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Novel Formulae of Certain Generalized Jacobi Polynomials

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Abstract: The main goal of this article is to investigate theoretically a kind of orthogonal polynomials, namely, generalized Jacobi polynomials ($GJPs$). These polynomials can be expressed as certain combinations of Legendre polynomials. Some basic formulas of these polynomials such as the power form representation and inversion formula of these polynomials are first introduced, and after that, some interesting formulas concerned with these polynomials are established. The formula of the derivatives of the moments of these polynomials is developed. As special cases of this formula, the moment and high-order derivative formulas of the $GJPs$ are deduced. New expressions for the high-order derivatives of the $GJPs$, but in terms of different symmetric and non-symmetric polynomials, are also established. These expressions lead to some interesting connection formulas between the $GJPs$ and some various polynomials.

Keywords: Legendre polynomials; generalized Jacobi polynomials; Chebyshev polynomials; linearization and connection coefficients; generalized hypergeometric functions

MSC: 11B37; 33C45

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1. Introduction

The importance of studies regarding different orthogonal polynomials has increased due to their numerous applications in many fields [1,2]. For example, this importance occurs in the theories of differential and integral equations (see, for example, [3–5]). Furthermore, in quantum physics and mathematical statistics, orthogonal polynomials have been demonstrated to be important. Among the most important orthogonal polynomials are the Jacobi polynomials. In fact, these polynomials have been investigated by a large number of authors from a theoretical point of view, see, for example [6–8]. From a numerical point of view, Jacobi polynomials occupy considerable interest in the area of solving differential and integral equations. One of the advantages of employing Jacobi polynomials is that several celebrated ones are among them. More precisely, the four kinds of Chebyshev polynomials, Legendre, and ultraspherical polynomials are particular polynomials of Jacobi polynomials. For some practical contributions concerned with Jacobi polynomials and their special classes of polynomials, one can refer to, for example, [9–13].

The idea of constructing different combinations of orthogonal polynomials to treat different types of differential equations was considered in the two leading papers of Shen [14,15]. In [14], the author considered orthogonal combinations of Legendre polynomials to numerically treat second- and fourth-order boundary value problems (BVPs) based on the application of the spectral Galerkin method. The main advantage of selecting such combinations is that their choices enable one to reduce the differential equations with their underlying conditions into systems of algebraic equations with special structures. Moreover, it has been shown that for particular types of differential equations, the resulting linear systems are diagonal, and this of course greatly simplifies the computational cost required for solving such types of differential equations (see [16]). Guo et al. in [17] considered a new type of orthogonal polynomials that are represented as a certain combination of Legendre polynomials, and they called them "$GJPs$". In [16], the authors found a closed formula for the $GJPs$. In addition, they employed such polynomials to find efficient spectral solutions for handling even-order BVPs. It is worth mentioning here that other combinations of orthogonal polynomials were proposed and employed for the numerical treatment of other types of differential equations. The author in [18] used another type of the $GJPs$ to treat the odd-order BVPs. This choice of basis functions led to systems with specially structured matrices. For some articles investigating the combinations of orthogonal polynomials, one can consult [19–23].

Obtaining explicit formulas for expressing the high-order derivatives of various special functions in general and of orthogonal polynomials, in particular, is an important target for many authors. This is because these expressions serve in the numerical solutions of different types of differential equations. For example, the authors in [24] developed high-order derivative formulas of Chebyshev polynomials of the third and fourth kind. In addition, these formulae were utilized to find spectral solutions for specific types of differential equations. These derivative formulas were generalized in [25] to find other formulas concerned with two certain non-symmetric classes of Jacobi polynomials that generalize the two classes of the third and fourth kind of Chebyshev polynomials. The high-order derivative formulas of other kinds of Chebyshev polynomials, namely Chebyshev polynomials of the fifth and sixth kinds were respectively derived in [26,27]. In [26], the authors utilized the derivative formulas of the fifth-kind Chebyshev polynomials to treat numerically the convection-diffusion equation, while in [27], the author found the high-order derivative formulas of the sixth-kind Chebyshev polynomials. Moreover, these formulas served to obtain a spectral solution of the one-dimensional Burgers' equation.

Among the important formulas concerned with any set of orthogonal polynomials are the connection and linearization formulas of such polynomials. The linearization and connection coefficients play significant roles in some applications, see, for example [28]. Therefore, there are numerous contributions regarding these problems, one can be referred to, for example, [29–33]. Furthermore, from a numerical point of view, the linearization coefficients are very useful in the numerical treatment of some non-linear differential equations. For example, the authors in [34] used some linearization formulas of Jacobi polynomials to solve the non-linear Riccati differential equation. Regarding the linearization and connection coefficients of various orthogonal polynomials, they are often expressed in terms of generalized hypergeometric functions of different arguments that can be reduced in some specific cases. Many approaches have been followed to find the connection and linearization coefficients of different special functions. For example, the author in [35] expressed a product of a certain two Jacobi polynomials in terms of a square of the ultraspherical polynomials. The linking coefficients are expressed as a terminating hypergeometric function of the form ${}_6F_5(1)$. Some linearization formulas can be extracted from the derived product formula. In addition, some linearization coefficients can be written in a form that is free of any hypergeometric forms. Other two approaches based on the connection and moments formulas were followed in [34] to obtain some linearization formulas of Jacobi polynomials. For some articles that deal with different algorithms to find connection and linearization coefficients of different orthogonal polynomials, one can consult, for example, [36–38].

The paper is structured as follows. Section 2 presents an overview of Legendre polynomials as well as the $GJPs$. Additionally, an account of some polynomials including some classical polynomials, and some of their interesting formulas are given in this section. Section 3 is confined to stating and proving two main theorems for the $GJPs$ that are considered the fundamental keys to deriving our desired results in the subsequent sections. The derivatives of the moments of the $GJPs$ are established in Section 4. As important special cases, the high-order derivatives of the $GJPs$ in terms of their original ones are also deduced in this section. Some other explicit moments' derivatives, derivatives and connection formulas of the $GJPs$ in terms of some other celebrated polynomials are given in Section 5. Some formulas that linearize the products of the $GJPs$ with some other celebrated polynomials are derived in Section 6. Finally, some conclusions are reported in Section 7.

2. Preliminaries and Some Interesting Properties of Some Special Functions

This section is confined to presenting some fundamental properties of some symmetric and non-symmetric polynomials. An overview of Legendre polynomials and certain combinations of them is given. Additionally, we give an account of some celebrated sequences of functions and polynomials. More specifically, some properties of generalized hypergeometric functions, Jacobi polynomials and their specific classes, generalized Jacobi polynomials, Hermite polynomials, and generalized Laguerre polynomials are given.

2.1. An Overview on Legendre Polynomials and Some Combinations

In this section, we give an overview of Legendre polynomials and some of their fundamental properties. In addition, some orthogonal sets of polynomials that are given as certain combinations of Legendre polynomials will also be accounted for.

It is well-known that the set of Legendre polynomials $\{P_j(x)\}_{j\geq 0}$ forms an orthogonal set of polynomials on $[-1,1]$ with respect to the unit weight function in the sense that:

$$\int_{-1}^{1} P_i(x)\, P_j(x)\, dx = \begin{cases} \dfrac{2}{2\,i+1}, & i=j, \\ 0, & i\neq j. \end{cases}$$

The power form representation of Legendre polynomials is [39]:

$$P_k(x) = 2^{-k} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m\,(2k-2m)!}{m!\,(k-2m)!\,(k-m)!}\, x^{k-2m}, \tag{1}$$

where $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.

The inversion formula of (1) is given by

$$x^k = 2^{-k}\,\sqrt{\pi}\,k! \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k-2m+\frac{1}{2}}{m!\,\Gamma\left(k-m+\frac{3}{2}\right)} P_{k-2m}(x), \quad k\geq 0.$$

Now, we provide an account of certain orthogonal combinations of Legendre polynomials. Consider the following two combinations that were selected as basis functions in [14]

$$\phi_k(x) = P_k(x) - P_{k+2}(x), \quad k\geq 0, \tag{2}$$

$$\psi_k(x) = P_k(x) - \frac{2\,(2k+5)}{2k+7}\, P_{k+2}(x) + \frac{2k+3}{2k+7}\, P_{k+4}(x), \quad k\geq 0. \tag{3}$$

In fact, the two combinations in (2) and (3) can also be written in terms of the ultraspherical polynomials with certain parameters. It can be shown that (see, [17])

$$\phi_k(x) = \frac{2k+3}{2}\,(1-x^2)\,C_k^{(\frac{3}{2})}(x), \quad k\geq 0,$$

$$\psi_k(x) = \frac{(2k+3)\,(2k+5)}{8}\,(1-x^2)^2\,C_k^{(\frac{5}{2})}(x), \quad k\geq 0,$$

where $C_k^{(\alpha)}(x)$ are the ultraspherical polynomials (see [35]). Therefore, it is evident that the polynomials $\phi_k(x)$ and $\psi_k(x)$ are respectively orthogonal polynomials with respect to the following weight functions: $w_1(x) = (1-x^2)^{-1}$ and $w_2(x) = (1-x^2)^{-2}$. In fact, the following two orthogonality relations are satisfied:

$$\int_{-1}^{1} w_1(x)\, \phi_i(x)\, \phi_j(x)\, dx = \begin{cases} \dfrac{2(2i+3)}{(i+1)(i+2)}, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\int_{-1}^{1} w_2(x)\, \psi_i(x)\, \psi_j(x)\, dx = \begin{cases} \dfrac{2(2i+3)^2(2i+5)}{(i+1)_4}, & i = j, \\ 0, & i \neq j, \end{cases}$$

where the symbol $(z)_m$ stands for the standard Pochhammer symbol, which has the definition $(z)_m = \dfrac{\Gamma(z+m)}{\Gamma(z)}$.

Remark 1. *The orthogonal polynomials $\phi_i(x)$ and $\psi_i(x)$ were utilized in Ref. [14] to solve the second- and fourth-order BVPs.*

2.2. An Overview of Hypergeometric Functions and Their Generalized Ones

According to Gauss, the ${}_2F_1(x)$ series is defined as [1]

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z \right) = 1 + \frac{a\,b}{c}\, z + \frac{a(a+1)\, b(b+1)}{c\,(c+1)}\, \frac{z^2}{2!} + \cdots = \sum_{\ell=0}^{\infty} \frac{(a)_\ell\, (b)_\ell}{(c)_\ell}\, \frac{z^\ell}{\ell!},$$

where c is an integer that is neither zero, nor negative.

The generalized hypergeometric function is defined by [1]

$${}_rF_s\left(\begin{matrix} p_1, p_2, \ldots, p_r \\ q_1, q_2, \ldots, q_s \end{matrix} \,\middle|\, z \right) = \sum_{\ell=0}^{\infty} \frac{(p_1)_\ell\, (p_2)_\ell\, \ldots (p_r)_\ell}{(q_1)_\ell\, (q_2)_\ell\, \ldots (q_s)_\ell}\, \frac{z^\ell}{\ell!},$$

in which both r and s are positive integers and no q_i, $1 \leq i \leq s$ is zero or a negative integer.

Remark 2. *Almost all famous functions and polynomials that can be represented in expressions involve hypergeometric functions or generalizations of them, which is why they are so crucial.*

2.3. An Overview on Jacobi Polynomials and Their Specific Classes

Using the following Rodrigues formula, we can construct the classical Jacobi polynomials: $P_m^{(\alpha,\beta)}(x),\ x \in [-1,1],\ m \geq 0$, and $\alpha > -1,\ \beta > -1$ (see Andrews et al. [1] and Rainville [39])

$$P_m^{(\alpha,\beta)}(x) = \frac{(-1)^m}{2^m\, m!} (1-x)^{-\alpha}\, (1+x)^{-\beta}\, \frac{d^m}{dx^m}\left[(1-x)^{\alpha+m} (1+x)^{\beta+m} \right].$$

Additionally, the following hypergeometric form may be used to represent them:

$$P_m^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_m}{m!}\, {}_2F_1\left(\begin{matrix} -m, m+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right).$$

The normalized Jacobi polynomials $R_m^{(\alpha,\beta)}(x)$ are usefully defined in the sense that $R_m^{(\alpha,\beta)}(1) = 1,\ m \geq 1$. Thus, here we have:

$$R_m^{(\alpha,\beta)}(x) = \frac{m!}{(\alpha+1)_m}\, P_m^{(\alpha,\beta)}(x),$$

and therefore, the normalized Jacobi polynomials $R_m^{(\alpha,\beta)}(x),\ x \in [-1,1],\ m \geq 0$, may be defined as (see, [35,40])

$$R_m^{(\alpha,\beta)}(x) = {}_2F_1\left(\begin{matrix} -m, m+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right).$$

We recall here that the following special polynomials are special ones of the Jacobi polynomials:

$$T_m(x) = R_m^{(-\frac{1}{2},-\frac{1}{2})}(x), \qquad U_m(x) = (m+1)\, R_m^{(\frac{1}{2},\frac{1}{2})}(x),$$
$$V_m(x) = R_m^{(-\frac{1}{2},\frac{1}{2})}(x), \qquad W_m(x) = (2m+1)\, R_m^{(\frac{1}{2},-\frac{1}{2})}(x),$$
$$P_m(x) = R_m^{(0,0)}(x), \qquad C_m^{(\alpha)}(x) = R_m^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x),$$

where $T_m(x)$, $U_m(x)$, $V_m(x)$, $W_m(x)$ denote, respectively, the first-, second-, third-, and fourth- kind Chebyshev polynomials, and $P_j(x)$ and $C_j^{(\alpha)}(x)$ are, respectively, the Legendre and ultraspherical polynomials. Note also that the polynomials $W_m(x)$ are related to the polynomials $V_m(x)$ by the following identity:

$$W_m(x) = (-1)^m\, V_m(-x). \tag{4}$$

We comment here that the ultraspherical polynomials are the well-known normalized Gegenbauer polynomials. They are given by

$$C_m^{(\alpha)}(x) = \frac{m!}{(2\alpha)_m}\, G_k^{(\alpha)}(x),$$

where $G_k^{(\alpha)}(x)$ are the well-known Gegenbauer polynomials.

The ultraspherical polynomials $C_m^{(\alpha)}(x)$ have a number of useful formulas, including a representation in power form and an inversion formula. Explicitly, we have (see [39]):

$$C_m^{(\alpha)}(x) = \frac{m!\,\Gamma(2\,\alpha+1)}{2\Gamma(\alpha+1)\,\Gamma(m+2\alpha)} \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^r\, 2^{m-2r}\,\Gamma(m-r+\alpha)}{r!\,(m-2r)!} x^{m-2r}, \quad m \geq 0,$$

and

$$x^m = \frac{2^{1-m}\,\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!\,(m-2r+\alpha)\,\Gamma(m-2r+2\alpha)}{r!\,(m-2r)!\,\Gamma(m-r+\alpha+1)}, \quad m \geq 0. \tag{5}$$

The class of polynomials $R_m^{(\alpha,\alpha+1)}(x)$ is one of the important classes that has recently been explored in [25]. This category extends the class of third-Chebyshev polynomials. The formulas for the power form and inversion of these polynomials are as follows [25]:

$$\begin{aligned} R_\ell^{(\alpha,\alpha+1)}(x) =& \frac{\ell!\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\ell+2\alpha+2)} \Bigg\{ \sum_{r=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{(-1)^r\, 2^{\ell-2r+2\alpha+1}\,\Gamma\big(\ell-r+\alpha+\frac{3}{2}\big)}{r!\,(\ell-2r)!}\, x^{\ell-2r} \\ &+ \sum_{r=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{(-1)^{r+1}\, 2^{\ell-2r+2\alpha}\,\Gamma\Big(\ell-r+\alpha+\frac{1}{2}\Big)}{r!\,(\ell-2r-1)!}\, x^{\ell-2r-1} \Bigg\}, \quad \ell \geq 0, \end{aligned}$$

and

$$\begin{aligned} x^\ell =& \frac{\sqrt{\pi}\,\ell!}{2^{\ell+2\alpha+1}\,\Gamma(\alpha+1)} \Bigg\{ \sum_{r=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{\Gamma(\ell-2r+2\alpha+2)}{r!\,(\ell-2r)!\,\Gamma\big(\ell-r+\alpha+\frac{3}{2}\big)}\, R_{\ell-2r}^{(\alpha,\alpha+1)}(x) \\ &+ \sum_{r=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{\Gamma(\ell-2r+2\alpha+1)}{r!\,(\ell-2r-1)!\,\Gamma\big(\ell-r+\alpha+\frac{3}{2}\big)}\, R_{\ell-2r-1}^{(\alpha,\alpha+1)}(x) \Bigg\}, \quad \ell \geq 0. \end{aligned} \tag{6}$$

It should be noted that the well-known four kinds all satisfy the same recurrence relation given by:

$$\phi_m(x) = 2\,x\,\phi_{m-1}(x) - \phi_{m-2}(x), \quad m \geq 2,$$

but with different initials.

We point out here that $\phi_{-j}(x)$, $j \geq 0$ are related with $\phi_j(x)$. The following identities hold [34]:

$$T_{-m}(x) = T_m(x), \quad U_{-m}(x) = -U_{m-2}(x),$$
$$V_{-m}(x) = V_{m-1}(x), \quad W_{-m}(x) = -W_{m-1}(x).$$

One can consult the helpful books of Andrews et al. [1] and Mason and Handscomb [2] for a thorough overview of Jacobi polynomials and their various classes.

2.4. Generalized Jacobi Polynomials

In their interesting paper [17], Guo et al. defined a family of generalized Jacobi polynomials where the indexes α and/or $\beta \leq -1$ are constrained to take negative integers. They also investigated the general cases for the parameters in the general cases in [41].

For $r, s \in \mathbb{Z}$ (the set of all integers), they defined in [17] the following generalized Jacobi polynomials $\tilde{J}_k^{(r,s)}(x)$ as:

$$\tilde{J}_k^{(r,s)}(x) = \begin{cases} (1-x)^{-r}\,(1+x)^{-s}\,P_{k-k_0}^{(-r,-s)}(x), & k_0 = -(r+s),\ r,s \leq -1, \\ (1-x)^{-r}\,P_{k-k_0}^{(-r,s)}(x), & k_0 = -r,\ r \leq -1,\ s > -1, \\ (1+x)^{-s}\,P_{k-k_0}^{(r,-s)}(x), & k_0 = -s,\ r > -1,\ s \leq -1, \\ P_{k-k_0}^{(r,s)}(x), & k_0 = 0,\ r,s > -1. \end{cases} \tag{7}$$

Due to the generalization to the Jacobi polynomials in (7), the authors in [17] called these polynomials "generalized Jacobi polynomials".

An important property of the GJPs is that for $r, s \in \mathbb{Z}$, and $r, s \geq 1$, we have

$$D^{\ell} J_k^{(-r,-s)}(1) = 0, \quad \ell = 0, 1, \ldots, r-1;$$
$$D^{j} J_k^{(-r,-s)}(-1) = 0, \quad j = 0, 1, \ldots, s-1.$$

The authors in [16] proved that the generalized polynomials $\tilde{J}_k^{(-n,-n)}(x)$ can be written as a combination of Legendre polynomials. More precisely, they proved the following identity:

$$\tilde{J}_k^{(-n,-n)}(x) = (1-x^2)^n\,P_k^{(n,n)}(x) = (k+1)_n \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} (4j+2k+1)\Gamma\left(j+k+\frac{1}{2}\right)}{2\,\Gamma\left(j+k+n+\frac{3}{2}\right)} P_{k+2j}(x). \tag{8}$$

Now, in order to generalize the combinations in (2) and (3), we define the following generalized Jacobi polynomials, "GJPs", which are the old ones multiplied by a suitable factor.

$$J_k^{(n)}(x) = \frac{\left(k+\frac{3}{2}\right)_n}{(k+1)_n}\,\tilde{J}_k^{(-n,-n)}(x). \tag{9}$$

and therefore, Formula (8) leads to the following identity for $J_k^n(x)$

$$J_k^n(x) = \frac{\Gamma\left(k+n+\frac{3}{2}\right)}{2\,\Gamma\left(k+\frac{3}{2}\right)} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}(4j+2k+1)\,\Gamma\left(j+k+\frac{1}{2}\right)}{\Gamma\left(j+k+n+\frac{3}{2}\right)}\,P_{2j+k}(x). \tag{10}$$

From relation (9), it is clear that the polynomials $J_k^n(x)$ are orthogonal on $[-1,1]$ with respect to the weight function: $w(x) = (1-x^2)^{-n}$, in the sense that:

$$\int_{-1}^{1} w(x)\, J_i^n(x)\, J_j^n(x)\, dx = \begin{cases} \dfrac{2^{2n-1}\,(2i+2n+1)\, i!\, \left(\Gamma\left(i+n+\frac{1}{2}\right)\right)^2}{\left(\Gamma\left(i+\frac{3}{2}\right)\right)^2 (i+2n)!}, & i=j, \\ 0, & i \neq j. \end{cases}$$

Remark 3. *These basis functions in* (8) *were generalized in* [16] *to handle BVPs of even orders.*

Remark 4. *For* $n = 1,2$, *the combination in* (10) *reduces respectively to the basis in* (2) *and* (3), *while the basis functions corresponding to* $n = 3$ *are given as*

$$J_k^3(x) = P_k(x) - \frac{3\,(2k+5)}{2k+9}\, P_{k+2}(x) + \frac{3\,(2k+3)}{2k+11}\, P_{k+4}(x) - \frac{(2k+3)(2k+5)}{(2k+9)(2k+11)}\, P_{k+6}(x), \quad k \geq 0.$$

The last basis functions were utilized in [42] *to solve the sixth-order BVPs.*

2.5. An Overview on Hermite and Generalized Laguerre Polynomials

The two classes of Hermite polynomials $\{H_i(x)\}_{i\geq 0}$ and generalized Laguerre polynomials $\left\{L_i^{(\alpha)}(x)\right\}_{i\geq 0}$ are orthogonal polynomials on, respectively, $(-\infty,\infty)$ and $(0,\infty)$ in the sense that:

$$\int_{-\infty}^{\infty} e^{-x^2}\, H_i(x)\, H_j(x)\, dx = \sqrt{\pi}\, 2^i\, i!\, \delta_{i,j},$$
$$\int_{-\infty}^{\infty} x^{\alpha}\, e^{-x}\, L_i^{(\alpha)}(x)\, L_j^{(\alpha)}(x)\, dx = \frac{\Gamma(i+\alpha+1)}{i!}\, \delta_{i,j},$$

and $\delta_{i,j}$ is the well-known Kronecker delta function.

The power form and inversion formulas of the Hermite polynomials are given by

$$H_i(x) = i! \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^m\, 2^{i-2m}}{m!\,(i-2m)!}\, x^{i-2m}, \quad i \geq 0,$$
$$x^i = \frac{i!}{2^i} \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{1}{m!(i-2m)!} H_{i-2m}(x), \quad i \geq 0, \tag{11}$$

while the power form and inversion formulas of the generalized Laguerre polynomials are given by

$$L_i^{(\alpha)}(x) = \frac{\Gamma(i+\alpha+1)}{i!} \sum_{k=0}^{i} \frac{(-1)^{i-k}\binom{i}{k}}{\Gamma(i+\alpha-k+1)}\, x^{i-k}, \quad i \geq 0,$$
$$x^i = i!\, \Gamma(1+i+\alpha) \sum_{k=0}^{i} \frac{(-1)^{i-k}}{k!\, \Gamma(i+\alpha-k+1)}\, L_{i-k}^{\alpha}(x), \quad i \geq 0. \tag{12}$$

2.6. An Overview on Two Generalized Classes of Polynomials

Among the important symmetric polynomials are the so-called generalized Fibonacci and generalized Lucas polynomials. These polynomials can be respectively constructed with the aid of the following two recurrence relations:

$$F_r^{A,B}(x) = A\, x\, F_{r-1}^{A,B}(x) + B\, F_{r-2}^{A,B}(x), \quad F_0^{A,B}(x) = 1,\ F_1^{A,B}(x) = A\, x, \quad r \geq 2, \tag{13}$$
$$L_r^{R,S}(x) = R\, x\, F_{r-1}^{R,S}(x) + S\, L_{r-2}^{R,S}(x), \quad F_0^{R,S}(x) = 2,\ F_1^{R,S}(x) = R\, x, \quad r \geq 2, \tag{14}$$

where the constants A, B, R, S are arbitrary non-zero real constants.

The two generalized classes of polynomials $F_r^{A,B}(x)$ and $L_r^{R,S}(x)$ can be represented, respectively, as [43]:

$$F_j^{A,B}(x) = \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} \tbinom{j-\ell}{\ell} B^\ell A^{j-2\ell} x^{j-2\ell}, \quad j \geq 0,$$

and

$$L_j^{R,S}(x) = j \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} \frac{S^\ell R^{j-2\ell} \binom{j-\ell}{\ell}}{j-\ell} x^{j-2\ell}, \quad j \geq 1,$$

where their inversion formulas are given by [43]

$$x^\ell = A^{-\ell} \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{(-B)^i \binom{\ell}{i} (\ell - 2i + 1)}{\ell - i + 1} F_{\ell-2i}^{A,B}(x), \quad \ell \geq 0, \tag{15}$$

$$x^\ell = R^{-\ell} \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} (-S)^i c_{\ell-2i} \binom{\ell}{i} L_{\ell-2i}^{R,S}(x), \quad \ell \geq 0, \tag{16}$$

where

$$c_j = \begin{cases} \frac{1}{2}, & j = 1, \\ 1, & j \geq 1. \end{cases}$$

Among the important properties of the two generalized polynomials, $F_k^{A,B}(x)$ and $L_k^{R,S}(x)$ are the moment formulas for these polynomials. Rewriting the two recurrence relations, (13) and (14), in the forms:

$$x F_k^{A,B}(x) = \frac{1}{A} F_{k+1}^{A,B}(x) - \frac{B}{A} F_{k-1}^{A,B}(x),$$

then it is easy to obtain the moment formulas for the generalized classes $F_k^{A,B}(x)$, and $L_k^{R,S}(x)$. The moment formulas for these polynomials are stated in the following lemma.

Lemma 1. *Let r and k be any two non-negative integers. The following moment formulas hold:*

$$x^r F_k^{A,B}(x) = \sum_{m=0}^{r} \binom{r}{m} A^{-r} (-B)^m F_{k+r-2m}^{A,B}(x),$$

$$x^r L_k^{A,B}(x) = \sum_{m=0}^{r} \binom{r}{m} R^{-r} (-S)^m L_{k+r-2m}^{A,B}(x).$$

Proof. The proofs can be easily obtained by induction based on the application of the two recurrence relations (13) and (14). □

3. New Essential Formulas of the $GJPs$

This section is confined to stating and proving two basic formulas concerned with the $GJPs$ which will be the core of most of the results of this paper. In this respect, the power form representation and inversion formulas of these polynomials are derived in detail. For the derivation of these two important formulas, the following three lemmas are needed in the sequel:

Lemma 2. *Let $k, m,$ and n be any non-negative integers. The following identity holds:*

$$\sum_{i=0}^{n} \frac{\binom{n}{i} (4i + 2k + 1) \Gamma\left(i + k + \frac{1}{2}\right) \Gamma\left(i + k - m + n + \frac{1}{2}\right)}{\Gamma\left(i + k + n + \frac{3}{2}\right) (i + m - n)!} = \frac{2 \Gamma\left(k - m + n + \frac{1}{2}\right)}{m!}.$$

Proof. If we set

$$S_{n,k,m} = \sum_{i=0}^{n} \frac{\binom{n}{i}\,(4i+2k+1)\,\Gamma\left(i+k+\frac{1}{2}\right)\Gamma\left(i+k-m+n+\frac{1}{2}\right)}{\Gamma\left(i+k+n+\frac{3}{2}\right)(i+m-n)!},$$

then the application of the celebrated algorithm of Zeilberger [44] shows that the following recurrence relation is satisfied by $S_{n,k,m}$:

$$(2k-2m+2n+1)S_{n,k,m} + 2\,S_{n+1,k,m} = 0, \quad S_{0,k,m} = \frac{2\,\Gamma\left(k-m+\frac{1}{2}\right)}{m!}.$$

The last recurrence relation can be easily solved to give

$$S_{n,k,m} = \frac{2\,\Gamma\left(k-m+n+\frac{1}{2}\right)}{m!}.$$

Lemma 2 is now proved. □

Lemma 3. *For all non-negative integers p, i, and n, one has*

$$\sum_{\ell=0}^{p} \frac{(-1)^{p-\ell}\,(4i-4\ell+2n+1)\,\Gamma\left(2i-\ell+n-p+\frac{1}{2}\right)}{\ell!\,(p-\ell)!\,\Gamma\left(2i-\ell+n+\frac{3}{2}\right)} = \begin{cases} 2, & p \geq 0, \\ 0, & p \geq 1. \end{cases}$$

Proof. The result is obvious for $p = 0$. For $p \geq 1$, to prove the result, it is enough to show that the following identity holds:

$$\sum_{\ell=0}^{p-1} \frac{(-1)^{p-\ell}(4i-4\ell+2n+1)\Gamma\left(2i-\ell+n-p+\frac{1}{2}\right)}{\ell!(p-\ell)!\Gamma\left(2i-\ell+n+\frac{3}{2}\right)} = \frac{-2}{p!\left(n+2i-2p+\frac{3}{2}\right)_p}. \tag{17}$$

For this purpose, let $r = p-1$, and set

$$G_{r,i,n} = \sum_{\ell=0}^{r} \frac{(-1)^{1-\ell+r}\,(4i-4\ell+2n+1)\,\Gamma(2i-\ell+n-r)}{\ell!\,(1-\ell+r-\frac{1}{2})!\,\Gamma\left(2i-\ell+n+\frac{3}{2}\right)}, \quad r \geq 0,$$

thanks to Zeilberger's algorithm again, it can be seen that the following recurrence relation of order one for $G_{r,i,n}$ is satisfied:

$$(r+2)(-4\,r+4\,i+2\,n-3)(-4\,r+4\,i+2\,n-5)\,G_{r+1,i,n} - 2(-2\,r+4\,i+2\,n-1)\,G_{r,i,n} = 0,$$
$$G_{0,i,n} = \frac{-4}{4\,i+2\,n-1},$$

that can be immediately solved to give

$$G_{r,i,n} = \frac{-2}{(r+1)!\left(n+2i-2r-\frac{1}{2}\right)_{r+1}},$$

and hence, Formula (17) can be obtained. □

Lemma 4. *For all non-negative integers p, i, and n, one has*

$$\sum_{\ell=0}^{i} \frac{(-1)^{i-\ell+p}\,(4i-4\ell+2n+1)\,\Gamma\left(i-\ell+n-p+\frac{1}{2}\right)}{\ell!\,\Gamma\left(2i-\ell+n+\frac{3}{2}\right)\,(i-\ell+p)!} = \frac{2\,(-1)^p\,\Gamma\left(n-p+\frac{1}{2}\right)\,(p)_i}{i!\,(i+p)!\,\Gamma\left(n+i+\frac{1}{2}\right)}.$$

Proof. The proof is similar to the proof of Lemma 2. □

We can now state and prove the following two key theorems.

Theorem 1. *The power form representation of the polynomials $J_k^n(x)$ is given by*

$$J_k^n(x) = \frac{\Gamma\left(k+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k+\frac{3}{2}\right)} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor + n} \frac{(-1)^{m+n}\, 2^{k-2m+2n}\, \Gamma\left(k-m+n+\frac{1}{2}\right)}{m!\,(k-2m+2n)!}\, x^{k-2m+2n}. \tag{18}$$

Proof. If we make use of the power form representation of Legendre polynomials in (1) along with relation (10), then $J_k^n(x)$ can be written in the following form:

$$\begin{aligned} J_k^n(x) = &\frac{\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \sum_{r=0}^{n} \frac{(-1)^r \left(k+2r+\frac{1}{2}\right)\binom{n}{r}\Gamma\left(k+r+\frac{1}{2}\right)}{2^{k+2r}\Gamma\left(k+n+r+\frac{3}{2}\right)} \times \\ &\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor + r} \frac{(-1)^m\,(2k-2m+4r)!}{m!\,(k-2m+2r)!\,(k-m+2r)!} x^{k+2r-2m}. \end{aligned}$$

After some lengthy manipulation, the last formula can be written alternatively in the following form:

$$\begin{aligned} J_k^n(x) = &\frac{\Gamma\left(k+n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(k+\frac{3}{2}\right)} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor + n} \frac{(-1)^{m+n} 2^{k-2m+2n-1}}{(k-2m+2n)!} \times \\ &\sum_{i=0}^{n} \frac{\binom{n}{i}\,(4i+2k+1)\,\Gamma\left(i+k+\frac{1}{2}\right)\Gamma\left(i+k-m+n+\frac{1}{2}\right)}{\Gamma\left(i+k+n+\frac{3}{2}\right)(i+m-n)!}\, x^{k-2m+2n}. \end{aligned} \tag{19}$$

The last formula can be reduced with the aid of Lemma 2. In such a case, Formula (19) can be transformed into

$$J_k^n(x) = \frac{\Gamma\left(k+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k+\frac{3}{2}\right)} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{m+n}\, 2^{k-2m+2n}\, \Gamma\left(k-m+n+\frac{1}{2}\right)}{m!\,(k-2m+2n)!}\, x^{k-2m+2n}.$$

This finalizes the proof of Theorem 1. □

Now, we give the inversion formula to Formula (18).

Theorem 2. *Let i and n be any two non-negative integers. The following formula is valid*

$$x^{i+2n} = \sqrt{\pi}(-1)^n\,(i+2n)! \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{1}{m!\,2^{i+2n}\,\left(i-2m+\frac{3}{2}\right)_{n-1}\Gamma\left(i-m+n+\frac{3}{2}\right)}\, J_{i-2m}^n(x) + \delta_i(x), \tag{20}$$

and $\delta_i(x)$ is given by

$$\delta_i(x) = \begin{cases} \frac{2\left(\frac{i}{2}+1\right)_n}{(n-1)!} \sum\limits_{\ell=0}^{n-1} \frac{(-1)^\ell\,\binom{n-1}{\ell} x^{-2\ell+2n-2}}{i+2\,\ell+2}, & i \text{ even}, \\ \frac{2\left(\frac{i+1}{2}\right)_n}{(n-1)!} \sum\limits_{\ell=0}^{n-1} \frac{(-1)^\ell\binom{n-1}{\ell} x^{-2\ell+2n-1}}{i+2\ell+1}, & i \text{ odd}. \end{cases} \tag{21}$$

Proof. The inversion Formula (20) can be split into the following two inversion formulas:

$$\begin{aligned}
x^{2i+2n} =&(-1)^n\, 4^{-i-n}\, \sqrt{\pi}\, (2(i+n))!\times \\
&\sum_{m=0}^{i} \frac{1}{m!\,\Gamma\left(2i-m+n+\frac{3}{2}\right)\left(2i-2m+\frac{3}{2}\right)_{n-1}}\, J_{2i-2m}^{n}(x) \\
&+\frac{(i+1)_n}{(n-1)!}\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}\binom{n-1}{\ell}\, x^{-2-2\ell+2n}}{i+\ell+1},
\end{aligned} \tag{22}$$

$$\begin{aligned}
x^{2i+2n+1} =&(-1)^n\, 2^{-1-2i-2n}\, \sqrt{\pi}\, (2i+2n+1)!\times \\
&\sum_{m=0}^{i} \frac{1}{m!\,\Gamma\left(\frac{5}{2}+2i-m+n\right)\left(\frac{5}{2}+2i-2m\right)_{n-1}}\, J_{2i-2m+1}^{n}(x) \\
&+\frac{(i+1)_n}{(n-1)!}\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}\binom{n-1}{\ell}\, x^{-1-2\ell+2n}}{i+\ell+1}.
\end{aligned} \tag{23}$$

The derivation of the two inversion Formulas (22) and (23) is similar. In the following, we are going to prove (22). Noting the identity:

$$\frac{(i+1)_n}{(n-1)!}\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}\binom{n-1}{\ell}\, x^{-1-2\ell+2n}}{i+\ell+1} = \frac{x^{2n-2}(i+1)_n}{(i+1)(n-1)!}\;{}_2F_1\left(\left.\begin{matrix} i+1, 1-n \\ i+2 \end{matrix}\,\right|\frac{1}{x^2}\right),$$

then, to prove (22), it is required to prove the validity of the following identity:

$$F_i(x) = x^{2i+2n} - \frac{x^{2n-2}(i+2)_n}{(n-1)!}\;{}_2F_1\left(\left.\begin{matrix} i+1, 1-n \\ i+2 \end{matrix}\,\right|\frac{1}{x^2}\right),$$

where

$$\begin{aligned}
F_i(x) =&(-1)^n\, 4^{-i-n}\, \sqrt{\pi}\, (2(i+n))!\times \\
&\sum_{m=0}^{i} \frac{1}{m!\,\Gamma\left(2i-m+n+\frac{3}{2}\right)\left(2i-2m+\frac{3}{2}\right)_{n-1}}\, J_{2i-2m}^{n}(x).
\end{aligned} \tag{24}$$

Now, with the aid of Theorem 1, the substitution by the power form representation of the polynomials $J_{2i-2m}^{n}(x)$ into the right-hand side of (24) enables one to write

$$\begin{aligned}
F_i(x) =&\left((-1)^n 2^{-2i-2n-1}(2(i+n))!\sum_{m=0}^{i}\frac{4i-4m+2n+1}{m!\,\Gamma\left(2i-m+n+\frac{3}{2}\right)}\right)\times \\
&\left(\sum_{\ell=0}^{i-m+n}\frac{(-1)^{\ell+n}2^{2(i-\ell-m+n)}\Gamma\left(2i-\ell-2m+n+\frac{1}{2}\right)}{\ell!(2(i-\ell-m+n))!}\right)x^{2i-2m-2\ell+2n}.
\end{aligned}$$

After some lengthy algebraic computations, the last formula can be converted into the following formula:

$$F_i(x) = \sum_1 + \sum_2,$$

where

$$\begin{aligned}
\sum_1 =&(2i+2n)!\times \\
&\sum_{p=0}^{i-1}\left(\frac{2^{-1-2p}}{(2i+2n-2p)!}\sum_{\ell=0}^{p}\frac{(-1)^{-\ell+p}(4i-4\ell+2n+1)\Gamma\left(2i-\ell+n-p+\frac{1}{2}\right)}{\ell!(p-\ell)!\Gamma\left(2i-\ell+n+\frac{3}{2}\right)}\right)x^{2i+2n-2p},
\end{aligned}$$

and

$$\sum_2 =(2i+2n)!\times$$
$$\sum_{p=0}^{n}\frac{2^{-1-2i-2p}}{(2n-2p)!}\left(\sum_{\ell=0}^{i}\frac{(-1)^{i-\ell+2n+p}(4i-4\ell+2n+1)\Gamma\left(i-\ell+n-p+\frac{1}{2}\right)}{\ell!(i-\ell+p)!\Gamma\left(2i-\ell+n+\frac{3}{2}\right)}\right)x^{2n-2p}.$$

Regarding $\sum_1$, and based on Lemma 3, we have

$$\sum_{\ell=0}^{p}\frac{(-1)^{p-\ell}\,(4i-4\ell+2n+1)\,\Gamma\left(2i-\ell+n-p+\frac{1}{2}\right)}{\ell!\,(p-\ell)!\,\Gamma\left(2i-\ell+n+\frac{3}{2}\right)}=\begin{cases}2, & p\geq 0,\\ 0, & p\geq 1,\end{cases}$$

and therefore, it is not difficult to show that:

$$\sum_1 = x^{2i+2n}.$$

Regarding $\sum_2$, and using Lemma 4, and after some computations, it can be transformed into the following one:

$$\sum_2 = \frac{(2i+2n)!}{i!\,\Gamma\left(i+n+\frac{1}{2}\right)}\sum_{p=0}^{n}\frac{(-1)^p\,2^{-2(i+p)}\,\Gamma\left(n-p+\frac{1}{2}\right)(p)_i}{(2n-2p)!\,(i+p)!}\,x^{2n-2p},$$

which can be written alternatively in the following hypergeometric representation:

$$\sum_2 = -\frac{x^{2n-2}(i+1)_n}{(i+1)(n-1)!}\,{}_2F_1\left(\begin{matrix}i+1,1-n\\ i+2\end{matrix}\,\middle|\,\frac{1}{x^2}\right).$$

This completes the proof of Formula (22). □

4. High-Order Derivatives of the Moments of the *GJPs*

This section is devoted to presenting a new formula that expresses the high-order derivatives of the moments of the *GJPs* in terms of their original polynomials. Several interesting formulas can be deduced as special cases of the derived formula.

Theorem 3. *Let k,q,r,n be non-negative integers with $k+2n+r\geq q$. The derivatives of the moments of the GJPs can be expressed in terms of the GJPs themselves as*

$$\begin{aligned}D^q(x^r J_k^n(x)) =& \frac{2^{q-r}\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)(k+2n+r)!}{\Gamma\left(k+\frac{3}{2}\right)(k+2n)!}\times\\ &\sum_{p=0}^{\left\lfloor\frac{k-q+r}{2}\right\rfloor}\frac{1}{p!\left(k-2p-q+r+\frac{3}{2}\right)_{n-1}\Gamma\left(k+n-p-q+r+\frac{3}{2}\right)}\times\\ &{}_4F_3\left(\begin{matrix}-p,-\frac{k}{2}-n,-\frac{k}{2}-n+\frac{1}{2},-k-n+p+q-r-\frac{1}{2}\\ -k-n+\frac{1}{2},-\frac{k}{2}-n-\frac{r}{2},-\frac{k}{2}-n-\frac{r}{2}+\frac{1}{2}\end{matrix}\,\middle|\,1\right)J_{k-q+r-2p}^n(x)\\ &+\sum_{m=0}^{\left\lfloor\frac{k}{2}\right\rfloor+n}E_{m,n,k,q,r}\,\delta_{k-2m-q+r}(x),\end{aligned}\tag{25}$$

where $E_{m,n,k,q,r}$ is given by

$$E_{m,n,k,q,r} = \frac{(-1)^{m+n}\, 2^{k-2m+2n-1}\,(2k+3)\,\Gamma\left(k-m+n+\frac{1}{2}\right)\,\left(k+\frac{5}{2}\right)_{n-1}}{\sqrt{\pi}\,m!\,(k-2m+2n)!} \times$$
$$(k-2m+2n-q+r+1)_q,$$

and $\delta_i(x)$ are given in (21).

Proof. The power form representation of $J_k^n(x)$ enables one to express the moments $\left(x^r\, J_k^n(x)\right)$ in the form

$$x^r\, J_k^n(x) = \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} B_{m,n,k}\, x^{k-2m+2n+r}, \tag{26}$$

and $B_{m,n,k}$ is given by

$$B_{m,n,k} = \frac{(-1)^{m+n}\, 2^{k-2m+2n-1}\,(2k+3)\,\Gamma\left(k-m+n+\frac{1}{2}\right)\left(k+\frac{5}{2}\right)_{n-1}}{\sqrt{\pi}\,m!\,(k-2m+2n)!}. \tag{27}$$

If (26) is differentiated q times with respect to x, then we obtain

$$\begin{aligned} D^q(x^r\, J_k^n(x)) &= \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} B_{m,n,k}\,(k-2m+2n+r-q+1)_q\, x^{k-2m+2n+r-q} \\ &= \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} E_{m,n,k,q,r}\, x^{k-2m+2n+r-q}, \end{aligned}$$

with

$$E_{m,n,k,q,r} = \frac{(-1)^{m+n}\, 2^{k-2m+2n-1}\,(2k+3)\,\Gamma\left(k-m+n+\frac{1}{2}\right)\,\left(k+\frac{5}{2}\right)_{n-1}}{\sqrt{\pi}\,m!\,(k-2m+2n)!} \times$$
$$(k-2m+2n-q+r+1)_q.$$

Making use of the inversion Formula (20) enables one to express $D^q\left(x^r\, J_k^n(x)\right)$ in the following form

$$D^q(x^r\, J_k^n(x)) = \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} E_{m,n,k,q,r} \left(\sum_{\ell=0}^{\left\lfloor \frac{k-q+r}{2} \right\rfloor - m} M_{\ell,k-2m-q+r}\, J^n_{k-2m-q+r-2\ell} + \delta_{k-2m-q+r}(x) \right), \tag{28}$$

where $M_{\ell,i}$ are given as

$$M_{\ell,i} = \frac{\sqrt{\pi}(-1)^n\,(i+2n)!}{\ell!\,2^{i+2n}\,\left(i-2\ell+\frac{3}{2}\right)_{n-1}\Gamma\left(i-\ell+n+\frac{3}{2}\right)}.$$

The arrangement of the right-hand side of Equation (28) yields

$$D^q(x^r\, J_k^n(x)) = \sum_{p=0}^{\left\lfloor \frac{k-q+r}{2} \right\rfloor} \sum_{t=0}^{p} E_{t,n,k,q,r}\, M_{p-t,k-q+r-2t}\, J^n_{k-q+r-2p}(x) + \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} E_{m,n,k,q,r}\, \delta_{k-2m-q+r}(x),$$

which is equivalent to the following relation

$$D^q\big(x^r\, J_k^n(x)\big) = \frac{2^{q-r}\,\Gamma\big(k+n+\frac{3}{2}\big)}{\Gamma\big(k+\frac{3}{2}\big)} \sum_{p=0}^{\left\lfloor \frac{k-q+r}{2} \right\rfloor} \frac{1}{\big(k-2p-q+r+\frac{3}{2}\big)_{n-1}} \times$$
$$\sum_{t=0}^{p} \frac{(-1)^t\,\Gamma\Big(k+n-t+\frac{1}{2}\Big)\,(k+2n+r-2t)!}{t!\,(p-t)!\,(k+2n-2t)!\,\Gamma\big(k+n-p-q+r-t+\frac{3}{2}\big)}\, J_{k-q+r-2p}^n(x)$$
$$+ \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} E_{m,n,k,q,r}\, \delta_{k-2m-q+r}(x).$$

The last formula can be written in the following hypergeometric form

$$D^q\big(x^r\, J_k^n(x)\big) = \frac{2^{q-r}\Gamma\Big(k+n+\frac{1}{2}\Big)\Gamma\big(k+n+\frac{3}{2}\big)(k+2n+r)!}{\Gamma\big(k+\frac{3}{2}\big)(k+2n)!} \times$$
$$\sum_{p=0}^{\left\lfloor \frac{k-q+r}{2} \right\rfloor} \frac{1}{p!\big(k-2p-q+r+\frac{3}{2}\big)_{n-1}\Gamma\big(k+n-p-q+r+\frac{3}{2}\big)} \times$$
$${}_4F_3\left(\begin{matrix} -p, -\frac{k}{2}-n, -\frac{k}{2}-n+\frac{1}{2}, -k-n+p+q-r-\frac{1}{2} \\ -k-n+\frac{1}{2}, -\frac{k}{2}-n-\frac{r}{2}, -\frac{k}{2}-n-\frac{r}{2}+\frac{1}{2} \end{matrix} \middle| 1 \right) J_{k-2m-q+r-2\,\ell}^n(x)$$
$$+ \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} E_{m,n,k,q,r}\, \delta_{k-2m-q+r}(x),$$

and, consequently, relation (25) can be obtained. □

Remark 5. *The expressions of the high-order derivatives of different orthogonal polynomials in terms of their original ones are very important. As an example, the derivative formulas of certain Jacobi polynomials in [25] were utilized to handle certain types of linear and non-linear even-order BVPs.*

Remark 6. *Some expressions of the high-order derivatives of the GJPs were utilized to treat even-order BVPs in [16]. The derivatives of the GJPs are expected to be used to treat some other differential equations based on utilizing suitable spectral methods. We plan to use these formulas in forthcoming work.*

Several interesting formulas can be deduced from Formula (25). To be more precise, the following formulas can be deduced as special cases of (25).

- The derivatives of the moments of Legendre polynomials.
- The moment's formula of the $GJPs$.
- The expression that gives the derivatives of the $GJPs$ in terms of their original ones.

Corollary 1. *Let k, q, r be non-negative integers with $k + r \geq q$. The derivatives of the moments of Legendre polynomials have the following expression in terms of Legendre polynomials:*

$$D^q\big(x^r\, P_k(x)\big) = \frac{2^{q-r}\,\Gamma\Big(k+\frac{1}{2}\Big)\,(k+r)!}{k!} \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q+r) \right\rfloor} \frac{k-2p-q+r+\frac{1}{2}}{p!\,\Gamma\big(k-p-q+r+\frac{3}{2}\big)} \times$$
$${}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-\frac{k}{2}, -\frac{k}{2}, -\frac{1}{2}-k+p+q-r \\ \frac{1}{2}-k, -\frac{k}{2}-\frac{r}{2}, \frac{1}{2}-\frac{k}{2}-\frac{r}{2} \end{matrix} \middle| 1 \right) P_{k-q+r-2p}(x).$$

Proof. The proof is a direct consequence of Theorem 3, only by setting $n = 0$. □

Corollary 2. *Let k, q, r, n be non-negative integers. The following moment's formula for the $GJPs$ is valid.*

$$x^r\, J_k^n(x) = \frac{2^{-r}\,\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)(k+2n+r)!}{\Gamma\left(k+\frac{3}{2}\right)(k+2n)!} \times \sum_{p=0}^{\left\lfloor\frac{k+r}{2}\right\rfloor} \frac{1}{p!\left(k-2p+r+\frac{3}{2}\right)_{n-1}\Gamma\left(k+n-p+r+\frac{3}{2}\right)} \times {}_4F_3\left(\begin{matrix} -p, -\frac{k}{2}-n, -\frac{k}{2}-n+\frac{1}{2}, -k-n+p-r-\frac{1}{2} \\ -k-n+\frac{1}{2}, -\frac{k}{2}-n-\frac{r}{2}, -\frac{k}{2}-n-\frac{r}{2}+\frac{1}{2} \end{matrix}\middle| 1\right) J_{k+r-2p}^n(x) + \sum_{m=0}^{\left\lfloor\frac{k}{2}\right\rfloor+n} B_{m,n,k}\, \delta_{k+r-2p}(x), \tag{29}$$

where $B_{m,n,k}$ is as given in (27).

Proof. Formula (29) can be immediately obtained by setting $q=0$ in Formula (25). □

Corollary 3. *Let k, q, n be non-negative integers with $k+2n \geq q$. The derivatives of the GJPs are connected with their original polynomials by the following formula:*

$$D^q(J_k^n(x)) = \frac{2^q\,\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \times \sum_{p=0}^{\left\lfloor\frac{k-q}{2}\right\rfloor} \frac{(-p-q+1)_p\,\Gamma\left(k-2p-q+\frac{3}{2}\right)}{p!\left(-k-n+\frac{1}{2}\right)_p \Gamma\left(k+n-2p-q+\frac{1}{2}\right)\Gamma\left(k+n-p-q+\frac{3}{2}\right)} J_{k-q-2p}^n + \sum_{m=0}^{\left\lfloor\frac{k}{2}\right\rfloor+n} \bar{E}_{m,n,k,q}\, \delta_{k-2m-q}(x), \tag{30}$$

where $\bar{E}_{m,n,k,q}$ are given as

$$\bar{E}_{m,n,k,q} = \frac{(-1)^{m+n}\, 2^{k-2m+2n}\,\Gamma\left(k+n+\frac{3}{2}\right)\Gamma\left(k-m+n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k+\frac{3}{2}\right) m!\,(k-2m+2n-q)!},$$

and $\delta_i(x)$ is as given in (21).

Proof. Setting $r=0$ in relation (25) gives the relation

$$D^q(J_k^n(x)) = \frac{2^q\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \times \sum_{p=0}^{\left\lfloor\frac{k-q}{2}\right\rfloor} \frac{\Gamma\left(k-2p-q+\frac{3}{2}\right)}{p!\,\Gamma\left(k+n-2p-q+\frac{1}{2}\right)\Gamma\left(k+n-p-q+\frac{3}{2}\right)} \times {}_2F_1\left(\begin{matrix} -p, -k-n+p+q-\frac{1}{2} \\ -k-n+\frac{1}{2} \end{matrix}\middle| 1\right) J_{k-q-2p}^n + \sum_{m=0}^{\left\lfloor\frac{k}{2}\right\rfloor+n} \bar{E}_{m,n,k,q}\, \delta_{k-2m-q}(x), \tag{31}$$

with

$$\bar{E}_{m,n,k,q} = \frac{(-1)^{m+n}\, 2^{k-2m+2n}\,\Gamma\left(k+n+\frac{3}{2}\right)\Gamma\left(k-m+n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k+\frac{3}{2}\right) m!\,(k-2m+2n-q)!}.$$

Now, with the aid of the Chu–Vandermonde identity, the appearing ${}_2F_1(1)$ in (31) can be summed to give

$$ {}_2F_1\left(\begin{matrix} -p, -k-n+p+q-\frac{1}{2} \\ -k-n+\frac{1}{2} \end{matrix} \middle| 1 \right) = \frac{(-p-q+1)_p}{\left(-k-n+\frac{1}{2}\right)_p}, $$

and therefore, relation (30) can be easily obtained. □

Corollary 4. *The derivatives of Legendre polynomials have the following expression in terms of their original polynomials:*

$$ D^q P_j(x) = 2^{q-1} \sum_{j=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(q)_j\,(2k-4j-2q+1)\,\Gamma\left(k-j+\frac{1}{2}\right)}{j!\,\Gamma\left(k-j-q+\frac{3}{2}\right)}\, P_{k-q-2j}(x). \tag{32} $$

Proof. The above formula can be easily obtained by setting $n=0$ in Formula (30). □

Remark 7. *Relation (32) agrees with that previously obtained in [45].*

5. Some Other Derivatives of Moments, Derivatives, and Connection Formulas for the *GJPs*

This section is devoted to deriving other new formulas concerned with the *GJPs*. Moments' derivatives formula of the *GJPs* in terms of the ultraspherical polynomials is established. Furthermore, the expressions linking the derivatives of the *GJPs* with the symmetric polynomials, namely, ultraspherical, generalized Fibonacci, and generalized Lucas polynomials, are given. The corresponding expressions with the Laguerre, generalized third-kind Chebyshev polynomials are also given. As direct consequences of the derivative formulas of the *GJPs*, some new connection formulas with some other polynomials are also deduced.

5.1. Derivatives of the Moments of the GJPs in Terms of Ultraspherical Polynomials

Theorem 4. *Let k, q, r, n be non-negative integers with $k+2n+r \geq q$. The derivatives of the moments of the GJPs can be expressed in terms of the ultraspherical polynomials $C_k^{(\lambda)}(x)$ as:*

$$ \begin{aligned} D^q\left(x^r J_k^n(x)\right) = {} & \frac{(-1)^n\, 2^{q-r-2\lambda+1}\,(k+2n+r)!\,\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{(k+2n)!\,\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)} \times \\ & \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q+r) \right\rfloor + n} \frac{(k+2n-2p-q+r+\lambda)\Gamma(k+2n-2p-q+r+2\lambda)}{p!\,(k+2n-2p-q+r)!\,\Gamma(k+2n-p-q+r+\lambda+1)} \times \\ & {}_4F_3\left(\begin{matrix} -p, -\frac{k}{2}-n, \frac{1}{2}-\frac{k}{2}-n, -k-2n+p+q-r-\lambda \\ \frac{1}{2}-k-n, -\frac{k}{2}-n-\frac{r}{2}, \frac{1}{2}-\frac{k}{2}-n-\frac{r}{2} \end{matrix} \middle| 1 \right) C_{k+2n+r-q-2p}^{(\lambda)}(x). \end{aligned} \tag{33} $$

Proof. If we begin with the power form representation in (18), then we can write

$$ \begin{aligned} D^q\left(x^r J_k^n(x)\right) = {} & \frac{(2k+3)\left(k+\frac{5}{2}\right)_{n-1}}{\sqrt{\pi}} \times \\ & \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor + n} \frac{(-1)^{m+n}\, 2^{k-2m+2n-1}\, \Gamma\left(k-m+n+\frac{1}{2}\right)\,(k-2m+2n-q+r+1)_q}{m!\,(k-2m+2n)!}\, x^{k-2m+2n-q+r}. \end{aligned} $$

Based on the inversion formula of the ultraspherical polynomials given in (5), we can write

$$D^{q}(x^{r}\,J_{k}^{n}(x)) = \frac{2^{q-r}\,\Gamma(\lambda+1)\,(2k+3)\,\left(k+\frac{5}{2}\right)_{n-1}}{\sqrt{\pi}\,\Gamma(2\,\lambda+1)}\times$$
$$\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor+n}\frac{(-1)^{m+n}\,\Gamma\left(k-m+n+\frac{1}{2}\right)\,(k-2m+2n+r)!}{m!\,(k-2m+2n)!}\times$$
$$\sum_{j=0}^{\lfloor\frac{1}{2}(k+r-q)\rfloor+n-m}\frac{(-2j+k-2m+2n-q+r+\lambda)\,\Gamma(-2j+k-2m+2n-q+r+2\lambda)}{j!\,\Gamma(1-2j+k-2m+2n-q+r)\,\Gamma(1-j+k-2m+2n-q+r+\lambda)}\times$$
$$C_{k-2m+2n+r-q-2j}^{(\lambda)}(x).$$

The last formula, after rearranging the terms, can be transformed into the following form:

$$D^{q}(x^{r}\,J_{k}^{n}(x)) = \frac{2^{q-r-2\lambda+1}\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\times$$
$$\sum_{p=0}^{\lfloor\frac{1}{2}(k+r-q)\rfloor+n}\frac{(k+2n-2p-q+r+\lambda)\Gamma(k+2n-2p-q+r+2\lambda)}{(k+2n-2p-q+r)!}\times$$
$$\sum_{\ell=0}^{p}\frac{(-1)^{\ell+n}\Gamma\left(k-\ell+n+\frac{1}{2}\right)\,(k-2\ell+2n+r)!}{\ell!\,(k-2\ell+2n)!\,(p-\ell)!\,\Gamma(k-\ell+2n-p-q+r+\lambda+1)}\,C_{k+2n+r-q-2p}^{(\lambda)}(x).$$

In hypergeometric form, the last formula can be written equivalently as

$$D^{q}(x^{r}\,J_{k}^{n}(x)) = \frac{(-1)^{n}\,2^{q-r-2\lambda+1}\,(k+2n+r)!\,\Gamma\left(k+n+\frac{1}{2}\right)\,\Gamma\left(k+n+\frac{3}{2}\right)}{(k+2n)!\,\Gamma\left(k+\frac{3}{2}\right)\,\Gamma\left(\lambda+\frac{1}{2}\right)}\times$$
$$\sum_{p=0}^{\lfloor\frac{1}{2}(k-q+r)\rfloor+n}\frac{(k+2n-2p-q+r+\lambda)\Gamma(k+2\,n-2\,p-q+r+2\lambda)}{p!\,(k+2n-2p-q+r)!\,\Gamma(k+2n-p-q+r+\lambda+1)}\times$$
$${}_4F_3\left(\begin{matrix}-p,-\frac{k}{2}-n,\frac{1}{2}-\frac{k}{2}-n,-k-2n+p+q-r-\lambda\\ \frac{1}{2}-k-n,-\frac{k}{2}-n-\frac{r}{2},\frac{1}{2}-\frac{k}{2}-n-\frac{r}{2}\end{matrix}\middle|1\right)C_{k+2n+r-q-2p}^{(\lambda)}(x).$$

This ends the proof of Theorem 4. □

The following corollary, which can be obtained as a special case of Theorem 4, exhibits the derivatives of the $GJPs$ in terms of the ultraspherical polynomials.

Corollary 5. *Let k,q,r,n be any non-negative integers with $k+2n\geq q$. Then, we have*

$$D^{q}J_{k}^{n}(x) = \frac{(-1)^{n}\,2^{q-2\lambda+1}\Gamma\left(k+n+\frac{1}{2}\right)\,\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\times$$
$$\sum_{p=0}^{\lfloor\frac{k-q}{2}\rfloor+n}\frac{(k+2n-2p-q+\lambda)\Gamma(k+2n-2p-q+2\lambda)\left(n-p-q+\lambda+\frac{1}{2}\right)_{p}}{p!\,(k+2n-2p-q)!\,\Gamma(k+2n-p-q+\lambda+1)\left(\frac{1}{2}-k-n\right)_{p}}\times \tag{34}$$
$$C_{k+2n-q-2p}^{(\lambda)}(x).$$

Proof. Setting $r=0$ in Formula (33) yields the following formula

$$D^q J_k^n(x) = \frac{(-1)^n 2^{q-2\lambda+1}\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\times$$
$$\sum_{p=0}^{\left\lfloor\frac{k-q}{2}\right\rfloor+n}\frac{(k+2n-2p-q+\lambda)\Gamma(k+2n-2p-q+2\lambda)}{p!(k+2n-2p-q)!\Gamma(k+2n-p-q+\lambda+1)}\times$$
$${}_2F_1\left(\begin{matrix}-p,-k-2n+p+q-\lambda\\ \frac{1}{2}-k-n\end{matrix}\,\middle|\,1\right) C_{k+2n-q-2p}^{(\lambda)}(x).$$

Making use of the well-known Chu–Vandermonde identity, it is easy to see that

$${}_2F_1\left(\begin{matrix}-p,-k-2n+p+q-\lambda\\ \frac{1}{2}-k-n\end{matrix}\,\middle|\,1\right) = \frac{\left(n-p-q+\lambda+\frac{1}{2}\right)_p}{\left(\frac{1}{2}-k-n\right)_p},$$

and accordingly, the following formula can be obtained

$$D^q J_k^n(x) = \frac{(-1)^n 2^{q-2\lambda+1}\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}\times$$
$$\sum_{p=0}^{\left\lfloor\frac{k-q}{2}\right\rfloor+n}\frac{(k+2n-2p-q+\lambda)\Gamma(k+2n-2p-q+2\lambda)\left(n-p-q+\lambda+\frac{1}{2}\right)_p}{p!\,(k+2n-2p-q)!\,\Gamma(k+2n-p-q+\lambda+1)\left(\frac{1}{2}-k-n\right)_p}\times$$
$$C_{k+2n-q-2p}^{(\lambda)}(x).$$

□

Noting that Chebyshev polynomials of the first and second kinds and Legendre polynomials are special classes of $C_k^{(\lambda)}(x)$, the following three formulas can be directly obtained.

Corollary 6. *Let k,n,q be any non-negative integers with $k+2n\geq q$. Then, we have*

$$D^q J_k^n(x) = \frac{(-1)^n 2^{q+1}\,\Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(k+\frac{3}{2}\right)}\times$$
$$\sum_{p=0}^{\left\lfloor\frac{k-q}{2}\right\rfloor+n}\frac{c_{k+2n-q-2p}\,\Gamma\left(n-q+\frac{1}{2}\right)}{p!\,\Gamma\left(n-p-q+\frac{1}{2}\right)(k+2n-p-q)!\left(\frac{1}{2}-k-n\right)_p}T_{k+2n-q-2p}(x),$$

with

$$c_r = \begin{cases}\frac{1}{2}, & r=0,\\ 1, & r\geq 1.\end{cases}$$

Proof. Setting $\lambda=0$ in (34) yields the desired result. □

Corollary 7. *Let k,n,q be any non-negative integers with $k+2n\geq q$. Then, we have*

$$D^q J_k^n(x) = \frac{(-1)^n\, 2^q\, \Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k+\frac{3}{2}\right)} \times \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor + n} \frac{(k+2n-2p-q+1)\,\Gamma\left(\frac{1}{2}-k-n\right)\Gamma\left(n-q+\frac{3}{2}\right)}{p!\,\Gamma\left(\frac{1}{2}-k-n+p\right)\Gamma\left(n-p-q+\frac{3}{2}\right)(k+2n-p-q+1)!}\, U_{k+2n-q-2p}(x).$$

Proof. Setting $\lambda = 1$ in (34) yields the desired result. □

Corollary 8. *Let k, n, q be any non-negative integers with $k+2n \geq q$. Then, we have*

$$D^q J_k^n(x) = \frac{(-1)^n\, 2^q\, \Gamma\left(k+n+\frac{1}{2}\right)\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} \times \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor + n} \frac{\left(k+2n-2p-q+\frac{1}{2}\right)(n-p-q+1)_p}{p!\,\Gamma\left(k+2n-p-q+\frac{3}{2}\right)\left(\frac{1}{2}-k-n\right)_p}\, P_{k+2n-q-2p}(x). \tag{35}$$

Proof. Setting $\lambda = \frac{1}{2}$ in (34) yields the desired result. □

Corollary 9. *For all non-negative integers k and n, the following formula is valid:*

$$D^n J_k^n(x) = \frac{(-2)^n\, \Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}\, P_{k+n}(x). \tag{36}$$

Proof. Setting $k = n$ in (35) yields (36). □

5.2. Derivative Formulas of the GJPs in Terms of Some Other Polynomials

In the following, we will give some explicit formulas of the derivatives of the *GJPs* in terms of some celebrated polynomials. The linking coefficients are expressed in terms of certain generalized hypergeometric functions that can be summed in some specific cases. The outlines of the proofs will be given.

Theorem 5. *Let k, n, q be any non-negative integers with $k+2n \geq q$. Then, we have*

$$D^q J_k^n(x) = \frac{(-1)^n\, 2^{q-2\alpha-1}\, \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\Gamma\left(k-2n+\frac{3}{2}\right)\Gamma(\alpha+1)} \times \left(\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+2)\,(n-p-q+\alpha+1)_p}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)\left(\frac{1}{2}-k+n\right)_p}\, R_{k-q-2p}^{(\alpha,\alpha+1)}(x) + \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+1)\,(n-p-q+\alpha+1)_p}{p!\,(k-2p-q-1)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)\left(\frac{1}{2}-k+n\right)_p}\, R_{k-q-2p-1}^{(\alpha,\alpha+1)}(x) \right). \tag{37}$$

Proof. The proof is based on making use of Formula (18) together with the inversion formula of the polynomials $R_m^{(\alpha,\alpha+1)}(x)$ given in (6). □

Remark 8. *Noting that Chebyshev polynomials of the third kind are special ones of the Jacobi polynomials $R_j^{(\alpha,\alpha+1)}(x)$ for the case corresponding to $\alpha = \frac{-1}{2}$, and noting the well-known identity*

$W_j(x) = (-1)^j\, V_j(-x)$, *we can obtain the derivative expressions of the GJPs in terms of Chebyshev polynomials of the third and fourth kinds. The following corollary exhibits these formulas.*

Corollary 10. *Let k, n, q be any non-negative integers with $k + 2n \geq q$. The expressions of $D^q J_k^n(x)$ in terms of the third and fourth kinds of Chebyshev polynomials are respectively given as*

$$D^q J_k^n(x) = \frac{(-1)^n\, 2^q\, \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\left(n-p-q+\frac{1}{2}\right)_p}{p!\,(k-p-q)!\left(\frac{1}{2}-k+n\right)_p} \times \left(V_{k-q-2\,p}(x) + V_{k-q-2p-1}(x)\right), \tag{38}$$

$$D^q J_k^n(x) = \frac{(-1)^n 2^q \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\left(n-p-q+\frac{1}{2}\right)_p}{p!(k-p-q)!\left(\frac{1}{2}-k+n\right)_p} \times \left(W_{k-q-2\,p}(x) - W_{k-q-2p-1}(x)\right). \tag{39}$$

Proof. Setting $\alpha = -\frac{1}{2}$ in Formula (37) yields Formula (38). Formula (39) can be easily obtained from Formula (38) by setting $(-x)$ instead of x and making use of Identity (4). □

Theorem 6. *Let k, n, q be any non-negative integers with $k + 2n \geq q$, and let $H_k(x)$ be the Hermite polynomial of degree k. The following formula applies:*

$$D^q J_k^n(x) = \frac{(-1)^n 2^q \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{1}{p!\,(k-2p-q)!} \times {}_1F_1\left(-p; \frac{1}{2}-k+n; -1\right) H_{k-q-2p}(x). \tag{40}$$

Proof. The proof is based on making use of Formula (18) together with the inversion formula of the Hermite polynomials given in (11). □

Theorem 7. *Let k, n, q be any non-negative integers with $k + 2n \geq q$, and let $L_k^{(\alpha)}(x)$ be the generalized Laguerre polynomial of degree k. The following formula applies:*

$$D^q J_k^n(x) = \gamma_{k,n} \left(\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} M_{k,q,\alpha} L_{k-q-2p}^{(\alpha)}(x) + \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \bar{M}_{k,q,\alpha} L_{k-q-2p-1}^{(\alpha)}(x) \right), \tag{41}$$

where

$$\gamma_{k,n} = \frac{2^k \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)\Gamma(k-q+\alpha+1)}{\sqrt{\pi}\Gamma\left(k-2n+\frac{3}{2}\right)},$$

$$M_{k,q,\alpha} = \frac{(-1)^{k+n-2p-q}}{(2p)!\Gamma(k-2p-q+\alpha+1)} \times {}_2F_3\left(\left. \begin{matrix} \frac{1}{2}-p, -p \\ -k+n+\frac{1}{2}, -\frac{k}{2}+\frac{q}{2}-\frac{\alpha}{2}, -\frac{k}{2}+\frac{q}{2}-\frac{\alpha}{2}+\frac{1}{2} \end{matrix} \right| \frac{1}{4} \right),$$

$$\bar{M}_{k,q,\alpha} = \frac{(-1)^{k+n-2p-q+1}}{(2p+1)!\Gamma(k-2p-q+\alpha)} \times {}_2F_3\left(\left. \begin{matrix} \frac{-1}{2}-p, -p \\ -k+n+\frac{1}{2}, -\frac{k}{2}+\frac{q}{2}-\frac{\alpha}{2}, -\frac{k}{2}+\frac{q}{2}-\frac{\alpha}{2}+\frac{1}{2} \end{matrix} \right| \frac{1}{4} \right).$$

Proof. The proof is based on making use of Formula (18) together with the inversion formula of the generalized Laguerre polynomials given in (12). □

Theorem 8. *Let k, n, q be any non-negative integers with $k + 2n \geq q$, and Let $F_k^{(A,B)}(x)$ be the generalized Fibonacci polynomial of degree k that is generated by (13). Then, we have*

$$D^q J_k^n(x) = \frac{2^k A^{-k+q}\, \Gamma\left(k-n+\frac{1}{2}\right) \Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\, \Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(-1)^{n+p+1} B^p (-1-k+2p+q)}{p!\,(k-p-q+1)!} \times \\ {}_2F_1\left(\begin{matrix} -p, -1-k+p+q \\ \frac{1}{2}-k+n \end{matrix} \middle| \frac{-A^2}{4\,B} \right) F_{k-q-2p}^{A,B}(x). \tag{42}$$

Proof. The proof is based on making use of Formula (18) together with the inversion Formula (15). □

Theorem 9. *Let k, n, q be any non-negative integers with $k + 2n \geq q$, and let $L_k^{(A,B)}(x)$ be the generalized Lucas polynomial of degree k that was generated by (14). Then, we have*

$$D^q J_k^n(x) = \frac{2^k R^{-k+q} \Gamma\left(k-n+\frac{1}{2}\right) \Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\, \Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{c_{k-q-2p}\, (-1)^{n+p}\, S^p}{p!\,(k-p-q)!} \times \\ {}_2F_1\left(\begin{matrix} -p, -k+p+q \\ \frac{1}{2}-k+n \end{matrix} \middle| \frac{-R^2}{4\,S} \right) L_{k-q-2p}^{R,S}(x). \tag{43}$$

Proof. The proof is based on making use of Formula (18) together with the inversion Formula (16). □

5.3. Connection Formulas between the GJPs and Some Other Polynomials

Here, we write some connection formulas between the *GJPs* and some other polynomials. In fact, these formulas are special ones of the derivative formulas that are introduced in Sections 5.1 and 5.2, only by setting $q = 0$ in the derivative formulas. In the following, we list some of these formulas.

Corollary 11. *The GJPs–Hermite connection formula is*

$$J_k^n(x) = \frac{(-1)^n\, \Gamma\left(k-n+\frac{1}{2}\right) \Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\, \Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{1}{p!(k-2p)!}\, {}_1F_1\left(-p; \frac{1}{2}-k+n; -1\right) H_{k-2p}(x). \tag{44}$$

Proof. Setting $q = 0$ in Formula (40) yields Formula (44). □

Corollary 12. *The GJPs–Laguerre connection formula is*

$$J_k^n(x) = \frac{2^k\, \Gamma\left(k-n+\frac{1}{2}\right) \Gamma\left(k-n+\frac{3}{2}\right) \Gamma(k+\alpha+1)}{\sqrt{\pi}\, \Gamma\left(k-2n+\frac{3}{2}\right)} \left(\sum_{p=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^{k+n}}{(2p)!\, \Gamma(k-2p+\alpha+1)} \times \right. \\ {}_2F_3\left(\begin{matrix} -p, \frac{1}{2}-p \\ -k+n+\frac{1}{2}, -\frac{k}{2}-\frac{\alpha}{2}, -\frac{k}{2}-\frac{\alpha}{2}+\frac{1}{2} \end{matrix} \middle| \frac{1}{4} \right) L_{k-2p}^{(\alpha)}(x) + \sum_{p=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^{k+n+1}}{(2p+1)!\Gamma(k-2p+\alpha)} \times \\ \left. {}_2F_3\left(\begin{matrix} -p, -\frac{1}{2}-p \\ -k+n+\frac{1}{2}, -\frac{k}{2}-\frac{\alpha}{2}, -\frac{k}{2}-\frac{\alpha}{2}+\frac{1}{2} \end{matrix} \middle| \frac{1}{4} \right) L_{k-2p-1}^{(\alpha)}(x) \right).$$

Proof. Setting $q = 0$ in Formula (41) yields the desired connection formula. □

Corollary 13. *The $GJPs$–Generaized Fibonacci connection formula is*

$$J_k^n(x) = \frac{2^k\, A^{-k}\, \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{n+p+1}\, B^p\,(-1-k+2p)}{p!\,(k-p+1)!} \times$$
$${}_2F_1\left(\left.\begin{array}{c} -p,-1-k+p \\ \frac{1}{2}-k+n \end{array}\right| \frac{-A^2}{4\,B}\right) F_{k-2p}^{A,B}(x).$$

Proof. Setting $q=0$ in Formula (42) yields the desired connection formula. □

Corollary 14. *The $GJPs$–Generalized Lucas connection formula is*

$$J_k^n(x) = \frac{2^k\, R^{-k}\, \Gamma\left(k-n+\frac{1}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(k-2n+\frac{3}{2}\right)} \sum_{p=0}^{\lfloor \frac{k}{2} \rfloor} \frac{c_{k-2p}\,(-1)^{n+p}\, S^p}{p!\,(k-p)!} \times$$
$${}_2F_1\left(\left.\begin{array}{c} -p,-k+p \\ \frac{1}{2}-k+n \end{array}\right| \frac{-R^2}{4\,S}\right) L_{k-2p}^{R,S}(x).$$

Proof. Setting $q=0$ in Formula (43) yields the desired connection formula. □

6. Some Linearization Formulas of the $GJPs$

This section is devoted to establishing some linearization formulas for the $GJPs$ with some other celebrated polynomials. In this respect, we will present the linearization formulas for the $GJPs$ with the following polynomials:

- The four kinds of Chebyshev polynomials;
- The generalized Fibonacci polynomials that were generated by (13);
- The generalized Lucas polynomials that were generated by (14).

The basic idea behind the derivation of our linearization formulas is based on making use of the power form representation of the $GJPs$ along with the moment's formula of the other polynomials.

Theorem 10. *For all non-negative integers i and j, the following linearization formula holds if $\phi_j(x)$ is any Chebyshev polynomial of the well-known four kinds*

$$J_i^n(x)\,\phi_j(x) = \frac{(-1)^n\,\Gamma\left(i+n+\frac{1}{2}\right)\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(i+\frac{3}{2}\right)(i+2n)!} \sum_{p=0}^{i+2n} \frac{\binom{i+2n}{p}\left(n-p+\frac{1}{2}\right)_p}{\left(\frac{1}{2}-i-n\right)_p}\,\phi_{j+i+2n-2p}(x).$$

Proof. The power form representation of $J_k^n(x)$ enables one to write

$$J_k^n(x)\,\phi_j(x) = \frac{\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(i+\frac{3}{2}\right)} \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor+n} \frac{(-1)^{m+n}\,2^{i-2m+2n}\,\Gamma\left(i-m+n+\frac{1}{2}\right)}{m!\,(i-2m+2n)!}\, x^{i-2m+2n}\,\phi_j(x). \tag{45}$$

The moment's formula of any kind of Chebyshev polynomials is given by

$$x^r\,\phi_j(x) = \frac{1}{2^r}\sum_{s=0}^{r}\binom{r}{s}\phi_{j+r-2s}(x).$$

The last moment formula is inserted into relation (45), then the following relation can be obtained

$$J_i^n(x)\,\phi_j(x) = \frac{\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(i+\frac{3}{2}\right)} \times \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor + n} \frac{(-1)^{m+n}\,\Gamma\left(i-m+n+\frac{1}{2}\right)}{m!\,(i-2m+2n)!} \sum_{s=0}^{i-2m+2n} \binom{i-2m+2n}{s} \phi_{j+i-2m+2n-2s}(x).$$

Some computations enable one to rewrite the last formula into the following alternative form:

$$J_i^n(x)\,\phi_j(x) = \frac{\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(i+\frac{3}{2}\right)} \sum_{p=0}^{i+2n} \sum_{\ell=0}^{p} \frac{(-1)^{\ell+n}\binom{i-2\ell+2n}{p-\ell}\Gamma\left(i-\ell+n+\frac{1}{2}\right)}{\ell!\,(i-2\ell+2n)!} \phi_{j+i+2n-2p}(x),$$

which can be written in the form

$$J_i^n(x)\,\phi_j(x) = \frac{(-1)^n\,\Gamma\left(i+n+\frac{1}{2}\right)\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(i+\frac{3}{2}\right)(i+2n)!} \sum_{p=0}^{i+2n} \binom{i+2n}{p} \times {}_2F_1\left(\begin{matrix} -p, -i-2n+p \\ \frac{1}{2}-i-n \end{matrix} \,\middle|\, 1\right) \phi_{j+i+2n-2p}(x).$$

Finally, Chu–Vandermonde's identity leads to the following linearization formula:

$$J_i^n(x)\,\phi_j(x) = \frac{(-1)^n\,\Gamma\left(i+n+\frac{1}{2}\right)\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(i+\frac{3}{2}\right)(i+2n)!} \sum_{p=0}^{i+2n} \frac{\binom{i+2n}{p}\left(n-p+\frac{1}{2}\right)_p}{\left(\frac{1}{2}-i-n\right)_p} \phi_{j+i+2n-2p}(x).$$

□

Theorem 11. *For all non-negative integers i and j, one has*

$$J_i^n(x)\,F_j^{A,B}(x) = \frac{(-1)^n\,2^{i+2n}\,A^{-i-2n}\,\Gamma\left(i+n+\frac{1}{2}\right)\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(i+\frac{3}{2}\right)(i+2n)!} \times \sum_{p=0}^{i+2n} (-B)^p \binom{i+2n}{p} {}_2F_1\left(\begin{matrix} -p, -i-2n+p \\ \frac{1}{2}-i-n \end{matrix} \,\middle|\, -\frac{A^2}{4B}\right) F_{j+i+2n-2p}^{A,B}(x).$$

Proof. Similar to the proof of Theorem 10. □

Theorem 12. *For all non-negative integers i and j, one has*

$$J_i^n(x)\,L_j^{R,S}(x) = \frac{(-1)^n\,2^{i+2n}\,R^{-i-2n}\,\Gamma\left(i+n+\frac{1}{2}\right)\Gamma\left(i+n+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(i+\frac{3}{2}\right)(i+2n)!} \times \sum_{p=0}^{i+2n} (-S)^p \binom{i+2n}{p} {}_2F_1\left(\begin{matrix} -p, -i-2n+p \\ \frac{1}{2}-i-n \end{matrix} \,\middle|\, -\frac{R^2}{4S}\right) L_{j+i+2n-2p}^{R,S}(x).$$

Proof. Similar to the proof of Theorem 10. □

7. Conclusions

In this paper, we investigated a type of orthogonal polynomials, namely, generalized Jacobi polynomials. These polynomials were written in certain combinations of Legendre polynomials. In fact, these polynomials are the natural basis for even-order boundary value problems. Several new formulas concerned with these types of polynomials were developed. Derivatives and connections of these polynomials with some other polynomials were displayed. To the best of our knowledge, most of the formulas in this article are new and offer new insights about these kinds of orthogonal polynomials that can be written as combinations of orthogonal polynomials. We are planning in the near future to utilize the generalized Jacobi orthogonal polynomials to treat several types of differential equations. In addition, we aim to investigate other types of combinations of polynomials in the future from both theoretical and practical points of view.

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References

1. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions*; Cambridge University Press: Cambridge, UK, 1999; Volume 71.
2. Mason, J.C.; Handscomb, D.C. *Chebyshev Polynomials*; CRC Press: Boca Raton, FL, USA, 2002.
3. Shen, J.; Tang, T.; Wang, L.L. *Spectral Methods: Algorithms, Analysis and Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2011; Volume 41.
4. Boyd, J.P. *Chebyshev and Fourier Spectral Methods*; Courier Corporation: North Chelmsford, MA, USA, 2001.
5. Hesthaven, J.S.; Gottlieb, S.; Gottlieb, D. *Spectral Methods for Time-Dependent Problems*; Cambridge University Press: Cambridge, UK, 2007; Volume 21.
6. Aloui, B.; Souissi, J. Jacobi polynomials and some connection formulas in terms of the action of linear differential operators. *Bull. Belg. Math. Soc. Simon Stevin* **2021**, *28*, 39–51. [CrossRef]
7. Conway, J.T. Indefinite integrals involving Jacobi polynomials from integrating factors. *Integral Transform. Spec. Funct.* **2021**, *32*, 801–811. [CrossRef]
8. Gil, A.; Segura, J.; Temme, N.M. Asymptotic expansions of Jacobi polynomials and of the nodes and weights of Gauss-Jacobi quadrature for large degree and parameters in terms of elementary functions. *J. Math. Anal. Appl.* **2021**, *494*, 124642. [CrossRef]
9. Singh, H.; Srivastava, H.M. Numerical simulation for fractional-order Bloch equation arising in nuclear magnetic resonance by using the Jacobi polynomials. *Appl. Sci.* **2020**, *10*, 2850. [CrossRef]
10. Singh, H.; Pandey, R.K.; Srivastava, H.M. Solving non-linear fractional variational problems using Jacobi polynomials. *Mathematics* **2019**, *7*, 224. [CrossRef]
11. Duangpan, A.; Boonklurb, R.; Treeyaprasert, T. finite integration method with shifted Chebyshev polynomials for solving time-fractional Burgers' equations. *Mathematics* **2019**, *7*, 1201. [CrossRef]
12. Jena, S.K.; Chakraverty, S.; Malikan, M. Application of shifted Chebyshev polynomial-based Rayleigh–Ritz method and Navier's technique for vibration analysis of a functionally graded porous beam embedded in Kerr foundation. *Eng. Comput.* **2021**, *37*, 3569–3589. [CrossRef]
13. Alsuyuti, M.M.; Doha, E.H.; Ezz-Eldien, S.S.; Bayoumi, B.I.; Baleanu, D. Modified Galerkin algorithm for solving multitype fractional differential equations. *Math. Methods Appl. Sci.* **2019**, *42*, 1389–1412. [CrossRef]
14. Shen, J. Efficient spectral-Galerkin method I. Direct solvers of second-and fourth-order equations using Legendre polynomials. *SIAM J. Sci. Comput.* **1994**, *15*, 1489–1505. [CrossRef]
15. Shen, J. Efficient spectral-Galerkin method II. Direct solvers of second-and fourth-order equations using Chebyshev polynomials. *SIAM J. Sci. Comput.* **1995**, *16*, 74–87. [CrossRef]
16. Doha, E.H.; Abd-Elhameed, W.M.; Bhrawy, A.H. New spectral-Galerkin algorithms for direct solution of high even-order differential equations using symmetric generalized Jacobi polynomials. *Collect. Math.* **2013**, *64*, 373–394. [CrossRef]
17. Guo, B.Y.; Shen, J.; Wang, L.L. Optimal spectral-Galerkin methods using generalized Jacobi polynomials. *J. Sci. Comput.* **2006**, *27*, 305–322. [CrossRef]
18. Abd-Elhameed, W.M. New spectral solutions for high odd-order boundary value problems via generalized Jacobi polynomials. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 1393–1412. [CrossRef]

19. Alfaro, M.; Marcellán, F.; Pena, A.; Rezola, M.L. When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials? *J. Comput. Appl. Math.* **2010**, *233*, 1446–1452. [CrossRef]
20. Grinshpun, Z. Special linear combinations of orthogonal polynomials. *J. Math. Anal. Appl.* **2004**, *299*, 1–18. [CrossRef]
21. Rahman, Q.I. Zeros of linear combinations of polynomials. *Canad. Math. Bull.* **1972**, *15*, 139–142. [CrossRef]
22. Marcellán, F.; Peherstorfer, F.; Steinbauer, R. Orthogonality properties of linear combinations of orthogonal polynomials. *Adv. Comput. Math.* **1996**, *5*, 281–295. [CrossRef]
23. Marcellán, F.; Peherstorfer, F.; Steinbauer, R. Orthogonality properties of linear combinations of orthogonal polynomials II. *Adv. Comput. Math.* **1997**, *7*, 401–428. [CrossRef]
24. Doha, E.H.; Abd-Elhameed, W.M.; Bassuony, M.A. On the coefficients of differentiated expansions and derivatives of Chebyshev polynomials of the third and fourth kinds. *Acta Math. Sci.* **2015**, *35*, 326–338. [CrossRef]
25. Abd-Elhameed, W.M.; Alkenedri, A.M. Spectral solutions of linear and nonlinear BVPs using certain Jacobi polynomials generalizing third-and fourth-kinds of Chebyshev polynomials. *CMES Comput. Model. Eng. Sci.* **2021**, *126*, 955–989. [CrossRef]
26. Abd-Elhameed, W.M.; Youssri, Y.H. New formulas of the high-order derivatives of fifth-kind Chebyshev polynomials: Spectral solution of the convection–diffusion equation. *Numer. Methods Partial. Differ. Equ.* **2021**. [CrossRef]
27. Abd-Elhameed, W.M. Novel expressions for the derivatives of sixth kind Chebyshev polynomials: Spectral solution of the non-linear one-dimensional Burgers' equation. *Fractal Fract.* **2021**, *5*, 53. [CrossRef]
28. Ruiz, J.S. Logarithmic potential of Hermite polynomials and information entropies of the harmonic oscillator eigenstates. *J. Math. Phys.* **1997**, *38*, 5031–5043. [CrossRef]
29. Popov, B.; Srivastava, H. Linearization of a product of two polynomials of different orthogonal systems. *Facta Univ. Ser. Math. Inform.* **2003**, *18*, 1–8.
30. Niukkanen, A. Clebsch-Gordan-type linearisation relations for the products of Laguerre polynomials and hydrogen-like functions. *J. Phy. A Math. Gen.* **1985**, *18*, 1399. [CrossRef]
31. Srivastava, H. A unified theory of polynomial expansions and their applications involving Clebsch-Gordan type linearization relations and Neumann series. *Astrophys. Space Sci.* **1988**, *150*, 251–266. [CrossRef]
32. Sánchez-Ruiz, J.; Dehesa, J.S. Some connection and linearization problems for polynomials in and beyond the Askey scheme. *J. Comput. Appl. Math.* **2001**, *133*, 579–591. [CrossRef]
33. Ahmed, H.M. Computing expansions coefficients for Laguerre polynomials. *Integral Transform. Spec. Funct.* **2021**, *32*, 271–289. [CrossRef]
34. Abd-Elhameed, W.M.; Badah, B.M. New approaches to the general linearization problem of Jacobi polynomials based on moments and connection formulas. *Mathematics* **2021**, *9*, 1573. [CrossRef]
35. Abd-Elhameed, W.M. New product and linearization formulae of Jacobi polynomials of certain parameters. *Integral Transforms Spec. Funct.* **2015**, *26*, 586–599. [CrossRef]
36. Abd-Elhameed, W.M.; Ali, A. New specific and general linearization formulas of some classes of Jacobi polynomials. *Mathematics* **2021**, *9*, 74. [CrossRef]
37. Koornwinder, T. Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula. *J. Lond. Math. Soc.* **1978**, *2*, 101–114. [CrossRef]
38. Dolgy, D.V.; Kim, D.S.; Kim, T.; Kwon, J. Connection problem for sums of finite products of Chebyshev polynomials of the third and fourth kinds. *Symmetry* **2018**, *10*, 617. [CrossRef]
39. Rainville, E.D. *Special Functions*; The Maximalan Company: New York, NY, USA, 1960.
40. Rahman, M. A non-negative representation of the linearization coefficients of the product of Jacobi polynomials. *Canad. J. Math.* **1981**, *33*, 915–928. [CrossRef]
41. Guo, B.Y.; Shen, J.; Wang, L.L. Generalized Jacobi polynomials/functions and their applications. *Appl. Numer. Math.* **2009**, *59*, 1011–1028. [CrossRef]
42. Doha, E.H.; Abd-Elhameed, W.M. Efficient solutions of multidimensional sixth-order boundary value problems using symmetric generalized Jacobi-Galerkin method. *Abstr. Appl. Anal.* **2012**, *2012*, 749370 . [CrossRef]
43. Abd-Elhameed, W.M.; Philippou, A.N.; Zeyada, N.A. Novel results for two generalized classes of Fibonacci and Lucas polynomials and their uses in the reduction of some radicals. *Mathematics* **2022**, *10*, 2342. [CrossRef]
44. Koepf, W. *Hypergeometric Summation*, 2nd ed.; Springer Universitext Series; Springer: Berlin/Heidelberg, Germany, 2014.
45. Doha, E.H. The coefficients of differentiated expansions and derivatives of ultraspherical polynomials. *Comput. Math. Appl.* **1991**, *21*, 115–122. [CrossRef]