Novel cosmological and black hole solutions in Einstein and higher-derivative gravity in two dimensions

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We consider cosmological and black hole solutions in the Einstein and higher-derivative gravity in two dimensions where the theory is formulated first in D dimensions. In the limit that D tends to 2 with simultaneous singular rescaling of the scalar curvature coupling constant as $1/(D-2)$, we get the novel Einstein and higher-derivative gravity. Due to non-trivial contribution of scalar curvature which is topological invariant in exactly two dimensions to gravitational equations in the two-dimensional limit one gets novel cosmological and black hole solutions. In particular, the de Sitter and radiation-dominated universe and the Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter black hole solutions are obtained and their properties are discussed.

PACS numbers:

I. INTRODUCTION

The study of dilatonic gravity at different dimensions attracts some attention for a variety of reasons (for review, see Ref. [1]). In the first place, this theory is closely related with string theory, where it appears as a sort of effective action. A lot of activity is connected with the study of two-dimensional dilatonic gravity, especially with its cosmology and black hole solutions. Of course, at early studies of two-dimensional dilatonic gravity it was repeated dozens times that the hope exists that through the study of easier two-dimensional models one can get some insights useful for the investigation of realistic four dimensional gravity. However, the fundamental reason for the consideration of two-dimensional dilatonic gravity is the fact that scalar curvature is topological invariant in two dimensions. As a result two-dimensional analog of General Relativity with cosmological constant does not give any dynamics due to the absence of consistent gravitational equations. The extra scalar degree of freedom suggests the way to overcome this difficulty and leads to one of the forms of dilatonic gravity.

In this paper we show that non-trivial two-dimensional analog of General Relativity may be formulated if we consider theory at D dimensions first. Then, the limit of D tends to 2 is considered while the coupling constant of curvature invariant is rescaled by $1/(D-2)$ term. In this way we get novel two-dimensional Einstein gravity which has cosmological and black hole solutions. Similar consideration for higher-derivative gravity in two dimensions also opens the window for appearance of new cosmology and black hole solutions. This is due to the fact that novel contributions to gravitational equations appear from this scalar curvature invariant in the limit of two dimensions. In fact, this strategy follows from similar consideration of Refs. [2, 3] where the Einstein-Gauss-Bonnet gravity was formulated in the limit from D to 4 dimensions with singular rescaling of the Gauss-Bonnet coupling constant as $1/(D-4)$.

II. TWO-DIMENSIONAL $R + \Lambda$ GRAVITY

A. Exactly two-dimensional model

We now consider the Einstein gravity in D space-time dimensions with a cosmological constant Λ

$$S_{\text{Einstein}} = \int d^D \sqrt{-g} \left(\frac{R}{\hat{\alpha}} + \Lambda \right) . \tag{1}$$

Here $\hat{\alpha}$ is a constant. Then the Einstein equation is given by

$$\frac{1}{\hat{\alpha}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{\Lambda}{2} g_{\mu\nu} , \tag{2}$$

which tells that the Ricci curvature $R_{\mu\nu}$ is covariantly constant and therefore proportional to $g_{\mu\nu}$. Multiplying Eq. (2) with $g^{\mu\nu}$, we obtain

$$\frac{1}{\tilde{\alpha}}\left(D-2\right)R = \frac{D\Lambda}{2}\,. \tag{3}$$

Thus, in two dimensions, $D=2$, there is no any solution, as is well-known. As it was mentioned this is due to fact that scalar curvature is topological invariant in two dimensions.

B.   The model in the limit of two dimensions

We now redefine the parameter $\tilde{\alpha}$ as

$$\tilde{\alpha} = \alpha\left(D-2\right)\,. \tag{4}$$

Then Eq. (3) has the following form

$$\frac{1}{\alpha}R = \frac{D\Lambda}{2}\,, \tag{5}$$

which has a solution in two dimensions,

$$R = R_0 \equiv \frac{D\alpha\Lambda}{2}\,. \tag{6}$$

When $R = R_0 > 0$, there is a solution describing the de Sitter space-time

$$ds^2 = -dt^2 + \mathrm{e}^{\frac{t^2}{l^2}}\sum_{i=1}^{D-1}\left(dx^i\right)^2\,,\quad R_0 = \frac{D(D-1)}{l^2}\,. \tag{7}$$

In the limit of two dimensions, that is, in the limit $D \to 2$, the metric (7) has the following form,

$$ds^2 = -dt^2 + \mathrm{e}^{\frac{t^2}{l^2}}dx^2\,,\quad R_0 = \frac{2}{l^2}\,. \tag{8}$$

Hence, anyway there exists cosmological solution even in two space-time dimensions.

On the other hand, as a static solution, we consider the solutions describing black hole.[1] If $R = R_0 > 0$, the Schwarzschild-de Sitter space-time is an exact solution and if $R = R_0 < 0$, the Schwarzschild-anti-de Sitter space-time is an exact solution,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{D-2}\,,\quad f(r) = 1 - \frac{2M}{r^{D-3}} - \lambda r^2\,,\quad R_0 = D(D-1)\lambda\,. \tag{9}$$

Then in the limit $D \to 2$, the metric in (9) has the following form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2\,,\quad f(r) = 1 - 2Mr - \lambda r^2\,,\quad \lambda = \frac{R_0}{2}\,. \tag{10}$$

When $R_0 > 0$, $\lambda\frac{1}{l^2}$ in (7) or (8). Then there may appear horizons, where $f(r) = 0$, at

$$r = r_\pm \equiv -\frac{M \pm \sqrt{M^2 + \lambda}}{\lambda}\,. \tag{11}$$

In order that there exists real and positive solution, we should require

$$0 > \lambda > -M^2\,. \tag{12}$$

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[1] It is interesting that in the $2D$ limit, the theory may be presented as scalar-tensor theory, see [4, 5].

Then $r = r_-$ corresponds to the inner horizon and $r = r_+$ to the outer horizon as for the charged black hole. The extremal limit, where the radii of the two horizons coincide with each other, is given by $M^2 = -\lambda$. Thus, we obtained a new type of black hole solution in two-dimensional General Relativity.

We now rewrite $f(r)$ in the following form,

$$f(r) = \frac{(r - r_+)(r - r_-)}{r_+ r_-}. \tag{13}$$

When $r \sim r_\pm$, by defining δr by $r = r_\pm \pm \delta r$, we find

$$f(r) = \frac{(r_+ - r_-)\,\delta r}{r_+ r_-}. \tag{14}$$

Then by the Wick rotating the time coordinate $t \to i\tau$, the metric (10) has the following form,

$$ds^2 = \frac{(r_+ - r_-)\,\delta r}{r_+ r_-} d\tau^2 + \frac{r_+ r_-}{(r_+ - r_-)\,\delta r} dr^2. \tag{15}$$

By defining a new coordinate ρ as

$$d\rho = \pm dr \sqrt{\frac{r_+ r_-}{(r_+ - r_-)\,\delta r}} \quad \text{that is,} \quad \rho = \pm 2\sqrt{\frac{r_+ r_- \delta r}{r_+ - r_-}} \quad \text{or} \quad \delta r = \frac{r_+ - r_-}{4 r_+ r_-}\rho^2, \tag{16}$$

the metric (15) can be rewritten as

$$ds^2 = \frac{(r_+ - r_-)^2}{4 r_+^2 r_-^2}\rho^2 d\tau^2 + d\rho^2. \tag{17}$$

In order to avoid the conical singularity in the Wick-rotated Euclidean space, we require the periodicity,

$$\frac{r_+ - r_-}{2 r_+ r_-}\tau \sim \frac{r_+ - r_-}{2 r_+ r_-}\tau + 2\pi, \tag{18}$$

which shows that black hole has the temperature T as

$$T = \frac{r_+ - r_-}{4\pi r_+ r_-} = \frac{\sqrt{M^2 + \lambda}}{\pi}. \tag{19}$$

There occurs also the Hawking radiation. Eq. (19) shows that the temperature T vanishes in the extremal limit $r_+ = r_-$ or $M^2 = -\lambda$. Note that it is difficult to define the thermodynamical quantities like entropy in two dimensions because the area of the black hole horizon naively vanishes. The area of $D - 2$ dimensional unit sphere is given by

$$S_{D-2} = \frac{\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{20}$$

Here Γ is the gamma function. When $D = 2$, the expression (20) gives $S_0 = 2$, which corresponds to the two points whose center is the origin on the straight line. We may assume that the entropy is proportional to S_{D-1}/G where G is Newton's gravitational constant. The action (1) tells that $\tilde{\alpha}$ can be regarded to be proportional to G and therefore the entropy should be proportional to $S_{D-1}/\tilde{\alpha}$. If we keep α in (4) to be finite, the entropy diverges in the limit of $D \to 2$ and therefore the entropy is ill-defined.

III. $R + R^2$ GRAVITY

Let us consider $F(R)$ gravity action without matter in D space-time dimensions (for general introduction, see Refs. [10–12])

$$S_{F(R)} = \frac{1}{2\kappa^2}\int d^D x \sqrt{-g} F(R). \tag{21}$$

Then the equation of motion for the $F(R)$ gravity is given by

$$0 = \frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \Box F'(R) + \nabla_\mu \nabla_\nu F'(R) \, . \tag{22}$$

If we assume that the Ricci curvature $R_{\mu\nu}$ is covariantly constant and the scalar curvature R is constant, Eq. (22) reduces to

$$0 = \frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) \, . \tag{23}$$

Multiplying $g^{\mu\nu}$, we obtain an algebraic equation for R,

$$0 = \frac{D}{2} F(R) - R F'(R) \, . \tag{24}$$

If Eq. (24) has a positive solution, $R = R_0 > 0$, the Schwarzschild-de Sitter space-time is an exact solution and if $R = R_0 < 0$, the Schwarzschild-anti-de Sitter space-time is an exact solution as in (9).

A. Exactly two-dimensional model

We now consider the following model

$$F(R) = \frac{R}{\tilde{\alpha}} + \beta R^2 \, . \tag{25}$$

It was shown long ago [13] that higher-derivatives terms for such model and its generalizations give constant curvature solution in the way similar to the Jackiw-Teitelboim gravity [6, 7]. Then Eq. (24) has the following form,

$$0 = \frac{(D-2) R}{2\tilde{\alpha}} + \frac{D-4}{\beta} R^2 \, , \tag{26}$$

In two dimensions, $D = 2$, the solution is given by

$$R = 0 \, . \tag{27}$$

If $R = 0$, Eq. (22) with (25) requires

$$R_{\mu\nu} = 0 \, . \tag{28}$$

First we consider the cosmological solution by assuming the FRW space-time

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{D-1} \left(dx^i \right)^2 \, . \tag{29}$$

In the D dimensional FRW space-time, the curvatures have the following expression,

$$R_{tt} = -(D-1) \left(\dot{H} + H^2 \right) \, , \quad R_{ij} = a^2 \left(\dot{H} + (D-1) H^2 \right) \delta_{ij} \, , \quad R = \left(2(D-1) \dot{H} + D(D-1) H^2 \right) \, , \tag{30}$$

Here $H = \frac{\dot{a}}{a}$. Especially in two dimensions, $D = 2$, the expressions (30) reduce to

$$R_{tt} = -\left(\dot{H} + H^2 \right) \, , \quad R_{ij} = a^2 \left(\dot{H} + H^2 \right) \delta_{ij} \, , \quad R = 2 \left(\dot{H} + H^2 \right) \, , \tag{31}$$

Then Eqs. (28) is satisfied if

$$0 = \dot{H} + H^2 \, . \tag{32}$$

whose solution is given by

$$H = \frac{1}{t - t_0} \, , \quad a = a_0 (t - t_0) \, , \tag{33}$$

with constants t_0 and a_0. The solution (33) corresponds to the radiation dominated universe in four dimensions but as a model in two dimensions, this solution gives a new type of cosmology.

We should note that there is a static solution (10) with $\lambda = 0$ even if $R = 0$ in (27),

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 \,, \quad f(r) = 1 - 2Mr \,, \tag{34}$$

which might look similar to the Schwarzschild solution and a horizon at $r = r_0 \equiv \frac{1}{2M}$ but the solution does not describe any black hole solution because the space-time signature is given by $(+, -, +, +)$ when $r > r_0$ although the signature is given by $(-, +, +, +)$ for the black hole solution. Hence the horizon at $r = r_0 \equiv \frac{1}{2M}$ is not the black hole horizon but the cosmological horizon. Note that there is no solution in the Einstein gravity in two dimensions as mentioned after Eq. (2) but there is anyway a solution (27), which is because the $F(R)$ gravity can be rewritten in the form of the scalar-tensor theory, which gives a solution. Therefore the solution (34) could be new type of static solution.

We can consider more complicated models of the sort introduced in Ref. [13] like

$$F(R) = \sum_i a_i R^{n_i} \,, \tag{35}$$

where a_i are coupling constants and powers n_i are some numbers. Then, such models have constant curvature solutions which give the de Sitter, Schwarzschild-de Sitter, or Schwarzschild-anti-de Sitter space-time.

B. The model in the limit of two dimensions

Instead of the model (25) by redefining the parameter $\tilde{\alpha}$ as in (4) we consider the following theory

$$F(R) = \frac{R}{\alpha (D - 2)} + \beta R^2 \,. \tag{36}$$

Then Eq. (24) has the following form,

$$0 = \frac{R}{2\alpha} + \frac{D - 4}{\beta} R^2 \,, \tag{37}$$

and we find non-trivial solution even in two dimensions,

$$R_0 = 0 \,, \quad -\frac{1}{2(D - 4)\alpha\beta} \,. \tag{38}$$

In the limit $D \to 2$, R_0 is finite. Hence, if $R_0 = 0$, we have the cosmological solution as in (33)

$$H = \frac{1}{t - t_0} \,, \quad a = a_0 (t - t_0) \,, \tag{39}$$

This cosmology corresponds to two-dimensional radiation-dominated universe. If $R_0 > 0$, the cosmological solution describing the de Sitter space-time,

$$ds^2 = -dt^2 + \mathrm{e}^{\frac{t^2}{l^2}} dx^2 \,, \quad \frac{1}{l^2} = \frac{R_0}{2} = \frac{1}{8\alpha\beta} \,, \tag{40}$$

and we also obtain the static space-time in (10)

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \,, \quad f(r) = 1 - \frac{2M}{r^{D-3}} - \lambda r^2 \,, \tag{41}$$

with

$$\lambda = 0 \,, \quad \frac{1}{8\alpha\beta} \,. \tag{42}$$

Then as in the previous section, this solution may give a new type of black hole space-time in two dimensions. When $\lambda \neq 0$, there may appear two horizons as in (11),

$$r = r_\pm \equiv -\frac{M \pm \sqrt{M^2 + \lambda}}{\lambda} \,. \tag{43}$$

This shows that theory (36) has a solution describing the black hole in the limit of $D \to 2$, whose situation is different from the model (25). The black hole has a temperature T as in (19),

$$T = \frac{\sqrt{M^2 + \lambda}}{\pi} \,, \tag{44}$$

and therefore there occurs the Hawking radiation. Hence, even in case of higher-derivative gravity in this limit from D to 2 with singular rescaling of coupling constant we get novel cosmological and black hole solutions. It is easy to see that this scheme may be easily generalized for more complicated versions of higher-derivative gravity [13]. Again, due to non-trivial contribution from R term the cosmological and the black hole dynamics will be changed.

IV. CONCLUSION AND DISCUSSION

We formulated the novel Einstein and R^2 gravity in two dimensions as the limit from D-dimensional theory to two-dimensional theory with simultaneous singular rescaling of gravitational coupling constant as $1/(D-2)$. As a result even the two-dimensional Einstein gravity becomes non-trivial theory which contains the de Sitter cosmology and the Schwarzschild-de Sitter or Schwarzschild-anti-de Sitter black hole solutions. This occurs due to non-trivial contribution of scalar curvature term in such limit to gravitational equations. For R^2 gravity we also got novel black hole solutions as well as the de Sitter universe and radiation-dominated universe cosmologies.

It would be interesting to further study this approach in related two-dimensional dilatonic gravities. Let us consider Callan-Giddings-Harvey-Strominger (CGHS) model [8] (see also [9]). The action is given by

$$S_{\mathrm{CGHS}} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[\mathrm{e}^{-2\phi} \left(R + 4 \left(\nabla \phi \right)^2 + 4\lambda^2 \right) - \frac{1}{2} \left(\nabla f \right)^2 \right] \,. \tag{45}$$

Here ϕ and f are dilaton and matter fields, respectively and λ^2 is a cosmological constant. This model has a solution describing the black hole in exactly two dimensions. By using the light-cone coordinates, $x^\pm = t \pm x$,

$$ds^2 = - \left(\frac{M}{\lambda} - \lambda^2 x^+ x^- \right) dx^+ dx^- \,. \tag{46}$$

Then there appears one horizon at $x^+ x^- = \frac{M}{\lambda^2}$, which is different from the models in this paper where there appear two horizons, (11) and (43). In [8], the black hole formation was discussed by using the exact shock wave solution and the Hawking radiation and back reaction of the metric is analyzed by using the trace-anomaly. We should note that the Hawking temperature is independent of the black hole mass and given by $T = \frac{1}{2\pi}$, which is also different from the temperature in the models of this paper, (19) and (44).

It might be interesting to extend the CGHS model (or the Jackiw-Teitelboim gravity with quantum matter fields, see for instance, Ref. [14]) by adding scalar curvature term with gravitational coupling constant and considering such theory in the same limit from D to 2 dimensions. Then again scalar curvature term gives non-trivial contribution to gravitational equations what can significantly change its dynamics. However, one should also account for quantum effects in such limit (say, conformal anomaly). Hence, such generalization requests careful investigation.

Acknowledgments

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