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Novel Results for Two Generalized Classes of Fibonacci and Lucas Polynomials and Their Uses in the Reduction of Some Radicals

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Abstract: The goal of this study is to develop some new connection formulae between two generalized classes of Fibonacci and Lucas polynomials. Hypergeometric functions of the kind ${}_2F_1(z)$ are included in all connection coefficients for a specific z. Several new connection formulae between some famous polynomials, such as Fibonacci, Lucas, Pell, Fermat, Pell–Lucas, and Fermat–Lucas polynomials, are deduced as special cases of the derived connection formulae. Some of the introduced formulae generalize some of those existing in the literature. As two applications of the derived connection formulae, some new formulae linking some celebrated numbers are given and also some newly closed formulae of certain definite weighted integrals are deduced. Based on using the two generalized classes of Fibonacci and Lucas polynomials, some new reduction formulae of certain odd and even radicals are developed.

Keywords: generalized Fibonacci and generalized Lucas numbers; Lucas and Fibonacci numbers; recurrence relation; radicals reduction

MSC: 11B83; 11B39

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1. Introduction

The Fibonacci and Lucas sequences are among the important number sequences. These sequences of numbers and their corresponding polynomials can be generated by recurrence relations, each of degree two. In fact, these sequences of polynomials and numbers are crucial in a variety of fields, such as number theory, probability, combinatorics, numerical analysis, and physics, so investigations of these sequences attract the attention of many mathematicians and scientists. From a theoretical point of view, there are several studies concerning Fibonacci and Lucas sequences, see, for instance, [1,2]. In addition, in the important books of Koshy [3] and Djordjevic and Milovanovic [4], the authors studied various sequences related to the Fibonacci and Lucas sequences and some of their generalizations.

Many authors devoted considerable attention to the generalizations of the Fibonacci and Lucas sequences of numbers and polynomials. For example, Fibonacci k-numbers were considered in [5]. In [6], the authors developed new identities for the two classes of $k-$Fibonacci and $k-$Lucas polynomials. Horadam in [7] is considered an important class of generalized numbers that involves four parameters. Abd-Elhameed et al. in [8] have investigated the same sequence of Horadam numbers and developed new identities of these numbers. The authors in [9] studied a type of generalized Fibonacci numbers. A type of generalized Fibonacci polynomials, namely, distance Fibonacci polynomials, was investigated in [10]. A four-parameter generalization of some special sequences was also considered in [11]. Some other studies regarding different types of generalized sequences can be found for example in [12–15].

If we have two polynomial sets $\{A_i(x)\}_{i\geq 0}$ and $\{B_j(x)\}_{j\geq 0}$, then to solve the connection problem between them, we have to find the connection coefficients $C_{i,j}$ in the equation:

$$A_i(x) = \sum_{j=0}^{i} C_{i,j}\, B_j(x). \tag{1}$$

The coefficients $C_{i,j}$ in (1) have prominent parts in several problems in mathematics as well as in mathematical physics. Due to this importance, the connection problems between various polynomials have been investigated by many authors. In this regard, Abd-Elhameed et al. in [16] solved the connection problems between Fibonacci polynomials and Chebyshev polynomials of first and second kinds. Some other studies concerning connection problems can be found in [17,18].

In the field of special functions and their applications, hypergeometric functions play an important role (see, for example, [19,20]). In fact, nearly all of mathematics' fundamental functions are hypergeometric or ratios of hypergeometric functions. Furthermore, numerous hypergeometric functions are widely used to express connection and linearization coefficients (see, for example, [21,22]).

Here are the main points of this article:

- We solve the connection problems between two certain classes of polynomials generalizing Fibonacci and Lucas polynomials. For a specific z, we prove that the obtained expressions involve hypergeometric functions of the form ${}_2F_1(z)$.
- We develop some applications of the introduced connection formulae. In this respect, two applications are presented. In the first, some new formulae between some celebrated numbers are given. In the second, some definite weighted integrals are evaluated in closed forms.
- We employ the two classes of generalized Fibonacci and generalized Lucas polynomials to obtain new reduction formulae of certain kinds of even and odd radicals.

The following is a list of the paper's contents. The two generalized Fibonacci and Lucas polynomials are discussed in the next section, which includes some essential properties and helpful relations. The development of new connection formulae between the two introduced generalized polynomials is the focus of Section 3. In Section 4, we establish some further connection equations between two polynomials from the same generalized polynomials class. Section 5 introduces two uses of the previously stated connection formulae. Section 6 is interested in using the two introduced generalized classes of polynomials to develop new expressions for particular even and odd radicals. In Section 7, new expressions for a few additional radicals are developed. We end the paper with some conclusions in Section 8.

2. Some Properties of Two Generalized Classes of Fibonacci and Lucas Polynomials

We discuss some properties of the two classes of generalized Fibonacci and Lucas polynomials in this section.

First, let a, b, r, and s be nonzero real numbers. Two classes generalizing Fibonacci and Lucas polynomials can be generated by means of the following two recurrence relations:

$$\phi_j^{a,b}(x) = a\,x\,\phi_{j-1}^{a,b}(x) + b\,\phi_{j-2}^{a,b}(x), \quad \phi_0^{a,b}(x) = 1,\ \phi_1^{a,b}(x) = a\,x, \quad j \geq 2, \tag{2}$$

and

$$\psi_j^{r,s}(x) = r\,x\,\psi_{j-1}^{r,s}(x) + s\,\psi_{j-2}^{r,s}(x), \quad \psi_0^{r,s}(x) = 2,\ \psi_1^{r,s}(x) = r\,x, \quad j \geq 2. \tag{3}$$

Note that for each $j \geq 0$, $\phi_j^{a,b}(x)$ and $\psi_j^{r,s}(x)$ are of degree j.

See also Philippou [23,24] and references therein for different generalized Fibonacci and Lucas polynomials. In addition, they are very useful in probability and reliability theory as well.

Binet's formula for $\phi_j^{a,b}(x)$ is given by [25],

$$\phi_j^{a,b}(x) = \frac{\left(a\,x + \sqrt{a^2\,x^2 + 4\,b}\right)^{j+1} - \left(a\,x - \sqrt{a^2\,x^2 + 4\,b}\right)^{j+1}}{2^{j+1}\sqrt{a^2\,x^2 + 4\,b}}, \quad j \geq 0, \tag{4}$$

while Binet's formulae for $\psi_j^{r,s}(x)$ is given by [26],

$$\psi_j^{r,s}(x) = \frac{\left(r\,x + \sqrt{r^2\,x^2 + 4\,s}\right)^{j} + \left(r\,x - \sqrt{r^2\,x^2 + 4\,s}\right)^{j}}{2^{j}}, \quad j \geq 0. \tag{5}$$

Two of the most useful properties of the two polynomials $\phi_j^{a,b}(x)$ and $\psi_j^{r,s}(x)$ are their power form representations. These expressions are given respectively in [25,26] in the following two combinatorial forms:

$$\phi_j^{a,b}(x) = \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \binom{j-m}{m}\, b^m\, a^{j-2m}\, x^{j-2m}, \quad j \geq 0, \tag{6}$$

and

$$\psi_j^{r,s}(x) = j \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{s^m\, r^{j-2m}\, \binom{j-m}{m}}{j-m}\, x^{j-2m}, \quad j \geq 1. \tag{7}$$

Formulae (6) and (7) yield the following functional formulae between the generalized Fibonacci and generalized Lucas polynomials and certain special classes of them. More precisely, we have

$$\phi_j^{a,b}(x) = b^{j/2} \phi_j^{1,1}\left(\frac{a\,x}{\sqrt{b}}\right), \quad j \geq 0,$$

and

$$\psi_j^{r,s}(x) = s^{j/2} \psi_j^{1,1}\left(\frac{r\,x}{\sqrt{s}}\right), \quad j \geq 1.$$

The inversion formulae of (6) and (7) are also of interest. The authors in [25] found the inversion formula of $\phi_j^{a,b}(x)$ in the form

$$x^j = a^{-j} \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(-b)^i \binom{j}{i} (j-2i+1)}{j-i+1}\, \phi_{j-2i}^{a,b}(x), \quad j \geq 0, \tag{8}$$

while the same authors proved in [26] that the inversion of $\psi_j^{r,s}(x)$ can be expressed as:

$$x^j = r^{-j} \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (-s)^i\, c_{j-2i} \binom{j}{i}\, \psi_{j-2i}^{r,s}(x), \quad j \geq 0,$$

where

$$c_j = \begin{cases} \frac{1}{2}, & j = 1, \\ 1, & j \geq 1. \end{cases} \tag{9}$$

The fundamental benefit of making use of the two generalized polynomials $\phi_j^{a,b}(x)$ and $\psi_j^{r,s}(x)$ is that several important classes of polynomials can be obtained as special cases of them. In fact, Fibonacci, Pell, Fermat, Chebyshev, and Dickson polynomials of the second kind are special kinds of $\phi_j^{a,b}(x)$, while Lucas, Pell–Lucas, Fermat–Lucas, Chebyshev, and Dickson polynomials of the first kind are special polynomials of $\psi_j^{r,s}(x)$. Table 1 displays the different celebrated special classes of the two generalized classes of polynomials.

Table 1. Special cases of the two generalizing classes.

Fibonacci polynomials	$F_{j+1}(x) = \phi_j^{1,1}(x)$
Pell polynomials	$P_{j+1}(x) = \phi_j^{2,1}(x)$
Fermat polynomials	$\mathcal{F}_{j+1}(x) = \phi_j^{3,-2}(x)$
Chebyshev polynomials of second kind	$U_j(x) = \phi_j^{2,-1}(x)$
Dickson polynomials of second kind	$E_j^{\alpha}(x) = \phi_j^{1,-\alpha}(x)$
Lucas polynomials	$L_j(x) = \psi_j^{1,1}(x)$
Pell–Lucas polynomials	$Q_j(x) = \psi_j^{2,1}(x)$
Fermat–Lucas polynomials	$f_j(x) = \psi_j^{3,-2}(x)$
Chebyshev polynomials of first kind	$T_j(x) = \frac{1}{2}\psi_j^{2,-1}(x)$
Dickson polynomials of first kind	$D_j^{\alpha}(x) = \psi_j^{1,-\alpha}(x)$

3. Connection Formulae Between the Two Generalized Classes of Fibonacci and Lucas Polynomials

This section concentrates on the development of new connection formulae between the two generalized polynomial classes considered in Section 2. Several connection formulae between some famous polynomials are also deduced as special cases. The connection coefficients are expressed in terms of ${}_2F_1(z)$ for certain z.

In this instance, we refer to the definition of the hypergeometric function ${}_2F_1(z)$ (see [19]),

$$ {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| z \right) = \sum_{r=0}^{\infty} \frac{(a_1)_r\,(a_2)_r}{(b_1)_r}\,\frac{z^r}{r!}, $$

where a_1, a_2, and b_1 are complex or real parameters, with b_1 not zero nor a negative integer.

Now, the following lemma is needed.

Lemma 1. *Let $a, b, r,$ and s be any nonzero real numbers and let*

$$ A_{m,i} = \frac{i\,s^m \left(\frac{r}{a}\right)^{i-2m} \binom{i-m}{m}}{i-m}\, {}_2F_1\left(\begin{matrix} -m, i-m \\ i-2m+2 \end{matrix} \middle| \frac{b\,r^2}{a^2\,s} \right). $$

$A_{m,i}$ fulfills the following recurrence relation:

$$ r\,A_{m,i-1} - r\,b\,A_{m-1,i-1} + a\,s\,A_{m-1,i-2} = a\,A_{m,i}. $$

Proof. Lemma 1 can be proved via straightforward lengthy computations in view of the definition of the hypergeometric function ${}_2F_1(z)$. □

Theorem 1. *The following connection formula applies for any non-negative integer i:*

$$ \phi_i^{a,b}(x) = \left(\frac{a}{r}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} c_{i-2m}\,(-s)^m \binom{i}{m}\, {}_2F_1\left(\begin{matrix} -m, m-i \\ -i \end{matrix} \middle| \frac{b\,r^2}{a^2\,s} \right) \psi_{i-2m}^{r,s}(x), \tag{10} $$

where the numbers c_j are those given in Equation (9).

Theorem 2. *The following connection formula applies for any non-negative integer i with $i \geq 1$:*

$$\psi_i^{r,s}(x) = i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{s^m \left(\frac{r}{a}\right)^{i-2m} \binom{i-m}{m}}{(i-m)} \, {}_2F_1\left(\begin{matrix} -m, i-m \\ i-2m+2 \end{matrix} \,\middle|\, \frac{b\, r^2}{a^2\, s} \right) \phi_{i-2m}^{a,b}(x). \tag{11}$$

Proof. Similar approaches can be used to prove Theorems 1 and 2. Therefore, it is sufficient to prove Theorem 2. We proceed by induction. For the starting value $i = 1$, it is easy to see that each of the two sides in (11) is equal to $(r\,x)$. Now, assume that (11) holds for all $j < i$, and we show that (11) is itself valid. If we start with the recurrence relation in (3) and apply the induction hypothesis twice, then we get

$$\psi_i^{a,b}(x) = r\,x \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} A_{m,i-1}\, \phi_{i-2m-1}^{a,b}(x) + s \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor -1} A_{m,i-2}\, \phi_{i-2m-2}^{a,b}(x), \tag{12}$$

where

$$A_{m,i} = \frac{i\, s^m \left(\frac{r}{a}\right)^{i-2m} \binom{i-m}{m}}{i-m} \, {}_2F_1\left(\begin{matrix} -m, i-m \\ i-2m+2 \end{matrix} \,\middle|\, \frac{b\, r^2}{a^2\, s} \right).$$

The recurrence relation in (2) implies that

$$x\, \phi_{i-2m-1}^{a,b}(x) = \frac{1}{a}\, \phi_{i-2m}^{a,b}(x) - \frac{b}{a}\, \phi_{i-2m-2}^{a,b}(x). \tag{13}$$

If we insert (13) into (12), then the following relation is obtained:

$$\begin{aligned} \psi_i^{a,b}(x) =& \frac{r}{a} \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} A_{m,i-1}\, \phi_{i-2m}^{a,b}(x) - \frac{r\,b}{a} \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} A_{m,i-1}\, \phi_{i-2m-2}^{a,b}(x) \\ &+ s \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor -1} A_{m,i-2}\, \phi_{i-2m-2}^{a,b}(x). \end{aligned}$$

After some algebraic calculations, the last relation can be transformed into the equivalent one:

$$\begin{aligned} \psi_i^{a,b}(x) =& \sum_{m=1}^{\lfloor \frac{i-1}{2} \rfloor} \left(\frac{r}{a} A_{m,i-1} - \frac{r\,b}{a} A_{m-1,i-1} + s\, A_{m-1,i-2} \right) \phi_{i-2m}^{a,b}(x) + \frac{r}{a}\, A_{0,i-1}\, \phi_i^{a,b}(x) \\ &- \frac{r\,b}{a}\, A_{\lfloor \frac{i-1}{2} \rfloor, i-1}\, \phi_{i-2\lfloor \frac{i-1}{2} \rfloor -2}^{a,b}(x) + s\, A_{\frac{i}{2}-1, i-2}\, \xi_i, \end{aligned} \tag{14}$$

where

$$\xi_i = \begin{cases} 1, & i \text{ even}, \\ 0, & i \text{ odd}. \end{cases}$$

In virtue of Lemma 1, one can rewrite (14) in the form

$$\begin{aligned} \psi_i^{a,b}(x) =& \sum_{m=1}^{\lfloor \frac{i-1}{2} \rfloor} A_{m,i}\, \phi_{i-2m}^{a,b}(x) + \frac{r}{a}\, A_{0,i-1}\, \phi_i^{a,b}(x) \\ &- \frac{r\,b}{a}\, A_{\lfloor \frac{i-1}{2} \rfloor, i-1}\, \phi_{i-2\lfloor \frac{i-1}{2} \rfloor -2}^{a,b}(x) + s\, A_{\frac{i}{2}-1, i-2}\, \xi_i. \end{aligned} \tag{15}$$

It is not difficult to see that (15) can be written alternatively as

$$\psi_i^{r,s}(x) = i\sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} \frac{s^m \left(\frac{r}{a}\right)^{i-2m} \binom{i-m}{m}}{i-m} \, {}_2F_1\left(\begin{matrix} -m, i-m \\ i-2m+2 \end{matrix} \middle| \frac{b\,r^2}{a^2\,s}\right) \phi_{i-2m}^{a,b}(x).$$

The proof of Theorem 2 is now complete. □

Next, we give a special case of Formula (10).

Corollary 1. *For $r = a, s = b$, the connection formula in (10) reduces to the following one:*

$$\phi_i^{a,b}(x) = \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,(-b)^m\,\psi_{i-2m}^{a,b}(x). \tag{16}$$

Proof. The substitution by $r = a$ and $s = b$ into Relation (10) yields the following relation:

$$\phi_i^{a,b}(x) = \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,(-b)^m \binom{i}{m} \, {}_2F_1\left(\begin{matrix} -m, m-i \\ -i \end{matrix} \middle| 1\right) \psi_{i-2m}^{r,s}(x).$$

With the aid of the Chu–Vandermond identity, it is easy to see that

$${}_2F_1\left(\begin{matrix} -m, m-i \\ -i \end{matrix} \middle| 1\right) = \frac{1}{\binom{i}{m}},$$

and consequently, Formula (16) can be easily obtained. □

Taking into consideration the special polynomials of the two classes $\phi_j^{a,b}(x)$ and $\psi_j^{r,s}(x)$ mentioned in Table 1, several connection formulae can be deduced as special cases of Corollary 1.

Corollary 2. *For a non-negative integer i, the following connection formulae hold:*

$$\begin{aligned} F_{i+1}(x) &= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,(-1)^m\,L_{i-2m}(x), \\ P_{i+1}(x) &= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,(-1)^m\,Q_{i-2m}(x), \\ \mathcal{F}_{i+1}(x) &= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,2^m\,f_{i-2m}(x), \\ U_i(x) &= 2\sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,T_{i-2m}(x), \\ E_{i+1}^{\alpha}(x) &= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,\alpha^m\,D_{i-2m}^{\alpha}(x). \end{aligned} \tag{17}$$

Remark 1. *The connection formula in (17) can be translated to the following trigonometric identity:*

$$2\,\sin\theta \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor} c_{i-2m}\,\cos(i-2m)\,\theta = \sin(i+1)\,\theta.$$

Corollary 3. *For $r = a$ and $s = b$, the connection formula in (11) reduces to the following connection formula:*

$$\psi_i^{a,b}(x) = \phi_i^{a,b}(x) + b\,\phi_{i-2}^{a,b}(x). \tag{18}$$

Proof. The substitution of $r = 1$ and $b = s$ into Relation (11) yields the following relation:

$$\psi_i^{a,b}(x) = i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{b^m \binom{i-m}{m}}{i-m}\, {}_2F_1\left(\begin{matrix} -m, i-m \\ i-2m+2 \end{matrix} \middle| 1 \right) \phi_{i-2m}^{a,b}(x). \tag{19}$$

Noting that

$${}_2F_1\left(\begin{matrix} -m, i-m \\ i-2m+2 \end{matrix} \middle| 1 \right) = \begin{cases} 1, & m = 0, \\ \frac{1}{i}, & m = 1, \\ 0, & \text{otherwise}, \end{cases}$$

it is easy to see that Formula (19) reduces to

$$\psi_i^{a,b}(x) = \phi_i^{a,b}(x) + b\,\phi_{i-2}^{a,b}(x).$$

□

Taking into consideration the special polynomials of the two classes $\phi_j^{a,b}(x)$ and $\psi_j^{r,s}(x)$, several simple connection formulae can be deduced as special cases of Relation (18).

Corollary 4. *For every non-negative integer i with $i \geq 1$, the following connection formulae hold:*

$$\begin{aligned} L_i(x) &= F_{i+1}(x) + F_{i-1}(x), & Q_i(x) &= P_{i+1}(x) + P_{i-1}(x), \\ f_i(x) &= \mathcal{F}_{i+1} - 2\,\mathcal{F}_{i-1}(x), & 2\,T_i(x) &= U_i(x) - U_{i-2}(x), \\ D_i^{\alpha}(x) &= E_i^{\alpha}(x) - \alpha\, E_{i-2}^{\alpha}(x). \end{aligned}$$

4. Connection Formulae Between Two Different Generalized Polynomials in the Same Class

This section is concerned with introducing other new connection formulae. We give connection formulae between two different generalized Fibonacci polynomials. Connection formulae between two generalized Lucas polynomials of different parameters are also given. We show again that all the connection coefficients are expressed in terms of ${}_2F_1(z)$ for a certain z.

Theorem 3. *The following connection formula applies for every non-negative integer i:*

$$\phi_i^{a,b}(x) = \left(\frac{a}{r}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-s)^m \binom{i}{m} (i-2m+1)}{i-m+1}\, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \middle| \frac{b\,r^2}{a^2\,s} \right) \phi_{i-2m}^{r,s}(x). \tag{20}$$

Theorem 4. *The following formula applies for every non-negative integer i:*

$$\psi_i^{a,b}(x) = \left(\frac{a}{r}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} c_{i-2m}\, (-s)^m \binom{i}{m}\, {}_2F_1\left(\begin{matrix} -m, m-i \\ 1-i \end{matrix} \middle| \frac{b\,r^2}{a^2\,s} \right) \psi_{i-2m}^{r,s}(x), \tag{21}$$

where the constants c_j are those given in (9).

Proof. Theorems 3 and 4 can be proved using similar procedures followed in the proof of Theorem 2, but we give here another strategy of proof which is built on making use of the power form representation of each of $\phi_i^{a,b}(x)$ and $\psi_i^{a,b}(x)$, $i \geq 0$, and their inversion formulae. Now, we are going to prove Theorem 3. Making use of the analytic form of $\phi_i^{a,b}(x)$ in (6) enables one to write

$$\phi_i^{a,b}(x) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} b^r\, a^{i-2r} \binom{i-r}{r} x^{i-2r}, \tag{22}$$

Expressing the term x^{i-2r} that appears in (22) in terms of the polynomials $\phi_j^{r,s}(x)$ using the inversion formula in (8) leads to the following formula:

$$\phi_i^{a,b}(x) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} b^r \binom{i-r}{r} \left(\frac{a}{r}\right)^{i-2r} \sum_{\ell=0}^{\lfloor \frac{i}{2} \rfloor - r} \frac{(-s)^\ell (i-2\ell-2r+1)\binom{i-2r}{\ell}}{i-\ell-2r+1}\, \phi_{i-2\ell-2r}^{r,s}(x).$$

Rearranging the right-hand side of the last relation and performing some lengthy manipulations enables one to obtain the following relation:

$$\phi_i^{a,b}(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (2m-i-1)\, \phi_{i-2m}^{r,s}(x) \left(\sum_{r=0}^{m} \frac{(-1)^{m+r}\, b^r \left(\frac{a}{r}\right)^{i-2r} s^{m-r} \binom{i-r}{r} \binom{i-2r}{m-r}}{m-i+r-1} \right). \tag{23}$$

It is not difficult to see that

$$\begin{aligned} &\sum_{r=0}^{m} \frac{(-1)^{m+r}\, b^r \left(\frac{a}{r}\right)^{i-2r} s^{m-r} \binom{i-r}{r} \binom{i-2r}{m-r}}{m-i+r-1} \\ &= \frac{(-1)^{m+1} s^m \left(\frac{a}{r}\right)^{i} \binom{i}{m}}{i-m+1}\, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \,\middle|\, \frac{b\, r^2}{a^2\, s} \right), \end{aligned}$$

and consequently, Relation (23) yields the connection formula

$$\phi_i^{a,b}(x) = \left(\frac{a}{r}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-s)^m \binom{i}{m} (i-2m+1)}{i-m+1}\, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \,\middle|\, \frac{b\, r^2}{a^2\, s} \right) \phi_{i-2m}^{r,s}(x).$$

□

Remark 2. *Several connection formulae can be deduced as special cases of the two connection formulae in (20) and (21). In fact, there are forty relations that can be obtained. In the following, we give some of these formulae.*

Corollary 5. *If we set $r = 2$ and $s = -1$ in Relation (20), then we obtain*

$$\phi_i^{a,b}(x) = \left(\frac{a}{2}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{\binom{i}{m} (i-2m+1)}{i-m+1}\, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \,\middle|\, \frac{4\, b}{-a^2} \right) U_{i-2m}(x). \tag{24}$$

Corollary 6. *If we set $a = 2$ and $b = -1$ in Relation (20), then we obtain*

$$U_i(x) = \left(\frac{2}{r}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-s)^m \binom{i}{m} (i-2m+1)}{i-m+1}\, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \,\middle|\, \frac{-r^2}{4\, s} \right) \phi_{i-2m}^{r,s}(x). \tag{25}$$

In particular, and if we set $a = b = 1$ in (24) and $r = s = 1$ in (25), then the following two relations are obtained:

$$F_{i+1}(x) = \left(\frac{1}{2}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{\binom{i}{m} (i-2m+1)}{i-m+1}\, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \,\middle|\, -4 \right) U_{i-2m}(x), \tag{26}$$

and

$$U_i(x) = 2^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^m \binom{i}{m} (i-2m+1)}{i-m+1} \, {}_2F_1\left(\begin{matrix} -m, -i+m-1 \\ -i \end{matrix} \middle| \frac{-1}{4} \right) F_{i-2m+1}(x). \quad (27)$$

Remark 3. *The two relations in* (26) *and* (27) *are in agreement with those developed in [16].*

Corollary 7. *If we set $r = 2$ and $s = -1$ in* (21), *then we obtain*

$$\psi_i^{a,b}(x) = 2 \left(\frac{a}{2}\right)^i \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} c_{i-2m} \binom{i}{m} \, {}_2F_1\left(\begin{matrix} -m, m-i \\ 1-i \end{matrix} \middle| \frac{-4\,b}{a^2} \right) T_{i-2m}(x).$$

5. Two Applications of the Introduced Connection Formulae

In this section, we present two applications of the connection formulae derived in Sections 3 and 4. In the first, we give some new relations between some celebrated numbers and in the second application, some new definite weighted integrals are given.

5.1. Formulae between Some Celebrated Numbers

Theorems 1–4 enable one to join two families of numbers, whether they either belong to the same class of generalized Fibonacci or Lucas numbers or they belong to two different such classes. With respect to the six families of numbers $\{F_i\}_{i\geq 0}, \{P_i\}_{i\geq 0}, \{\mathcal{F}_i\}_{i\geq 0}, \{L_i\}_{i\geq 0}$, $\{Q_i\}_{i\geq 0}$, and $\{f_i\}_{i\geq 0}$ defined in Table 1, there are thirty relations linking them. All the coefficients between them involve ${}_2F_1(z)$ for a certain z. In six cases only, the appearing ${}_2F_1(z)$ can be summed. They are given explicitly in the following corollary.

Corollary 8. *For every non-negative integer i, the following identities are valid:*

$$F_{i+1} = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} c_{i-2m} (-1)^m L_{i-2m}, \qquad P_{i+1} = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} c_{i-2m} (-1)^m Q_{i-2m},$$

$$\mathcal{F}_{i+1} = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} c_{i-2m} \, 2^m f_{i-2m}, \qquad L_i = F_{i+1} + F_{i-1},$$

$$Q_i = P_{i+1} + P_{i-1}, \qquad f_i = \mathcal{F}_{i+1} - 2\,\mathcal{F}_{i-1}.$$

Now, with respect to the remaining twenty-four relations among the celebrated sequences of numbers, they can be deduced easily by setting $x = 1$ in the relations of Theorems 1–4, taking into consideration the different special cases of the two generalized polynomials $\{\phi_i^{a,b}(x)\}_{i\geq 0}$ and $\{\psi_i^{a,b}(x)\}_{i\geq 0}$.

5.2. Some Definite Weighted Integrals

In this section, and based on the connection formulae developed in Sections 3 and 4, some definite weighted integrals are given in terms of certain hypergeometric functions of the type ${}_2F_1(z)$. Some of these results are given in the following two corollaries.

Corollary 9. *For all non-negative integers i, j with $j \geq i$, the following two integral formulae hold:*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, \phi_i^{a,b}(x) \, T_j(x) \, dx = \begin{cases} \pi \left(\frac{a}{2}\right)^i (-1)^{i+j} \binom{i}{\frac{i-j}{2}} \, {}_2F_1\left(\begin{matrix} -\left(\frac{i+j}{2}\right) \; \frac{j-i}{2} \\ -i \end{matrix} \middle| \frac{-4\,b}{a^2} \right), & \text{if } (i+j) \text{ even}, \\ 0, & \text{otherwise}, \end{cases} \quad (28)$$

and

$$\int_{-1}^{1} \sqrt{1-x^2}\, \psi_i^{r,s}(x)\, U_j(x)\, dx = \begin{cases} \dfrac{\pi\, i \left(\frac{r}{2}\right)^j \binom{\frac{i+j}{2}}{\frac{i-j}{2}} s^{\frac{i-j}{2}}}{i+j}\; {}_2F_1\left(\begin{matrix} \frac{j-i}{2}, \frac{i+j}{2} \\ j+2 \end{matrix} \,\middle|\, \frac{-r^2}{4\,s} \right), & \textit{if } (i+j) \textit{ even}, \\ 0, & \textit{otherwise}. \end{cases} \tag{29}$$

Proof. If we set $r = 2$ and $s = -1$ in Relation (10), then after making use of the orthogonality relation of $T_j(x)$, Relation (28) can be obtained. Relation (29) can be obtained by setting $a = 2$ and $b = -1$ in Relation (11) and making use of the orthogonality relation of $U_j(x)$. □

6. Reduction of Some Odd and Even Radicals

In this section, we introduce some new reduction formulae of certain odd and even radicals through the employment of the two classes of generalized Fibonacci and Lucas polynomials introduced in Section 2. The problems of the reduction of radicals are of interest. There are considerable contributions regarding the reduction of various kinds of radicals (see, for example, [27–32]). First, the following lemma is basic in the sequel.

Lemma 2. *For every non-negative integer k, the following formula is valid:*

$$(a^2\,x^2 + 4\,b)\left(\phi_{k-1}^{a,b}(x)\right)^2 - \left(\psi_k^{a,b}(x)\right)^2 = 4\,(-1)^{k+1}\,b^k. \tag{30}$$

Proof. The two Binet's formulae in (4) and (5) can be rewritten in the form

$$\phi_{k-1}^{a,b}(x) = \frac{(\alpha(x))^k - (\beta(x))^k}{\alpha(x) - \beta(x)}, \tag{31}$$

and

$$\psi_k^{a,b}(x) = (\alpha(x))^k + (\beta(x))^k, \tag{32}$$

where

$$\alpha(x) = \frac{a\,x + \sqrt{a^2\,x^2 + 4\,b}}{2}, \qquad \beta(x) = \frac{a\,x - \sqrt{a^2\,x^2 + 4\,b}}{2}, \tag{33}$$

and consequently, we have

$$\begin{aligned} (a^2\,x^2 + 4\,b)\left(\phi_{k-1}^{a,b}(x)\right)^2 - \left(\psi_k^{a,b}(x)\right)^2 &= \left\{(\alpha(x))^k - (\beta(x))^k\right\}^2 \\ -\left\{(\alpha(x))^k + (\beta(x))^k\right\}^2 &= -4(\alpha(x)\,\beta(x))^k = 4\,(-1)^{k+1}\,b^k. \end{aligned}$$

Lemma 2 is now proved. □

6.1. New Formulae of Some Odd Radicals

We state and prove the following theorem in which we show how to reduce some odd radicals using the two generalized Fibonacci and Lucas polynomials.

Theorem 5. *Let k be any positive odd integer. Then for every $x \in \mathbb{R}$ such that $(ax)^2 \geq -4b$, the following two identities hold:*

$$\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} + \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} + b^k}} + \sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} - \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} + b^k}} = a\,x, \tag{34}$$

and

$$\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2}+\sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4}+b^k}}-\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2}-\sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4}+b^k}}=\sqrt{a^2\,x^2+4\,b}. \tag{35}$$

Proof. The idea of the proof is built on making use of the two Binet's formulae in (31) and (32). From (33), it is clear that

$$\alpha(x)+\beta(x)=a\,x, \tag{36}$$

$$\alpha(x)-\beta(x)=\sqrt{a^2\,x^2+4\,b}. \tag{37}$$

Inserting the identity in (37) into Relation (31), the following relation is obtained:

$$\sqrt{a^2\,x^2+4\,b}\,\phi_{k-1}^{a,b}(x)=(\alpha(x))^k-(\beta(x))^k.$$

Now, the last relation along with Relation (32) yields the following two relations:

$$(\alpha(x))^k=\frac{1}{2}\left\{\psi_k^{a,b}(x)+\sqrt{a^2\,x^2+4\,b}\,\phi_{k-1}^{a,b}(x)\right\}, \tag{38}$$

and

$$(\beta(x))^k=\frac{1}{2}\left\{\psi_k^{a,b}(x)-\sqrt{a^2\,x^2+4\,b}\,\phi_{k-1}^{a,b}(x)\right\}. \tag{39}$$

Next, and based on the identity in (30), Relations (38) and (39) can be written alternatively as

$$(\alpha(x))^k=\frac{1}{2}\left\{\psi_k^{a,b}(x)+\sqrt{\left(\psi_k^{a,b}(x)\right)^2+4\,b^k}\right\},$$

and

$$(\beta(x))^k=\frac{1}{2}\left\{\psi_k^{a,b}(x)-\sqrt{\left(\psi_k^{a,b}(x)\right)^2+4\,b^k}\right\}.$$

Finally, we observe that the last two relations lead to the following interesting identity,

$$\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2}+\sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4}+b^k}}\pm\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2}-\sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4}+b^k}}=\alpha(x)\pm\beta(x),$$

which can be split using Relations (36) and (37) to give the two identities in (34) and (35). This proves Theorem 5. □

Corollary 10. *Let k be any positive odd integer. Then for nonzero real numbers a and b with $b\geq -1/4$, there are infinite numbers of expressions of unit radicals in the sense that*

$$\sqrt[k]{\frac{\psi_k^{a,b}(\frac{1}{a})}{2}+\sqrt{\frac{\left(\psi_k^{a,b}(\frac{1}{a})\right)^2}{4}+b^k}}+\sqrt[k]{\frac{\psi_k^{a,b}(\frac{1}{a})}{2}-\sqrt{\frac{\left(\psi_k^{a,b}(\frac{1}{a})\right)^2}{4}+b^k}}=1. \tag{40}$$

Proof. Setting $x=\frac{1}{a}$ in (34) yields Formula (40). □

As a consequence of Theorem 5 and taking into consideration the special polynomials of $\psi_i^{a,b}(x)$ in Table 1, we obtain some new reduction formulae of some odd radicals involving Lucas, Pell–Lucas, Fermat–Lucas, Chebyshev, and Dickson polynomials of the first kinds. The following corollaries display these formulae.

Corollary 11. *For every positive integer i, the following reduction formulae hold:*

1. *For all $x \in \mathbb{R}$, one has*

$$\sqrt[2i+1]{\frac{L_{2i+1}(x)}{2}+\sqrt{\frac{L_{2i+1}^2(x)}{4}+1}}+\sqrt[2i+1]{\frac{L_{2i+1}(x)}{2}-\sqrt{\frac{L_{2i+1}^2(x)}{4}+1}}=x,$$
$$\sqrt[2i+1]{\frac{L_{2i+1}(x)}{2}+\sqrt{\frac{L_{2i+1}^2(x)}{4}+1}}-\sqrt[2i+1]{\frac{L_{2i+1}(x)}{2}-\sqrt{\frac{L_{2i+1}^2(x)}{4}+1}}=\sqrt{x^2+4}.$$

2. *For all $x \in \mathbb{R}$, one has*

$$\sqrt[2i+1]{\frac{Q_{2i+1}(x)}{2}+\sqrt{\frac{Q_{2i+1}^2(x)}{4}+1}}+\sqrt[2i+1]{\frac{Q_{2i+1}(x)}{2}-\sqrt{\frac{Q_{2i+1}^2(x)}{4}+1}}=2\,x,$$
$$\sqrt[2i+1]{\frac{Q_{2i+1}(x)}{2}+\sqrt{\frac{Q_{2i+1}^2(x)}{4}+1}}-\sqrt[2i+1]{\frac{Q_{2i+1}(x)}{2}-\sqrt{\frac{Q_{2i+1}^2(x)}{4}+1}}=2\sqrt{x^2+1}.$$

3. *For all $x \in \mathbb{R} \setminus (-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3})$, one has*

$$\sqrt[2i+1]{\frac{f_{2i+1}(x)}{2}+\sqrt{\frac{f_{2i+1}^2(x)}{4}-2^{2i+1}}}+\sqrt[2i+1]{\frac{f_{2i+1}(x)}{2}-\sqrt{\frac{f_{2i+1}^2(x)}{4}-2^{2i+1}}}=3\,x,$$
$$\sqrt[2i+1]{\frac{f_{2i+1}(x)}{2}+\sqrt{\frac{f_{2i+1}^2(x)}{4}-2^{2i+1}}}-\sqrt[2i+1]{\frac{f_{2i+1}(x)}{2}-\sqrt{\frac{f_{2i+1}^2(x)}{4}-2^{2i+1}}}=\sqrt{9\,x^2-8}.$$

4. *For all $x \in \mathbb{R} \setminus (-1, 1)$, one has*

$$\sqrt[2i+1]{T_{2i+1}(x)+\sqrt{T_{2i+1}^2(x)-1}}+\sqrt[2i+1]{T_{2i+1}(x)-\sqrt{T_{2i+1}^2(x)-1}}=2\,x,$$
$$\sqrt[2i+1]{T_{2i+1}(x)+\sqrt{T_{2i+1}^2(x)-1}}-\sqrt[2i+1]{T_{2i+1}(x)-\sqrt{T_{2i+1}^2(x)-1}}=2\sqrt{x^2-1}.$$

5. *Either for all $x \in \mathbb{R}$ whenever $\alpha < 0$ or for all $x \in \mathbb{R} \setminus (-2\sqrt{\alpha}, 2\sqrt{\alpha})$ whenever $\alpha > 0$, one has*

$$\sqrt[2i+1]{\frac{D_{2i+1}^{\alpha}(x)}{2}+\sqrt{\frac{(D_{2i+1}^{\alpha}(x))^2}{4}-\alpha^{2i+1}}}+\sqrt[2i+1]{\frac{D_{2i+1}^{\alpha}(x)}{2}-\sqrt{\frac{(D_{2i+1}^{\alpha}(x))^2}{4}-\alpha^{2i+1}}}=x,$$
$$\sqrt[2i+1]{\frac{D_{2i+1}^{\alpha}(x)}{2}+\sqrt{\frac{(D_{2i+1}^{\alpha}(x))^2}{4}-\alpha^{2i+1}}}-\sqrt[2i+1]{\frac{D_{2i+1}^{\alpha}(x)}{2}-\sqrt{\frac{(D_{2i+1}^{\alpha}(x))^2}{4}-\alpha^{2i+1}}}=\sqrt{x^2-4\,\alpha}.$$

If we put $x = 1$ in the first three items of the above corollary, we get the following new interesting reduction formulae of some odd radicals involving Lucas, Pell–Lucas, and Fermat–Lucas numbers.

Corollary 12. *For every non-negative integer i, one has*

$$\sqrt[2i+1]{\frac{L_{2i+1}}{2}+\sqrt{\frac{L_{2i+1}^2}{4}+1}}+\sqrt[2i+1]{\frac{L_{2i+1}}{2}-\sqrt{\frac{L_{2i+1}^2}{4}+1}}=1,$$
$$\sqrt[2i+1]{\frac{L_{2i+1}}{2}+\sqrt{\frac{L_{2i+1}^2}{4}+1}}-\sqrt[2i+1]{\frac{L_{2i+1}}{2}-\sqrt{\frac{L_{2i+1}^2}{4}+1}}=\sqrt{5},$$
$$\sqrt[2i+1]{\frac{Q_{2i+1}}{2}+\sqrt{\frac{Q_{2i+1}^2}{4}+1}}+\sqrt[2i+1]{\frac{Q_{2i+1}}{2}-\sqrt{\frac{Q_{2i+1}^2}{4}+1}}=2,$$
$$\sqrt[2i+1]{\frac{Q_{2i+1}}{2}+\sqrt{\frac{Q_{2i+1}^2}{4}+1}}-\sqrt[2i+1]{\frac{Q_{2i+1}}{2}-\sqrt{\frac{Q_{2i+1}^2}{4}+1}}=2\sqrt{2},$$
$$\sqrt[2i+1]{\frac{f_{2i+1}}{2}+\sqrt{\frac{f_{2i+1}^2}{4}-2^{2i+1}}}+\sqrt[2i+1]{\frac{f_{2i+1}}{2}-\sqrt{\frac{f_{2i+1}^2}{4}-2^{2i+1}}}=3,$$
$$\sqrt[2i+1]{\frac{f_{2i+1}}{2}+\sqrt{\frac{f_{2i+1}^2}{4}-2^{2i+1}}}-\sqrt[2i+1]{\frac{f_{2i+1}}{2}-\sqrt{\frac{f_{2i+1}^2}{4}-2^{2i+1}}}=1.$$

In the following example, we give the reduction formulae of a few radicals as an application of Corollary 11.

Example 1. *The following identities are valid:*

1. $\sqrt[3]{7+5\sqrt{2}}+\sqrt[3]{7-5\sqrt{2}}=$
$$\sqrt[3]{\frac{L_3(2)}{2}+\sqrt{\frac{(L_3(2))^2}{4}+1}}+\sqrt[3]{\frac{L_3(2)}{2}-\sqrt{\frac{(L_3(2))^2}{4}+1}}=2.$$
2. $\sqrt[3]{7+5\sqrt{2}}-\sqrt[3]{7-5\sqrt{2}}=$
$$\sqrt[3]{\frac{L_3(2)}{2}+\sqrt{\frac{(L_3(2))^2}{4}+1}}-\sqrt[3]{\frac{L_3(2)}{2}-\sqrt{\frac{(L_3(2))^2}{4}+1}}=2\sqrt{2}.$$
3. $\sqrt[3]{117+37\sqrt{10}}+\sqrt[3]{117-37\sqrt{10}}=$
$$\sqrt[3]{\frac{Q_3(3)}{2}+\sqrt{\frac{(Q_3(3))^2}{4}+1}}+\sqrt[3]{\frac{Q_3(3)}{2}-\sqrt{\frac{(Q_3(3))^2}{4}+1}}=6.$$
4. $\sqrt[5]{115896+19876\sqrt{34}}+\sqrt[5]{115896-19876\sqrt{34}}=$
$$\sqrt[5]{\frac{f_5(4)}{2}+\sqrt{\frac{(f_5(4))^2}{4}-2^5}}+\sqrt[5]{\frac{f_5(4)}{2}-\sqrt{\frac{(f_5(4))^2}{4}-2^5}}=12.$$
5. $\sqrt[5]{47525+19402\sqrt{6}}+\sqrt[5]{47525-19402\sqrt{6}}=$
$$\sqrt[5]{T_5(5)+\sqrt{(T_5(5))^2-1}}+\sqrt[5]{T_5(5)-\sqrt{(T_5(5))^2-1}}=10.$$

6.2. New Reduction Formulae of Some Even Radicals

This section is devoted to presenting the new formulae of some even radicals.

Theorem 6. *Let k be any positive even integer. Then for every real number with $(ax)^2 \geq -4b$, the following two identities hold:*

$$\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} + \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} - b^k}} - \sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} - \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} - b^k}} = \begin{cases} |a\,x|, & b > 0, \\ \sqrt{a^2\,x^2 + 4\,b}, & b < 0, \end{cases} \tag{41}$$

and

$$\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} + \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} - b^k}} + \sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} - \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} - b^k}} = \begin{cases} \sqrt{a^2\,x^2 + 4\,b}, & b > 0, \\ |a\,x|, & b < 0. \end{cases} \tag{42}$$

Proof. Let k be a positive even integer. In view of Equation (4), for all real numbers x with $(ax)^2 \geq -4b$, $\phi_{k-1}^{a,b}(x) \geq 0$ if and only if $a\,x \geq 0$.

Now, Equation (30) may be written as $\sqrt{a^2x^2 + 4b}\ |\ \phi_{k-1}\ | \ = \ \sqrt{\psi_k^2 - 4\,b^k}$. Using similar procedures followed in the proof of Theorem 5 leads to the following identity,

$$\sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} + \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} - b^k}} \mp \sqrt[k]{\frac{\psi_k^{a,b}(x)}{2} - \sqrt{\frac{\left(\psi_k^{a,b}(x)\right)^2}{4} - b^k}} = \big||\alpha(x)| \mp |\beta(x)|\big|,$$

and consequently, the identities in (41) and (42) can be obtained. □

As consequences of Theorem 6, we give some reduction formulae of certain even radicals. These formulae are shown in the corollary below.

Corollary 13. *For every positive integer i and for every real number x, the following formulae hold:*

1. *For all $x \in \mathbb{R}$, one has*

$$\sqrt[2i]{\frac{L_{2i}(x)}{2} + \sqrt{\frac{L_{2i}^2(x)}{4} - 1}} - \sqrt[2i]{\frac{L_{2i}(x)}{2} - \sqrt{\frac{L_{2i}^2(x)}{4} - 1}} = |x|,$$

$$\sqrt[2i]{\frac{L_{2i}(x)}{2} + \sqrt{\frac{L_{2i}^2(x)}{4} - 1}} + \sqrt[2i]{\frac{L_{2i}(x)}{2} - \sqrt{\frac{L_{2i}^2(x)}{4} - 1}} = \sqrt{x^2 + 4}.$$

2. *For all $x \in \mathbb{R}$, one has*

$$\sqrt[2i]{\frac{Q_{2i}(x)}{2} + \sqrt{\frac{Q_{2i}^2(x)}{4} - 1}} - \sqrt[2i]{\frac{Q_{2i}(x)}{2} - \sqrt{\frac{Q_{2i}^2(x)}{4} - 1}} = 2\,|x|,$$

$$\sqrt[2i]{\frac{Q_{2i}(x)}{2} + \sqrt{\frac{Q_{2i}^2(x)}{4} - 1}} + \sqrt[2i]{\frac{Q_{2i}(x)}{2} - \sqrt{\frac{Q_{2i}^2(x)}{4} - 1}} = 2\sqrt{x^2 + 1}.$$

3. *For all $x \in \mathbb{R} \setminus (-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3})$, one has*

$$\sqrt[2i]{\frac{f_{2i}(x)}{2} + \sqrt{\frac{f_{2i}^2(x)}{4} - 2^{2i}}} - \sqrt[2i]{\frac{f_{2i}(x)}{2} - \sqrt{\frac{f_{2i}^2(x)}{4} - 2^{2i}}} = \sqrt{9\,x^2 - 8},$$

$$\sqrt[2i]{\frac{f_{2i}(x)}{2} + \sqrt{\frac{f_{2i}^2(x)}{4} - 2^{2i}}} + \sqrt[2i]{\frac{f_{2i}(x)}{2} - \sqrt{\frac{f_{2i}^2(x)}{4} - 2^{2i}}} = 3\,|x|.$$

4. *For all $x \in \mathbb{R} \setminus (-1,1)$, one has*

$$\sqrt[2i]{T_{2i}(x)+\sqrt{T_{2i}^2(x)-1}}-\sqrt[2i]{T_{2i}(x)+\sqrt{T_{2i}^2(x)-1}}=2\sqrt{x^2-1},$$
$$\sqrt[2i]{T_{2i}(x)+\sqrt{T_{2i}^2(x)-1}}+\sqrt[2i]{T_{2i}(x)+\sqrt{T_{2i}^2(x)-1}}=2\,|x|.$$

In the following example, we give the formulae of some specific even radicals based on Corollary 13.

Example 2. *The following identities are valid:*

1. $\sqrt[4]{17+12\sqrt{2}}-\sqrt[4]{17-12\sqrt{2}}=$
$$\sqrt[4]{\frac{L_4(2)}{2}+\sqrt{\frac{(L_4(2))^2}{4}-1}}-\sqrt[4]{\frac{L_4(2)}{2}-\sqrt{\frac{(L_4(2))^2}{4}-1}}=2.$$
2. $\sqrt[4]{5201+1020\sqrt{26}}-\sqrt[4]{5201-1020\sqrt{26}}=$
$$\sqrt[4]{\frac{Q_4(5)}{2}+\sqrt{\frac{(Q_4(5))^2}{4}-1}}-\sqrt[4]{\frac{Q_4(5)}{2}-\sqrt{\frac{(Q_4(5))^2}{4}-1}}=10.$$
3. $\sqrt[6]{\frac{83448209}{2}+\frac{4010265}{2}\sqrt{433}}+\sqrt[6]{\frac{83448209}{2}-\frac{4010265}{2}\sqrt{433}}=$
$$\sqrt[6]{\frac{f_6(7)}{2}+\sqrt{\frac{(f_6(7))^2}{4}-2^6}}+\sqrt[6]{\frac{f_6(7)}{2}-\sqrt{\frac{(f_6(7))^2}{4}-2^6}}=21.$$
4. $\sqrt[8]{708158977+408855776\sqrt{3}}+\sqrt[8]{708158977-408855776\sqrt{3}}=$
$$\sqrt[8]{T_8(7)+\sqrt{(T_8(7))^2-1}}+\sqrt[8]{T_8(7)-\sqrt{(T_8(7))^2-1}}=14.$$

7. Some Other Radical Formulae

The purpose of this section is to establish two other reduction formulae for some radicals. In the sequel, the following lemma is required.

Lemma 3. *For every positive integer j and every $x \in \mathbb{R}^*$, one has*

$$\psi_j^{a,b}\left(\frac{1}{a}(x-\tfrac{b}{x})\right)=\begin{cases}x^j-\dfrac{b^j}{x^j}, & j\text{ odd},\\[2ex] x^j+\dfrac{b^j}{x^j}, & j\text{ even}.\end{cases}$$

Proof. Binet's formula in (5) implies that

$$\begin{aligned}
&\psi_j^{a,b}\left(\frac{1}{a}(x-\tfrac{b}{x})\right)\\
&=\frac{1}{2^j}\left\{\left(\left(x-\tfrac{b}{x}\right)+\sqrt{\left(x-\tfrac{b}{x}\right)^2+4b}\right)^j+\left((x-\tfrac{b}{x})-\sqrt{(x-\tfrac{b}{x})^2+4b}\right)^j\right\}\\
&=\frac{1}{2^j}\left\{\left((x-\tfrac{b}{x})+\sqrt{(x+\tfrac{b}{x})^2}\right)^j+\left((x-\tfrac{b}{x})-\sqrt{(x+\tfrac{b}{x})^2}\right)^j\right\}\\
&=\frac{1}{2^j}\left\{\left(\left(x-\tfrac{b}{x}\right)+\left|x+\tfrac{b}{x}\right|\right)^j+\left((x-\tfrac{b}{x})-\left|x+\tfrac{b}{x}\right|\right)^j\right\}\\
&=\frac{1}{2^j}\left\{(2\,x)^j+\left(\tfrac{-2\,b}{x}\right)^j\right\}=\begin{cases}x^j-\frac{b^j}{x^j}, & j\text{ odd},\\ x^j+\frac{b^j}{x^j}, & j\text{ even}.\end{cases}
\end{aligned}$$

The proof is now complete. □

Theorem 7. *If j is an odd positive integer and $b \in \mathbb{R}$, then for every $x \in \mathbb{R}^*$, the following identity applies:*

$$\sqrt[j]{\frac{\psi_j^{a,b}(\frac{1}{a}(x-\frac{b}{x}))}{2}+\sqrt{\frac{\left(\psi_j^{a,b}(\frac{1}{a}(x-\frac{b}{x}))\right)^2}{4}+b^j}}= \begin{cases} x, & x>0,\ b>0,\\ \frac{-b}{x}, & x<0,\ b>0,\\ x, & x\in[-\sqrt{-b},0)\cup[\sqrt{-b},\infty),\ b<0,\\ \frac{-b}{x}, & x\in(-\infty,-\sqrt{-b})\cup(0,\sqrt{-b}),\ b<0. \end{cases} \tag{43}$$

Proof. From Lemma 3, we have

$$\sqrt[j]{\frac{\psi_j^{a,b}(\frac{1}{a}(x-\frac{b}{x}))}{2}+\sqrt{\frac{\left(\psi_j^{a,b}(\frac{1}{a}(x-\frac{b}{x}))\right)^2}{4}+b^j}}=\sqrt[j]{\frac{(x^j-\frac{b^j}{x^j})}{2}+\sqrt{\frac{(x^j-\frac{b^j}{x^j})^2}{4}+b^j}}$$

$$=\sqrt[j]{\frac{(x^j-\frac{b^j}{x^j})}{2}+\sqrt{\frac{(x^j+\frac{b^j}{x^j})^2}{4}}}=\sqrt[j]{\frac{(x^j-\frac{b^j}{x^j})}{2}+\frac{|x^j+\frac{b^j}{x^j}|}{2}},$$

and consequently, Relation (43) can be obtained. □

The following results are direct consequences of Theorem 7.

Corollary 14. *If j is an odd positive integer, then the following identities are true for Lucas, Pell–Lucas, Fermat–Lucas, and Chebyshev polynomials of the first kind:*

$$\sqrt[j]{\frac{L_j(x-\frac{1}{x})}{2}+\sqrt{\frac{\left(L_j(x-\frac{1}{x})\right)^2}{4}+1}}=\begin{cases} x, & x>0,\\ \frac{-1}{x}, & x<0,\end{cases}$$

$$\sqrt[j]{\frac{Q_j(\frac{1}{2}(x-\frac{1}{x}))}{2}+\sqrt{\frac{\left(Q_j(\frac{1}{2}(x-\frac{1}{x}))\right)^2}{4}+1}}=\begin{cases} x, & x>0,\\ \frac{-1}{x}, & x<0,\end{cases}$$

$$\sqrt[j]{\frac{f_j(\frac{1}{3}(x+\frac{2}{x}))}{2}+\sqrt{\frac{\left(f_j(\frac{1}{3}(x+\frac{2}{x}))\right)^2}{4}-2^j}}=\begin{cases} x, & x\in[\sqrt{2},\infty)\cup[-\sqrt{2},0),\\ \frac{2}{x}, & x\in(-\infty,-\sqrt{2})\cup(0,\sqrt{2}),\end{cases}$$

$$\sqrt[j]{T_j(\tfrac{1}{2}(x+\tfrac{1}{x}))+\sqrt{\left(T_j(\tfrac{1}{2}(x+\tfrac{1}{x}))\right)^2-1}}=\begin{cases} x, & x\in[1,\infty)\cup[-1,0),\\ \frac{1}{x}, & x\in(-\infty,-1)\cup(0,1).\end{cases}$$

Example 3. *As an application of Corollary 14, which is a corollary of Theorem 7, we readily get (for $j=5$ and $x=\frac{-1}{3}$) the following odd reduction formula:*

$$\sqrt[5]{\frac{-1889569}{486}+\sqrt{\frac{3570463447489}{236196}}}=\sqrt[5]{\frac{f_5(\frac{-19}{9})}{2}+\sqrt{\frac{(f_5(\frac{-19}{9}))^2}{4}-32}}=-\frac{1}{3}.$$

The following theorem exhibits the counterpart result of Theorem 7 for even radicals.

Theorem 8. *If j is an even positive integer, then the following identity applies for every nonzero real number x:*

$$\sqrt[j]{\frac{\psi_j^{a,b}(\frac{1}{a}(x-\frac{b}{x}))}{2}+\sqrt{\frac{\left(\psi_j^{a,b}(\frac{1}{a}(x-\frac{b}{x}))\right)^2}{4}-b^j}}=\begin{cases}|x|, & x\in\mathbb{R}\setminus(-\sqrt{|b|},\sqrt{|b|}),\\ |\frac{b}{x}|, & x\in(-\sqrt{|b|},\sqrt{|b|}).\end{cases}$$

Example 4. *The dedication for Paper [31] reads: "In memory of Ramanujan on the*

$$\sqrt[6]{32(\tfrac{146410001}{48400})^3-6(\tfrac{146410001}{48400})+\sqrt{(32(\tfrac{146410001}{48400})^3-6(\tfrac{146410001}{48400}))^2-1}}\quad th$$

anniversary of his birth."

This radical was calculated by Osler in [30] using Cardan polynomials. In the following few lines, we show that it can be evaluated with the aid of Theorem 8. In fact, if we note that

$$32\left(\tfrac{146410001}{48400}\right)^3-6\left(\tfrac{146410001}{48400}\right)=\frac{1}{2}\psi_3^{1,-1}\left((110)^2+\tfrac{1}{(110)^2}\right),$$

then we have

$$\sqrt[6]{32(\tfrac{146410001}{48400})^3-6(\tfrac{146410001}{48400})+\sqrt{(32(\tfrac{146410001}{48400})^3-6(\tfrac{146410001}{48400}))^2-1}}=$$
$$\sqrt[6]{\frac{\psi_3^{1,-1}\left((110)^2+\frac{1}{(110)^2}\right)}{2}+\sqrt{\frac{\left(\psi_3^{1,-1}\left((110)^2+\frac{1}{(110)^2}\right)\right)^2}{4}-1}}=\sqrt{(110)^2}=110.$$

8. Conclusions

In this paper, we established several new connection formulae between some classes of polynomials generalizing the celebrated Fibonacci and Lucas polynomials. Some of these formulae generalize respective known ones. Two applications of the derived connection formulae were presented. A number of new expressions between the celebrated Fibonacci, Pell, Fermat, Lucas, Pell–Lucas, and Fermat–Lucas numbers were deduced. As a very important utilization of the two generalized Fibonacci and Lucas polynomials, some new reduction formulae of certain odd and even radicals were given. We have deduced as special cases several reduction formulae of certain radicals involving Lucas, Pell–Lucas, Fermat–Lucas, Chebyshev, and Dickson polynomials of the first and second kinds. Numerous examples were given to apply the reduction of radicals. As far as we know, most of the formulae in this paper are new and they are thought to be crucial and useful. Furthermore, we do believe that other generalizations of Fibonacci and Lucas sequences of numbers and polynomials can be considered. In the near future, we hope to consider and investigate some of these generalized sequences.

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