{s}) = \mathfrak{g}(\mathfrak{r}, \mathfrak{r}) = \bot$, $v(\mathfrak{s}, P) = \bot$ and $v(\mathfrak{r}, P) = c$. Moreover, let $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$, where $\mathfrak{S}' = \{\mathfrak{s}', \mathfrak{r}'\}$, $\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') = c$, $\mathfrak{g}'(\mathfrak{r}', \mathfrak{s}') = \mathfrak{g}'(\mathfrak{s}', \mathfrak{s}') = \mathfrak{g}'(\mathfrak{r}', \mathfrak{r}') = \bot$, $v'(\mathfrak{s}', P) = \bot$ and $v'(\mathfrak{r}', P) = \top$. It is easy to check that the following function Z is a weak bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$:

$$Z(a) = \begin{cases} \{\langle \mathfrak{s}, \mathfrak{s}' \rangle, \langle \mathfrak{r}, \mathfrak{r}' \rangle\} & \text{if } a \leq c \\ \{\langle \mathfrak{s}, \mathfrak{s}' \rangle\} & \text{otherwise} \end{cases}$$

Therefore, $\mathfrak{M}, \mathfrak{s} \underline{\leftrightsquigarrow}_\top \mathfrak{M}', \mathfrak{s}'$. Suppose now, for the sake of contradiction, that $\mathfrak{M}, \mathfrak{s} \underline{\leftrightarrow}_\top \mathfrak{M}', \mathfrak{s}'$ and let B be a $\top$-bisimulation such that $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in B$. The forth condition implies that $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in B$ (since the only possible transition from state $\mathfrak{s}'$ is to state $\mathfrak{r}'$). Since $\top \wedge v(\mathfrak{r}, P) \neq \top \wedge v'(\mathfrak{r}', P)$, B violates the base condition (contradiction). Consequently, $\mathfrak{s}$ and $\mathfrak{s}'$ are weakly $\top$-bisimilar but not (strongly) $\top$-bisimilar.

3.2.2 Weak bisimulation and truth invariance

In this subsection, we state and prove the basic truth invariance result; the related result for strong bisimulation follows immediately.

Theorem 3.11 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ two states and $t \in \mathcal{H}$ a truth value ($t \neq \bot$). If $\mathfrak{M}, \mathfrak{s} \underline{\leftrightsquigarrow}_t \mathfrak{M}', \mathfrak{s}'$, then $t \wedge v(\mathfrak{s}, X) = t \wedge v'(\mathfrak{s}', X)$ for every $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$.*

Proof. We first prove that the theorem holds in the case that t is join-irreducible. The proof runs by induction on the formation of X. If $X \in \Phi$, the result follows by the base condition and if $X \in \mathcal{H}$ is a propositional constant it is trivial. If $X = A \vee B$, then

$$\begin{aligned}
t \wedge v(\mathfrak{s}, A \vee B) &= t \wedge \Big(v(\mathfrak{s}, A) \vee v(\mathfrak{s}, B)\Big) && \text{(Def. 2.4)}\\
&= \Big(t \wedge v(\mathfrak{s}, A)\Big) \vee \Big(t \wedge v(\mathfrak{s}, B)\Big) && \text{(distributivity)}\\
&= \Big(t \wedge v'(\mathfrak{s}', A)\Big) \vee \Big(t \wedge v'(\mathfrak{s}', B)\Big) && \text{(inductive hypothesis)}\\
&= t \wedge \Big(v'(\mathfrak{s}', A) \vee v'(\mathfrak{s}', B)\Big) && \text{(distributivity)}\\
&= t \wedge v'(\mathfrak{s}', A \vee B) && \text{(Def. 2.4)}
\end{aligned}$$

If $X = A \wedge B$ the result is obtained similarly, using the idempotency of the meet operation in lattices. If $X = A \supset B$, then

$$\begin{aligned}
t \wedge v(\mathfrak{s}, A \supset B) &= t \wedge \Big(v(\mathfrak{s}, A) \Rightarrow v(\mathfrak{s}, B)\Big) && \text{(Def. 2.4)}\\
&= t \wedge \bigg(\Big(t \wedge v(\mathfrak{s}, A)\Big) \Rightarrow \Big(t \wedge v(\mathfrak{s}, B)\Big)\bigg) && \text{(Lemma 2.2(6))}\\
&= t \wedge \bigg(\Big(t \wedge v'(\mathfrak{s}', A)\Big) \Rightarrow \Big(t \wedge v'(\mathfrak{s}', B)\Big)\bigg) && \text{(inductive hypothesis)}\\
&= t \wedge \Big(v'(\mathfrak{s}', A) \Rightarrow v'(\mathfrak{s}', B)\Big) && \text{(Lemma 2.2(6))}\\
&= t \wedge v'(\mathfrak{s}', A \supset B) && \text{(Def. 2.4)}
\end{aligned}$$

For the case $X = \Box A$, the proof is split into two inequalities. Let $\mathfrak{R} = \{\mathfrak{r} \in \mathfrak{S} \mid t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot\}$. Notice that for any state $\mathfrak{q}$ in $\mathfrak{S} - \mathfrak{R}$, it is $\mathfrak{g}(\mathfrak{s}, \mathfrak{q}) \Rightarrow v(\mathfrak{q}, A) = \bot \Rightarrow v(\mathfrak{q}, A) = \top$ by Lemma 2.2(2), which implies that $\mathfrak{q}$ does not play any role in the computation of $v(\mathfrak{s}, \Box A)$. We also denote by $\mathfrak{R}'$ the set that contains all the states $\mathfrak{r}'$ specified in the definition of the forth condition.

$$
\begin{array}{rcll}
t \wedge v(\mathfrak{s}, \Box A) & = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \Rightarrow v(\mathfrak{r}, A) \Big) & \text{(Def. 2.4)} \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{S}} \Big(t \wedge \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \Rightarrow v(\mathfrak{r}, A) \Big) \Big) & \text{(idempotency)} \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{S}} \Big(t \wedge \Big((t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})) \Rightarrow v(\mathfrak{r}, A) \Big) \Big) & \text{(Lemma 2.2(7))} \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{S}} \Big((t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})) \Rightarrow v(\mathfrak{r}, A) \Big) & \text{(idempotency)} \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{R}} \Big((t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})) \Rightarrow v(\mathfrak{r}, A) \Big) & \text{(Lemma 2.2(2), properties of } \wedge) \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{R}} \Big((\bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} c) \Rightarrow v(\mathfrak{r}, A) \Big) & \text{(Lemma (3.7))} \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{R}} \bigwedge_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} \Big(c \Rightarrow v(\mathfrak{r}, A) \Big) & \text{(Lemma 2.2(5))} \\
& = & t \wedge \bigwedge_{\mathfrak{r} \in \mathfrak{R}} \bigwedge_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} \Big(c \Rightarrow (c \wedge v(\mathfrak{r}, A)) \Big) & \text{(Lemma 2.2(4))} \\
& = & t \wedge \bigwedge_{\mathfrak{r}' \in \mathfrak{R}'} \bigwedge_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} \Big(c \Rightarrow (c \wedge v'(\mathfrak{r}', A)) \Big) & \text{(def. of } \mathfrak{R}', \text{ induct. hypothesis)} \\
& = & t \wedge \bigwedge_{\mathfrak{r}' \in \mathfrak{R}'} \bigwedge_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} \Big(c \Rightarrow v'(\mathfrak{r}', A) \Big) & \text{(Lemma 2.2(4))} \\
& \geq & t \wedge \bigwedge_{\mathfrak{r}' \in \mathfrak{R}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \Rightarrow v'(\mathfrak{r}', A) \Big) & \text{(Lemma 2.2(3), Def. of } \mathfrak{R}') \\
& \geq & t \wedge \bigwedge_{\mathfrak{r}' \in \mathfrak{S}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \Rightarrow v'(\mathfrak{r}', A) \Big) & (\mathfrak{R}' \subseteq \mathfrak{S}') \\
& = & t \wedge v'(\mathfrak{s}', \Box A) & \text{(Def. 2.4)}
\end{array}
$$

The proof of the other inequality ($\leq$) runs in a completely symmetric way by the back condition. For the case of the other modal operator ($X = \Diamond A$), we provide below the argument for the first direction ($\leq$):

$$
\begin{array}{rcll}
t \wedge v(\mathfrak{s}, \Diamond A) & = & t \wedge \bigvee_{\mathfrak{r} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge v(\mathfrak{r}, A)\Big) & \text{(Def. 2.4)} \\
& = & t \wedge \bigvee_{\mathfrak{r} \in \mathfrak{S}} \Big(t \wedge \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge v(\mathfrak{r}, A)\Big)\Big) & \text{(idempotency, distributivity)} \\
& = & t \wedge \bigvee_{\mathfrak{r} \in \mathfrak{S}} \Big((t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})) \wedge v(\mathfrak{r}, A)\Big) & \text{(associativity)} \\
& = & t \wedge \bigvee_{\mathfrak{r} \in \mathfrak{R}} \Big((t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})) \wedge v(\mathfrak{r}, A)\Big) & \text{(properties of } \bot) \\
& = & t \wedge \bigvee_{\mathfrak{r} \in \mathfrak{R}} \Big(\big(\bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} c\big) \wedge v(\mathfrak{r}, A)\Big) & \text{(Lemma (3.7))} \\
& = & t \wedge \bigvee_{\mathfrak{r} \in \mathfrak{R}} \bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))} \Big(c \wedge v(\mathfrak{r}, A)\Big) & \text{(distributivity)} \\
& \leq & t \wedge \bigvee_{\mathfrak{r}' \in \mathfrak{R}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge v'(\mathfrak{r}', A)\Big) & \text{(def. of } \mathfrak{R}', \text{ induct. hypothesis)} \\
& \leq & t \wedge \bigvee_{\mathfrak{r}' \in \mathfrak{S}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge v'(\mathfrak{r}', A)\Big) & (\mathfrak{R}' \subseteq \mathfrak{S}') \\
& = & t \wedge v'(\mathfrak{s}', \Diamond A) & \text{(Def. 2.4)}
\end{array}
$$

The other direction is symmetric and the proof for that case in which t is join-irreducible is complete.

Suppose now that t is not join-irreducible. Then:

$$
\begin{array}{rcll}
t \wedge v(\mathfrak{s}, X) & = & \Big(\bigvee_{c \in D_{\mathcal{H}}(t)} c\Big) \wedge v(\mathfrak{s}, X) & \text{(Lemma (3.7))} \\
& = & \bigvee_{c \in D_{\mathcal{H}}(t)} \Big(c \wedge v(\mathfrak{s}, X)\Big) & \text{(distributivity)} \\
& = & \bigvee_{c \in D_{\mathcal{H}}(t)} \Big(c \wedge v'(\mathfrak{s}', X)\Big) & (c \text{ is join-irreducible}) \\
& = & \Big(\bigvee_{c \in D_{\mathcal{H}}(t)} c\Big) \wedge v'(\mathfrak{s}', X) & \text{(distributivity)} \\
& = & t \wedge v'(\mathfrak{s}', X) & \text{(Lemma (3.7))}
\end{array}
$$

∎

EF-type games for weak-bisimulation We can define a variant of the Ehrenfeucht-Fraissé game for the weak-bisimulation, by modifying the game proposed in the previous subsection for the t-bisimulation. One important difference is that the truth value under consideration changes as the game proceeds; Player I has one more choice to make in each round. Therefore, the current configuration consists of two pointed models together with the current truth value. In each round, in which the current configuration is $(\mathfrak{M}, \mathfrak{s}; \mathfrak{M}', \mathfrak{s}'; t)$

- **Player I:** [**i**] selects one of the $\mathcal{H}$-models, chooses a t-compatible successor $\mathfrak{r}$ of the marked element and moves the marker along the edge (labelled with a_I) to its target. [**ii**] picks a join-irreducible value $c \leq t \wedge a_I$ and sets the current truth value to c.
- **Player II:** responds with a move of the marker in the other $\mathcal{H}$-model along an edge labelled with a_{II} to a state $\mathfrak{r}'$, such that $c \leq a_{II}$ and the labels of the propositional letters in $\mathfrak{r}$ and $\mathfrak{r}'$ are c-equivalent.

The new configuration after this round is $(\mathfrak{M}, \mathfrak{r}; \mathfrak{M}', \mathfrak{r}'; c)$.

The number of rounds can be finite or infinite. Player II loses the game that starts from the configuration $(\mathfrak{M}, \mathfrak{s}_0; \mathfrak{M}', \mathfrak{s}_0'; t)$, if at a certain round cannot respond with an appropriate move or if the initial states $\mathfrak{s}_0$ and $\mathfrak{s}_0'$ are not t-equivalent. Again, we can think that Player I challenges the claim of bisimilarity, which is defended by Player II.

Proposition 3.12 *Let $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v\rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v'\rangle$ be $\mathcal{H}$-models and $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ be two states. The following statements hold:*

1. *Player II has a winning strategy for the infinite game that starts from the configuration $(\mathfrak{M}, \mathfrak{s}_0; \mathfrak{M}', \mathfrak{s}_0'; t)$ iff $\mathfrak{s}_0 \leftrightarroweq_t \mathfrak{s}_0'$.*
2. *If Player II has a winning strategy in the n-round game that starts from the configuration $(\mathfrak{M}, \mathfrak{s}_0; \mathfrak{M}', \mathfrak{s}_0'; t)$, then modal truth is t-invariant for the transition between $\mathfrak{s}_0$ and $\mathfrak{s}_0'$, for every formula up to modal depth n.*

Proof. The first item follows from the definition of weak bisimulation and the corresponding game. The proof of the second item runs by induction on n and is actually a restatement of the proof of Theorem 3.11. ∎

3.2.3 t-image-finite $\mathcal{H}$-models and weak bisimulations

One of the fundamental questions in the bisimulation-based analysis of modal languages, concerns the identification of cases in which the converse of Theorem 3.11 is true. Much obviously, it is not always true: the classical counterexample of two tree models, both with a finite branch for each natural number, one of which possesses an infinite branch, suffices (cf. [BdRV01, Chapter 2.2]). The simplest example of Hennessy-Milner classes of modal models (classes in which modal equivalence is itself a bisimulation relation) is the class of *image-finite* models, in which each state has only a finite number of successors. It is natural to consider a straightforward many-valued analog of this notion by considering $\mathcal{H}$-models in which for each state $\mathfrak{s}$, the set of t-compatible successors of $\mathfrak{s}$ is finite and check whether in this case t-invariance implies t-bisimilarity. Formally, the notion of t-image-finite $\mathcal{H}$-models is defined as follows.

Definition 3.13 (t-image-finite $\mathcal{H}$-models) An $\mathcal{H}$-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ is called *t-image-finite* if for every $\mathfrak{s} \in \mathfrak{S}$, the set $\mathfrak{S}^t_{\mathfrak{s}} = \{\mathfrak{r} \in \mathfrak{S} \mid t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot\}$ is finite.

The following theorem states that for t-image-finite $\mathcal{H}$-models, t-invariance implies t-bisimilarity.

Theorem 3.14 *Let $t \in \mathcal{H}$ be a truth value ($t \neq \bot$), $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be t-image-finite $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ and $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ two states. If modal truth is t-invariant for the transition between $\mathfrak{s}$ and $\mathfrak{s}'$, then $\mathfrak{M}, \mathfrak{s} \leftrightarrows_t \mathfrak{M}', \mathfrak{s}'$.*

Proof. Define the function Z from $\mathcal{H} - \{\bot\}$ to $2^{\mathfrak{S} \times \mathfrak{S}'}$ so that for every $\mathfrak{s} \in \mathfrak{S}$, every $\mathfrak{s}' \in \mathfrak{S}'$ and every $d \in \mathcal{H} - \{\bot\}$, $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(d)$ iff modal truth is d-invariant for the transition between $\mathfrak{s}$ and $\mathfrak{s}'$. We will prove that Z is a weak bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

We first prove that Z satisfies the consistency condition. Suppose that $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t_1 \vee t_2)$. Then for every formula $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $(t_1 \vee t_2) \wedge v(\mathfrak{s}, X) = (t_1 \vee t_2) \wedge v'(\mathfrak{s}', X)$. By taking the meet of both sides of this equation with t_1, we obtain $t_1 \wedge v(\mathfrak{s}, X) = t_1 \wedge v'(\mathfrak{s}', X)$. Therefore, truth is t_1-invariant for the transition between $\mathfrak{s}$ and $\mathfrak{s}'$, which implies $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t_1)$. Similarly we obtain $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t_2)$. Thus, $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t_1) \cap Z(t_2)$.

Conversely, suppose that $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t_1) \cap Z(t_2)$. Then for every formula $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $t_1 \wedge v(\mathfrak{s}, X) = t_1 \wedge v'(\mathfrak{s}', X)$ and $t_2 \wedge v(\mathfrak{s}, X) = t_2 \wedge v'(\mathfrak{s}', X)$, which imply, using the distributivity of $\mathcal{H}$, that $(t_1 \vee t_2) \wedge v(\mathfrak{s}, X) = (t_1 \vee t_2) \wedge v'(\mathfrak{s}', X)$. Therefore, truth is $(t_1 \vee t_2)$-invariant for the transition between $\mathfrak{s}$ and $\mathfrak{s}'$, which implies $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t_1 \vee t_2)$.

Thus, Z satisfies the consistency condition. It is easy to see that the base condition also holds. We next show that Z satisfies the forth condition; the proof for the back condition is completely symmetric.

Suppose for the sake of contradiction that Z does not satisfy the forth condition. This means that there exist a join-irreducible value $d \in I_{\mathcal{H}}$, a pair $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(d)$, a state $\mathfrak{r} \in \mathfrak{S}$ such that $d \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$ and a join-irreducible value $c \in D_{\mathcal{H}}(d \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}))$, such that for every $\mathfrak{r}' \in \mathfrak{S}'$, if $c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$ then $\langle \mathfrak{r}, \mathfrak{r}' \rangle \notin Z(c)$.

Since $\mathfrak{M}'$ is t-image finite and $c \leq t$, the set $\mathfrak{R}' = \{\mathfrak{r}' \in \mathfrak{S}' \mid c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')\}$ is finite. We will show that $\mathfrak{R}'$ is non-empty. We have:

$$c \leq d \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \leq d \wedge \bigvee_{\mathfrak{q} \in \mathfrak{S}} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{q}) \wedge \top\big) = d \wedge v(\mathfrak{s}, \Diamond\top) = d \wedge v'(\mathfrak{s}', \Diamond\top)$$
$$= d \wedge \bigvee_{\mathfrak{r}' \in \mathfrak{S}'} \big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge \top\big) = \bigvee_{\mathfrak{r}' \in \mathfrak{S}'} \big(d \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')\big)$$

By Lemma 2.2(8), there exists some $\mathfrak{r}' \in \mathfrak{S}'$ such that $c \leq d \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$. Thus, $\mathfrak{R}'$ is non-empty.

Suppose that $\mathfrak{R}' = \{\mathfrak{r}'_1, \mathfrak{r}'_2, \ldots, \mathfrak{r}'_k\}$. Then for every i, $1 \leq i \leq k$, $\langle \mathfrak{r}, \mathfrak{r}_i{}' \rangle \notin Z(c)$, which implies that there exists a formula X_i such that $c \wedge v(\mathfrak{r}, X_i) \neq c \wedge v'(\mathfrak{r}_i{}', X_i)$. We will define a new formula Y_i such that $c \wedge v(\mathfrak{r}, Y_i) = c$ and $c \wedge v'(\mathfrak{r}_i{}', Y_i) < c$. Let $a_i = c \wedge v(\mathfrak{r}, X_i)$ and $b_i = c \wedge v'(\mathfrak{r}_i{}', X_i)$. We consider two cases:

Case 1: $a_i \leq b_i$. Define $Y_i = X_i \supset a_i$. Then, $c \wedge v(\mathfrak{r}, Y_i) = c \wedge v(\mathfrak{r}, X_i \supset a_i) = c \wedge (v(\mathfrak{r}, X_i) \Rightarrow a_i) = c \wedge ((c \wedge v(\mathfrak{r}, X_i)) \Rightarrow (c \wedge a_i)) = c \wedge (a_i \Rightarrow a_i) = c$ (the third equality follows from Lemma 2.2(6)). Similarly we obtain $c \wedge v'(\mathfrak{r_i}', Y_i) = c \wedge (b_i \Rightarrow a_i)$. Suppose, for the sake of contradiction, that $c \wedge (b_i \Rightarrow a_i) = c$. Then, $c \leq (b_i \Rightarrow a_i)$, which implies $c \wedge b_i \leq a_i$ (by Lemma 2.2(1)). Since $b_i = c \wedge b_i$, we obtain $a_i = b_i$ (contradiction). Therefore, $c \wedge v'(\mathfrak{r_i}', Y_i) < c$.

Case 2: $a_i \not\leq b_i$. Define $Y_i = a_i \supset X_i$. Then, $c \wedge v(\mathfrak{r}, Y_i) = c \wedge (a_i \Rightarrow a_i) = c$ and $c \wedge v'(\mathfrak{r_i}', Y_i) = c \wedge (a_i \Rightarrow b_i)$. Suppose, for the sake of contradiction, that $c \wedge (a_i \Rightarrow b_i) = c$. Then, $c \leq (a_i \Rightarrow b_i)$, which implies $c \wedge a_i \leq b_i$ (by Lemma 2.2(1)). Since $a_i = c \wedge a_i$, we obtain $a_i \leq b_i$ (contradiction). Therefore, $c \wedge v'(\mathfrak{r_i}', Y_i) < c$.

Let $Y = \bigwedge_{i=1}^{k} Y_i$. It is easy to see that $c \wedge v(\mathfrak{r}, Y) = c$ and $c \wedge v'(\mathfrak{r_i}', Y) < c$, which imply that $c \leq v(\mathfrak{r}, Y)$ and $c \not\leq v'(\mathfrak{r_i}', Y)$, for every i, $1 \leq i \leq k$. Moreover,

$$v(\mathfrak{s}, \Diamond Y) = \bigvee_{\mathfrak{q} \in \mathfrak{S}} \Big(\mathfrak{g}(\mathfrak{s}, \mathfrak{q}) \wedge v(\mathfrak{q}, Y) \Big) \geq \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge v(\mathfrak{r}, Y) \geq c \wedge c = c \qquad \text{(i)}$$

Since $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(d)$, it is $d \wedge v(\mathfrak{s}, \Diamond Y) = d \wedge v'(\mathfrak{s}', \Diamond Y)$, which implies (since it is also $c \leq d$) that $c \wedge v(\mathfrak{s}, \Diamond Y) = c \wedge v'(\mathfrak{s}', \Diamond Y)$. Then, (i) implies that $c \wedge v'(\mathfrak{s}', \Diamond Y) = c$. Therefore:

$$\begin{aligned} c &\leq v'(\mathfrak{s}', \Diamond Y) \\ &= \bigvee_{\mathfrak{r}' \in \mathfrak{S}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge v'(\mathfrak{r}', Y) \Big) \\ &= \bigvee_{\mathfrak{r}' \in \mathfrak{R}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge v'(\mathfrak{r}', Y) \Big) \vee \bigvee_{\mathfrak{r}' \in \mathfrak{S}' - \mathfrak{R}'} \Big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge v'(\mathfrak{r}', Y) \Big) \\ &\leq \bigvee_{\mathfrak{r}' \in \mathfrak{R}'} v'(\mathfrak{r}', Y) \vee \bigvee_{\mathfrak{r}' \in \mathfrak{S}' - \mathfrak{R}'} \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \end{aligned}$$

Since c is a join-irreducible element, Lemma 2.2(8)) implies that either there exists an $\mathfrak{r}' \in \mathfrak{R}'$ such that $c \leq v'(\mathfrak{r}', Y)$ or there exists an $\mathfrak{r}' \in \mathfrak{S}' - \mathfrak{R}'$ such that $c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$, which is a contradiction.

Consequently, Z satisfies the forth condition, and the proof is complete. ■

For t-image-finite models, the converse of Proposition 3.12(*2*) also holds.

Proposition 3.15 *Let $t \in \mathcal{H}$ be a truth value ($t \neq \bot$), $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be t-image-finite $\mathcal{H}$-models and $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ be two states. If modal truth is t-invariant for the transition between $\mathfrak{s}_0$ and $\mathfrak{s}_0'$ for every formula up to modal depth n, then Player II has a winning strategy in the n-round weak bisimulation game that starts from the configuration $(\mathfrak{M}, \mathfrak{s}_0; \mathfrak{M}', \mathfrak{s}_0'; t)$.*

Proof. The proof runs by induction on n and is actually a restatement of the proof of Theorem 3.14. ■

3.2.4 Bisimulation in Models based on Linear Heyting Algebras

In this subsection we consider $\mathcal{H}$-models, in which $\mathcal{H}$ is a Linear Heyting Algergbra (LHA for short), that is, a lattice in which the truth values are totally ordered. In LHAs the definitions of the join, meet and pseudo-complement operations can be simplified as follows:

$$a \vee b = \max(a, b)$$

$$a \wedge b = \min(a, b)$$

$$(a \Rightarrow b) = \begin{cases} \top, & \text{if } a \leq b \\ b, & \text{otherwise.} \end{cases}$$

Moreover, in every LHA, every element different from $\perp$ is join-irreducible: if $c = a \vee b$, then $c = \max(a, b)$, which obviously implies that either $c = a$ or $c = b$. Using the above properties of LHAs, the definitions of t-bisimulation and weak-bisimulation become simpler:

Definition 3.16 (Weak bisimulation for Linear Heyting Algebras) Let $\mathcal{H}$ be a LHA. Given two $\mathcal{H}$-models $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$, a function Z from $\mathcal{H} - \{\perp\}$ to $2^{\mathfrak{S} \times \mathfrak{S}'}$ is a *weak bisimulation between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if it satisfies the following properties*:*

- for every $t_1, t_2 \in \mathcal{H} - \{\perp\}$

  (*consistency*) $t_1 \leq t_2$ implies $Z(t_2) \subseteq Z(t_1)$

- for every value $t \in \mathcal{H}$ $(t \neq \perp)$ and any pair $\langle \mathfrak{s}, \mathfrak{s}' \rangle \in Z(t)$

  (*base*) $\min(t, v(\mathfrak{s}, P)) = \min(t, v'(\mathfrak{s}', P))$, for every $P \in \Phi$

  (*forth*) for every $\mathfrak{r} \in \mathfrak{S}$ such that $\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \perp$ there exists an $\mathfrak{r}' \in \mathfrak{S}'$ such that $\min(t, \mathfrak{g}(\mathfrak{s}, \mathfrak{r})) \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in Z(\min(t, \mathfrak{g}(\mathfrak{s}, \mathfrak{r})))$

  (*back*) for every $\mathfrak{r}' \in \mathfrak{S}'$ such that $\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \neq \perp$ there exists an $\mathfrak{r} \in \mathfrak{S}$ such that $\min(t, \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')) \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r})$ and $\langle \mathfrak{r}, \mathfrak{r}' \rangle \in Z(\min(t, \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')))$

A similar simplified definition can be given for the notion of t-bisimulation. The t-bisimulation games can be formulated in a simple way for the class of languages built on finite linear orders. Assuming further that truth values are colours, linearly ordered, the game can be described in an easy way that provides also an element of fun.

3.3 Characteristic Formulas

Characteristic formulae aim at capturing the bisimulation game, in a purely syntactic fashion [GO07]. So, in the case that the set of **propositional variables** Φ as well as the **truth space** $\mathcal{H}$ are **finite**, *the weak-bisimulation game can be expressed in terms of a modal formula in* $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. More specifically, for a fixed state $\mathfrak{s}$ of a model $\mathfrak{M}$, we can construct a formula $X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}$, which is satisfied in $\mathfrak{s}$, and for every state $\mathfrak{s}'$ of any model $\mathfrak{M}'$, $X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}$ is satisfied in $\mathfrak{s}'$ iff Player II has a winning strategy in the n-round weak-bisimulation game starting from the configuration $(\mathfrak{M},\mathfrak{s};\mathfrak{M}',\mathfrak{s}';t)$. The definition of $X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}$ is actually based on a formula $Y^{n}_{[\mathfrak{M},\mathfrak{s}]}$ with the following property: Player II has a winning strategy in the n-round game starting from $(\mathfrak{M},\mathfrak{s};\mathfrak{M}',\mathfrak{s}';t)$ iff t is a lower bound for the truth value of $Y^{n}_{[\mathfrak{M},\mathfrak{s}]}$ in $\mathfrak{s}'$.

Definition 3.17 Let Φ be a finite set of propositional variables, $\mathcal{H}$ be a finite Heyting algebra, $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v\rangle$ be a $\mathcal{H}$-model for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s}$ a state in $\mathfrak{S}$ and n a non-negative integer. We define the formula $Y^{n}_{[\mathfrak{M},\mathfrak{s}]}$ recursively:

$$Y^{0}_{[\mathfrak{M},\mathfrak{s}]} = \bigwedge_{P\in\Phi} \big((P \supset v(\mathfrak{s},P)) \wedge (v(\mathfrak{s},P) \supset P)\big)$$

$$Y^{n+1}_{[\mathfrak{M},\mathfrak{s}]} = Y^{0}_{[\mathfrak{M},\mathfrak{s}]} \wedge \bigwedge_{\mathfrak{r}\in\mathfrak{S}} \big((\mathfrak{g}(\mathfrak{s},\mathfrak{r}) \supset \Diamond Y^{n}_{[\mathfrak{M},\mathfrak{r}]}\big) \wedge \Box \bigvee_{\mathfrak{r}\in\mathfrak{S}} \big((\mathfrak{g}(\mathfrak{s},\mathfrak{r}) \wedge Y^{n}_{[\mathfrak{M},\mathfrak{r}]}\big)$$

For every $t \in \mathcal{H}$, we define $X^{n,t}_{[\mathfrak{M},\mathfrak{s}]} = t \supset Y^{n}_{[\mathfrak{M},\mathfrak{s}]}$.

Notice that, although the set $\{\mathfrak{r} \mid \mathfrak{g}(\mathfrak{s},\mathfrak{r}) \neq \bot\}$ may be infinite, it is easy to prove (by induction on n) that for every n there exists a set of formulas S_n such that for every $\mathfrak{r} \in \mathfrak{S}$, $Y^{n}_{[\mathfrak{M},\mathfrak{r}]}$ is modally equivalent to a formula in S_n. Thus, although the definition of $Y^{n+1}_{[\mathfrak{M},\mathfrak{s}]}$ involves infinite disjunctions and conjunctions, it is easy to obtain an equivalent finite formula, by replacing each $Y^{n}_{[\mathfrak{M},\mathfrak{r}]}$ with an equivalent formula in S_n and then using the idempotency property of meet and join operations. Therefore, $Y^{n}_{[\mathfrak{M},\mathfrak{s}]}$ is a well defined formula.

The following theorem and corollaries, demonstrate the properties of $Y^{n}_{[\mathfrak{M},\mathfrak{s}]}$ and $X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}$.

Theorem 3.18 *Let Φ be a finite set of propositional variables, $\mathcal{H}$ be a finite Heyting algebra, $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v\rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v'\rangle$ be $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$ two states and $t \in \mathcal{H}$ a truth value ($t \neq \bot$). Then, Player II has a winning strategy in the n-round weak-bisimulation game starting from the configuration $(\mathfrak{M},\mathfrak{s};\mathfrak{M}',\mathfrak{s}';t)$ iff $t \leq v'(\mathfrak{s}', Y^{n}_{[\mathfrak{M},\mathfrak{s}]})$.*

Proof. The proof is by induction on n. For the basis case ($n = 0$), we have

$$v'(\mathfrak{s}', Y^{0}_{[\mathfrak{M},\mathfrak{s}]}) = \bigwedge_{P\in\Phi} \big((v'(\mathfrak{s}',P) \Rightarrow v(\mathfrak{s},P)) \wedge (v(\mathfrak{s},P) \Rightarrow v'(\mathfrak{s}',P))\big) \qquad \text{(ii)}$$

If Player II has a winning strategy in the 0-round game from $(\mathfrak{M},\mathfrak{s};\mathfrak{M}',\mathfrak{s}';t)$, then for every $P \in \Phi$, it is $t \wedge v(\mathfrak{s},P) = t \wedge v'(\mathfrak{s}',P)$. Thus, $t \wedge v(\mathfrak{s},P) \leq v'(\mathfrak{s}',P)$, which implies $t \leq v(\mathfrak{s},P) \Rightarrow v'(\mathfrak{s}',P)$. Similarly we get $t \leq v'(\mathfrak{s}',P) \Rightarrow v(\mathfrak{s},P)$. From (ii) we obtain $t \leq v'(\mathfrak{s}', Y^{0}_{[\mathfrak{M},\mathfrak{s}]})$.

Conversely, if $t \leq v'(\mathfrak{s}', Y^{0}_{[\mathfrak{M},\mathfrak{s}]})$, then (ii) implies that for every $P \in \Phi$, it is $t \leq v(\mathfrak{s},P) \Rightarrow v'(\mathfrak{s}',P)$ and $t \leq v'(\mathfrak{s}',P) \Rightarrow v(\mathfrak{s},P)$. The first inequality implies $t \wedge v(\mathfrak{s},P) \leq$

$v'(\mathfrak{s}', P)$; since it is also $t \wedge v(\mathfrak{s}, P) \leq t$, we obtain $t \wedge v(\mathfrak{s}, P) \leq t \wedge v'(\mathfrak{s}', P)$. Similarly, from the second inequality we obtain $t \wedge v'(\mathfrak{s}', P) \leq t \wedge v(\mathfrak{s}, P)$. Therefore, $t \wedge v(\mathfrak{s}, P) = t \wedge v'(\mathfrak{s}', P)$, which implies that Player II has a winning strategy in the 0-round game from $(\mathfrak{M}, \mathfrak{s}; \mathfrak{M}', \mathfrak{s}'; t)$. This completes the proof for the basis case.

For the inductive step, suppose that the statement holds for n. For the 'only if' direction, suppose that Player II has a winning strategy in the $(n+1)$-round game from $(\mathfrak{M}, \mathfrak{s}; \mathfrak{M}', \mathfrak{s}'; t)$. Using the same arguments as in the basis case, we get

$$t \leq v'(\mathfrak{s}', Y^0_{[\mathfrak{M},\mathfrak{s}]}). \tag{iii}$$

Consider now any state $\mathfrak{r} \in \mathfrak{S}$ such that $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$ and any join-irreducible value $c \leq t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})$. Since Player II has a response in the case that Player I selects the state $\mathfrak{r}$ and the truth value c in his first move, there exists a state $\mathfrak{r}'_c \in \mathfrak{S}'$, such that $c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}'_c)$ and Player II has a winning strategy in the n-round game from $(\mathfrak{M}, \mathfrak{r}; \mathfrak{M}', \mathfrak{r}'_c; c)$. Using the inductive hypothesis, we have $c \leq v'(\mathfrak{r}'_c, Y^n_{[\mathfrak{M},\mathfrak{r}]})$. Therefore,

$$v'(\mathfrak{s}', \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]}) \geq \bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s},\mathfrak{r}))} \big(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}'_c) \wedge v'(\mathfrak{r}'_c, Y^n_{[\mathfrak{M},\mathfrak{r}]})\big) \geq \bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}(\mathfrak{s},\mathfrak{r}))} c = t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})$$

which implies $t \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \Rightarrow v'(\mathfrak{s}', \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]}) = v'(\mathfrak{s}', \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \supset \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]})$. Consequently,

$$t \leq \bigwedge_{\mathfrak{r} \in \mathfrak{S}} v'(\mathfrak{s}', \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \supset \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]}) \tag{iv}$$

Consider now any state $\mathfrak{r}' \in \mathfrak{S}'$ such that $t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \neq \bot$ and any join-irreducible value $c \leq t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$. Since Player II has a response in the case that Player I selects the state $\mathfrak{r}'$ and the truth value c in his first move, there exists a state $\mathfrak{r}_c \in \mathfrak{S}$, such that $c \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r}_c)$ and Player II has a winning strategy in the n-round game from $(\mathfrak{M}, \mathfrak{r}_c; \mathfrak{M}', \mathfrak{r}'; c)$. Using the inductive hypothesis, we have $c \leq v'(\mathfrak{r}', Y^n_{[\mathfrak{M},\mathfrak{r}_c]})$. Therefore,

$$v'(\mathfrak{r}', \bigvee_{\mathfrak{r} \in \mathfrak{S}} (\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge Y^n_{[\mathfrak{M},\mathfrak{r}]})) \geq \bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}'(\mathfrak{s}',\mathfrak{r}'))} \big(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}_c) \wedge v'(\mathfrak{r}', Y^n_{[\mathfrak{M},\mathfrak{r}_c]})\big) \geq \bigvee_{c \in D_{\mathcal{H}}(t \wedge \mathfrak{g}'(\mathfrak{s}',\mathfrak{r}'))} c = t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$$

which implies $t \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \Rightarrow v'(\mathfrak{r}', \bigvee_{\mathfrak{r} \in \mathfrak{S}} (\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge Y^n_{[\mathfrak{M},\mathfrak{r}]}))$. Consequently,

$$t \leq v'(\mathfrak{s}', \Box \bigvee_{\mathfrak{r} \in \mathfrak{S}} (\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge Y^n_{[\mathfrak{M},\mathfrak{r}]})) \tag{v}$$

From (iii), (iv) and (v), we obtain $t \leq v'(\mathfrak{s}', Y^{n+1}_{[\mathfrak{M},\mathfrak{s}]})$. This completes the proof of the 'only if' direction of the inductive step.

For the 'if' direction, suppose that $t \leq v'(\mathfrak{s}', Y^{n+1}_{[\mathfrak{M},\mathfrak{s}]})$. Then,

$$t \leq v'(\mathfrak{s}', Y^0_{[\mathfrak{M},\mathfrak{s}]}) \tag{vi}$$

Moreover, for every state $\mathfrak{r} \in \mathfrak{S}$ it is $t \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \Rightarrow v'(\mathfrak{s}', \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]})$, which implies

$$t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \leq v'(\mathfrak{s}', \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]}) \tag{vii}$$

and for every $\mathfrak{r}' \in \mathfrak{S}'$ it is $t \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \Rightarrow v'(\mathfrak{r}', \bigvee_{\mathfrak{r}\in\mathfrak{S}}(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge Y^n_{[\mathfrak{M},\mathfrak{r}]}))$, which implies

$$t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \leq v'(\mathfrak{r}', \bigvee_{\mathfrak{r}\in\mathfrak{S}}(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge Y^n_{[\mathfrak{M},\mathfrak{r}]})) \tag{viii}$$

Using the same arguments as in the basis case, (vi) implies that $t \wedge v(\mathfrak{s}, P) = t \wedge v'(\mathfrak{s}', P)$, for every $P \in \Phi$, that is, Player II does not lose the game in the initial configuration.

Suppose that Player I moves the marker from $\mathfrak{s}$ to a state $\mathfrak{r}$ such that $t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \neq \bot$ and chooses a join-irreducible value $c \leq t \wedge \mathfrak{g}(\mathfrak{s}, \mathfrak{r})$. Then (vii) implies that

$$c \leq v'(\mathfrak{s}', \Diamond Y^n_{[\mathfrak{M},\mathfrak{r}]}) = \bigvee_{\mathfrak{r}'\in\mathfrak{S}'} \left(\mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \wedge v'(\mathfrak{r}', Y^n_{[\mathfrak{M},\mathfrak{r}]})\right)$$

Since c is a join-irreducible element of $\mathcal{H}$, by Lemma 2.2(8) there exists some $\mathfrak{r}'_c \in \mathfrak{S}'$ such that $c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}'_c) \wedge v'(\mathfrak{r}'_c, Y^n_{[\mathfrak{M},\mathfrak{r}]})$, which implies $c \leq \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}'_c)$ and $c \leq v'(\mathfrak{r}'_c, Y^n_{[\mathfrak{M},\mathfrak{r}]})$. The latter inequality implies, using the inductive hypothesis, that Player II has a winning strategy for the n-round game, from the configuration $(\mathfrak{M}, \mathfrak{r}; \mathfrak{M}', \mathfrak{r}'_c; c)$; thus she also has a winning strategy for the $(n+1)$-round game: she moves the marker from $\mathfrak{s}'$ to $\mathfrak{r}'_c$ and then follows the winning strategy for the n-round game.

On the other hand, suppose that Player I moves the marker from $\mathfrak{s}'$ to a state $\mathfrak{r}'$ such that $t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}') \neq \bot$ and chooses a join-irreducible value $c \leq t \wedge \mathfrak{g}'(\mathfrak{s}', \mathfrak{r}')$. Then (viii) implies that

$$c \leq v'(\mathfrak{r}', \bigvee_{\mathfrak{r}\in\mathfrak{S}}(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge Y^n_{[\mathfrak{M},\mathfrak{r}]})) = \bigvee_{\mathfrak{r}\in\mathfrak{S}} \left(\mathfrak{g}(\mathfrak{s}, \mathfrak{r}) \wedge v'(\mathfrak{r}', Y^n_{[\mathfrak{M},\mathfrak{r}]})\right)$$

Since c is a join-irreducible element of $\mathcal{H}$, there exists some $\mathfrak{r}_c \in \mathfrak{S}$ such that $c \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r}_c) \wedge v'(\mathfrak{r}', Y^n_{[\mathfrak{M},\mathfrak{r}_c]})$, which implies $c \leq \mathfrak{g}(\mathfrak{s}, \mathfrak{r}_c)$ and $c \leq v'(\mathfrak{r}', Y^n_{[\mathfrak{M},\mathfrak{r}_c]})$. The latter inequality implies, using the inductive hypothesis, that Player II has a winning strategy for the n-round game, from the configuration $(\mathfrak{M}, \mathfrak{r}_c; \mathfrak{M}', \mathfrak{r}'; c)$; thus she also has a winning strategy for the $(n+1)$-round game: she moves the marker from $\mathfrak{s}$ to $\mathfrak{r}_c$ and then follows the winning strategy for the n-round game.

This completes the proof of the 'if' direction of the inductive step. ∎

Corollary 3.19 *Let Φ be a finite set of propositional variables, $\mathcal{H}$ be a finite Heyting algebra, $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ be a $\mathcal{H}$-model for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, and $\mathfrak{s} \in \mathfrak{S}$ be a state. Then, $v(\mathfrak{s}, Y^n_{[\mathfrak{M},\mathfrak{s}]}) = \top$. Moreover, for every $t \in \mathcal{H}$, $v(\mathfrak{s}, X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}) = \top$.*

Proof. Suppose that the weak-bisimulation game is played on two copies of $\mathfrak{M}$. Then Player II has a winning strategy for the n-round game from $(\mathfrak{M}, \mathfrak{s}; \mathfrak{M}, \mathfrak{s}; \top)$: she always performs the same move as Player I. Therefore, Theorem 3.18 implies $v(\mathfrak{s}, Y^n_{[\mathfrak{M},\mathfrak{s}]}) = \top$ and Lemma 2.2(2) implies $v(\mathfrak{s}, X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}) = \top$. ∎

Corollary 3.20 *Let Φ be a finite set of propositional variables, $\mathcal{H}$ be a finite Heyting algebra, $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\mathfrak{M}' = \langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ be $\mathcal{H}$-models for $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$, $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s}' \in \mathfrak{S}'$*

two states and $t \in \mathcal{H}$ a truth value ($t \neq \bot$). Then, Player II has a winning strategy in the n-round game for weak-bisimulation from the initial configuration $(\mathfrak{M}, \mathfrak{s}; \mathfrak{M}', \mathfrak{s}'; t)$ *iff* $v'(\mathfrak{s}', X^{n,t}_{[\mathfrak{M},\mathfrak{s}]}) = \top$.

PROOF. It follows from Theorem 3.18 and Lemma 2.2(2). ∎

Remark. In the beginning of this subsection we required that Φ and $\mathcal{H}$ are finite. If Φ is infinite, then $Y^0_{[\mathfrak{M},\mathfrak{s}]}$ is obviously an infinite formula; this implies that $Y^n_{[\mathfrak{M},\mathfrak{s}]}$ is also infinite for every n. If Φ is finite and $\mathcal{H}$ is infinite, then $Y^0_{[\mathfrak{M},\mathfrak{s}]}$ remains finite (since only finitely many truth values are used locally in $\mathfrak{s}$), but it is possible that $Y^n_{[\mathfrak{M},\mathfrak{s}]}$ is infinite. However, if in addition $\mathfrak{M}$ is image-finite, then $Y^n_{[\mathfrak{M},\mathfrak{s}]}$ is finite for every n. Thus, in defining characteristic formulas, we may require image-finite models instead of finite Heyting algebras.

At this point, it should be noted that it is not possible to construct characteristic formulas for the t-bisimulation game, even for Heyting algebras with only three elements and for models with only two states. The reason for this is that modal equivalence does not imply t-bisimilarity. Therefore, it is possible that Player II does not have a winning strategy in the t-bisimulation game, but no modal formula in $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ can express this fact. This scenario is demonstrated in the following example.

Example 3.21 Consider again the two models $\mathfrak{M}$ and $\mathfrak{M}'$ of Example 3.10. Suppose that $\bot, c, \top$ are the only elements of $\mathcal{H}$ (that is, $\mathcal{H}$ is a LHA with $\bot \leq c \leq \top$). We can prove by induction on the formation of X, that $v(\mathfrak{r}', X) = \bot$ iff $v'(\mathfrak{r}', X) = \bot$, for every $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. Then, using the above fact, we can prove by a similar induction that $v(\mathfrak{s}', X) = v'(\mathfrak{s}', X)$, for every $X \in L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$.

We have shown in Example 3.10 that $\mathfrak{s}$ and $\mathfrak{s}'$ are not $\top$-bisimilar, which implies that Player II does not have a winning strategy for the $\top$-bisimulation game played on $\mathfrak{M}, \mathfrak{s}$ and $\mathfrak{M}', \mathfrak{s}'$. On the other hand, Player II has a winning strategy for the $\top$-bisimulation game played on two copies of $\mathfrak{M}, \mathfrak{s}$. Since $\mathfrak{s}$ and $\mathfrak{s}'$ are modally equivalent, we conclude that no formula in $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$ can capture the $\top$-bisimulation game.

4 Multiple-Expert Semantics

The perspective on Heyting-valued modal logics exposed in this section, is presented in full detail in [Fit92, Sect. 1, 3 & 5]. We briefly review below this facet of Fitting's many-valued modal languages and interpret our definitions and results in this alternative context. Note that, in the very recent expository paper [Fit09], this perspective ('*to identify the truth value of a formula with the set of those agents who say the formula is true*') is coherently presented, although basically at the level of Boolean algebras.

In this section, we confine ourselves in the class of many-valued modal languages built on *finite* HAs. Each language of this class can be reformulated in a way that is of interest to KR situations involving many interrelated experts. We note, however, that our results carried out in the previous sections hold in the broader class of complete Heyting algebras.

A *multiple-expert modal model* is a structure $\langle \mathcal{E}, \mathfrak{S}, \{R_e\}_{e\in\mathcal{E}}, \{v_e\}_{e\in\mathcal{E}}, D\rangle$, such that:

- $\mathcal{E}$ is a finite set of experts.
- D is a partial-order dominance relation on $\mathcal{E}$.
- $\mathfrak{S}$ is a common set of worlds.
- For each $e \in \mathcal{E}$, $\langle \mathfrak{S}, R_e, v_e\rangle$ is a (two-valued) modal model, such that
  (D_1) if $R_e(\mathfrak{s}_1, \mathfrak{s}_2)$ and $D(e, f)$, then $R_f(\mathfrak{s}_1, \mathfrak{s}_2)$, and
  (D_2) for any propositional variable P, if $v_e(\mathfrak{s}, P)$ and $D(e, f)$, then $v_f(\mathfrak{s}, P)$.

The valuations v_e are then properly extended to all modal formulae, so that (D_2) above is preserved.

We are interested now in finding the experts' 'consensus', that is, in elegantly calculating the modal formulae on which our experts agree. This problem can be reformulated as one involving a many-valued language, where sets of experts who agree on the truth of an epistemic statement can be seen as generalized truth values. Note however an important point: by (D_1) and (D_2), **not every set of experts is an 'admissible' generalized truth value**. The 'admissible' sets of experts are those which are dominance-closed, that is, upwards-closed in the order D. The set of all admissible sets of experts form a finite Heyting algebra $\mathcal{H}$ when ordered under set inclusion. We can thus produce an $\mathcal{H}$-model $\langle \mathfrak{S}, \mathfrak{g}, v\rangle$ as follows:

- For $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$, $\mathfrak{g}(\mathfrak{s}, \mathfrak{t}) := \{e \in \mathcal{E} : \mathfrak{s} R_e \mathfrak{t}\}$.
- For a propositional variable P, $v(\mathfrak{s}, P) := \{e \in \mathcal{E} : v_e(\mathfrak{s}, P) = 1\}$.

It can then be proved that for any modal formulae X,

$$v(\mathfrak{s}, X) = \{e \in \mathcal{E} : v_e(\mathfrak{s}, X) = 1\}.$$

We have thus provided a translation of the multiple-expert situation into a many-valued modal model of the language $L^{\mathcal{H}}_{\Box\Diamond}(\Phi)$. The other translation is also feasible. Both translations are presented in [Fit92, Sect. 5].

We are now in the position to express the meaning of our results in this alternative setting. It suffices to observe that in the finite Heyting algebra $\mathcal{H}$ of the 'admissible' subsets of experts, meet is set intersection and join is set union. Moreover, a join irreducible element c is the set that consists of an expert e_c and all the experts that dominate e_c; in other words, it is the minimum admissible set that contains e_c. Assume that Z is a weak bisimulation between two $\mathcal{H}$-models $\langle \mathfrak{S}, \mathfrak{g}, v\rangle$ and $\langle \mathfrak{S}', \mathfrak{g}', v'\rangle$ and let $\mathfrak{s} \in \mathfrak{S}$, $\mathfrak{s}' \in \mathfrak{S}'$ be two states. If c is a join-irreducible value then $Z(c)$ represents the bisimulation relation "from the expert's e_c point of view". The consistency property guarantees that for an arbitrary set of experts t, the pair of states $\langle \mathfrak{s}, \mathfrak{s}'\rangle$ belongs to $Z(t)$ iff every expert in t thinks that these states are related by Z. Therefore, in the case of weak bisimulation we have a sharp description:

- If c is a join-irreducible value, then the states $\mathfrak{s}$ and $\mathfrak{s}'$ are weakly c-bisimilar iff the states $\mathfrak{s}$ and $\mathfrak{s}'$ are bisimilar in the models $\langle \mathfrak{S}, R_{e_c}, v_{e_c} \rangle$ and $\langle \mathfrak{S}', R_{e_c}, v_{e_c} \rangle$.

- The states $\mathfrak{s}$ and $\mathfrak{s}'$ are weakly t-bisimilar iff for every expert e in t the states $\mathfrak{s}$ and $\mathfrak{s}'$ in the corresponding models $\langle \mathfrak{S}, R_e, v_e \rangle$ and $\langle \mathfrak{S}', R_e, v_e \rangle$ are bisimilar.

As one might expect from Example 3.10, an analogous statement *does not* hold for the notion of t-bisimulation, as the example below shows.

Example 4.1 *Let $\mathcal{E} = \{e, f\}$ be the set of experts, $D = \emptyset$, $\mathfrak{S} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3\}$ and $\mathfrak{S}' = \{\mathfrak{s}_4, \mathfrak{s}_5\}$ two sets of states, P the unique propositional variable, and:*

- *$R_e(\mathfrak{s}_1, \mathfrak{s}_2)$ and $R_e(\mathfrak{s}_4, \mathfrak{s}_5)$ hold, and R_e fails for any other pair of states,*

- *$R_f(\mathfrak{s}_1, \mathfrak{s}_3)$ and $R_f(\mathfrak{s}_4, \mathfrak{s}_5)$ hold, and R_f fails for any other pair of states,*

- *$v_e(\mathfrak{s}_2, P) = v_e(\mathfrak{s}_5, P) = 1$ and $v_e(x, P) = 0$ for every $x \in \{\mathfrak{s}_1, \mathfrak{s}_3, \mathfrak{s}_4\}$,*

- *$v_f(\mathfrak{s}_3, P) = v_f(\mathfrak{s}_5, P) = 1$ and $v_f(x, P) = 0$ for every $x \in \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_4\}$.*

Then, for the expert e the modal models $\langle \mathfrak{S}, R_e, v_e \rangle$ and $\langle \mathfrak{S}', R_e, v_e \rangle$ are bisimilar, for the expert f the modal models $\langle \mathfrak{S}, R_f, v_f \rangle$ and $\langle \mathfrak{S}', R_f, v_f \rangle$ are bisimilar, but the corresponding $\mathcal{H}$-models $\langle \mathfrak{S}, \mathfrak{g}, v \rangle$ and $\langle \mathfrak{S}', \mathfrak{g}', v' \rangle$ are not $\{e, f\}$-bisimilar.

We can, however, interpret the conditions of the t-bisimulation in the multiple-expert scenario as follows:

- The *base* condition of Def. 3.2 says that moving back and forth between t-bisimilar states does not affect the belief of any expert from the set t, for any propositional letter P.

- The *forth* condition of Def. 3.2 says that any transition in the first model that involves experts from the fixed set t can be matched with a transition in the second model where all the relevant experts from t are also involved; more experts can also be involved; what we require concerns only those in t.

- Similarly for the *back* condition.

Finally, the meaning of Theorems 3.3 and 3.11 is that the bisimulation relation between states of models guarantees the invariance of the epistemic consensus of some experts from a predefined fixed set t. It is also easy to give an equivalent definition of the EF-type games of bisimulation, in terms of the epistemic agreement of the experts.

5 Conclusions - Future Work

In this paper, we have examined the notion of bisimulation for Fitting's Heyting-valued modal languages and attempted to establish its basic facts. As for future work: there exists a matrix-based approach to bisimulations developed in [Fit03], where a computational test is presented for examining frame and model bisimulations. It is noted in [Fit09] that our weak bisimulations from Section 3.2 seem to be the Heyting algebra counterpart of the Boolean-based bisimulations from [Fit03]. So, the most immediate direction of future research is to establish the exact relationship between our approach and the one in [Fit03] and try to transfer the matrix methods of [Fit03] to the Heyting algebra case.

In general, with respect to the logical content of bisimulation techniques, perhaps the most interesting question in the broad area of multiple-valued modal logic is the identification of *bisimilarity notions in bilattice-valued modal languages* (see [Fit06] for an exposition and [Fit09] for some basic questions). From the Computer Science applications perspective, it seems that the area of distributed systems might provide a field of applications for this family of logics; the consensus of many interacting agents is a premium notion there and it is intensively investigated.

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