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Novel Analysis of the Fractional-Order System of Non-Linear Partial Differential Equations with the Exponential-Decay Kernel

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Abstract: This article presents a homotopy perturbation transform method and a variational iterative transform method for analyzing the fractional-order non-linear system of the unsteady flow of a polytropic gas. In this method, the Yang transform is combined with the homotopy perturbation transformation method and the variational iterative transformation method in the sense of Caputo–Fabrizio. A numerical simulation was carried out to verify that the suggested methodologies are accurate and reliable, and the results are revealed using graphs and tables. Comparing the analytical and actual solutions demonstrates that the proposed approaches are effective and efficient in investigating complicated non-linear models. Furthermore, the proposed methodologies control and manipulate the achieved numerical solutions in a very useful way, and this provides us with a simple process to adjust and control the convergence regions of the series solution.

Keywords: variational iterative method; Caputo–Fabrizio derivative; Yang transform; homotopy perturbation method; polytropic gas equation

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1. Introduction

In the last few centuries, fractional derivative have been applied to solve several physical models, and these representations have been very good at modeling the real world. Numerous fundamental definitions of fractional derivatives were introduced by Atangana–Baleanu, Caputo–Fabrizio, Liouville–Caputo, Riemann–Liouville, Riesz, Weyl, Grunwald–Letnikov, and Hadamard, among others [1–6]. Numerous non-linear problems have been established and widely applied in applied sciences such as biology, mathematics, chemistry, and numerous classical mechanics fields such as fluid mechanics, condensed matter physics, transport phenomena, theoretical physics, and non-linear optics over the last few decades. When dealing with linear problems, it is hard to determine the actual conclusion of non-linear equations, which is vital for defining the features and attributes of physical events [7–9]. Several useful techniques have been used to investigate non-linear fractional partial differential equations (PDEs), for instance the variational iterative transformation technique [10], the natural Adomian decomposition technique [11–14], the homotopy perturbation transformation technique [15,16], the homotopy analysis transformation technique [17,18], the q-homotopy analysis transformation technique [19–21], the finite element technique [22,23], the finite difference technique [24], the reduced differential transformation technique [25–27], and so on.

Here, we stud a fractional gas dynamics system model that describes the evolution of the two-dimensional unsteady flow of a perfect gas. In astrophysics, the polytropic gas is defined as [28]:

$$P = h\rho^{1+(1/m)},$$

in which $\rho = \vec{U}/\vec{V}$ represents the energy density, $\vec{U}$ represents the total energy of the gas, $\vec{V}$ represents the container volume, h represents a constant, and m represents the polytropic index. Two examples of such gases are degenerated electron gas and adiabatic gas. Polytropic gases are studied extensively in astrophysics and cosmology [29], and these gases can act similarly to dark energy [30]. Consider the following set of gas dynamic equations that describe the behavior of an unsteady flow of any ideal gas [31,32]:

$$
\begin{aligned}
&{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{U} + \mathcal{U}_{\varepsilon}\mathcal{U} + \mathcal{V}\mathcal{U}_{\mathfrak{b}} + \frac{\mathcal{W}_{\varepsilon}}{\mathcal{X}} = 0 \\
&{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{V} + \mathcal{U}\mathcal{V}_{\varepsilon} + \mathcal{V}\mathcal{V}_{\mathfrak{b}} + \frac{\mathcal{W}_{\mathfrak{b}}}{\mathcal{X}} = 0 \\
&{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{X} + \mathcal{U}\mathcal{X}_{\varepsilon} + \mathcal{V}\mathcal{X}_{\mathfrak{b}} + \mathcal{X}\mathcal{U}_{\varepsilon} + \mathcal{X}\mathcal{V}_{\mathfrak{b}} = 0 \\
&{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{W} + \mathcal{U}\mathcal{W}_{\varepsilon} + \mathcal{V}\mathcal{W}_{\mathfrak{b}} + \tau\mathcal{W}\mathcal{U}_{\varepsilon} + \tau\mathcal{W}\mathcal{V}_{\mathfrak{b}} = 0
\end{aligned}
$$

under initial conditions:

$$
\begin{aligned}
&\mathcal{U}(\varepsilon,\mathfrak{b},0) = a(\varepsilon,\mathfrak{b}), \quad \mathcal{V}(\varepsilon,\mathfrak{b},0) = b(\varepsilon,\mathfrak{b}) \\
&\mathcal{X}(\varepsilon,\mathfrak{b},0) = c(\varepsilon,\mathfrak{b}) \text{ and } \mathcal{W}(\varepsilon,\mathfrak{b},0) = d(\varepsilon,\mathfrak{b})
\end{aligned}
$$

where $\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q})$ and $\mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q})$ are the velocity components, $\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q})$ is the density, $\mathcal{W}(\varepsilon,\mathfrak{b},0)$ is the pressure, q is time, and τ is the ratio of the heat capacity, which denotes the adiabatic index.

In 1999, the homotopy perturbation method (HPM) was constructed by [33], which merges the homotopy method and the classic perturbation technique, which has been widely utilized in linear and non-linear problems [34–36]. The HPM is significant because it eliminates the necessity for a small parameter in the problems, therefore avoiding the difficulties associated with the standard perturbation methods. The primary aim of this study was to apply the HPM to the analysis of fractional-order non-linear gas dynamic equations by applying a newly presented integral transform called the "Yang transform" [37]. We gain a power-series-form result in the setting of a quick convergence series, where some termsare necessary to achieve very efficient solutions. There is no requirement for a linearization or discretization of non-linear problems, and only a few of these methods can produce a result that can be calculated fast.

J.H. He was the first to suggest the variational iteration method. The approach provides the results as a quickly converging consecutive approximation, which may yield a precise solution if one exists [38–40]. It was discovered that a few terms can be employed for numerical purposes for concrete problems where a precise answer is not possible. Afshan et al. proposed a new modification for the variational iteration technique in [41]. The primary impetus for this modification was the combination of the functional corrections of the variational iteration technique and the Laplace transform.

2. Preliminaries and Concepts

Definition 1. *If the Caputo–Fabrizio (CF) derivative is defined as [42]:*

$$
{}^{CF}D_{\mathfrak{q}}^{\wp}[\mathbb{P}(\mathfrak{q})] = \frac{N(\wp)}{1-\wp}\int_0^{\mathfrak{q}} \mathbb{P}'(\varrho)K(\mathfrak{q},\varrho)d\varrho, \quad n-1<\wp\leq n \tag{1}
$$

$N(\wp)$ is the normalization function with $N(0) = N(1) = 1$.

$$
{}^{CF}D_{\mathfrak{q}}^{\wp}[\mathbb{P}(\mathfrak{q})] = \frac{N(\wp)}{1-\wp}\int_0^{\mathfrak{q}} [\mathbb{P}(\mathfrak{q}) - \mathbb{P}(\varrho)]K(\mathfrak{q},\varrho)d\varrho. \tag{2}
$$

Definition 2. *The fractional CF integral is expressed as [42]:*

$$
{}^{CF}I_{\mathfrak{q}}^{\wp}[\mathbb{P}(\mathfrak{q})] = \frac{1-\wp}{N(\wp)}\mathbb{P}(\mathfrak{q}) + \frac{\wp}{N(\wp)}\int_0^{\mathfrak{q}} \mathbb{P}(\varrho)d\varrho, \quad \mathfrak{q}\geq 0, \wp\in(0,1]. \tag{3}
$$

Definition 3. *For $N(\wp) = 1$, we show the following solution of the Laplace transform of the CF derivative [42]:*

$$L\left[{}^{CF}D_{\mathfrak{q}}^{\wp}[\mathbb{P}(\mathfrak{q})]\right] = \frac{vL[\mathbb{P}(\mathfrak{q}) - \mathbb{P}(0)]}{v + \wp(1-v)}. \tag{4}$$

Definition 4. *The Yang transform of $\mathbb{P}(\mathfrak{q})$ is expressed as [37]:*

$$\mathbb{Y}[\mathbb{P}(\mathfrak{q})] = \chi(v) = \int_0^\infty \mathbb{P}(\mathfrak{q})e^{-\frac{\mathfrak{q}}{v}}d\mathfrak{q}. \ \ \mathfrak{q} > 0 \tag{5}$$

3. Remarks

The Yang transform of several valuable terms is described as:

$$\begin{aligned} \mathbb{Y}[1] &= v, \\ \mathbb{Y}[\mathfrak{q}] &= v^2, \\ \mathbb{Y}[\mathfrak{q}^i] &= \Gamma(i+1)v^{i+1}. \end{aligned} \tag{6}$$

Lemma 1. *Let the Laplace transform of $\mathbb{P}(\mathfrak{q})$ is $F(v)$, then $\chi(v) = F(1/v)$ [43].*

Proof. From Equation (5), we can obtain the Yang transform by putting $\mathfrak{q}/v = \zeta$ as:

$$L[\mathbb{P}(\mathfrak{q})] = \chi(v) = v\int_0^\infty \mathbb{P}(v\zeta)e^{\zeta}d\zeta. \ \ \zeta > 0. \tag{7}$$

Since $L[\mathbb{P}(\mathfrak{q})] = F(v)$, this implies that:

$$F(v) = L[\mathbb{P}(\mathfrak{q})] = \int_0^\infty \mathbb{P}(\mathfrak{q})e^{-v\mathfrak{q}}d\mathfrak{q}. \tag{8}$$

Putting $\mathfrak{q} = \zeta/v$ in (8), we obtain:

$$F(v) = \frac{1}{v}\int_0^\infty \mathbb{P}\left(\frac{\zeta}{v}\right)e^{\zeta}d\zeta. \tag{9}$$

Thus, from Equation (7), we obtain:

$$F(v) = \chi\left(\frac{1}{v}\right). \tag{10}$$

Furthermore, from Equations (5) and (8), we obtain:

$$F\left(\frac{1}{v}\right) = \chi(v). \tag{11}$$

The links (10) and (11) represent the duality connection among the Laplace and Yang transforms. □

Lemma 2. *Let $\mathbb{P}(\mathfrak{q})$ be a continuous function; then, the Yang transform of the CF derivatives of $\mathbb{P}(\mathfrak{q})$ is defined by [43]:*

$$\mathbb{Y}[\mathbb{P}(\mathfrak{q})] = \frac{\mathbb{Y}[\mathbb{P}(\mathfrak{q}) - v\mathbb{P}(0)]}{1 + \wp(v-1)}. \tag{12}$$

Proof. The fractional Laplace transform of CF is defined as:

$$L[\mathbb{P}(\mathfrak{q})] = \frac{L[v\mathbb{P}(\mathfrak{q}) - \mathbb{P}(0)]}{v + \wp(1-v)}. \tag{13}$$

Additionally, we have the relationship between the Yang and Laplace properties, namely $\chi(v) = F(1/v)$. To achieve the required answer, we put $1/v$ for v in Equation (13), and we obtain:

$$\begin{aligned}\mathbb{Y}[\mathbb{P}(\mathfrak{q})] &= \frac{\frac{1}{v}\mathbb{Y}[\mathbb{P}(\mathfrak{q}) - \mathbb{P}(0)]}{\frac{1}{v} + \wp(1 - \frac{1}{v})}, \\ \mathbb{Y}[\mathbb{P}(\mathfrak{q})] &= \frac{\mathbb{Y}[\mathbb{P}(\mathfrak{q}) - v\mathbb{P}(0)]}{1 + \wp(v - 1)}.\end{aligned} \tag{14}$$

As a result, the proof is complete. □

4. Road Map of the Suggested Method

The analysis of non-linear fractional PDEs via the HPTM: Consider the non-linear function $N(\mathcal{U}(\varphi, \mathfrak{q}))$ and the linear fractional $L(\mathcal{U}(\varphi, \mathfrak{q}))$ as [43]:

$$\begin{cases} {}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{U}(\varepsilon, \mathfrak{q}) + L(\mathcal{U}(\varepsilon, \mathfrak{q})) + N(\mathcal{U}(\varepsilon, \mathfrak{q})) = g(\varepsilon, \mathfrak{q}), \\ \mathcal{U}(\varepsilon, 0) = h(\varepsilon), \end{cases} \tag{15}$$

where the term $g(\varepsilon, \mathfrak{q})$ defines the source function. Applying the Yang transform to Equation (15), we obtain:

$$\begin{aligned}\frac{\mathbb{Y}[\mathcal{U}(\varepsilon, \mathfrak{q}) - v\mathcal{U}(\varepsilon, 0)]}{1 + \wp(v - 1)} &= -\mathbb{Y}[L(\mathcal{U}(\varepsilon, \mathfrak{q})) + N(\mathcal{U}(\varepsilon, \mathfrak{q}))] + \mathbb{Y}[g(\varepsilon, \mathfrak{q})], \\ \mathbb{Y}[\mathcal{U}(\varepsilon, \mathfrak{q})] = vh(\varepsilon) - (1 + \wp(v - 1))&[\mathbb{Y}[L(\mathcal{U}(\varepsilon, \mathfrak{q})) + N(\mathcal{U}(\varepsilon, \mathfrak{q}))] + \mathbb{Y}[g(\varepsilon, \mathfrak{q})].\end{aligned} \tag{16}$$

Implementing the inverse Yang transform, we obtain:

$$\mathcal{U}(\varepsilon, \mathfrak{q}) = \mathcal{U}(\varepsilon, 0) - \mathbb{Y}^{-1}[(1 + \wp(v - 1))[\mathbb{Y}[L(\mathcal{U}(\varepsilon, \mathfrak{q})) + N(\mathcal{U}(\varepsilon, \mathfrak{q}))] + \mathbb{Y}[g(\varepsilon, \mathfrak{q})]]. \tag{17}$$

Now, we use the homotopy perturbation technique:

$$\mathcal{U}(\varepsilon, \mathfrak{q}) = \sum_{i=0}^{\infty} \rho^i \mathcal{U}_i(\varepsilon, \mathfrak{q}). \tag{18}$$

We decompose the non-linear function $N(\mathcal{U}(\varepsilon, \mathfrak{q}))$ as:

$$N(\mathcal{U}(\varepsilon, \mathfrak{q})) = \sum_{i=0}^{\infty} \rho^i H_i(\mathcal{U}), \tag{19}$$

where $H_i(\mathcal{U})$ shows He's polynomials and is determined with the help of the formula:

$$H_i(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \cdots, \mathcal{U}_i) = \frac{1}{\Gamma(i+1)} \frac{\partial^i}{\partial \rho^i}\left[N\left(\sum_{i=0}^{\infty} \rho^i \mathcal{U}_i\right)\right]_{\rho=0}, \quad i = 1, 2, 3, \cdots. \tag{20}$$

Putting Equations (18) and (19) in Equation (17), we obtain:

$$\sum_{i=0}^{\infty} \rho^i \mathcal{U}_i(\varepsilon, \mathfrak{q}) = \mathcal{U}(\varepsilon, \mathfrak{q}) - \rho\left(\mathbb{Y}^{-1}\left[(1 + \wp(v - 1))\mathbb{Y}\left[L\sum_{i=0}^{\infty} \rho^i \mathcal{U}_i(\varepsilon, \mathfrak{q}) + N\sum_{i=0}^{\infty} \rho^i H_i(\mathcal{U})\right]\right]\right) \tag{21}$$

By analyzing the coefficient of ρ in (21), we derive the following terms:

$$\begin{aligned}
\rho^0 : \mathcal{U}_0(\varepsilon,\mathfrak{q}) =& \mathcal{U}(\varepsilon,\mathfrak{q}),\\
\rho^1 : \mathcal{U}_1(\varepsilon,\mathfrak{q}) =& \mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}[L(\mathcal{U}_0(\varepsilon,\mathfrak{q}))+H_0(\mathcal{U})]],\\
\rho^2 : \mathcal{U}_2(\varepsilon,\mathfrak{q}) =& \mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}[L(\mathcal{U}_1(\varepsilon,\mathfrak{q}))+H_1(\mathcal{U})]],\\
\rho^3 : \mathcal{U}_3(\varepsilon,\mathfrak{q}) =& \mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}[L(\mathcal{U}_2(\varepsilon,\mathfrak{q}))+H_2(\mathcal{U})]],\\
&\vdots\\
\rho^i : \mathcal{U}_i(\varepsilon,\mathfrak{q}) =& \mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}[L(\mathcal{U}_i(\varepsilon,\mathfrak{q}))+H_i(\mathcal{U})]].
\end{aligned} \tag{22}$$

The achieved result of Equation (15) can be defined as follows:

$$\mathcal{U}(\varepsilon,\mathfrak{q}) = \mathcal{U}_0(\varepsilon,\mathfrak{q}) + \mathcal{U}_1(\varepsilon,\mathfrak{q}) + \cdots . \tag{23}$$

5. Error Analysis and Convergence

The following theorems are foundational to address the original models (15), the convergence, and the error analysis.

Theorem 1. *Let $\mathcal{U}(\varepsilon,\mathfrak{q})$ be the exact solution of (15), and let $\mathcal{U}_i(\varepsilon,\mathfrak{q}) \in H$ and $\sigma \in (0,1)$, where H defines the Hilbert space. Then, the obtained solution $\sum_{i=0}^{\infty}\mathcal{U}_i(\varepsilon,\mathfrak{q})$ will be convergent $\mathcal{U}(\varepsilon,\mathfrak{q})$ if $\mathcal{U}_i(\varepsilon,\mathfrak{q}) \leq \mathcal{U}_{i-1}(\varepsilon,\mathfrak{q})\ \forall i > A$, i.e., for any $\omega > 0 \exists A > 0$, such that $||\mathcal{U}_{i+n}(\varepsilon,\mathfrak{q})|| \leq \beta, \forall i,n \in N$.*

Proof. We make a sequence of $\sum_{i=0}^{\infty}\mathcal{U}_i(\varepsilon,\mathfrak{q})$.

$$\begin{aligned}
C_0(\varepsilon,\mathfrak{q}) =& \mathcal{U}_0(\varepsilon,\mathfrak{q}),\\
C_1(\varepsilon,\mathfrak{q}) =& \mathcal{U}_0(\varepsilon,\mathfrak{q}) + \mathcal{U}_1(\varepsilon,\mathfrak{q}),\\
C_2(\varepsilon,\mathfrak{q}) =& \mathcal{U}_0(\varepsilon,\mathfrak{q}) + \mathcal{U}_1(\varepsilon,\mathfrak{q}) + \mathcal{U}_2(\varepsilon,\mathfrak{q}),\\
C_3(\varepsilon,\mathfrak{q}) =& \mathcal{U}_0(\varepsilon,\mathfrak{q}) + \mathcal{U}_1(\varepsilon,\mathfrak{q}) + \mathcal{U}_2(\varepsilon,\mathfrak{q}) + \mathcal{U}_3(\varepsilon,\mathfrak{q}),\\
&\vdots\\
C_i(\varepsilon,\mathfrak{q}) =& \mathcal{U}_0(\varepsilon,\mathfrak{q}) + \mathcal{U}_1(\varepsilon,\mathfrak{q}) + \mathcal{U}_2(\varepsilon,\mathfrak{q}) + \cdots + \mathcal{U}_i(\varepsilon,\mathfrak{q})
\end{aligned} \tag{24}$$

To obtain the proper result, we must show that $C_i(\varepsilon,\mathfrak{q})$ forms a "Cauchy sequence." Consider the following:

$$\begin{aligned}
||C_{i+1}(\varepsilon,\mathfrak{q}) - C_i(\varepsilon,\mathfrak{q})|| = ||\mathcal{U}_{i+1}(\varepsilon,\mathfrak{q})|| \leq \sigma||\mathcal{U}_i(\varepsilon,\mathfrak{q})|| \leq \sigma^2||\mathcal{U}_{i-1}(\varepsilon,\mathfrak{q})|| \leq \sigma^3||\mathcal{U}_{i-2}(\varepsilon,\mathfrak{q})|| \cdots\\
\leq \sigma_{i+1}||\mathcal{U}_0(\varepsilon,\mathfrak{q})||.
\end{aligned} \tag{25}$$

For $i,n \in N$, we obtain:

$$\begin{aligned}
||C_i(\varepsilon,\mathfrak{q}) - C_n(\varepsilon,\mathfrak{q})|| =& ||\mathcal{U}_{i+n}(\varepsilon,\mathfrak{q})|| = ||C_i(\varepsilon,\mathfrak{q}) - C_{i-1}(\varepsilon,\mathfrak{q}) + (C_{i-1}(\varepsilon,\mathfrak{q}) - C_{i-2}(\varepsilon,\mathfrak{q}))\\
&+ (C_{i-2}(\varepsilon,\mathfrak{q}) - C_{i-3}(\varepsilon,\mathfrak{q})) + \cdots + (C_{n+1}(\varepsilon,\mathfrak{q}) - C_n(\varepsilon,\mathfrak{q}))||\\
\leq& ||C_i(\varepsilon,\mathfrak{q}) - C_{i-1}(\varepsilon,\mathfrak{q})|| + ||(C_{i-1}(\varepsilon,\mathfrak{q}) - C_{i-2}(\varepsilon,\mathfrak{q}))||\\
&+ ||(C_{i-2}(\varepsilon,\mathfrak{q}) - C_{i-3}(\varepsilon,\mathfrak{q}))|| + \cdots + ||(C_{n+1}(\varepsilon,\mathfrak{q}) - C_n(\varepsilon,\mathfrak{q}))||\\
\leq& \sigma^i||\mathcal{U}_0(\varepsilon,\mathfrak{q})|| + \sigma^{i-1}||\mathcal{U}_0(\varepsilon,\mathfrak{q})|| + \cdots + \sigma^{i+1}||\mathcal{U}_0(\varepsilon,\mathfrak{q})||\\
=& ||\mathcal{U}_0(\varepsilon,\mathfrak{q})||(\sigma^i + \sigma^{i-1} + \sigma^{i+1})\\
=& ||\mathcal{U}_0(\varepsilon,\mathfrak{q})||\frac{1-\sigma^{i-n}}{1-\sigma^{i+1}}\sigma^{n+1}.
\end{aligned} \tag{26}$$

Since $0 < \sigma < 1$ and $\mathcal{U}_0(\varepsilon, \mathfrak{q})$ is bounded, take $\beta = 1 - \sigma/(1 - \sigma_{i-n})\sigma^{n+1}||\mathcal{U}_0(\varepsilon, \mathfrak{q})||$, and we obtain Thus, $\{\mathcal{U}_i(\varepsilon, \mathfrak{q})\}_{i=0}^{\infty}$ forms a "Cauchy sequence" in H. That is, the following sequence $\{\mathcal{U}_i(\varepsilon, \mathfrak{q})\}_{i=0}^{\infty}$ is a convergent sequences with the limits $\lim_{i\to\infty} \mathcal{U}_i(\varepsilon, \mathfrak{q}) = \mathcal{U}(\varepsilon, \mathfrak{q})$ for $\exists \mathcal{U}(\varepsilon, \mathfrak{q}) \in \mathcal{H}$. As a result, the proof is complete. □

Theorem 2. *Let $\sum_{h=0}^{k} \mathcal{U}_h(\varepsilon, \mathfrak{q})$ be finite and $\mathcal{U}(\varepsilon, \mathfrak{q})$ show the obtained series result. Let $\sigma > 0$ such that $||\mathcal{U}_{h+1}(\varepsilon, \mathfrak{q})|| \leq ||\mathcal{U}_h(\varepsilon, \mathfrak{q})||$; therefore, the following analysis yields the maximum absolute error.*

$$||\mathcal{U}(\varepsilon, \mathfrak{q}) - \sum_{h=0}^{k} \mathcal{U}_h(\varepsilon, \mathfrak{q})|| < \frac{\sigma^{k+1}}{1-\sigma}||\mathcal{U}_0(\varepsilon, \mathfrak{q})||. \tag{27}$$

Proof. Since $\sum_{h=0}^{k} \mathcal{U}_h(\varepsilon, \mathfrak{q})$ is finite, this implies that $\sum_{h=0}^{k} \mathcal{U}_h(\varepsilon, \mathfrak{q}) < \infty$.

Consider:

$$\begin{aligned} ||\mathcal{U}(\varepsilon, \mathfrak{q}) - \sum_{h=0}^{k} \mathcal{U}_h(\varepsilon, \mathfrak{q})|| =& || \sum_{h=k+1}^{\infty} \mathcal{U}_h(\varepsilon, \mathfrak{q})|| \\ &\leq \sum_{h=k+1}^{\infty} ||\mathcal{U}_h(\varepsilon, \mathfrak{q})|| \\ &\leq \sum_{h=k+1}^{\infty} \sigma^h ||\mathcal{U}_0(\varepsilon, \mathfrak{q})|| \\ &\leq \sigma^{k+1}(1 + \sigma + \sigma^2 + \cdots)||\mathcal{U}_0(\varepsilon, \mathfrak{q})|| \\ &\leq \frac{\sigma^{k+1}}{1-\sigma}||\mathcal{U}_0(\varepsilon, \mathfrak{q})||. \end{aligned} \tag{28}$$

As a result, the proof is complete. □

6. The General Discussion of the VITM

In this portion, we describe the VITM solution for the fractional-order PDEs.

$$\begin{aligned} &{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{U}(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{L}_1(\mathcal{U}, \mathcal{V}) + \mathcal{N}_1(\mathcal{U}, \mathcal{V}) - \mathcal{G}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) = 0, \\ &{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{V}(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{L}_2(\mathcal{U}, \mathcal{V}) + \mathcal{N}_2(\mathcal{U}, \mathcal{V}) - \mathcal{G}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) = 0, \quad m-1 < \wp \leq m, \end{aligned} \tag{29}$$

with the initial conditions being:

$$\mathcal{U}(\varepsilon, \mathfrak{b}, 0) = g_1(\varepsilon, \mathfrak{b}), \quad \mathcal{V}(\varepsilon, \mathfrak{b}, 0) = g_2(\varepsilon, \mathfrak{b}), \tag{30}$$

where is $D_{\mathfrak{q}}^{\wp} = \frac{\partial^{\wp}}{\partial \mathfrak{q}^{\wp}}$ the fractional Caputo operator of $\wp$, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{N}_1$, $\mathcal{N}_2$ are linear and non-linear terms, respectively, and $\mathcal{G}_1$,$\mathcal{G}_2$ are the source functions.
The Yang transform is applied to Equation (29), and we have:

$$\begin{aligned} &\mathbb{Y}[D_{\mathfrak{q}}^{\wp}\mathcal{U}(\varepsilon, \mathfrak{b}, \mathfrak{q})] + \mathbb{Y}[\mathcal{L}_1(\mathcal{U}, \mathcal{V}) + \mathcal{N}_1(\mathcal{U}, \mathcal{V}) - \mathcal{G}_1(\varepsilon, \mathfrak{b}, \mathfrak{q})] = 0, \\ &\mathbb{Y}[D_{\mathfrak{q}}^{\wp}\mathcal{V}(\varepsilon, \mathfrak{b}, \mathfrak{q})] + \mathbb{Y}[\mathcal{L}_2(\mathcal{U}, \mathcal{V}) + \mathcal{N}_2(\mathcal{U}, \mathcal{V}) - \mathcal{G}_2(\varepsilon, \mathfrak{b}, \mathfrak{q})] = 0. \end{aligned} \tag{31}$$

Applying the differentiation property of the Yang transform, we obtain:

$$\begin{aligned} &\mathbb{Y}[\mathcal{U}(\varepsilon, \mathfrak{b}, \mathfrak{q})] = -\mathbb{Y}[\mathcal{L}_1(\mathcal{U}, \mathcal{V}) + \mathcal{N}_1(\mathcal{U}, \mathcal{V}) - \mathcal{G}_1(\varepsilon, \mathfrak{b}, \mathfrak{q})], \\ &\mathbb{Y}[\mathcal{V}(\varepsilon, \mathfrak{b}, \mathfrak{q})] = -\mathbb{Y}[\mathcal{L}_2(\mathcal{U}, \mathcal{V}) + \mathcal{N}_2(\mathcal{U}, \mathcal{V}) - \mathcal{G}_2(\varepsilon, \mathfrak{b}, \mathfrak{q})], \end{aligned} \tag{32}$$

$$\begin{aligned}\mathbb{Y}[\mathcal{U}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q})] =&\mathbb{Y}[\mathcal{U}_m(\varepsilon,\mathfrak{b},\mathfrak{q})]+\lambda(s)\bigg[\frac{1}{(1+\wp(s-1))}\mathcal{U}_m(\varepsilon,\mathfrak{b},\mathfrak{q})\\&-\mathbb{Y}[\mathcal{G}_1(\varepsilon,\mathfrak{b},\mathfrak{q})]-\mathbb{Y}\{\mathcal{L}_1(\mathcal{U},\mathcal{V})+\mathcal{N}_1(\mathcal{U},\mathcal{V})\}],\\ \mathbb{Y}[\mathcal{V}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q})] =&\mathbb{Y}[\mathcal{V}_m(\varepsilon,\mathfrak{b},\mathfrak{q})]+\lambda(s)\bigg[\frac{1}{(1+\wp(s-1))}\mathcal{V}_m(\varepsilon,\mathfrak{b},\mathfrak{q})\\&-\mathbb{Y}[\mathcal{G}_2(\varepsilon,\mathfrak{b},\mathfrak{q})]-\mathbb{Y}\{\mathcal{L}_2(\mathcal{U},\mathcal{V})+\mathcal{N}_2(\mathcal{U},\mathcal{V})\}].\end{aligned} \tag{33}$$

The Lagrange multiplier is as:

$$\lambda(s) = -(1+\wp(s-1)) \tag{34}$$

Using the inverse Yang transform $\mathbb{Y}^{-1}$ in Equation (33):

$$\begin{aligned}\mathcal{U}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q}) =&\mathcal{U}_m(\varepsilon,\mathfrak{b},\mathfrak{q})-\mathbb{Y}^{-1}\bigg[(1+\wp(s-1))\bigg[\frac{1}{(1+\wp(s-1))}\mathcal{U}_m(\varepsilon,\mathfrak{b},\mathfrak{q})\\&-\mathbb{Y}[\mathcal{G}_1(\varepsilon,\mathfrak{b},\mathfrak{q})]-\mathbb{Y}\{\mathcal{L}_1(\mathcal{U},\mathcal{V})+\mathcal{N}_1(\mathcal{U},\mathcal{V})\}]],\\ \mathcal{V}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q}) =&\mathcal{V}_m(\varepsilon,\mathfrak{b},\mathfrak{q})-\mathbb{Y}^{-1}\bigg[(1+\wp(s-1))\bigg[\frac{1}{(1+\wp(s-1))}\mathcal{V}_m(\varepsilon,\mathfrak{b},\mathfrak{q})\\&-\mathbb{Y}[\mathcal{G}_2(\varepsilon,\mathfrak{b},\mathfrak{q})]-\mathbb{Y}\{\mathcal{L}_2(\mathcal{U},\mathcal{V})+\mathcal{N}_2(\mathcal{U},\mathcal{V})\}]]\end{aligned} \tag{35}$$

The initial value can be found as:

$$\begin{aligned}\mathcal{U}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathbb{Y}^{-1}[s\{\mathcal{U}(\varepsilon,\mathfrak{b},0)\}],\\ \mathcal{V}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathbb{Y}^{-1}[s\{\mathcal{V}(\varepsilon,\mathfrak{b},0)\}].\end{aligned} \tag{36}$$

7. Applications

7.1. Example 1

Consider the fractional scheme of the non-linear gas equations:

$$\begin{aligned}&{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{U}+\mathcal{U}\frac{\partial\mathcal{U}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{U}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\varepsilon}=0,\\ &{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{V}+\mathcal{U}\frac{\partial\mathcal{V}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}=0,\\ &{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{X}+\mathcal{U}\frac{\partial\mathcal{X}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{X}}{\partial\mathfrak{b}}+\mathcal{X}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)=0,\\ &{}^{CF}D_{\mathfrak{q}}^{\wp}\mathcal{W}+\mathcal{U}\frac{\partial\mathcal{W}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}+\mathfrak{q}\mathcal{W}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)=0,\end{aligned} \tag{37}$$

with the initial conditions:

$$\begin{aligned}&\mathcal{U}(\varepsilon,\mathfrak{b},0)=e^{\varepsilon+\mathfrak{b}},\ \ \mathcal{V}(\varepsilon,\mathfrak{b},0)=-1-e^{\varepsilon+\mathfrak{b}},\\ &\mathcal{X}(\varepsilon,\mathfrak{b},0)=e^{\varepsilon+\mathfrak{b}},\ \ \mathcal{W}(\varepsilon,\mathfrak{b},0)=c,\end{aligned} \tag{38}$$

where c is the real constant.

7.2. Case 1

First, we solve this system with the help of the **HPTM**.

Now, applying the Yang transform of Equation (37), we obtain:

$$\begin{aligned}
&\frac{1}{(1+\wp(s-1))}\mathbb{Y}[\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q})]=\frac{s}{(1+\wp(s-1))}\mathcal{U}(\varepsilon,\mathfrak{b},0)-\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{U}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{U}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\varepsilon}\right\},\\
&\frac{1}{(1+\wp(s-1))}\mathbb{Y}[\mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q})]=\frac{s}{(1+\wp(s-1))}\mathcal{V}(\varepsilon,\mathfrak{b},0)-\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{V}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}\right\},\\
&\frac{1}{(1+\wp(s-1))}\mathbb{Y}[\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q})]=\frac{s}{(1+\wp(s-1))}\mathcal{X}(\varepsilon,\mathfrak{b},0)-\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{X}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{X}}{\partial\mathfrak{b}}+\mathcal{X}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)\right\},\\
&\frac{1}{(1+\wp(s-1))}\mathbb{Y}[\mathcal{W}(\varepsilon,\mathfrak{b},\mathfrak{q})]=\frac{s}{(1+\wp(s-1))}\mathcal{W}(\varepsilon,\mathfrak{b},0)-\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{W}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}+\mathfrak{q}\mathcal{W}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)\right\},
\end{aligned}\tag{39}$$

$$\begin{aligned}
&\mathbb{Y}[\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q})]=s\mathcal{U}(\varepsilon,\mathfrak{b},0)-(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{U}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{U}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\varepsilon}\right\},\\
&\mathbb{Y}[\mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q})]=s\mathcal{U}(\varepsilon,\mathfrak{b},0)-(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{V}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}\right\},\\
&\mathbb{Y}[\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q})]=s\mathcal{U}(\varepsilon,\mathfrak{b},0)-(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{X}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{X}}{\partial\mathfrak{b}}+\mathcal{X}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)\right\},\\
&\mathbb{Y}[\mathcal{W}(\varepsilon,\mathfrak{b},\mathfrak{q})]=s\mathcal{U}(\varepsilon,\mathfrak{b},0)-(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{W}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}+\mathfrak{q}\mathcal{W}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)\right\}.
\end{aligned}\tag{40}$$

Taking the inverse Yang transform, we obtain:

$$\begin{aligned}
&\mathbb{Y}[\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q})]=e^{\varepsilon+\mathfrak{b}}-\mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{U}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{U}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\varepsilon}\right\}\right],\\
&\mathbb{Y}[\mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q})]=-1-e^{\varepsilon+\mathfrak{b}}-\mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{V}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}+\frac{1}{\mathcal{X}}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}\right\}\right],\\
&\mathbb{Y}[\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q})]=e^{\varepsilon+\mathfrak{b}}-\mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{X}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{X}}{\partial\mathfrak{b}}+\mathcal{X}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)\right\}\right],\\
&\mathbb{Y}[\mathcal{W}(\varepsilon,\mathfrak{b},\mathfrak{q})]=c-\mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\mathcal{U}\frac{\partial\mathcal{W}}{\partial\varepsilon}+\mathcal{V}\frac{\partial\mathcal{W}}{\partial\mathfrak{b}}+\mathfrak{q}\mathcal{W}\left(\frac{\partial\mathcal{U}}{\partial\varepsilon}+\frac{\partial\mathcal{V}}{\partial\mathfrak{b}}\right)\right\}\right].
\end{aligned}\tag{41}$$

Implementing the HPM in Equation (41), we can obtain:

$$\begin{aligned}
\sum_{\kappa=0}^{\infty}p^{\kappa}\mathcal{U}_{\kappa}(\varepsilon,\mathfrak{b})=&e^{\varepsilon+\mathfrak{b}}-p\left[\mathbb{Y}^{-1}\left\{(1+\wp(s-1))\mathbb{Y}\left\{\sum_{\kappa=0}^{\infty}p^{\kappa}H_{\kappa}(\mathcal{U})\right\}\right\}\right],\\
\sum_{\kappa=0}^{\infty}p^{\kappa}\mathcal{V}_{\kappa}(\varepsilon,\mathfrak{b})=&-1-e^{\varepsilon+\mathfrak{b}}-p\left[\mathbb{Y}^{-1}\left\{(1+\wp(s-1))\mathbb{Y}\left\{\sum_{\kappa=0}^{\infty}p^{\kappa}H_{\kappa}(\mathcal{V})\right\}\right\}\right],\\
\sum_{\kappa=0}^{\infty}p^{\kappa}\mathcal{X}_{\kappa}(\varepsilon,\mathfrak{b})=&e^{\varepsilon+\mathfrak{b}}-p\left[\mathbb{Y}^{-1}\left\{(1+\wp(s-1))\mathbb{Y}\left\{\sum_{\kappa=0}^{\infty}p^{\kappa}H_{\kappa}(\mathcal{X})\right\}\right\}\right],\\
\sum_{\kappa=0}^{\infty}p^{\kappa}\mathcal{W}_{\kappa}(\varepsilon,\mathfrak{b})=&c-p\left[\mathbb{Y}^{-1}\left\{(1+\wp(s-1))\mathbb{Y}\left\{\sum_{\kappa=0}^{\infty}p^{\kappa}H_{\kappa}(\mathcal{W})\right\}\right\}\right],
\end{aligned}\tag{42}$$

where $H_{\kappa}(\mathcal{U})$, $H_{\kappa}(\mathcal{V})$, $H_{\kappa}(\mathcal{X})$, and $H_{\kappa}(\mathcal{W})$ are He's polynomials, which signify the nonlinear terms. The first few terms of He's polynomials are suggested as:

$$H_0(\mathcal{U}) = \mathcal{U}_0\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{U}_0}{\partial \mathfrak{b}} + \frac{1}{\mathcal{X}_0}\frac{\partial \mathcal{W}_0}{\partial \varepsilon},$$
$$H_0(\mathcal{V}) = \mathcal{U}_0\frac{\partial \mathcal{V}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}} + \frac{1}{\mathcal{X}_0}\frac{\partial \mathcal{W}_0}{\partial \mathfrak{b}},$$
$$H_0(\mathcal{X}) = \mathcal{U}_0\frac{\partial \mathcal{X}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{X}_0}{\partial \mathfrak{b}} + \mathcal{X}_0\left(\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}}\right),$$
$$H_0(\mathcal{W}) = \mathcal{U}_0\frac{\partial \mathcal{W}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{W}_0}{\partial \mathfrak{b}} + \mathfrak{q}\mathcal{W}\left(\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}}\right).$$

$$H_1(\mathcal{U}) = \mathcal{U}_1\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{U}_0\frac{\partial \mathcal{U}_1}{\partial \varepsilon} + \mathcal{V}_1\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{U}_1}{\partial \varepsilon} + \frac{1}{\mathcal{X}_0}\left(\mathcal{X}_0\frac{\partial \mathcal{W}_1}{\partial \varepsilon} - \mathcal{X}_1\frac{\partial \mathcal{W}_0}{\partial \varepsilon}\right),$$
$$H_1(\mathcal{V}) = \mathcal{U}_1\frac{\partial \mathcal{V}_0}{\partial \varepsilon} + \mathcal{U}_0\frac{\partial \mathcal{V}_1}{\partial \varepsilon} + \mathcal{V}_1\frac{\partial \mathcal{V}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{V}_1}{\partial \varepsilon} + \frac{1}{\mathcal{X}_0}\left(\mathcal{X}_0\frac{\partial \mathcal{W}_1}{\partial \mathfrak{b}} - \mathcal{X}_1\frac{\partial \mathcal{W}_0}{\partial \mathfrak{b}}\right),$$
$$H_1(\mathcal{X}) = \mathcal{U}_1\frac{\partial \mathcal{X}_0}{\partial \varepsilon} + \mathcal{U}_0\frac{\partial \mathcal{X}_1}{\partial \varepsilon} + \mathcal{V}_1\frac{\partial \mathcal{X}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{X}_1}{\partial \varepsilon} + \mathcal{X}_1\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{X}_0\frac{\partial \mathcal{U}_1}{\partial \varepsilon} + \mathcal{X}_1\frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}} + \mathcal{X}_0\frac{\partial \mathcal{V}_1}{\partial \mathfrak{b}},$$
$$H_1(\mathcal{W}) = \mathcal{U}_1\frac{\partial \mathcal{W}_0}{\partial \varepsilon} + \mathcal{U}_0\frac{\partial \mathcal{W}_1}{\partial \varepsilon} + \mathcal{V}_1\frac{\partial \mathcal{W}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{W}_1}{\partial \varepsilon} + \mathcal{W}_1\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{W}_0\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{W}_1\frac{\partial \mathcal{W}_0}{\partial \mathfrak{b}} + \mathcal{W}_0\frac{\partial \mathcal{V}_1}{\partial \mathfrak{b}}.$$

On both sides of the comparison coefficient of p, we have:

$$p^0 : \mathcal{U}_0(\varepsilon, \mathfrak{b}, \mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}, \;\; p^0 : \mathcal{V}_0(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -1 - e^{\varepsilon+\mathfrak{b}},$$
$$p^0 : \mathcal{X}_0(\varepsilon, y, \mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}, \;\; p^0 : \mathcal{W}_0(\varepsilon, y, \mathfrak{q}) = c. \tag{43}$$

$$p^1 : \mathcal{U}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_0(\mathcal{U})\}] = e^{\varepsilon+\mathfrak{b}}\Big\{1 + \wp\mathfrak{q} - \wp\Big\},$$
$$p^1 : \mathcal{V}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_0(\mathcal{V})\}] = -e^{\varepsilon+\mathfrak{b}}\Big\{1 + \wp\mathfrak{q} - \wp\Big\},$$
$$p^1 : \mathcal{X}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_0(\mathcal{X})\}] = e^{\varepsilon+\mathfrak{b}}\Big\{1 + \wp\mathfrak{q} - \wp\Big\},$$
$$p^1 : \mathcal{W}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_0(\mathcal{W})\}] = 0,$$

$$p^2 : \mathcal{U}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_1(\mathcal{U})\}] = e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\Big\},$$
$$p^2 : \mathcal{V}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_1(\mathcal{V})\}] = -e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\Big\},$$
$$p^2 : \mathcal{X}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_1(\mathcal{X})\}] = e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\Big\},$$
$$p^2 : \mathcal{W}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) = -\mathbb{Y}^{-1}[(1 + \wp(s-1))\mathbb{Y}\{H_1(\mathcal{W})\}] = 0,$$
$$\vdots$$

The suggested problem is a series form result defined as results in Figures 1–3:

$$\mathcal{U}_m(\varepsilon, \mathfrak{b}, \mathfrak{q}) = \mathcal{U}_0(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{U}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{U}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{U}_3(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \cdots,$$
$$\mathcal{V}_m(\varepsilon, \mathfrak{b}, \mathfrak{q}) = \mathcal{V}_0(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{V}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{V}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{V}_3(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \cdots,$$
$$\mathcal{X}_m(\varepsilon, \mathfrak{b}, \mathfrak{q}) = \mathcal{X}_0(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{X}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{X}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{X}_3(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \cdots,$$
$$\mathcal{W}_m(\varepsilon, \mathfrak{b}, 0) = \mathcal{W}_0(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{W}_1(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{W}_2(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \mathcal{W}_3(\varepsilon, \mathfrak{b}, \mathfrak{q}) + \cdots,$$

$$
\begin{aligned}
\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q}) =& e^{\varepsilon+\mathfrak{b}} + e^{\varepsilon+\mathfrak{b}}\Big\{1+\wp\mathfrak{q}-\wp\Big\} + e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)2\wp\mathfrak{q}+(1-\wp)^2+\frac{\wp^2\mathfrak{q}^2}{2}\Big\}+\\
& e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)^2 3\wp\mathfrak{q}+(1-\wp)^3+\frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2}+\frac{\wp^3\mathfrak{q}^3}{3!}\Big\}+\cdots,\\
\mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q}) =& -1-e^{\varepsilon+\mathfrak{b}} - e^{\varepsilon+\mathfrak{b}}\Big\{1+\wp\mathfrak{q}-\wp\Big\} - e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)2\wp\mathfrak{q}+(1-\wp)^2+\frac{\wp^2\mathfrak{q}^2}{2}\Big\}-\\
& e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)^2 3\wp\mathfrak{q}+(1-\wp)^3+\frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2}+\frac{\wp^3\mathfrak{q}^3}{3!}\Big\}-\cdots,\\
\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q}) =& e^{\varepsilon+\mathfrak{b}} + e^{\varepsilon+\mathfrak{b}}\Big\{1+\wp\mathfrak{q}-\wp\Big\} + e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)2\wp\mathfrak{q}+(1-\wp)^2+\frac{\wp^2\mathfrak{q}^2}{2}\Big\}+\\
& e^{\varepsilon+\mathfrak{b}}\Big\{(1-\wp)^2 3\wp\mathfrak{q}+(1-\wp)^3+\frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2}+\frac{\wp^3\mathfrak{q}^3}{3!}\Big\}+\cdots,\\
\mathcal{W}(\varepsilon,\mathfrak{b},\mathfrak{q}) =& c+0+\cdots
\end{aligned}
$$

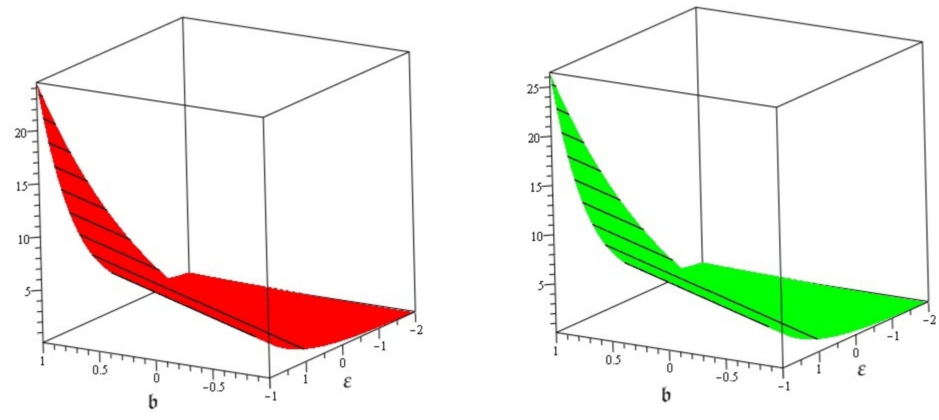

Figure 1. The solution graph of the HPTM/VITM of $\mathcal{U}(\varepsilon,\mathfrak{b})$ of Example 1 at $\wp = 1$ (**left**) and 0.8 (**right**).

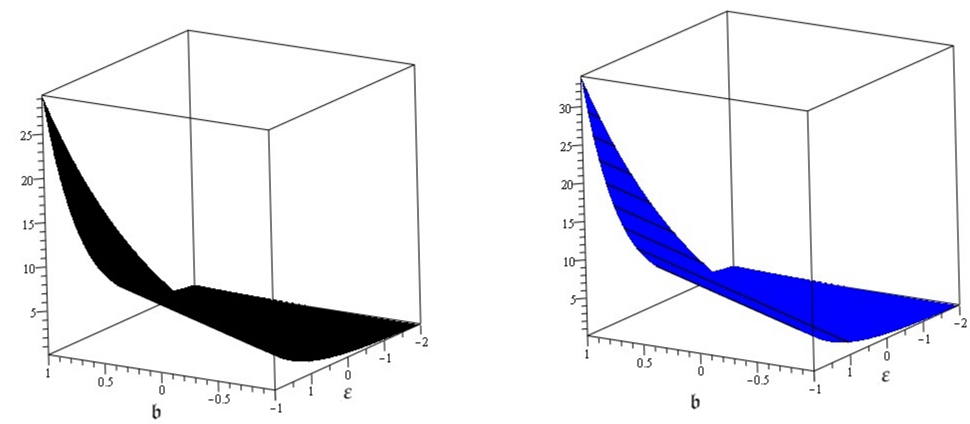

Figure 2. The solution graph of the HPTM/VITM of $\mathcal{U}(\varepsilon,\mathfrak{b})$ of Example 1 at $\wp = 0.6$ (**left**) and 0.4 (**right**).

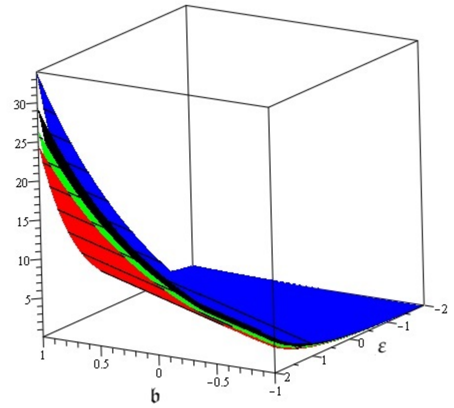

Figure 3. The graph of the HPTM/VITM of $\mathcal{U}(\varepsilon, \mathfrak{b})$ of Example 1 at various fractional orders.

7.3. Case 2

The approximate result by the **VITM**:
According to the iteration technique Equation (37), we obtain:

$$\begin{aligned}
\mathcal{U}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{U}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{U}_m}{\partial \mathfrak{q}} + \mathcal{U}_m\frac{\partial \mathcal{U}_m}{\partial \varepsilon} + \mathcal{V}_m\frac{\partial \mathcal{U}_m}{\partial \mathfrak{b}} + \frac{1}{\mathcal{X}_m}\frac{\partial \mathcal{W}_m}{\partial \varepsilon}\right\}\right],\\
\mathcal{V}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{V}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{V}_m}{\partial \mathfrak{q}} + \mathcal{U}_m\frac{\partial \mathcal{V}_m}{\partial \varepsilon} + \mathcal{V}_m\frac{\partial \mathcal{V}_m}{\partial \mathfrak{b}} + \frac{1}{\mathcal{X}_m}\frac{\partial \mathcal{W}_m}{\partial \mathfrak{b}}\right\}\right],\\
\mathcal{X}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{X}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{U}_m}{\partial \mathfrak{q}} + \mathcal{U}_m\frac{\partial \mathcal{X}_m}{\partial \varepsilon} + \mathcal{V}_m\frac{\partial \mathcal{X}_m}{\partial \mathfrak{b}}\right.\right.\\
&\left.\left. + \mathcal{X}_m\left(\frac{\partial \mathcal{U}_m}{\partial \varepsilon} + \frac{\partial \mathcal{V}_m}{\partial \mathfrak{b}}\right)\right\}\right],\\
\mathcal{W}_{m+1}(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{W}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{V}_m}{\partial \mathfrak{q}} + \mathcal{U}_m\frac{\partial \mathcal{W}_m}{\partial \varepsilon} + \mathcal{V}_m\frac{\partial \mathcal{W}_m}{\partial \mathfrak{b}}\right.\right.\\
&\left.\left. + \mathfrak{q}\mathcal{W}_m\left(\frac{\partial \mathcal{U}_m}{\partial \varepsilon} + \frac{\partial \mathcal{V}_m}{\partial \mathfrak{b}}\right)\right\}\right],
\end{aligned} \tag{44}$$

with the initial sources:

$$\begin{aligned}
&\mathcal{U}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}, \;\; \mathcal{V}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) = -1 - e^{\varepsilon+\mathfrak{b}},\\
&\mathcal{X}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}, \;\; \mathcal{W}_0(\varepsilon,\mathfrak{b},0) = c.
\end{aligned} \tag{45}$$

For $m = 0, 1, 2, \cdots$

$$\begin{aligned}
\mathcal{U}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{U}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{U}_0}{\partial \mathfrak{q}} + \mathcal{U}_0\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{U}_0}{\partial \mathfrak{b}} + \frac{1}{\mathcal{X}_0}\frac{\partial \mathcal{W}_0}{\partial \varepsilon}\right\}\right],\\
\mathcal{V}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{V}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{V}_0}{\partial \mathfrak{q}} + \mathcal{U}_0\frac{\partial \mathcal{V}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}} + \frac{1}{\mathcal{X}_0}\frac{\partial \mathcal{W}_0}{\partial \mathfrak{b}}\right\}\right],\\
\mathcal{X}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{X}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{U}_0}{\partial \mathfrak{q}} + \mathcal{U}_0\frac{\partial \mathcal{X}_0}{\partial \varepsilon_0} + \mathcal{V}_0\frac{\partial \mathcal{X}_0}{\partial \mathfrak{b}_0} + \mathcal{X}_0\left(\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}}\right)\right\}\right],\\
\mathcal{W}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) &= \mathcal{W}_0(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial \mathcal{V}_0}{\partial \mathfrak{q}} + \mathcal{U}_0\frac{\partial \mathcal{W}_0}{\partial \varepsilon} + \mathcal{V}_0\frac{\partial \mathcal{W}_0}{\partial \mathfrak{b}} + \mathfrak{q}\mathcal{W}_0\left(\frac{\partial \mathcal{U}_0}{\partial \varepsilon} + \frac{\partial \mathcal{V}_0}{\partial \mathfrak{b}}\right)\right\}\right],\\
\mathcal{U}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) &= e^{\varepsilon+\mathfrak{b}}\left\{1+\left\{1+\wp\mathfrak{q}-\wp\right\}\right\}, \qquad \mathcal{V}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) = -1 - e^{\varepsilon+\mathfrak{b}}\left\{1+\left\{1+\wp\mathfrak{q}-\wp\right\}\right\},\\
\mathcal{X}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) &= e^{\varepsilon+\mathfrak{b}}\left\{1+\left\{1+\wp\mathfrak{q}-\wp\right\}\right\}, \qquad \mathcal{W}_1(\varepsilon,\mathfrak{b},0) = c+0.
\end{aligned}$$

$$\mathcal{U}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{U}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{U}_1}{\partial\mathfrak{q}} + \mathcal{U}_1\frac{\partial\mathcal{U}_1}{\partial\varepsilon} + \mathcal{V}_1\frac{\partial\mathcal{U}_1}{\partial\mathfrak{b}} + \frac{1}{\mathcal{X}_1}\frac{\partial\mathcal{W}_1}{\partial\varepsilon}\right\}\right],$$
$$\mathcal{V}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{V}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{V}_1}{\partial\mathfrak{q}} + \mathcal{U}_1\frac{\partial\mathcal{V}_1}{\partial\varepsilon} + \mathcal{V}_1\frac{\partial\mathcal{V}_1}{\partial\mathfrak{b}} + \frac{1}{\mathcal{X}_1}\frac{\partial\mathcal{W}_1}{\partial\mathfrak{b}}\right\}\right],$$
$$\mathcal{X}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{X}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{U}_1}{\partial\mathfrak{q}} + \mathcal{U}_1\frac{\partial\mathcal{X}_1}{\partial\varepsilon_0} + \mathcal{V}_1\frac{\partial\mathcal{X}_1}{\partial\mathfrak{b}_1} + \mathcal{X}_1\left(\frac{\partial\mathcal{U}_1}{\partial\varepsilon} + \frac{\partial\mathcal{V}_1}{\partial\mathfrak{b}}\right)\right\}\right],$$
$$\mathcal{W}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{W}_1(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{V}_1}{\partial\mathfrak{q}} + \mathcal{U}_1\frac{\partial\mathcal{W}_1}{\partial\varepsilon} + \mathcal{V}_1\frac{\partial\mathcal{W}_1}{\partial\mathfrak{b}} + \mathfrak{q}\mathcal{W}_1\left(\frac{\partial\mathcal{U}_1}{\partial\varepsilon} + \frac{\partial\mathcal{V}_1}{\partial\mathfrak{b}}\right)\right\}\right],$$
$$\mathcal{U}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\}\right\},$$
$$\mathcal{V}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = -1 - e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\}\right\},$$
$$\mathcal{X}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\}\right\},$$
$$\mathcal{W}_2(\varepsilon,\mathfrak{b},0) = c + 0.$$
$$\mathcal{U}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{U}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{U}_2}{\partial\mathfrak{q}} + \mathcal{U}_2\frac{\partial\mathcal{U}_2}{\partial\varepsilon} + \mathcal{V}_2\frac{\partial\mathcal{U}_2}{\partial\mathfrak{b}} + \frac{1}{\mathcal{X}_2}\frac{\partial\mathcal{W}_2}{\partial\varepsilon}\right\}\right],$$
$$\mathcal{V}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{V}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{V}_2}{\partial\mathfrak{q}} + \mathcal{U}_2\frac{\partial\mathcal{V}_2}{\partial\varepsilon} + \mathcal{V}_2\frac{\partial\mathcal{V}_2}{\partial\mathfrak{b}} + \frac{1}{\mathcal{X}_2}\frac{\partial\mathcal{W}_2}{\partial\mathfrak{b}}\right\}\right],$$
$$\mathcal{X}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{X}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{U}_2}{\partial\mathfrak{q}} + \mathcal{U}_2\frac{\partial\mathcal{X}_2}{\partial\varepsilon_0} + \mathcal{V}_2\frac{\partial\mathcal{X}_2}{\partial\mathfrak{b}_2} + \mathcal{X}_2\left(\frac{\partial\mathcal{U}_2}{\partial\varepsilon} + \frac{\partial\mathcal{V}_2}{\partial\mathfrak{b}}\right)\right\}\right],$$
$$\mathcal{W}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = \mathcal{W}_2(\varepsilon,\mathfrak{b},\mathfrak{q}) - \mathbb{Y}^{-1}\left[(1+\wp(s-1))\mathbb{Y}\left\{\frac{1}{(1+\wp(s-1))}\frac{\partial\mathcal{V}_2}{\partial\mathfrak{q}} + \mathcal{U}_2\frac{\partial\mathcal{W}_2}{\partial\varepsilon} + \mathcal{V}_2\frac{\partial\mathcal{W}_2}{\partial\mathfrak{b}} + \mathfrak{q}\mathcal{W}_2\left(\frac{\partial\mathcal{U}_2}{\partial\varepsilon} + \frac{\partial\mathcal{V}_2}{\partial\mathfrak{b}}\right)\right\}\right],$$
$$\mathcal{U}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{q} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2} + \frac{\wp^3\mathfrak{q}^3}{3!}\right\}\right\},$$
$$\mathcal{V}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = -1 - e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{q} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2} + \frac{\wp^3\mathfrak{q}^3}{3!}\right\}\right\},$$
$$\mathcal{X}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{q} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2} + \frac{\wp^3\mathfrak{q}^3}{3!}\right\}\right\},$$
$$\mathcal{W}_3(\varepsilon,\mathfrak{b},\mathfrak{q}) = c + 0.$$
$$\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q}) = \sum_{m=0}^{\infty}\mathcal{U}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{q} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2} + \frac{\wp^3\mathfrak{q}^3}{3!}\right\} + \cdots.\right\},$$
$$\mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q}) = \sum_{m=0}^{\infty}\mathcal{V}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) = -1 - e^{\varepsilon+\mathfrak{b}}\left\{1 + \left\{1+\wp\mathfrak{q}-\wp\right\} + \left\{(1-\wp)2\wp\mathfrak{q} + (1-\wp)^2 + \frac{\wp^2\mathfrak{q}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{q} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2} + \frac{\wp^3\mathfrak{q}^3}{3!}\right\} + \cdots.\right\},$$

$$\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q}) = \sum_{m=0}^{\infty}\mathcal{X}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}}\bigg\{1+\Big\{1+\wp\mathfrak{q}-\wp\Big\}+\Big\{(1-\wp)2\wp\mathfrak{q}+(1-\wp)^2+\frac{\wp^2\mathfrak{q}^2}{2}\Big\}+$$
$$\Big\{(1-\wp)^2 3\wp\mathfrak{q}+(1-\wp)^3+\frac{3\wp^2(1-\wp)\mathfrak{q}^2}{2}+\frac{\wp^3\mathfrak{q}^3}{3!}\Big\}+\cdots.\bigg\},$$

$$\mathcal{W}(\varepsilon,\mathfrak{b},\mathfrak{q}) = \sum_{m=0}^{\infty}\mathcal{W}_m(\varepsilon,\mathfrak{b},\mathfrak{q}) = c+0.$$

The exact result of Equation (41) at $\wp = 1$ is results in Figures 4–9:

$$\begin{aligned}&\mathcal{U}(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}+\mathfrak{q}},\ \ \mathcal{V}(\varepsilon,\mathfrak{b},\mathfrak{q}) = -1-e^{\varepsilon+\mathfrak{b}+\mathfrak{q}},\\ &\mathcal{X}(\varepsilon,\mathfrak{b},\mathfrak{q}) = e^{\varepsilon+\mathfrak{b}+\mathfrak{q}},\ \ \mathcal{W}(\varepsilon,\mathfrak{b},\mathfrak{q}) = c.\end{aligned} \quad (46)$$

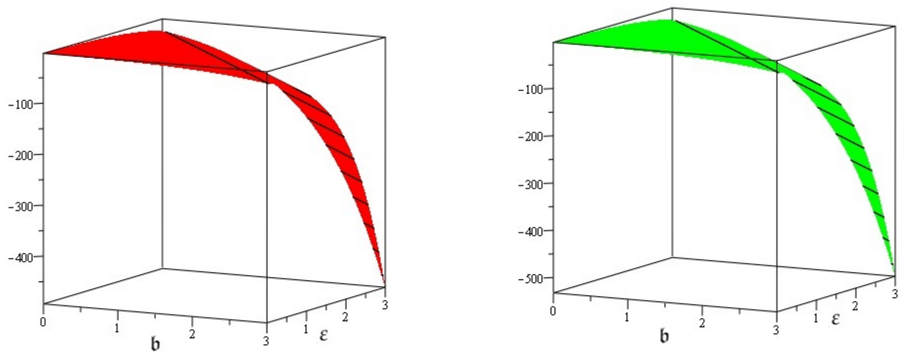

Figure 4. The solution graph of the HPTM/VITM of $\mathcal{V}(\varepsilon,\mathfrak{b})$ of Example 1 at $\wp = 1$ (**left**) and 0.8 (**right**).

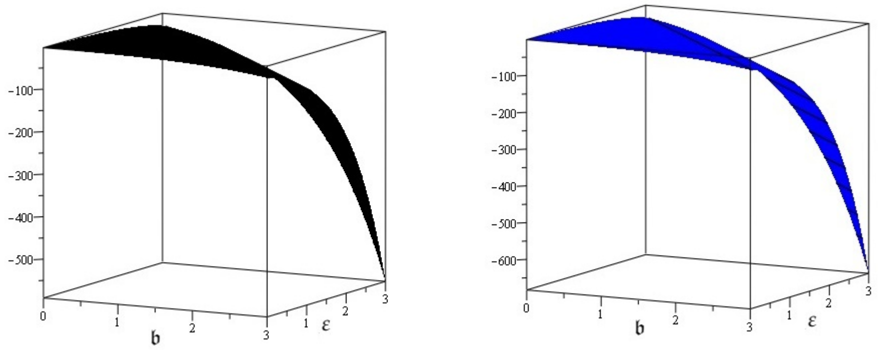

Figure 5. The solution graph of the HPTM/VITM of $\mathcal{V}(\varepsilon,\mathfrak{b})$ of Example 1 at $\wp = 0.6$ (**left**) and 0.4 (**right**).

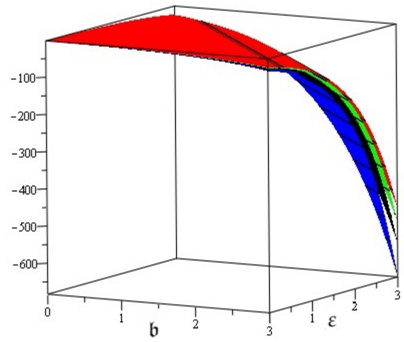

Figure 6. The solution graph of the HPTM/VITM of $\mathcal{V}(\varepsilon, \mathfrak{b})$ of Example 1 at different fractional orders.

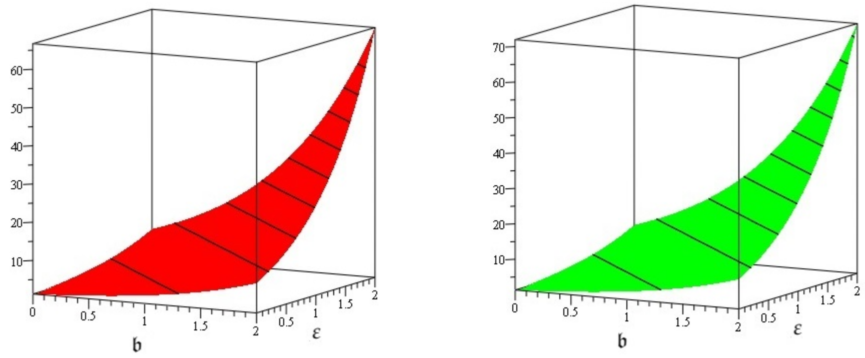

Figure 7. The solution graph of the HPTM/VITM of $\mathcal{X}(\varepsilon, \mathfrak{b})$ of Example 1 at $\wp = 1$ (**left**) and 0.8 (**right**).

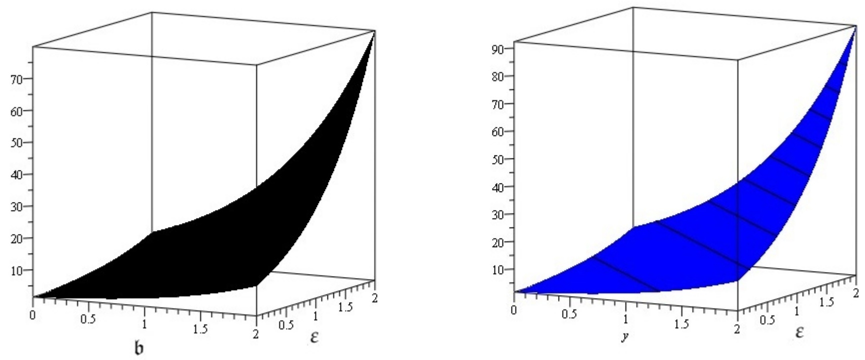

Figure 8. The solution graph of the HPTM/VITM of $\mathcal{X}(\varepsilon, \mathfrak{b})$ of Example 1 at $\wp = 0.6$ (**left**) and 0.4 (**right**).

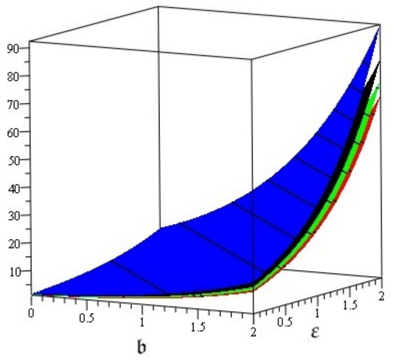

Figure 9. The solution graphof the HPTM/VITM of $\mathcal{X}(\varepsilon, \mathfrak{b})$ of example 1 at different fractional orders.

8. Results and Discussion

In this section, we analysis the solution-figures of problem which have been investigated by applying homotopy perturbation transformation method and the variational iterative transformation method in the sense of Caputo-Fabrizio. Figure 1, represents the three-dimensional solution-figures for variables $\mathcal{U}$ of example 1 at fractional order $\wp = 1$ and 0.8, respectively in Figure 2 at $\wp = 0.6$ and 0.4. It is observed that homotopy perturbation transformation method and the variational iterative transformation method solution-figures are identical and close contact with each other. In Figure 3, show that the different fractional order graph of $\wp$. In similar way in Figure 4 represents the three-dimensional solution-figures for variables $\mathcal{V}$ of example 1 at fractional order $\wp = 1$ and 0.8, respectively in Figure 5 at $\wp = 0.6$ and 0.4. In Figure 6, show that the different fractional order graph of $\wp$. The same graphs of the suggested methods are attained and confirmed the applicability of the present techniques. In Figures 7–9, the homotopy perturbation transformation method and the variational iterative transformation method solutions are plotted in three dimensional at fractional-order $\wp = 1, 0.8, 0.6$ and 0.4 of example 1. The convergence phenomenon of the fractional-solutions towards integer-solution is observed. The same accuracy is achieved by using the present techniques.

9. Conclusions

This paper used the homotopy perturbation transform methodology and the variational iterative transform method to provide numerical fractional-order solutions for a non-linear system of the unsteady flow of a polytropic gas, which is widely employed in applied mathematics as a challenge for spatial effects. The procedures provide a succession of convergent findings in physical models. The findings of this research are predicted to be beneficial in the analysis of complex non-linear physical issues in the future. These strategies have extremely basic and clear computations. As a solution, we may deduce that these approaches can be applied to a wide range of non-linear fractional-order systems of PDEs.

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