Infinite Dimensional Analysis, Quantum Probability and Related Topics
Vol. 1, No. 2 (1998) 201–220

NOTIONS OF INDEPENDENCE RELATED TO THE FREE GROUP

LUIGI ACCARDI,* YUKIHIRO HASHIMOTO and NOBUAKI OBATA
Graduate School of Polymathematics, Nagoya University,
Nagoya 464-8602, Japan

Received 4 November 1997

The central limit problem for algebraic probability spaces associated with the Haagerup states on the free group with countably many generators leads to a new form of statistical independence in which the singleton condition is not satisfied. This circumstance allows us to obtain nonsymmetric distributions from the central limit theorems deduced from this notion of independence. In the particular case of the Haagerup states, the role of the Gaussian law is played by the Ullman distribution. The limit process is explicitly realized on the finite temperature Boltzmannian Fock space. The role of entangled ergodic theorems in the proof of the central limit theorems is discussed.

1. Introduction

The notion of *statistical independence* is basic for both classical and quantum probability and many papers in the past years have been devoted to the study of various aspects of this notion. As usual, in quantum probability there are several inequivalent ways to generalize the classical statistical independence to a noncommutative context, see e.g., Refs. 16 and 19. In this paper we shall not survey these different approaches, but rather adopt a pragmatic point of view that *a good notion of statistical independence is one that allows nontrivial central limit theorems.*

In recent years the free group F_∞ with countably many generators $\{g_j : j \in \mathbf{N}\}$ has played an important role in the enrichment of the notions of statistical independence with the introduction of the notion of *free independence* due to Voiculescu.[21] This group shall also be the starting point of this paper. Let $\mathcal{A}$ denote the group $*$-algebra of F_∞, i.e. the complex polynomial algebra with identity generated by $\{g_j, g_j^{-1}\,;\, j \in \mathbf{N}\}$ where the involution is defined by $g_j^* = g_j^{-1}$. In this paper we will use the following notations: for $\varepsilon = \pm 1$ and $j \in \mathbf{N}$ we put

$$\alpha = (j, \varepsilon)\,, \qquad \alpha^* = (j, -\varepsilon)\,, \qquad g_\alpha = g_j^\varepsilon\,. \tag{1.1}$$

A product

$$x = g_{\alpha_1} \cdots g_{\alpha_k}\,, \qquad k \geq 1\,, \tag{1.2}$$

*Also at Centro Vito Volterra, Università di Roma "Tor Vergata", Roma 00133, Italia.

is called a *reduced word* if

$$\alpha_i \neq \alpha_{i+1}^* \,. \tag{1.3}$$

In that case k is called the *length* of x and denoted by $|x|$. The identity has length zero by definition: $|e| = 0$. Following Figà-Talamanca and Picardello[9] (see also Chiswell[7] and Lyndon[18]), a state φ on $\mathcal{A}$ is called a *Haagerup state* if

(i) $\varphi(e) = 1$ and $|\varphi(g_j)| \leq 1$ for all g_j;
(ii) $\varphi(g_j^{-1}) = \overline{\varphi(g_j)}$;
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$ for any $x, y \in F_\infty$ with $|xy| = |x| + |y|$.

Examples of Haagerup states are given by the one-parameter family of states φ_γ, $0 \leq \gamma \leq 1$, defined by

$$\varphi_\gamma(x) = \gamma^{|x|}\,, \qquad x \in F_\infty \,. \tag{1.4}$$

Throughout this paper we understand tacitly that $0^0 = 1$, hence φ_0 is the tracial state on $\mathcal{A}$ defined by:

$$\varphi_0(x) = \begin{cases} 1\,, & \text{if} \quad x = e, \\ 0, & \text{if} \quad x \in F_\infty, x \neq e. \end{cases} \tag{1.5}$$

Voiculescu proved a central limit theorem showing that the random variables

$$Q_N = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (g_j + g_j^{-1}) \tag{1.6}$$

converge to a semi-circle random variable. Realizing that the deep reason for this convergence is to be found in a statistical property of the generators $g^{\pm 1}$ with respect to the tracial state φ_0, he abstracted a new general notion of statistical independence called *free independence* and proved a general free central limit theorem with convergence to semi-circle random vaiables. These results established the role of the semi-circle law for free independence as an analog of the Gaussian for classical independence. By means of the quantum probabilistic technique originated by Giri and von Waldenfels,[10] Speicher[20] extended the Voiculescu central limit theorem to the fully quantum case where one considers separately the two sums

$$a_N^+ = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j \,, \qquad a_N^- = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j^{-1} \,, \tag{1.7}$$

and one looks for the limit of the *mixed momenta:*

$$\lim_{N\to\infty} \varphi_0(a_N^{\varepsilon_1} \cdots a_N^{\varepsilon_k})\,, \qquad k \in \mathbf{N}\,, \quad \varepsilon_1, \ldots, \varepsilon_k \in \{\pm\}\,, \tag{1.8}$$

which clearly contains much more information than the limit (1.6).

On the other hand, by a direct, combinatorial and analytical method, Hashimoto[12] proved that the limit of the rescaled centered expectation values

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}} \left((Q_N - \overline{Q}_N)^k\right)$$

with respect to the Haagerup state exists for any natural integer k and for $0 \leq \lambda \leq 1$, where $\overline{Q}_N$ is the mean value of Q_N with respect to $\varphi_{\lambda/\sqrt{N}}$. Moreover, the limit is the kth moment of an absolutely continuous probability measure whose density

$$u_\lambda(t) = \frac{1}{2\pi} \chi_{[-2-\lambda, 2-\lambda]}(t) \, \frac{\sqrt{(2+\lambda+t)(2-\lambda-t)}}{1-\lambda t} \tag{1.9}$$

belongs to the Ullman family of probability measures which was introduced in connection with variational problems of potential theory. In fact, as shown by Hiai and Petz,[13] the distributions in this family are characterized, among the measures supported in the full positive half-line, by the property of maximizing, under constraints on moments, the logarithmic energy which, as recently shown by Voiculescu[22] (for related results see Ben Arous and Guionnet[3]) can naturally be interpreted as a free entropy. Beyond potential theory the Ullman distribution also emerged naturally in quantum probability and in physics: Bozejko, Leinert and Speicher[6] obtained, from a limit theorem on quantum convolutions of classical measures, a measure different from the Ullman distribution (1.9) only for an additional atomic part. In the half-planar approximation to the large-N limit of quantum chromodynamics, an explicit formula for the n-point Green functions (correlations of the field operators) was derived (formula (3.23) of Ref. 2). The Ullman distribution appears from this formula by restricting ourselves to: (i) zero space–time dimensions, i.e. the space is reduced to a single point; (ii) the interaction Hamiltonian V_{int} is linear in the field operator.

The emergence of the Ullman distribution in many different contexts naturally raises the following problems:

Problem 1. Just as Voiculescu showed that *free independence* (i.e. φ_0-independence) underlies the Wigner semi-circle law, can one show that there is some kind of φ_γ-independence underlying the Ullman distribution? If such a notion exists, it cannot be the free independence because one easily verifies (cf. Sec. 4 below) that the subalgebras $\mathcal{A}_j$ are not free with respect to the Haagerup state φ_γ if $\gamma \neq 0$.

Problem 2. Is there a fully quantum central limit theorem extending the Hashimoto result in the same sense as Speicher extended the Voiculescu result? In other words, defining a_N and a_N^+ by (1.7) and $\tilde{a}_N, \tilde{a}_N^+$ by their centered versions, we ask the existence of the limit:

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\varepsilon_1} \cdots \tilde{a}_N^{\varepsilon_k})\,, \qquad k \in \mathbf{N}\,, \quad \varepsilon_1, \ldots, \varepsilon_k \in \{\pm\}\,. \tag{1.10}$$

Problem 3. How can one describe explicitly the GNS space of the limit? More precisely, from the reconstruction theorem[1] we know that, if the limit (1.10) exists, then there exist an algebraic probability space $\{\mathcal{A}_\lambda, \psi_\lambda\}$ and two random variables a_λ, a_λ^+, such that

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\varepsilon_1}\cdots\tilde{a}_N^{\varepsilon_k}) = \psi_\lambda(a_\lambda^{\varepsilon_1}\cdots a_\lambda^{\varepsilon_k}) = \langle \Phi_\lambda, a_\lambda^{\varepsilon_1}\cdots a_\lambda^{\varepsilon_k}\Phi_\lambda\rangle\,, \tag{1.11}$$

where $\{\mathcal{H}_\lambda, \Phi_\lambda\}$ is the GNS space of $\{\mathcal{A}_\lambda, \psi_\lambda\}$ and a_λ, a_λ^+ are identified with operators on $\mathcal{H}_\lambda$. The problem is to realize $\{\mathcal{H}_\lambda,\ a_\lambda,\ a_\lambda^+, \Phi_\lambda\}$ concretely, for example, on some tensor algebra.

Problem 4. Can we abstract from this φ_γ-independence a general algebraic notion of independence which still guarantees the validity of a central limit theorem?

In this paper we solve the above problems and show that the notion of independence, underlying the central limit theorems we are going to prove, has a peculiar feature with respect to all the notions of independence considered up to now in the proofs of the central limit theorems by the method of momenta: it does not satisfy the *singleton condition*, see Sec. 2 below. This fact has a deep implication: it accounts for the appearance of *non-symmetric* limit laws such as the Ullman distribution arising from some central limit theorems. The solution of Problem 3 (see also Sec. 6) shows that the limit process a_λ, a_λ^+ can be realized not on the usual Boltzmannian Fock space but on its *finite temperature* analog, which was introduced by Fagnola[8] to establish the free Lévy martingale representation theorem. *A posteriori* this is clear since, before the limit, the generators g_j, g_j^{-1} are completely symmetric with respect to any Haagerup states φ_γ, while a and a^+ are not symmetric in the Boltzmannian Fock representation. Finally, in Sec. 7 we abstract the notion of *singleton independence* which allows the presence of singletons, and prove a corresponding central limit theorem.

The results obtained in this paper naturally suggest the conjecture that, underlying any central limit theorem arising in the harmonic analysis on discrete groups or, more generally, on graphs (cf. Ref. 14 for the latest developements in this direction), there should be an appropriate notion of *independence* or of *weak dependence* and the corresponding fully quantum central limit theorem in the sense described above. This conjecture is supported by several examples and more detailed analysis shall appear elsewhere.

2. The Singleton Condition

In Secs. 2 and 3 we discuss the role of the singleton condition in the proof of the central limit theorems and we observe why this condition leads to *symmetric* random variables. Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{C}$ a C^*-algebra with norm $|\cdot|$, and $E : \mathcal{A} \to \mathcal{C}$ a real linear map.

Definition 2.1. Assume that we are given a finite or countably infinite set of sequences

$$(b_n^{(1)})_{n=1}^\infty\,,\ (b_n^{(2)})_{n=1}^\infty, \ldots$$

of elements in $\mathcal{A}$ with mean $E(b_n^{(j)}) = 0$. We say that the set of sequences satisfies the *singleton condition* with respect to E if for any choice of $j \geq 1$, $j_1, \ldots, j_k \in \mathbf{N}$, and $n_1, \ldots, n_k \in \mathbf{N}$

$$E\left(b_{n_1}^{(j_1)}\cdots b_{n_k}^{(j_k)}\right) = 0 \tag{2.1}$$

holds whenever there exists an index n_s which is different from all others, i.e. such that $n_s \neq n_t$ for $s \neq t$.

The condition of $E(b_n^{(j)}) = 0$ is, in fact, a consequence of (2.1) and hence redundant in the above definition. We put it just for clarity. The singleton condition is equivalent to the usual independence in the classical case and follows from the free independence.

The role of the upper suffices of $(b_n^{(j)})$ is illustrated by a typical situation in many concrete central limit theorems. Let $\{\mathcal{A}, \varphi, \mathcal{B}, (j_n)\}$ be an algebraic stochastic process with state algebra $\mathcal{B}$. Consider the case where $\mathcal{C} = \mathbf{C}$ and $E = \varphi$. Let $\{b^{(1)}, b^{(2)}, \ldots\} \subset \mathcal{B}$ be a set of algebraic generators and for each $j = 1, 2, \ldots$ and $n \in \mathbf{N}$ define

$$b_n^{(j)} = j_n(b^{(j)}) - \varphi(j_n(b^{(j)}))1_{\mathcal{A}}\,.$$

Then $(b_n^{(1)}), (b_n^{(2)}), \ldots$ are sequences of elements in $\mathcal{A}$ with mean zero.

Definition 2.2. We say that sequences $(b_n^{(1)}), (b_n^{(2)}), \ldots$ of elements of $\mathcal{A}$ satisfy the condition of *boundedness of the mixed momenta* if for each $k \in \mathbf{N}$ there exists a positive constant $\nu_k \geq 0$ such that

$$\left|E\left(b_{n_1}^{(j_1)}\cdots b_{n_k}^{(j_k)}\right)\right| \leq \nu_k \tag{2.2}$$

for any choice of $n_1, \ldots, n_k$ and $j_1, \ldots, j_k$.

Given a sequence $b = (b_n)_{n=0}^\infty \subset \mathcal{A}$, we put

$$S_N(b) = \sum_{n=1}^N b_n\,. \tag{2.3}$$

Lemma 2.3. *Let $(b_n^{(1)}), (b_n^{(2)}), \ldots$ be sequences of elements of $\mathcal{A}$ satisfying the condition of boundedness of the mixed momenta. Then, for any $\alpha > 0$ it holds that*

$$\lim_{N\to\infty} E\left(\frac{S_N(b^{(1)})}{N^\alpha}\cdot\frac{S_N(b^{(2)})}{N^\alpha}\cdots\frac{S_N(b^{(k)})}{N^\alpha}\right)$$
$$= \lim_{N\to\infty} N^{-\alpha k} \sum_{\alpha k \leq p \leq k}\ \sum_{\substack{\pi:\{1,\ldots,k\}\to\{1,\ldots,p\}\\ \text{surjective}}}\ \sum_{\substack{\sigma:\{1,\ldots,p\}\to\{1,\ldots,N\}\\ \text{order-preserving}}} E\left(b_{\sigma\circ\pi(1)}^{(1)}\cdots b_{\sigma\circ\pi(k)}^{(k)}\right), \tag{2.4}$$

in the sense that one limit exists if and only if the other does and in this case they are equal. (The limit is understood in the sense of norm convergence in $\mathcal{C}$.)

Proof. Expanding the product explicitly by means of (2.3), we obtain

$$E\left(\frac{S_N(b^{(1)})}{N^\alpha}\cdot\frac{S_N(b^{(2)})}{N^\alpha}\cdots\frac{S_N(b^{(k)})}{N^\alpha}\right)=N^{-\alpha k}\sum_{j_1,\dots,j_k=1}^{N}E\left(b^{(1)}_{j_1}\cdots b^{(k)}_{j_k}\right). \tag{2.5}$$

Note that the sum may be taken over all mappings $j:\{1,\dots,k\}\to\{1,\dots,N\}$. We shall split the sum according to the cardinality of the range of j. Suppose that j has a range of p elements, $1\le p\le k$. Then there exist a unique surjective map $\pi:\{1,\dots,k\}\to\{1,\dots,p\}$ and a unique order-preserving map $\sigma:\{1,\dots,p\}\to\{1,\dots,N\}$ such that $j=\sigma\circ\pi$. Then (2.5) becomes

$$N^{-\alpha k}\sum_{1\le p\le k}\ \sum_{\substack{\pi:\{1,\dots,k\}\to\{1,\dots,p\}\\ \text{surjective}}}\ \sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order-preserving}}}E\left(b^{(1)}_{\sigma\circ\pi(1)}\cdots b^{(k)}_{\sigma\circ\pi(k)}\right). \tag{2.6}$$

For the assertion (2.4) it is sufficient to show that, whenever $p<\alpha k$, one has

$$\lim_{N\to\infty}N^{-\alpha k}\sum_{\substack{\pi:\{1,\dots,k\}\to\{1,\dots,p\}\\ \text{surjective}}}\ \sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order-preserving}}}E\left(b^{(1)}_{\sigma\circ\pi(1)}\cdots b^{(k)}_{\sigma\circ\pi(k)}\right)=0. \tag{2.7}$$

Let $C_{k,p}$ be the number of surjective maps $\{1,\dots,k\}\to\{1,\dots,p\}$ and $I_N(p)$ the number of order preserving maps $\{1,\dots,p\}\to\{1,\dots,N\}$. Then, it follows from (2.2) that

$$\begin{aligned}&\left|N^{-\alpha k}\sum_{\substack{\pi:\{1,\dots,k\}\to\{1,\dots,p\}\\ \text{surjective}}}\ \sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order-preserving}}}E\left(b^{(1)}_{\sigma\circ\pi(1)}\cdots b^{(k)}_{\sigma\circ\pi(k)}\right)\right|\\ &\qquad\le N^{-\alpha k}\nu_k C_{k,p}I_N(p).\end{aligned} \tag{2.8}$$

Then (2.7) follows from the obvious identity:

$$\lim_{N\to\infty}N^{-p}I_N(p)=\lim_{N\to\infty}N^{-p}\binom{N}{p}=\frac{1}{p!}.$$

□

Lemma 2.4. *Notations and assumptions being as in Lemma 2.3, assume that the sequences $(b^{(j)}_n)$ satisfies the singleton condition with respect to E. Then the limit*

$$\lim_{N\to\infty}E\left(\frac{S_N(b^{(1)})}{N^\alpha}\cdot\frac{S_N(b^{(2)})}{N^\alpha}\cdots\frac{S_N(b^{(k)})}{N^\alpha}\right)=0 \tag{2.9}$$

takes place if $\alpha>1/2$ or if $\alpha=1/2$ and k is odd. If $\alpha=1/2$ and $k=2n$, the left-hand side of (2.9) is equal to the limit

$$\lim_{N\to\infty}N^{-n}\sum_{\substack{\pi:\{1,\dots,2n\}\to\{1,\dots,n\}\\ 2-1\text{ map}}}\ \sum_{\substack{\sigma:\{1,\dots,n\}\to\{1,\dots,N\}\\ \text{order-preserving}}}E\left(b^{(1)}_{\sigma\circ\pi(1)}\cdots b^{(2n)}_{\sigma\circ\pi(2n)}\right). \tag{2.10}$$

Moreover, the following Gaussian bound takes place:

$$\limsup_{N\to\infty}\left|E\left(\frac{S_N(b^{(1)})}{N^{1/2}}\cdot\frac{S_N(b^{(2)})}{N^{1/2}}\cdots\frac{S_N(b^{(2n)})}{N^{1/2}}\right)\right|\le\frac{(2n)!}{2^n n!}\nu_{2n}. \tag{2.11}$$

Proof. We use the same notation as in the proof of Lemma 2.3. For each surjective map $\pi:\{1,\dots,k\}\to\{1,\dots,p\}$ put $S_j=\pi^{-1}(j)$, $1\le j\le p$. If $|S_j|=1$ for some j,

$$\sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order-preserving}}}E\left(b^{(1)}_{\sigma\circ\pi(1)}\cdots b^{(k)}_{\sigma\circ\pi(k)}\right)=0$$

by the singleton condition. Suppose that $|S_j|\ge 2$ for all j. Then

$$k=\sum_{j=1}^{p}|S_j|\ge 2p. \tag{2.12}$$

This condition is incompatible with $p\ge\alpha k$ if $\alpha>1/2$ or if $\alpha=1/2$ and k is odd. Therefore, in view of the estimate (2.8), the left-hand side of (2.9) is equal to zero.

Suppose that $\alpha=1/2$ and k is even, say, $k=2n$. Then the limit on the left-hand side of (2.5) exists if and only if the limit on the right-hand side exists and the right-hand side is reduced to (2.10). Finally, (2.10) is dominated in norm by

$$\lim_{N\to\infty}N^{-n}\nu_{2n}\left|\{\pi:\{1,\dots,n\}\to\{1,\dots,2n\},\ 2\text{–}1\text{ map}\}\right|I_N(n)=\frac{(2n)!}{2^n n!}\nu_{2n}, \tag{2.13}$$

as desired. □

3. Entangled Ergodic Theorems

Here we give a sufficient condition for the existence of the limit (2.6) in terms of *entangled ergodic theorems*. In order to clarify the connections between CLT and this notion it is convenient to introduce the following

Definition 3.1. Let $(S_1,\dots,S_p)$ be a partition of $\{1,\dots,k\}$ and put

$$\underline{s}_j=\min\{s\in S_j\},\qquad \overline{s}_j=\max\{s\in S_j\}.$$

Then S_j is called *non-crossing* if for any $h=1,\dots,p$,

$$(\underline{s}_j,\overline{s}_j)\cap(\underline{s}_h,\overline{s}_h)\ne\emptyset\Leftrightarrow(\underline{s}_j,\overline{s}_j)\subseteq(\underline{s}_h,\overline{s}_h)\quad\text{or}\quad(\underline{s}_h,\overline{s}_h)\subseteq(\underline{s}_j,\overline{s}_j).$$

The set S_j is said to belong to the *non-crossing component* of a partition if, whenever $(\underline{s}_h,\overline{s}_h)\subseteq(\underline{s}_j,\overline{s}_j)$ it follows that S_h is non-crossing. The partition $(S_1,\dots,S_p)$ is called *totally crossing* if no two consecutive indices belong to the same set S_j.

Definition 3.2. Let $\mathcal{A}$ and E be as in Definition 2.1. For each $j \in \mathbf{N}$ let $(b_n^{(j)})$ be a sequence of elements of $\mathcal{A}$. These sequences are said to satisfy the *entangled ergodic theorem* with respect to E if for any $n \in \mathbf{N}$ and any totally crossing pair partition

$$\{1, \dots, 2n\} = \bigcup_{k=1}^{n} \{i_k, j_k\}, \qquad 1 = i_1 < i_2 < \dots < i_n, \quad i_k < j_k,$$

the limit

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{\alpha_1,\dots,\alpha_n=1}^{N} E\left(b_{\alpha_1}^{(i_1)} \cdots b_{\alpha_1}^{(j_1)} \cdots b_{\alpha_n}^{(i_n)} \cdots b_{\alpha_n}^{(j_n)}\right) \tag{3.1}$$

exists in $\mathcal{C}$.

Remark. The *entanglement* is due to the non-commutativity. If $b_n^{(i)}$ commutes with $b_m^{(j)}$ for $i \neq j$ (and any m, n), the limit is reduced to a limit of usual ergodic averages:

$$\lim_{N\to\infty} E\left\{\left(\frac{1}{N}\sum_{\alpha_1=1}^{N} b_{\alpha_1}^{(i_1)} b_{\alpha_1}^{(j_1)}\right) \cdots \left(\frac{1}{N}\sum_{\alpha_n=1}^{N} b_{\alpha_n}^{(i_n)} b_{\alpha_n}^{(j_n)}\right)\right\}.$$

Theorem 3.3. *Under the assumptions of Lemma 2.3 suppose that the algebra $\mathcal{C}$ is the complex numbers and that the mean covariance*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\alpha=1}^{N} E\left(b_\alpha^{(\mu)} b_\alpha^{(\nu)}\right) \tag{3.2}$$

exists for any μ and ν. Then the CLT holds if and only if the entangled ergodic theorem holds.

Proof. Notice that the pair partitions of the set $\{1, \dots, 2n\}$ are in one-to-one correspondence with the pairs of injective maps $i : \{1, \dots, n\} \to \{1, \dots, 2n\}$ and $j : \{1, \dots, n\} \to \{1, \dots, 2n\} \setminus \{i_1, \dots, i_n\}$ such that $i_k < j_k (k = 1, \dots, n)$. This is because a pair partition is uniquely determined once one knows, for any element i_k of any pair, the element j_k paired to it. In this section we shall identify the two notions. The limit (2.6) can be written

$$\sum_{\substack{i:\{1,\dots,n\}\to\{1,\dots,2n\} \\ j:\{1,\dots,n\}\to\{1,\dots,2n\}\setminus\{i_1,\dots,i_n\} \\ \text{injective } i_k<j_k}} \lim_{N\to\infty} \frac{1}{N^n} \sum_{\alpha_1,\dots,\alpha_n=1}^{N} E\left(b_{\alpha_1}^{(i_1)} \cdots b_{\alpha_1}^{(j_1)} \cdots b_{\alpha_n}^{(i_n)} \cdots b_{\alpha_n}^{(j_n)}\right).$$

Using the singleton condition, it is easy to convince oneself that this limit is equal to

$$\left(\prod_{k\in\max(\pi)} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E(b_n^{(i_k)} b_n^{(j_k)})\right) \cdot \lim \frac{1}{N^m} \sum_{\alpha_1,\dots,\alpha_m=1}^{N} E\left(b_{\alpha_1}^{(x_1)} \cdots b_{\alpha_{2m}}^{(x_{2m})}\right), \tag{3.3}$$

where the product in (3.3) is extended to all pairs (i_k, j_k) belonging to the maximal non-crossing component of the pair partition π and $x_1, \dots, x_{2m}$ are the indices defining the totally crossing pair partition associated to $\pi = \{i_1, j_1, \dots, i_n, j_n\}$. Since we have assumed that the limit (3.2) exists, the statement follows because the remaining limit has precisely the form (3.1). □

Corollary 3.4. *For a state E satisfying the singleton condition and the uniform boundedness of the moments (2.3), the CLT holds if any one of the following conditions is satisfied:*

(i) (*q-commutation relations*) *for each i, $j \in \mathbf{N}$, $i \neq j$, there exists complex numbers q_{ij} such that $b_m^{(i)} b_n^{(j)} = q_{ij} b_n^{(j)} b_m^{(i)}$ for any m, $n \in \mathbf{N}$;*
(ii) (*Symmetry*) *for any pair partition π (as above) the expectation value is independent of $\alpha_1, \dots, \alpha_n$;*
(iii) (*Pair partition freeness*) *for any totally crossing pair partition $\pi = \{i_1, j_1, \dots, i_n, j_n\}$ one has*

$$E(b_{\alpha_1}^{(i_1)} \cdots b_{\alpha_1}^{(j_1)} \cdots b_{\alpha_n}^{(i_n)} \cdots b_{\alpha_n}^{(j_n)}) = 0.$$

Proof. It is clear that each of the conditions (i), (ii), (iii) implies the existence of the limit (3.1) hence, by Theorem 3.3 the CLT. □

That the same stationarity condition, which guarantees the validity of the usual ergodic theorem, is also sufficient for the validity of the entangled ergodic theorem in the general case was conjectured by us on the basis of several examples and some indications of the proof were given in the case in which one could prove *a priori* that only the non-crossing pair partitions are relevant in the limit. A first step towards the proof of the entangled ergodic theorem has been done by Liebscher.[17]

4. Properties of the Haagerup States

In this section we investigate some basic properties of the Haagerup states which are essential to prove the associated CLT. Let F_∞ be the free group on countably infinite generators $\{g_n : n \in \mathbf{N}\}$. Denote by $\mathcal{A}$ the group $*$-algebra of F_∞ and for each $n \in \mathbf{N}$ by $\mathcal{A}_n$ the $*$-subalgebra generated by g_n (and g_n^{-1}). For $0 \le \gamma \le 1$ we denote by φ_γ the Haagerup state defined by

$$\varphi_\gamma(w) = \gamma^{|w|}, \qquad w \in F_\infty.$$

The two sequences $\{(g_n), (g_n^{-1})\}$ satisfy the singleton condition with respect to φ_γ if and only if $\gamma = 0$. In fact, the algebras $\mathcal{A}_n$ are free independent with respect to φ_0 and hence the singleton condition is satisfied. On the other hand, since

$$\varphi_\gamma(g_1 g_2 g_1^{-1}) = \gamma^3, \qquad \varphi_\gamma(g_2)\varphi_\gamma(g_1 g_1^{-1}) = \gamma,$$

the singleton condition is not satisfied whenever $\gamma \neq 0, 1$. In particular, the algebras $\mathcal{A}_n$ are not free with respect to φ_γ, $\gamma \neq 0, 1$.

We shall see that φ_γ satisfies a weak analog of the singleton condition. To this goal it is convenient to introduce the following notations: for $\alpha = (j, \varepsilon) \in \mathbf{N} \times \{\pm\}$ we put

$$g_\alpha = g_j^\varepsilon\,, \qquad g_\alpha^* = g_{\alpha^*}\,, \qquad \alpha^* = (j, -\varepsilon)\,.$$

When the state φ_γ under consideration is fixed, we write for simplicity

$$\tilde{g}_\alpha = g_\alpha - \gamma\,.$$

Then $\varphi_\gamma(\tilde{g}_\alpha) = 0$.

Definition 4.1. (i) A product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ is called *separable* at k, $1 \le k \le m$, if $\alpha_p \neq \alpha_q^*$ whenever $1 \le p \le k < q \le m$.
(ii) $\tilde{g}_{\alpha_k}$ is called a *singleton* in the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ if $\tilde{g}_{\alpha_k} \neq \tilde{g}_{\alpha_l}^*$ for any $l \neq k$.
(iii) Let $\tilde{g}_{\alpha_k}$ be a singleton in the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$. It is called *outer* if $\tilde{g}_{\alpha_p} \neq \tilde{g}_{\alpha_q}^*$ for any $p < k < q$.
(iv) A singleton $\tilde{g}_{\alpha_k}$ is called *inner* if $\tilde{g}_{\alpha_p} = \tilde{g}_{\alpha_q}^*$ for some $p < k < q$.

Example. Consider the product $\tilde{g}_1 \tilde{g}_2 \tilde{g}_1^{-1} \tilde{g}_3 \tilde{g}_2$, where the first $\tilde{g}_2$ is an inner singleton while $\tilde{g}_3$ and the last $\tilde{g}_2$ are outer singletons.

Proposition 4.2. *If $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ is separable at k, then*

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = \varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \varphi(\tilde{g}_{\alpha_k+1} \cdots \tilde{g}_{\alpha_m})\,.$$

Proof. Explicit computation in terms of $\tilde{g}_\alpha = g_\alpha - \gamma$ leads to

$$\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k} = \sum_{l=0}^{k} \sum_{1 \le p_1 \le \cdots \le p_l \le k} g_{\alpha_{p_1}} \cdots g_{\alpha_{p_l}} (-\gamma)^{k-l}\,,$$

$$\tilde{g}_{\alpha_{k+1}} \cdots \tilde{g}_{\alpha_m} = \sum_{l'=0}^{m-k} \sum_{k+1 \le q_1 \le \cdots \le q_{l'} \le m} g_{\alpha_{q_1}} \cdots g_{\alpha_{q_{l'}}} (-\gamma)^{m-k-l'}\,.$$

Then

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = \sum_{l=0}^{k} \sum_{l'=0}^{m-k} \sum_{1 \le p_1 \le \cdots \le p_l \le k} \sum_{k+1 \le q_1 \le \cdots \le q_{l'} \le m} \times \varphi_\gamma(g_{\alpha_{p_1}} \cdots g_{\alpha_{p_l}} g_{\alpha_{q_1}} \cdots g_{\alpha_{q_{l'}}}) (-\gamma)^{k-l} (-\gamma)^{m-k-l'}\,.$$

Since $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ is separable at k by assumption, it follows that

$$|g_{\alpha_{p_1}} \cdots g_{\alpha_{p_l}} \cdot g_{\alpha_{q_1}} \cdots g_{\alpha_{q_{l'}}}| = |g_{\alpha_{p_1}} \cdots g_{\alpha_{p_l}}| + |g_{\alpha_{q_1}} \cdots g_{\alpha_{q_{l'}}}|\,,$$

therefore,

$$\varphi_\gamma(g_{\alpha_{p_1}} \cdots g_{\alpha_{p_l}} \cdot g_{\alpha_{q_1}} \cdots g_{\alpha_{q_{l'}}}) = \varphi_\gamma(g_{\alpha_{p_1}} \cdots g_{\alpha_{p_l}}) \cdot \varphi_\gamma(g_{\alpha_{q_1}} \cdots g_{\alpha_{q_{l'}}})\,,$$

whence the result follows by resumming the right-hand side of (4.1). □

Corollary 4.3. *If $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ has an outer singleton, then*

$$\varphi(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = 0\,.$$

Proof. If $\tilde{g}_{\alpha_k}$ is an outer singleton, by applying twice Proposition 4.2 we find

$$\begin{aligned}\varphi(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) &= \varphi(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \varphi(\tilde{g}_{\alpha_{k+1}} \cdots \tilde{g}_{\alpha_m}) \\ &= \varphi(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_{k-1}}) \varphi(\tilde{g}_{\alpha_k}) \varphi(\tilde{g}_{\alpha_{k+1}} \cdots \tilde{g}_{\alpha_m}) \\ &= 0\,.\end{aligned}$$

□

Corollary 4.4. *For any $m \ge 1$,*

$$\varphi\big((\tilde{g}_1 + \cdots + \tilde{g}_N)^m\big) = \varphi\big((\tilde{g}_1^{-1} + \cdots + \tilde{g}_N^{-1})^m\big) = 0\,.$$

Proof. The statement follows from Corollary 4.3 because in the expansion

$$(\tilde{g}_1 + \cdots + \tilde{g}_N)^m = \sum_{j_1, \ldots, j_m = 1}^{N} \tilde{g}_{j_1} \cdots \tilde{g}_{j_m}$$

every $\tilde{g}_{j_l}$ in each product $\tilde{g}_{j_1} \cdots \tilde{g}_{j_m}$ is an outer singleton. □

Lemma 4.5. *Assume that a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ includes no singleton at all or no outer singletons. Let s be the number of inner singletons in the product and let*

$$p = |\{g_j \,:\, \text{there exist } 1 \le k, l \le m \text{ such that } \alpha_k = (j, +),\ \alpha_l = (j, -)\}|\,.$$

Then

$$s \le m - 2 \quad \text{and} \quad p \le \frac{m - s}{2}\,. \tag{4.1}$$

Proof. Since there is no outer singleton, there exist at least two factors $\tilde{g}_{\alpha_k}$ and $\tilde{g}_{\alpha_l}$ with $\alpha_k^* = \alpha_l$. Hence $m \ge 2$ and $s \le m - 2$. If $\tilde{g}_{\alpha_l}$ is not a singleton, there exists at least one element $\tilde{g}_{\alpha_k}$ such as $\alpha_k^* = \alpha_l$ and then $j_k = j_l$ $(k \neq l)$. Therefore $2p + s \le m$. □

Definition 4.6. Assume that a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ contains $s \ge 0$ inner singletons and no outer singletons. Let $\alpha_{j_1}, \ldots, \alpha_{j_s}$ be the suffices which correspond the singletons and denote the rest by $\beta_1, \ldots, \beta_{m-s}$ in order. We say that the product satisfies (NCI) if $g_{\beta_1} \cdots g_{\beta_{m-s}} = e$.

The term (NCI) stands for *non-crossing pair-partition with inner singletons.*

Lemma 4.7. *If the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ consists only of non-crossing pair partitions and s inner singletons then*

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = (-\gamma)^s + (-\gamma)^{s+1} P(\gamma), \tag{4.2}$$

where P is a polynomial. If the (NCI) condition is not satisfied then

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = (-\gamma)^{s+1} P(\gamma). \tag{4.3}$$

Proof. Suppose that the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ includes s inner singletons $\{\nu_j\}$. Expand the product

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = \sum_{k=0}^{m} \sum_{\beta_1, \ldots, \beta_{m-k}} (-\gamma)^k \varphi_\gamma(g_{\beta_1} \cdots g_{\beta_{m-k}}).$$

Let s_1 be the number of $g_{\alpha_{\nu_j}}$'s which appear in the product

$$g_{\beta_1} \cdots g_{\beta_{m-k}} \quad \text{and} \quad h = |g_{\beta_1} \cdots g_{\beta_{m-k}}|.$$

Obviously $h \geq s_1$ and $k \geq s - s_1$. Now, suppose that $s_1 \neq 0$. Then $h > s_1$ or $k > s - s_1$ holds: since $h = s_1$ implies that all elements except the s_1 singletons are canceled and that $g_{\beta_1} \cdots g_{\beta_{m-k}}$ consists of a non-crossing pair partition with s_1 outer singletons which are inner in $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$. Let $g_{\alpha_{\nu_j}}$ be one of such outer singletons. Then there exist at least two elements p and q with $p < \nu_j < q$ and $\alpha_p = \alpha_q^*$ which are not included in $g_{\beta_1} \cdots g_{\beta_{m-k}}$. Hence $k \geq s - s_1 + 2 > s - s_1$. Thus, any term $(-\gamma)^k \varphi_\gamma(g_{\beta_1} \cdots g_{\beta_{m-k}})$ which includes at least one inner singleton ν_j gives $(-\gamma)^k \gamma^h = (-1)^k \gamma^{s'}$ where $s' = k + h > s$.

Finally, we consider a term $(-\gamma)^s \varphi_\gamma(g_{\beta_1} \cdots g_{\beta_{m-s}})$ without inner singletons $\{\nu_j\}$. We see that this term gives $(-\gamma)^s$ if and only if $g_{\beta_1} \cdots g_{\beta_{m-s}} = e$, that is, the product consists of a non-crossing pair partition. This completes the proof. □

5. The Central Limit Theorem for the Haagerup States

In this section we prove a central limit theorem for

$$a_N^+ = \frac{\tilde{g}_1 + \cdots + \tilde{g}_N}{\sqrt{N}} \quad \text{and} \quad a_N^- = \frac{\tilde{g}_1^{-1} + \cdots + \tilde{g}_N^{-1}}{\sqrt{N}}$$

with respect to the Haagerup states. To obtain a meaningful limit we need the rescaling $\gamma = \lambda/\sqrt{N}$. In fact, let us consider, for instance, the expectation value of the product $a_N^+ a_N^+ a_N^-$

$$\begin{aligned}
\varphi_\gamma(a_N^+ a_N^+ a_N^-) &= \frac{1}{\sqrt{N^3}} \sum_{j_1, j_2, j_3} \varphi_\gamma(\tilde{g}_{j_1} \tilde{g}_{j_2} \tilde{g}_{j_3}^{-1}) = \frac{1}{\sqrt{N^3}} \sum_{j_1, j_2} \varphi_\gamma(\tilde{g}_{j_1} \tilde{g}_{j_2} \tilde{g}_{j_1}^{-1}) \\
&= \frac{1}{\sqrt{N^3}} \sum_{j_1, j_2} \left(\varphi_\gamma(\tilde{g}_{j_1} g_{j_2} \tilde{g}_{j_1}^{-1}) - \gamma \varphi_\gamma(\tilde{g}_{j_1} \tilde{g}_{j_1}^{-1})\right) \\
&= \frac{1}{\sqrt{N^3}} \sum_{j_1, j_2} (-\gamma(1 - \gamma^2)) = \sqrt{N} \gamma (\gamma^2 - 1).
\end{aligned}$$

Then, obviously, we obtain a finite limit when $\gamma = O(N^\alpha)$ with $\alpha \leq -1/2$, and it is nontrivial only when $\alpha = -1/2$.

Let $T_N = [\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}]$ be the set of products $\tilde{g}_{\alpha'_1} \cdots \tilde{g}_{\alpha'_m}$ with the following property:

$$\exists \text{ permutation } \tau \text{ on } \{1, \ldots, N\} \text{ s.t. } \alpha'_k = \tau(\alpha_k) \equiv (\tau(j_k), \varepsilon_k), \qquad k = 1, \ldots, m. \tag{5.1}$$

When relation (5.1) holds, we say that $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ and $\tilde{g}_{\alpha'_1} \cdots \tilde{g}_{\alpha'_m}$ are equivalent.

Proposition 5.1. *Given a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ and let $T_N = [\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}]$ be the set defined above. Let s be the number of inner singletons in the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ and put*

$$p = |\{g_j : \textit{there exist } 1 \leq k, l \leq m \textit{ such that } \alpha_k = (j, +),\ \alpha_l = (j, -)\}|.$$

Then,

$$\lim_{N \to \infty} \sum_{\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m} \in T_N} N^{m/2} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = \begin{cases} (-\lambda)^s, & \textit{if the product satisfies (NCI)} \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. It is sufficient to discuss the case where there is no outer singleton in the product. It then follows from Lemma 4.5 that $|\{\alpha_1, \ldots, \alpha_m\}| = s + p \leq (m+s)/2$ and there are exactly $\binom{N}{s+p}$ ways to choose indices $\alpha_1, \ldots, \alpha_m$ with these properties.

Notice that, if a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ satisfies the (NCI) condition of Definition 4.6, then all the other products $\tilde{g}_{\alpha'_1} \cdots \tilde{g}_{\alpha'_m} \in T_N$ also satisfies (NCI). Moreover, in this case $m = 2p + s$ where s is the number of inner singletons and p is the number of pairs in the product. Therefore $m + s$ is even and $s + p = (s+m)/2$.

It follows from Lemma 4.7 that

$$\lim_{N\to\infty} \sum_{\tilde g_{\alpha_1}\cdots \tilde g_{\alpha_m}\in T_N} \frac{1}{\sqrt{N^m}}\varphi_{\lambda/\sqrt{N}}(\tilde g_{\alpha_1}\cdots \tilde g_{\alpha_m})$$

$$= \lim_{N\to\infty} \frac{1}{\sqrt{N^m}} N^{(m+s)/2}\left\{\left(\frac{-\lambda}{\sqrt{N}}\right)^s + \left(\frac{-\lambda}{\sqrt{N}}\right)^{s+1} P\left(\frac{\lambda}{\sqrt{N}}\right)\right\}$$

$$= (-\lambda)^s .$$

In all the other cases, it follows from Lemma 4.7 and $s+p \le (s+m)/2$ that

$$\lim_{N\to\infty} \sum_{\alpha_1\cdots\alpha_m\in T_N} \frac{1}{\sqrt{N^m}}\varphi_{\lambda/\sqrt{N}}(\tilde g_{\alpha_1}\cdots \tilde g_{\alpha_m})$$

$$= \lim_{N\to\infty} \frac{1}{\sqrt{N^m}}\binom{N}{s+p}\left(\frac{-\lambda}{\sqrt{N}}\right)^{s+1} P\left(\frac{\lambda}{\sqrt{N}}\right) = 0 . \qquad \square$$

Theorem 5.2. *Let $NCI_m(s,\varepsilon)$ be the set of all equivalence classes of products $\tilde g_{\alpha_1}\cdots \tilde g_{\alpha_m}$ with the index $\varepsilon = (\varepsilon_1,\dots,\varepsilon_m)$, which consist of $p=(m-s)/2$ non-crossing pairs and s inner singletons. Then, in the notation (1.11), we have*

$$\psi_\lambda(a^{\varepsilon_1}\cdots a^{\varepsilon_m}) = \sum_{s=0}^{m-2} (-\lambda)^s \cdot |\mathrm{NCI}_m(s,\varepsilon)| . \tag{5.2}$$

Proof. From Proposition 5.1 each class with s singletons gives to the limit the same contribution $(-\lambda)^s$. This immediately implies (5.2). $\square$

Remark. The distribution of the *field operator* $\{(a^+ + a^-)/\sqrt{2}\}$ with respect to the limit state ψ_λ was obtained in Ref. 12 and is given by the following

Theorem 5.3. *Let $\mu_{\gamma,N}$ be the distribution of $\{(a_N^+ + a_N^-)/\sqrt{2}\}$ with respect to the Haagerup state φ_γ. Assume that $\gamma \approx \lambda(2N)^\alpha$ as $N\to\infty$. Then,*

(i) *If $\alpha < -1/2$, $d\mu_{\gamma,N}$ converges weakly to the normalized semi-circle distribution:*

$$\lim_{N\to\infty} d\mu_{\gamma,N}(x) = \frac{1}{2\pi}\chi_{[-2,2]}(x)\sqrt{4-x^2}\, dx .$$

(ii) *If $\alpha = -1/2$ and $0\le\lambda\le 1$, $d\mu_{\gamma,k}$ converges weakly to a probability measure with a parameter λ:*

$$\lim_{N\to\infty} d\mu_{\gamma,N}(x) = \frac{1}{2\pi}\chi_{[-2-\lambda,2-\lambda]}(x)\frac{\sqrt{(2+\lambda+x)(2-\lambda-x)}}{1-\lambda x}dx .$$

6. Identification of the Limit Process

Let

$$\Gamma(\mathbf{C})_\nu = \mathbf{C}\oplus\bigoplus_{n=1}^{\infty}\mathbf{C}^{\otimes n} \quad \left(=\bigoplus_{n=0}^{\infty}\mathbf{C}\right), \qquad \nu = L, R,$$

denote two copies of the full Fock spaces on $\mathbf{C}$ with free creations a_ν^+ and free annihilation a_ν. Let $\mathcal{H} = \bigoplus_{m,n=0}^{\infty}\mathcal{H}_{m,n}$ be the free product $\Gamma(\mathbf{C})_L * \Gamma(\mathbf{C})_R$, that is, the (m,n)-particle space $\mathcal{H}_{m,n}$ is the complex linear span of the set of vectors $\{a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi\}$ which satisfy the following conditions:

$$|\{j \mid \nu_j = L\}| = m , \qquad |\{j \mid \nu_j = R\}| = n$$

and the scalar product is given by

$$\left\langle a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi,\, a_{\nu_1'}^+\cdots a_{\nu_l'}^+\Phi\right\rangle_{\mathcal{H}} = \begin{cases} 1, & \text{if } (\nu_1,\dots,\nu_k) = (\nu_1',\dots,\nu_l'),\\ 0, & \text{otherwise}.\end{cases}$$

The actions of the creation operators

$$L^+ := a_L^+ * 1 : \mathcal{H}_{m,n}\to\mathcal{H}_{m+1,n}, \quad R^+ := 1 * a_R^+ : \mathcal{H}_{m,n}\to\mathcal{H}_{m,n+1}$$

are given respectively by

$$L^+ a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi = a_L^+ a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi ,$$

$$R^+ a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi = a_R^+ a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi ,$$

and the action of the annihilation

$$L = a_L * 1 : \mathcal{H}_{m,n}\to\mathcal{H}_{m-1,n}, \quad R = 1 * a_R : \mathcal{H}_{m,n}\to\mathcal{H}_{m,n-1}$$

is given by

$$L a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi = \begin{cases} a_{\nu_2}^+\cdots a_{\nu_k}^+\Phi, & \text{if } \nu_1 = L \text{ and } k\ge 2,\\ \Phi, & \text{if } \nu_1 = L \text{ and } k = 1,\\ 0, & \text{otherwise},\end{cases}$$

$$R a_{\nu_1}^+\cdots a_{\nu_k}^+\Phi = \begin{cases} a_{\nu_2}^+\cdots a_{\nu_k}^+\Phi, & \text{if } \nu_1 = R \text{ and } k\ge 2,\\ \Phi, & \text{if } \nu_1 = R \text{ and } k = 1,\\ 0, & \text{otherwise}.\end{cases}$$

Let $P:\mathcal{H}\to\mathcal{H}$ be the orthogonal projection onto $\mathcal{H}_{0,0}^{\perp}$. Put

$$A_\lambda^- = L^+ + R + \lambda P , \qquad A_\lambda^+ = L + R^+ + \lambda P ,$$

where $\lambda\ge 0$ is a constant.

Theorem 6.1. *The limit process (a^+, a^-, ψ_λ) is represented in $\mathcal{H}$. That is, all its correlations are given by*

$$\psi_\lambda(a^{\varepsilon_1}\cdots a^{\varepsilon_m}) = \langle \Phi, A_\lambda^{\varepsilon_1}\cdots A_\lambda^{\varepsilon_m}\Phi\rangle_{\mathcal{H}}\,.$$

Proof. In Theorem 5.2 we have seen that the ψ_λ-correlators are completely determined by the cardinalities of the sets NCI_m. Therefore all we need to do is to establish a bijective correspondence between NCI_m-partitions associated with $a^{\varepsilon_1}\cdots a^{\varepsilon_m}$ and terms in the expansion of

$$\langle\phi, A_\lambda^{\varepsilon_1}\cdots A_\lambda^{\varepsilon_m}\phi\rangle = \sum_{B_{\nu_1}^{\varepsilon_1},\dots,B_{\nu_m}^{\varepsilon_m}} \langle \phi, B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}\phi\rangle\,,$$

where $B_R^- = L^+$, $B_L^- = R$, $B_R^+ = R^+$, $B_L^+ = L$ and $B_0^- = B_0^+ = -\lambda P$. In a product $B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}$, we call $(B_{\nu_p}^{\varepsilon_p}, B_{\nu_q}^{\varepsilon_q})$ $(p<q)$ a *pair* if $B_{\nu_p}^{\varepsilon_p} = L$ and $B_{\nu_q}^{\varepsilon_q} = L^+$ or $B_{\nu_p}^{\varepsilon_p} = R$ and $B_{\nu_q}^{\varepsilon_q} = R^+$. If $B_{\nu_p}^{\varepsilon_p} = -\lambda P$ we call it a *singleton*. From the definition of $\mathcal{H}, A_\lambda^+, A_\lambda$ we see easily that $\langle\phi, B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}\phi\rangle \neq 0$ if and only if $B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}$ forms a non-crossing pair partition with s inner singletons $(0\le s\le m-2)$. In this case,

$$\langle\phi, B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}\phi\rangle = (-\lambda)^s\,.$$

□

Therefore we obtain the desired bijective correspondence.

7. Singleton Independence

In our proof of the central limit theorem for the Haagerup states in Sec. 4, we have used some very specific properties of these states which allowed explicit computations. In this section we show that, for the validity of the CLT alone, much less is required. This allows to generalize our result to $*$-algebras.

Definition 7.1. Let $\mathcal{A}$ be a $*$-algebra and let $S = \{g_n, g_n^*\,;\, n\in\mathbf{N}\}$ be a countable subset of $\mathcal{A}$. Put $g^+ = g$ and $g^- = g^*$ and use the notation (1.1). Assume that we are given a family of states φ_γ, $\gamma\ge 0$, on $\mathcal{A}$ such that $\varphi_\gamma(g_\alpha) = \gamma$ for any g_α. The sequence $\{g_n\}$ is called *singleton-independent* with respect to φ_γ if

$$|\varphi_\gamma(g_{\alpha_1}\cdots g_{\alpha_k})| \le \gamma c_k |\varphi_\gamma(g_{\alpha_1}\cdots \hat{g}_{\alpha_s}\cdots g_{\alpha_k})|\,, \tag{7.1}$$

whenever α_s is a singleton for $(\alpha_1,\dots,\alpha_k)$. If we take $\gamma = 0$, the usual singleton condition is related (where we put $0^0 = 1$). Condition (7.1) implies that

$$|\varphi_\gamma(g_{\alpha_1}\cdots g_{\alpha_m})| \le C_m\gamma^s \tag{7.2}$$

whenever $g_{\alpha_1}\cdots g_{\alpha_k}$ has s singletons.

Remark. Conditions (7.1), (7.2) are easily verified for the Haagerup state but there are also other states which satisfy the singleton independence. For example, Figà-Talamanca and Picardello studied a family of unitary representations of the free group with N generators, called the principal series, and obtained positive definite functions. A special one is given by

$$\varphi(g) = \left(1 + |g|\frac{N-1}{N}\right)(2N-1)^{-|g|/2}\,,$$

where N is the number of generators of finitely generated free group F_N. This state also satisfies the singleton independence.

In the notations introduced at the beginning of Sec. 4 we define the sums

$$S_N^\varepsilon = \sum_{j=1}^N \tilde{g}_j^\varepsilon\,, \qquad \varepsilon = \pm 1\,,$$

and, for fixed $k\in\mathbf{N}$ and $\varepsilon_1,\dots,\varepsilon_k\in\{\pm1\}$ we consider the product

$$S_N^{\varepsilon_1}\cdots S_N^{\varepsilon_k} = \sum_{j_1,\dots,j_k=1}^N \tilde{g}_{j_1}^{\varepsilon_1}\cdots\tilde{g}_{j_k}^{\varepsilon_k} = \sum_{j_1,\dots,j_k}\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}\,.$$

Put $I_k = \{(1,\varepsilon_1),\dots,(k,\varepsilon_k)\}$ and consider α as a function $\alpha: I_k\to\{1,\dots,N\}$. For given α let

$$p = |\alpha(I_k)|\,,$$

where $|\cdot|$ denotes cardinality. We denote by $\alpha(I_k) = \{\overline{\alpha}_1,\dots,\overline{\alpha}_p\}$ its range (with $\overline{\alpha}_i\neq\overline{\alpha}_j$) and put

$$S_j = \alpha^{-1}(\overline{\alpha}_j)\,, \qquad j = 1,\dots,p\,,$$

$$\mathcal{P}_{k,p} = \{(S_1,\dots,S_p)\,;\ \text{partition of } I_k \text{ of cardinality } p\}\,,$$

$$[S_1,\dots,S_p] = \{\alpha\,;\ \alpha|_{S_j} = \alpha(S_j) = \text{const. and } \alpha(S_i)\neq\alpha(S_j) \text{ if } i\neq j\}\,.$$

With these notations our goal is to study the large-N asymptotics of the rescaled expectation values

$$\frac{1}{N^{k/2}}\varphi_{\lambda/\sqrt{N}}(S_N^{\varepsilon_1}\cdots S_N^{\varepsilon_k}) = \frac{1}{N^{k/2}}\sum_{p=1}^k\ \sum_{(S_1,\dots,S_p)\in\mathcal{P}_{k,p}}\ \sum_{\alpha\in[S_1,\dots,S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})\,. \tag{7.3}$$

Lemma 7.2. *Given $s = 0,1,\dots,k$, denote*

$$\mathcal{P}_{k,p}^s = \{(S_1,\dots,S_p) \text{ which have exactly } s \text{ singletons}\}\,.$$

Then it holds that $p\le (k+s)/2$. Moreover, if $p<(k+s)/2$ then

$$\lim_{N\to\infty}\frac{1}{N^{k/2}}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{k,p}^s}\ \sum_{\alpha\in[S_1,\dots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}) = 0\,.$$

Proof. For $(S_1, \dots, S_p) \in \mathcal{P}^s_{k,p}$ we have

$$k = \sum_{j=1}^{p} |S_j| = \sum_{j\in\{1,\dots,p\},|S_j|\geq 2} |S_j| + s \geq 2(p-s) + s = 2p - s$$

and this proves that inequality (2.3) implies that the sum is dominated by a constant times

$$\frac{1}{N^{k/2+s/2}} |\mathcal{P}^s_{k,p}| \frac{\lambda^s}{p!} N^p \to 0\,.$$

□

From Lemma 7.2 we conclude that the only nontrivial contributions to the limit of (7.3) come from those partitions $(S_1, \dots, S_p) \in \mathcal{P}^s_{k,p}$ satisfying

$$p = \frac{k+s}{2} \Leftrightarrow k = 2p - s\,. \tag{7.4}$$

Lemma 7.3. *Let $(S_1, \dots, S_p) \in \mathcal{P}^s_{k,p}$ be such that (7.4) is satisfied. Then*

$$|S_j| = 1 \text{ or } |S_j| = 2\,.$$

Proof. If for some j (with no loss of generality we can put $j = 1$)

$$|S_j| = |S_1| \geq 3\,,$$

then

$$k = 3 + \sum_{j\geq 2, |S_j|\geq 2} |S_j| + s \geq 3 + 2(p - s - 1) + s = 3 + 2p - 2s - 2 + s = 2p - s + 1$$

which is incompatible with (7.4). □

For any partition $(S_1, \dots, S_p)$ of $\{1, \dots, k\}$ with s singletons and $|S_j| = 1$ or 2 for $j = 1, \dots, p$, denote $(\tilde{S}_1, \dots, \tilde{S}_{p-s})$ the set of all S_j's with $|S_j| = 2$. We say that $(\tilde{S}_1, \dots, \tilde{S}_{p-s})$ is the pair partition associated with $(S_1, \dots, S_p)$. The pair partition associated to a 2–1 map $\beta : \{1, \dots, 2p\} \to \{1, \dots, p\}$ shall be called *negligible* if

$$|\varphi_\gamma(g_{\beta_1} \cdots g_{\beta_{2p}})| \leq c\gamma\,. \tag{7.5}$$

Lemma 7.4. *Suppose that φ_γ satisfies condition (7.5). Fix $s = 0, \dots, k$ and let $\tilde{\mathcal{P}}_{k,1,2,s}$ denote the set of all partitions $(S_1, \dots, S_p)$ with s singletons such that $|S_j| = 1$ or 2 and such that the associated pair partition is negligible. Then*

$$\lim_{N\to\infty} \frac{1}{N^{k/2}} \sum_{(S_1,\dots,S_p)\in\tilde{\mathcal{P}}_{k,1,2,s}} \sum_{\alpha\in[S_1,\dots,S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) = 0\,. \tag{7.6}$$

Proof. Iterating (7.1), we see that the sum (7.6) is majorized by

$$\frac{C}{N^{\frac{k+s}{2}}} \sum_{(S_1,\dots,S_p)\in\tilde{\mathcal{P}}_{k,1,2,s}} \sum_{\alpha\in[S_1,\dots,S_p]} |\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{k-s}})|\,, \tag{7.7}$$

where $(\beta_1, \dots, \beta_{k-s})$ is obtained from $(\alpha_1, \dots, \alpha_k)$ by removing the singletons. Since the pair partition associated to $(S_1, \dots, S_p)$ is negligible, by (7.5)

$$|\varphi_\gamma(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{k-s}})| \leq c \cdot \frac{\lambda}{\sqrt{N}}\,,$$

the sum (7.7) is majorized by a constant times

$$\frac{c}{N^{\frac{k+s}{2}}} |\tilde{\mathcal{P}}_{k,1,2,s}| \cdot \frac{\lambda}{\sqrt{N}} \cdot N^p\,. \tag{7.8}$$

As we have shown that $p = (k + s)/2$, (7.8) is majorized by $c/\sqrt{N} \to 0$. □

Summing up, we come to

Theorem 7.5. *In the notations of Definition 7.1 suppose that the states φ_γ satisfy conditions (7.1) and (7.5) for $\gamma \in [0, \bar{\gamma}]$, $\bar{\gamma} > 0$. Then one has*

$$\lim_{N\to\infty} \frac{1}{N^{k/2}} \varphi_{\lambda/\sqrt{N}}(S_N^{\varepsilon_1} \cdots S_N^{\varepsilon_k})$$
$$= \lim_{N\to\infty} N^{-k/2} \sum_{1\leq s\leq k} \sum_{\substack{\alpha:\text{negligible pair partiton}\\ \text{with } s \text{ singletons}}} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k})\,. \tag{7.9}$$

Remark. The existence of the limit (7.9) is guaranteed by conditions of the same type as in Corollary 3.4.

Remark. One easily verifies condition (7.1) for the Haagerup states. For these states the negligible partitions are precisely the crossing ones. Other examples shall be considered elsewhere.

Acknowledgments

The authors are grateful to Profs. Denes Petz and Volkmar Liebscher for several remarks which allowed one to improve the presentation of this paper and for interesting discussions on the entangled ergodic theorems. They also thank Prof. Marek Bożejko for stimulating discussions.

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Infinite Dimensional Analysis, Quantum Probability and Related Topics
Vol. 1, No. 2 (1998) 221–246

CENTRAL LIMIT THEOREMS AND ASYMPTOTIC SPECTRAL ANALYSIS ON LARGE GRAPHS*

AKIHITO HORA

Department of Environmental and Mathematical Sciences,
Faculty of Environmental Science and Technology,
Okayama University, Okayama 700, Japan

Received 29 July 1997

Regarding the adjacency matrix of a graph as a random variable in the framework of algebraic or noncommutative probability, we discuss a central limit theorem in which the size of a graph grows in several patterns. Various limit distributions are observed for some Cayley graphs and some distance-regular graphs. To obtain the central limit theorem of this type, we make combinatorial analysis of mixed moments of noncommutative random variables on one hand, and asymptotic analysis of spectral structure of the graph on the other hand.

1. Introduction

1.1. *Theme*

In this paper we discuss a central limit theorem (CLT) for the adjacency matrix (or the Laplacian) of a graph in connection with asymptotic spectral analysis of the graph. The asymptotic is concerned with a certain infinite volume limit in which the size of a graph grows. We regard an adjacency matrix as a random variable in the same manner with the framework of noncommutative probability. In the simplest setting of noncommutative probability (often referred to as quantum probability or more generally as algebraic probability), a usual probability space is replaced by a pair $(\mathcal{A}, \phi)$ of (possibly noncommutative) algebra $\mathcal{A}$ and unital (i.e. $\phi(1) = 1$) linear functional ϕ on $\mathcal{A}$. $\mathcal{A}$ corresponds to the algebra of measurable functions and hence an element in $\mathcal{A}$ is regarded as a noncommutative random variable. The distribution of $X \in \mathcal{A}$ under ϕ is determined as an element in the dual of $\mathbf{C}[x]$ (the polynomials in one variable) by

$$\mathbf{C}[x] \ni f \longmapsto \phi(f(X)) \in \mathbf{C}\,.$$

Considered in the context of a C^*-algebra, the distribution of a self-adjoint element in $\mathcal{A}$ would be realized as a probability on $\mathbf{R}$.

*Supported by Grant-in-Aid for Scientific Research (No. 09740108), The Ministry of Education, Science, Sports and Culture, Japan.