JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 32, No. 2, March–April 2009

Engineering Notes

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Novel Expressions of Equations of Relative Motion and Control in Keplerian Orbits

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DOI: 10.2514/1.38210

I. Introduction

THE autonomous control of relative motion between multiple spacecraft in orbit is one of the most essential technologies in near-future space programs. For instance, autonomous rendezvous and docking was used in resupplying the Mir space station [1] and has been identified as a key technology in many future space programs, such as assembly of modular systems in orbit [2], robotic sample return from planets [3], etc. However, in most space programs, the rendezvous and docking is currently achieved by manual operations.

In recent years, a number of research works have been published which propose relative motion controls in orbits [4–12]. Some of them are focusing on the spacecraft formation flying problem of maintaining the relative orbit of a cluster of spacecraft [4–9]. In this problem, a cluster of spacecraft is supposed to maintain their formation over the *entire* life span of the spacecraft, and thus prediction and control of relative motion is significantly sensitive to relative orbit modeling errors [9]. Therefore, the effect of minor perturbations (J_2 perturbation, atmospheric, and solar drags, etc.) and its modeling are one of the main interests in these works, and the feedback control laws were generally synthesized in a very long time scale.

On the other hand, some research works have been published with a focus on dynamics and controls in the rendezvous and docking phase. In this problem, the maneuvers are conducted in a relatively short period (a few orbits) compared to the lifetime, and the distance between the two spacecraft is relatively small. Therefore, a relative orbit description expressed in the rotating Hill frame, in which the effects of minor perturbations other than the gravitational force from the (point mass) Earth are ignored, is sufficient to guide the spacecraft to a successful docking. Kluever [10] proposed a continuous feedback control law that guides a chaser (or a deputy) to dock with a target (or a chief) with a desired approaching direction and speed. However, it used simple Clohessy–Wiltshire equations [13], which are valid only for a circular orbit and also uses the assumption that the deputy is on the chief's orbit plane. More recently, Karlgaard [11] proposed a continuous feedback controller for rendezvous navigation in elliptical orbit. In this work, the full equations of relative motion (see [9]) were converted into full nonlinear equations in a spherical coordinate system, and then a navigation control law was derived using the simple feedback linearization method. Singla et al. [12] also proposed an adaptive output feedback control law based on full nonlinear equations of relative motion. Their control law did not use velocity feedback, and thus was less sensitive to the measurement noise. It was also adaptive to the unknown mass of the deputy spacecraft.

As far as we understand, however, most previous works (including [10–12]) based on the relative motion equations in the Hill frame have used purely mathematical approaches in synthesis of their control laws, and the physical properties of the relative motion have been overlooked. As Slotine and Li have mentioned in their works [14,15], some physical properties, such as energy conservation, may hold significant potential in analysis and control design for multi-input nonlinear mechanical systems. For example, a multilink manipulator robot [14,15] and attitude dynamics of a spacecraft [15,16] have been studied with physical insight. In these works, the multivariable equations of motion are written in the general form of a second-order differential equation, which is the so-called robot equation in the robot control context [17]. Expressing dynamics of systems in this generic form has several advantages. First, the equation can be easily derived by applying Lagrange's equation. Second, there are physical properties which are extremely useful in designing advanced control schemes. Third, and most importantly, its form is so general that it can represent various kinds of dynamic systems, and thus a control technique developed based on this generic form for one application can be easily applied to other applications with minimal modifications.

In this Note, the relative motion between two spacecraft is studied with emphasis on physical insights. The full equations of relative motion in Keplerian orbits are converted into the general form of a second-order differential equation, and then feedback control laws are proposed for different control objectives. Besides the control laws presented in this Note, other various control design methods, which have already been developed for other applications (such as robot manipulator or spacecraft control), can also be easily applied for the relative motion control, thanks to the use of the generic form.

II. Relative Equations of Motion

Let us consider the relative motion of a active deputy spacecraft to a passive chief spacecraft in a general Keplerian orbit. The local-vertical–local-horizontal (LVLH) frame with (x, y, z) axes whose origin is fixed to the chief spacecraft is used to describe the relative motion. The $+z$ axis is directed toward the center of the Earth, the $+y$ axis is directed toward the negative orbit normal, and the $+x$ axis is defined as $\mathbf{i} = \mathbf{j} \times \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the basis vector of the LVLH frame. Thus, in a circular orbit, the $+x$ axis is along the velocity vector of the chief spacecraft. The scalar $r_c > 0$ refers to the radius of the chief from the gravity center, and $r \triangleq \sqrt{x^2 + y^2 + z^2} \geq 0$ refers to the range of the deputy from the chief.

It is assumed that distance between these two spacecraft is small compared with the chief orbit radius, that is $r_c \gg r$, and no disturbances except control forces are acting on these spacecraft. These assumptions are generally made in the literature on the

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autonomous rendezvous and docking control for a short time period. Then, the relative equations of motion can be described in LVLH frame as follows (see [9]):

$$\ddot{x} - z\dot{\omega} - 2\dot{z}\omega - x\left(\omega^2 - \frac{\mu}{r_c^3}\right) = a_x \tag{1a}$$

$$\ddot{y} + \frac{\mu}{r_c^3} y = a_y \tag{1b}$$

$$\ddot{z} - z\left(\omega^2 + 2\frac{\mu}{r_c^3}\right) + x\dot{\omega} + 2\dot{x}\omega = a_z \tag{1c}$$

where a_x, a_y, and a_z are control accelerations of the deputy spacecraft, ω is the time derivative of the latitude of the chief spacecraft, and μ is the gravitational parameter. The spherical coordinate version of Eq. (1) is also available in [11,18,19].

Equation (1) is a highly coupled (between x and z axes motions) multi-input multi-output dynamic equation. Its spherical coordinate version is even more complicated and fully nonlinear. Many of the previous works based on Eq. (1) or its spherical coordinate version in literature used the feedback linearization method which cancels nonlinear and/or coupling terms. These methods are purely mathematical and do not exploit any physical properties that the relative motion dynamics may have.

In the present Note, we rewrite Eq. (1) into the second-order differential equation in the general form of

$$H(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \tag{2}$$

where $\mathbf{q} \in \mathbb{R}^3$ is the generalized coordinates vector, $H \in \mathbb{R}^{3\times3}$ is the (symmetric positive definite) inertia matrix, $C\dot{\mathbf{q}}$ is a nonlinear vector of Coriolis and centripetal forces, $\mathbf{g} \in \mathbb{R}^3$ is the gravity vector, and $\mathbf{u}$ is the control force. The terms in Eq. (2) can be accelerations (instead of forces) if the equation is properly divided by its inertia.

In Eq. (2), there is a physical property that the matrix $\dot{H} - 2C$ is *skew-symmetric*, which can be viewed as a matrix expression of *energy conservation*,

$$\dot{\mathbf{q}}^T(\mathbf{u} - \mathbf{g}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\dot{\mathbf{q}}^T H \dot{\mathbf{q}}] \tag{3}$$

This property is extremely useful in designing advanced control schemes [15].

From a fact that the specific kinetic energy per unit mass is $\frac{1}{2}\dot{\mathbf{q}}^T H \dot{\mathbf{q}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ when $\mathbf{q} = [x, y, z]^T$, we can define the inertia matrix H as

$$H(\mathbf{q}) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

In general, the skew symmetry of the matrix $\dot{H} - 2C$ can be written as $\dot{H} = C + C^T$ [15], and thus C can be decomposed as $C_{\mathrm{sym}} + C_{\mathrm{skew}}$, where $C_{\mathrm{sym}}(=\frac{1}{2}\dot{H})$ and C_{skew} are symmetric and skew-symmetric matrices, respectively. Therefore, collecting the terms with $\dot{\mathbf{q}} = [\dot{x}, \dot{y}, \dot{z}]^T$ in Eq. (1), one can have the C matrix as

$$C = C_{\mathrm{skew}} = \begin{bmatrix} 0 & 0 & -2\omega \\ 0 & 0 & 0 \\ 2\omega & 0 & 0 \end{bmatrix} \tag{5}$$

since $C_{\mathrm{sym}} = \frac{1}{2}\dot{H} = 0$ from Eq. (4).

The remaining terms are collected as the term $\mathbf{g}$ as follows:

$$\mathbf{g} = \begin{bmatrix} -z\dot{\omega} - x(\omega^2 - \frac{\mu}{r_c^3}) \\ \frac{\mu}{r_c^3} y \\ -z(\omega^2 + 2\frac{\mu}{r_c^3}) + x\dot{\omega} \end{bmatrix} \tag{6}$$

The vector $\mathbf{g}$ can be decomposed into two parts as $\mathbf{g} = \mathbf{g}_c + \mathbf{g}_{\mathrm{nc}}$, where $\mathbf{g}_c$ is the conservative acceleration and $\mathbf{g}_{\mathrm{nc}}$ is the nonconservative acceleration. The conservative acceleration $\mathbf{g}_c$ can be defined as a gradient of a virtual potential function U as follows:

$$\mathbf{g}_c = \nabla U = \left[\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right]^T = \begin{bmatrix} -x(\omega^2 - \frac{\mu}{r_c^3}) \\ \frac{\mu}{r_c^3} y \\ -z(\omega^2 + 2\frac{\mu}{r_c^3}) \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} U &\triangleq \frac{1}{2}\frac{\mu}{r_c^3}(x^2 + y^2 - 2z^2) - \frac{\omega^2}{2}(x^2 + z^2) \\ &= \frac{1}{2}\frac{\mu}{r_c^3}\{-(e\cos f)x^2 + y^2 - (3 + e\cos f)z^2\} \end{aligned} \tag{8}$$

since $\omega^2 = (\mu/r_c^3)(1 + e\cos f)$, where e is the eccentricity of the chief orbit and f is the chief's true anomaly. The nonconservative acceleration $\mathbf{g}_{\mathrm{nc}}$ is then

$$\mathbf{g}_{\mathrm{nc}} = [-z\dot{\omega}, 0, x\dot{\omega}]^T \tag{9}$$

Finally, the control input $\mathbf{u}$ is defined as $\mathbf{u} \triangleq \mathbf{a} = [a_x, a_y, a_z]^T$.

Using a similar method, we can also derive a spherical coordinate version in the form of Eq. (2), and it is presented in the Appendix. This version is more complicated and fully nonlinear but might be more compatible to the radar navigation system.

Remark 1: Some of the vectors of the equations of relative motion in the form of Eq. (2) are not real corresponding force/acceleration but virtual ones. For instance, the gravitational vector $\mathbf{g}$ defined in Eq. (6) is not necessarily the same as the real gravitational force exerted by the Earth. They appear due to the use of rotating/translating Hill frame and the relative position/velocity state variables.

III. Feedback Controller Design: Case Studies

In this section, we design feedback control laws which control the relative motion of the deputy spacecraft with respect to the chief spacecraft in three different scenarios.

A. Case 1: Rendezvous in Circular Orbit

In this case, we revisit a classic problem which designs feedback control for executing a rendezvous to a chief spacecraft in a circular orbit. Here, the equations of relative motion reduce to the Clohessy–Wiltshire equations. In this scenario, the desired reference position is $\mathbf{q}_d = [0, 0, 0]^T$. Since $\dot{\omega} = 0$ and $\omega^2 = \mu/r_c^3$ with a circular orbit, the nonconservative acceleration $\mathbf{g}_{\mathrm{nc}}$ becomes zero, and thus $\mathbf{g}$ is a pure conservative acceleration:

$$\mathbf{g} = \mathbf{g}_c = \nabla U = [0, \omega^2 y, -3\omega^2 z]^T \tag{10}$$

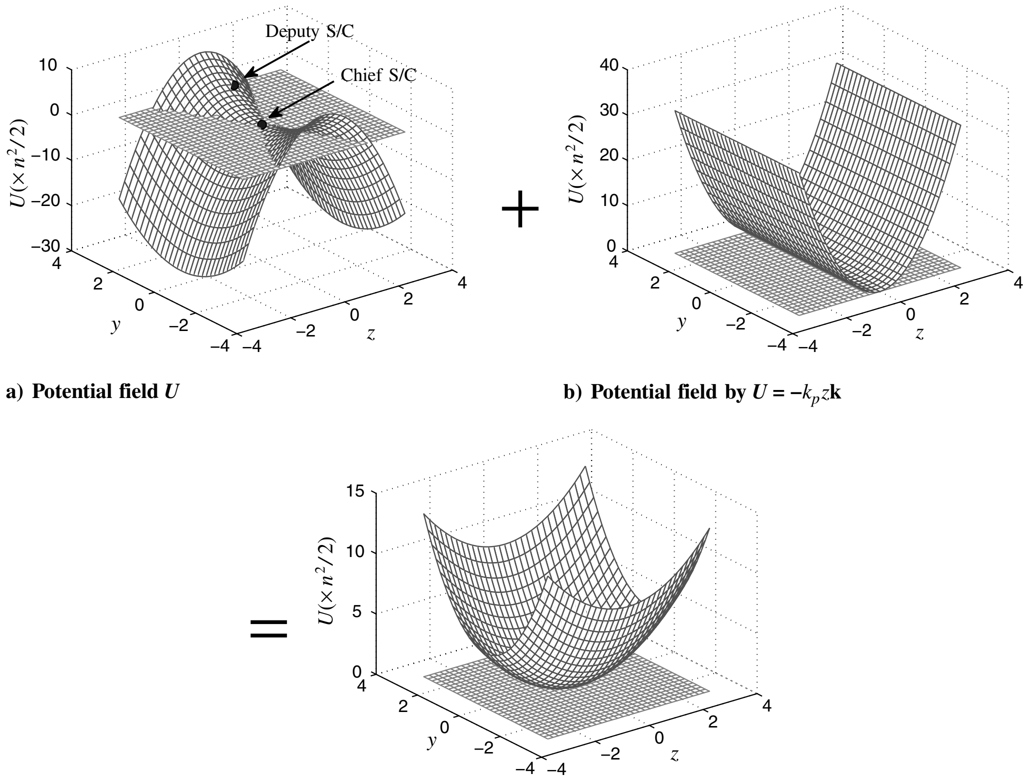
where

$$U = \tfrac{1}{2}\omega^2(y^2 - 3z^2) \tag{11}$$

Figure 1a shows the three-dimensional shape of the potential function U in the yz frame. As shown in this figure, the potential field of U has a saddlelike shape, thus the relative motion driven *only* by the virtual gravity acceleration $\mathbf{g}$ is unstable. The gradient of U is directed toward the origin along the y axis, but outward along the z axis. To make the relative motion by the gravity force about the chief spacecraft to be stable, we need to "bend" the potential field "upward" along the z axis using a control acceleration, so that the resultant field has a concave shape. From this argument, a feedback control law is proposed as follows.

Proposition 1: The relative motion in a circular orbit can be stabilized to the origin by a proportional-derivative (PD) feedback control law

$$\mathbf{u} = -K_p\mathbf{q} - K_d\dot{\mathbf{q}} \tag{12}$$



a) Potential field U

b) Potential field by $U = -k_p z\mathbf{k}$

c) Superposed potential field

Fig. 1 Potential fields and their superposition.

where $K_p = K_{p1} + K_{p2}$, $K_{p1} = \mathrm{diag}\{[k_{px1}, k_{py1}, k_{pz1}]\}$, $k_{px1} > 0$, $k_{py1} \geq 0$, $k_{pz1} \geq 0$, $K_{p2} = \mathrm{diag}\{[0, 0, k_{pz2}]\}$, $k_{pz2} > 3\omega^2$, and $K_d > 0$.

Proof: Let us define a Lyapunov function V as

$$V \triangleq \tfrac{1}{2}\dot{\mathbf{q}}^T H \dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{q}^T K_p \mathbf{q} + U = \tfrac{1}{2}\dot{\mathbf{q}}^T H \dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{q}^T K_{p1} \mathbf{q} + \tfrac{1}{2}k_{pz2} z^2 + \tfrac{1}{2}\omega^2 (y^2 - 3z^2) \geq 0 \tag{13}$$

then, from Eq. (3), the time derivative of V can be written as

$$\dot{V} = \dot{\mathbf{q}}^T (\mathbf{u} + K_p \mathbf{q}) \tag{14}$$

since $\dot{U} = \dot{\mathbf{q}}^T \nabla U = \dot{\mathbf{q}}^T \mathbf{g}$. Under the proposed control law (12), it follows that

$$\dot{V} = -\dot{\mathbf{q}}^T K_d \dot{\mathbf{q}} \leq 0 \tag{15}$$

Then, it can be easily shown that $\mathbf{q} = 0$ is globally asymptotically stable using the invariant set theorem [20]. □

It is noticeable that the Lyapunov function V is defined as a total mechanical energy, that is, a sum of the kinetic energy $\frac{1}{2}\dot{\mathbf{q}}^T H \dot{\mathbf{q}}$ and the potential energy $\frac{1}{2}\mathbf{q}^T K_p \mathbf{q} + U$. Also, notice that the control law (12) has a form of the PD control, whose proportional term is composed of two terms: $-K_{p1}\mathbf{q}$ and $-k_{pz} z\mathbf{k}$. Whereas the former term ensures the static stability, the latter term constructs a potential field, as shown in Fig. 1b. By superposing this (artificial) potential field with the gravitational potential field U, we can have a concave potential field, as shown in Fig. 1c, which makes the relative motion stable. It is noticeable that one can set $k_{py1} = 0$, which can be interpreted that proportional control in the y-axis direction is not needed thanks to the gradient of U toward the origin along the y axis.

The derivative control term $-K_d\dot{\mathbf{q}}$ has a role of a virtual damper which dissipates the total mechanical energy, and $\dot{V}$ is the power dissipated by the virtual damper. In fact, one does not need the exact information of $\dot{\mathbf{q}}$, which is usually susceptible to measurement noise. A control law that replaces the derivative control term $-K_d\dot{\mathbf{q}}$ with $\mathbf{u}_d$ where $\dot{\mathbf{q}}^T \mathbf{u}_d < 0$ will successfully stabilize the relative motion. It is also noticeable that the proposed control law (12) does not require the exact value of ω for the chief spacecraft orbit, which is needed in the previous works for a feedback linearization [10], but only needs the upper limit of ω.

For a case of noncircular orbits, one cannot use such a simple control law (12), because the potential function U defined in Eq. (8) varies as the chief spacecraft moves along its orbit, that is, $\dot{U} \neq \dot{\mathbf{q}}^T \nabla U$. However, one can still easily design a control law for this case using the control law presented in the next section.

Remark 2: The use of the potential fields in the analysis/synthesis in this section (and later) only yields a sufficient condition for the stability, but does not give any necessary condition. Even if the shape of the potential field is not concave, it is possible that the relative motion is stable.

B. Case 2: Constant Relative Position Regulation

The control objective in this case study is to regulate the deputy to a constant relative position with respect to the chief in a Keplerian orbit. This goal is required when the two spacecraft need to keep their relative position to operate space missions or to communicate to each other.

Proposition 2: The relative motion in a Keplerian orbit can be regulated to the constant desired position $\mathbf{q}_d$ by a feedback control law

$$\mathbf{u} = \mathbf{g}(\mathbf{q}) - K_p \tilde{\mathbf{q}} - K_d \dot{\mathbf{q}} \tag{16}$$

where $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}_d$, $K_p > 0$, and $K_d > 0$.

Proof: Define a Lyapunov function

$$V \triangleq \tfrac{1}{2}\dot{\mathbf{q}}^T H(\mathbf{q}) \dot{\mathbf{q}} + \tfrac{1}{2}\tilde{\mathbf{q}}^T K_d \tilde{\mathbf{q}} \tag{17}$$

then, the time derivative of V under the feedback law (16) becomes

$$\dot{V} = \dot{\mathbf{q}}^T (-C\dot{\mathbf{q}} - \mathbf{g} + \mathbf{u} + C\dot{\mathbf{q}} + K_p \tilde{\mathbf{q}}) = -\dot{\mathbf{q}}^T K_d \dot{\mathbf{q}} \leq 0 \tag{18}$$

and so the Lyapunov stability is proven. The global asymptotic stability can be easily shown using the invariant set theorem [17,20]. □

Compared to Eq. (12), the proposed control law (16) can be applied not only for circular orbits but also for general Keplerian orbits. It can also be applied for a case with $\mathbf{q}_d \neq 0$. At a cost of these advantages, we cannot enjoy the use of the gravity field U. The Lyapunov function defined in Eq. (17) does not contain the gravitational potential function, which means the designed control law needs to contain the gravitational term $\mathbf{g}(\mathbf{q})$ as shown in the proposed control law (16). This term makes the potential field "flat" by canceling the (virtual) gravity force (not the real gravity force exerted by the Earth), and then the proportional error feedback $-K_p\tilde{\mathbf{q}}$ forms a concave field centered at the desired position $\mathbf{q}_d$. However, it is also noticeable that the proposed control law still does not need to cancel the Coriolis/centrifugal forces which need to be canceled in conventional feedback linearization methods.

Another remark is that the control law that is needed to keep the deputy at the constant relative set position $\mathbf{q}_d$ is $\mathbf{u} = \mathbf{g}(\mathbf{q}_d)$.

C. Case 3: Relative Motion Tracking Control

In this case, we derive a relative motion tracking control law that makes the deputy track a given reference relative trajectory. The control law can be used to guide a deputy to perform a docking to the chief along a given approach direction and relative speed.

There have been proposed many sorts of tracking control schemes for systems whose dynamic equations can be written in the form of Eq. (2). Many of them can be considered as special cases of the class of "computed-torque controllers," which is, at the same time, a special application of feedback linearization of nonlinear systems [17]. A summary of the various computed-torque (and computed-torque-like) controllers is given in [17] (p. 151). For example, a PD computed-torque controller can be given as

$$\mathbf{u} = H(\mathbf{q})(\ddot{\mathbf{q}}_d - K_d\dot{\mathbf{e}} - K_p\mathbf{e}) + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) \tag{19}$$

where a preplanned desired trajectory is given in terms of known bounded functions $\mathbf{q}_d(t)$, $\dot{\mathbf{q}}_d(t)$, and $\ddot{\mathbf{q}}_d(t)$, and the tracking error is defined as $\mathbf{e} \triangleq \mathbf{q} - \mathbf{q}_d$. Using this control law, which was originally developed for robot control, we can easily achieve the relative motion tracking control.

IV. Numerical Examples

In this section, numerical simulations using the proposed control laws are presented. The simulations were conducted, not based on the equations of relative motion Eq. (1), but on Newton's law of universal gravitation between each spacecraft and the point-mass Earth. For case 1, we assume that the chief spacecraft is in a circular Earth orbit with an altitude of 340 km, which is the nominal altitude of the International Space Station. The deputy spacecraft is initially located at $[1, 1, 1\ \text{km}]^T$ in the LVLH frame, and the relative velocity with respect to the LVLH frame is zero. The control gains in Eq. (12) are $K_p = 1 \times 10^4\mathbf{I}$ and $K_d = 0.1\mathbf{I}$. The simple PD control law (12) shows the convergence of the relative motion to zero, as shown in Fig. 2.

The second numerical example shows the performance of the control law (16) for case 2. The chief spacecraft is initially located at the perigee (which is at an altitude of 340 km) of its elliptical orbit with the eccentricity $e = 0.44$. The deputy spacecraft is initially located at the same relative position with the previous example and zero relative velocity. The control gains are identical to those of the previous example, and the desired relative position is $\mathbf{q}_d = [100, 100, 100\ \text{m}]^T$ in the LVLH frame. Figure 3 shows that the control law (16) successfully regulates the relative position of the deputy to the constant desired position.

In a real rendezvous and docking mission in space, it is desired that the deputy spacecraft approach the chief spacecraft along a prescribed docking axis. The third example shows how the tracking control law (19) achieves this control objective. The initial states are the as same as in the second example, and the desired trajectory is given so that, for first hour, the deputy maneuvers toward the docking axis while keeping the range to the chief constant, and then for next hour, the deputy approaches the chief along the docking axis which is chosen as $+x$ axis. (A docking in $+x$ axis is commonly referred to as a "V-bar" docking in the literature.) Fig. 4 shows the resultant

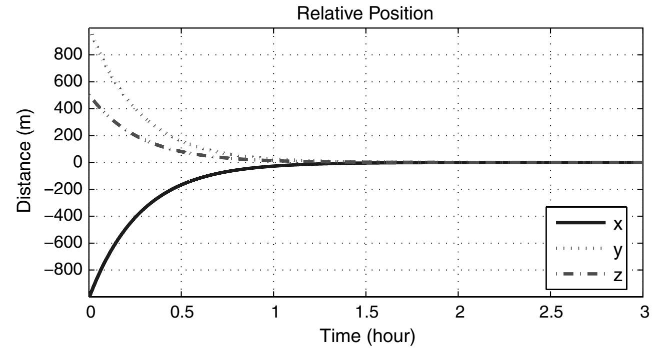


Fig. 2 Rendezvous maneuver in a circular orbit by PD control law (12).

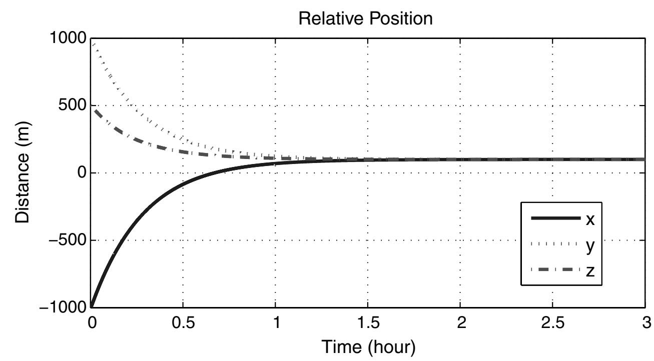
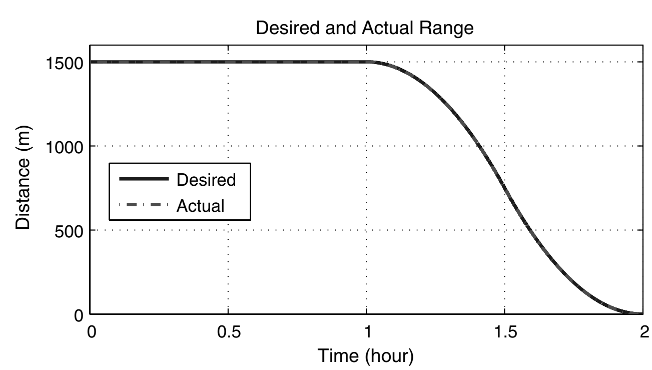
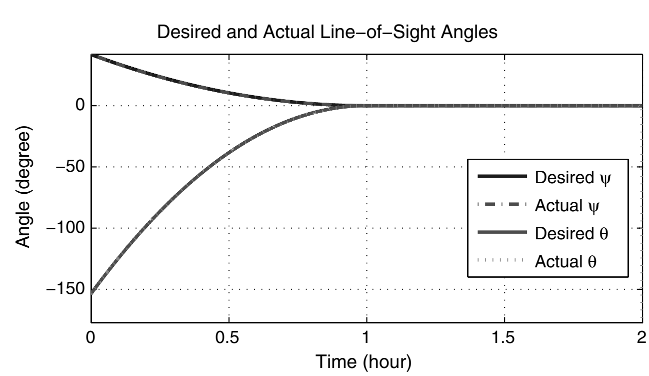


Fig. 3 Relative position regulation in an elliptical orbit by control law (16).



a) Range



b) Line-of-sight angles

Fig. 4 Autonomous rendezvous and docking along $+x$ axis.

trajectories of the relative motion expressed in the line-of-sight spherical coordinate, which is defined in the Appendix. There are very small transient errors (which would be shown in magnified plots) during the first few minutes because the deputy's initial velocity is different than the desired one, but the deputy immediately maneuvers as prescribed and successfully docks along $+x$ axis.

V. Conclusions

In this Note, new expressions of the equations of relative motion in general Keplerian orbit are presented, both in a rectangular frame and a spherical frame. It is shown, by introducing the virtual potential field, that a simple PD control law without canceling the coupled terms can perform a rendezvous (without directional constraint) in a circular orbit. Control laws for relative position regulation and tracking in Keplerian orbits are also presented. Besides the proposed control laws, one may use various advanced control laws, such as adaptive and/or robust controls which have already been developed for robot manipulators and spacecraft attitude controls, etc. In addition, control law design using the expression in a spherical coordinate can also be a potential future study, because the spherical frame is more compatible with the radar system for spacecraft navigation.

Appendix: Expression in Spherical Coordinate System

Let us introduce a spherical coordinate vector $\mathbf{q}_s = [r, \psi, \theta]^T$ which has a relation that

$$x = r\cos\psi\cos\theta, \qquad y = r\sin\psi, \qquad z = -r\cos\psi\sin\theta \quad \text{(A1)}$$

then, the relative motion with the assumption $r_c \gg r$ can be described as follows (see [18,19]):

$$\ddot{r} - r\dot{\psi}^2 - r(\dot{\theta} - \omega)^2\cos^2\psi = \frac{\mu}{r_c^3}(-r + 3r\sin^2\theta\cos^2\psi) + a_{x2} \quad \text{(A2a)}$$

$$\begin{aligned} r\ddot{\psi} + 2\dot{r}\,\dot{\psi} + r(\dot{\theta} - \omega)^2\sin\psi\cos\psi \\ = \frac{\mu}{r_c^3}(-3r\sin^2\theta\cos\psi\sin\psi) + a_{y2} \end{aligned} \quad \text{(A2b)}$$

$$\begin{aligned} r(\ddot{\theta} - \dot{\omega}) + 2\dot{r}(\dot{\theta} - \omega) - 2r\dot{\psi}(\dot{\theta} - \omega)\tan\psi \\ = \frac{\mu}{r_c^3}(3r\cos\theta\sin\theta) - \frac{a_{z2}}{\cos\psi} \end{aligned} \quad \text{(A2c)}$$

where a_{x2}, a_{y2}, and a_{z2} are control accelerations along the basis of the line-of-sight frame. Using the similar method in Sec. II, the equations of relative motion in a spherical coordinate can be written as follows:

$$H_s(\mathbf{q}_s)\ddot{\mathbf{q}}_s + C_s(\mathbf{q}_s, \dot{\mathbf{q}}_s)\dot{\mathbf{q}}_s + \mathbf{g}_{s,c}(\mathbf{q}_s) + \mathbf{g}_{s,\mathrm{nc}}(\mathbf{q}_s) = \mathbf{u}_s \quad \text{(A3)}$$

where

$$H_s(\mathbf{q}_s) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\cos^2\psi \end{bmatrix} \quad \text{(A4)}$$

$$\mathbf{g}_{s,c} \triangleq \nabla U_s = \begin{bmatrix} \frac{\partial U_s}{\partial r} \\ \frac{\partial U_s}{\partial \psi} \\ \frac{\partial U_s}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -r\left\{\omega^2\cos^2\psi - \frac{\mu}{r_c^3}(1 - 3\sin^2\theta\cos^2\psi)\right\} \\ r^2\left\{\omega^2\cos\psi\sin\psi + \frac{\mu}{r_c^3}(3\sin^2\theta\cos\psi\sin\psi)\right\} \\ -r^2\frac{\mu}{r_c^3}(3\sin\theta\cos\theta\cos^2\psi) \end{bmatrix} \quad \text{(A6)}$$

$$\mathbf{g}_{s,\mathrm{nc}}(\mathbf{q}_s) \triangleq [0, 0, r^2\dot{\omega}\cos^2\psi]^T \quad \text{(A7)}$$

where the potential function U_s in the spherical coordinate system is defined as

$$\begin{aligned} U_s &\triangleq -\frac{1}{2}r^2\left\{\omega^2\cos^2\psi - \frac{\mu}{r_c^3}(1 - 3\sin^2\theta\cos^2\psi)\right\} \\ &= -\frac{1}{2}\frac{\mu r^2}{r_c^3}\{(1 + e\cos f + 3\sin^2\theta)\cos^2\psi - 1\} \end{aligned} \quad \text{(A8)}$$

and the control input vector $\mathbf{u}_s \triangleq [a_{x2}, ra_{y2}, -ra_{z2}\cos\psi]^T$. It can be easily shown that $U(x, y, z) = U_s(r, \psi, \theta)$ for given e and f.

Acknowledgments

The authors would like to thank Panagiotis Tsiotras for providing motivation for this work and for helpful discussions. The authors also thank anonymous reviewers for valuable comments and suggestions.

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$$C_s(\dot{\mathbf{q}}_s, \mathbf{q}_s) \triangleq \begin{bmatrix} 0 & -r\dot{\psi} & -r\dot{\theta}\cos^2\psi + 2r\omega\cos^2\psi \\ r\dot{\psi} & r\dot{r} & r^2\dot{\theta}\sin\psi\cos\psi - 2r^2\omega\sin\psi\cos\psi \\ r\dot{\theta}\cos^2\psi - 2r\omega\cos^2\psi & -r^2\dot{\theta}\sin\psi\cos\psi + 2r^2\omega\sin\psi\cos\psi & r\dot{r}\cos^2\psi - r^2\dot{\psi}\cos\psi\sin\psi \end{bmatrix} \quad \text{(A5)}$$

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