Novel Symmetric Numerical Methods for Solving Symmetric Mathematical Problems

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Abstract—The mathematical model for many problems is arising in different industries of natural science, basically formulated using differential, integral and integro-differential equations. The investigation of these equations is conducted with the help of numerical integration theory. It is commonly known that a class of problems can be solved by applying numerical integration. The construction of the quadrature formula has a direct relation with the computation of definite integrals. The theory of definite integrals is used in geometry, physics, mechanics and in other related subjects of science. In this work, the existence and uniqueness of the solution of above-mentioned equations are investigated. By this way, the domain has been defined in which the solution of these problems is equivalent. All proposed four problems can be solved using one and the same methods. We define some domains in which the solution of one of these problems is also the solution of the other problems. Some stable methods with the degree p<=8 are constructed to solve some problems, and obtained results are compared with other known methods. In addition, symmetric methods are constructed for comparing them with other well-known methods in some symmetric and asymmetric mathematical problems. Some of our constructed methods are compared with Gauss methods. In addition, symmetric methods are constructed for comparing them with other well-known methods in some symmetric and asymmetric mathematical problems. Some of our constructed methods are compared with Gauss methods. On the intersection of multistep and

hybrid methods have been constructed multistep methods and have been proved that these methods are more exact than others. And also has been shown that, hybrid methods constructed here are more exact than Gauss methods. Noted that constructed here hybrid methods preserves the properties of the Gauss method.

Keywords—Initial-value problem, ODE, Volterra integral equation, integro-differential equation of Volterra type, Symmetric multistep methods, multistep methods of hybrid type, Systems theory and systems engineering.

I. Introduction

New efficient methods are investigated in this work to solve the proposed four problems. For this aim, we propose the intersection of the multistep methods with the hybrid, and forward-jumping (advanced) methods. Using the method of unknown coefficients, one can construct stable methods with high exactness having the new properties. Such efficient methods can be constructed by using the domain of intersection for the solution of above-mentioned problems in extended form, and by estimating the errors received in this case. The studied problems in this work have different properties. Therefore, the basic properties for all considering problems are defined in order to construct special methods for solving the proposed problems. For example, by comparison of these methods with our constructed method which has been applied to solve the initial-value problem for the ODE and calculate the definite integrals. By using the methods, which have been applied to determine the solution of these problems, can be successfully applied to solve some problems of seismology (see for example [1,2]). By using the named methods, one can solve the same related problems with the investigation of energy (see [3-5]). One of the actual directions in solving of Volterra integral equation is the use of non-polynomial splines (see [6]). In [7] has been given the new way for solving Volterra integral equation, which represents the scientific interest in Applied Mathematics.One of the same methods in one case can be taken as an implicit and in other cases as an explicit. The hybrid methods that have some advantages for solving the initial-value problem of ODE arise some difficulties related to the calculation of the values of the solution of studied problems at the hybrid points. However, in the calculation of definite integrals, there are no difficulties. From this situation, some recommendations are given for the application of our constructed methods to solve the investigated problems. To define the values of the order of exactness, some relationships by which one can determine the maximal value for the order of exactness for the stable and unstable methods (see for example [8]). These results can be taken as the development of Dahlquist theory (see for example [9]). To illustrate, the obtained results are constructed symmetric stable methods with the order $p \le 9$ and some of them have been applied to solve the model problems.

The numerical solution of Volterra Integral equation of second kind was fundamentally investigated by the multistep methods of hybrid and forward-jumping (advanced) types. Some events such as the eclipses of the moon or sun, and the processes associated with them, in the Middle Ages, have forced scientists to study the motion of celestial bodies. For this aim, the mathematical model is formulated for the named problems, and the obtained ODE of the first and second orders is solved where many works of the famous scientists as Newton, Leibniz, Euler, Dalamber, Klero, Cowell, Adams, Runge, and Kutta have been dedicated to this topic. The named problems are investigated with the help of power series, which called as the analytic-numerical methods. By showing the disadvantages of these methods, Euler constructed the direct method to solve the initial-value problem for ODEs. This numerical method has been developed by Adams and Runge-Kutta in the result of which a class of multistep and one-step methods has been constructed. The investigated multistep methods with constant coefficients are considered with some modifications to solve the initial-value problem for both ODE and Volterra integro-differential equation in order to solve the Volterra integral equation. In addition, the computation of double integrals is studied by using the methods which have been applied to compute the definite integrals. For solving the initial-value problem for ODE, Euler and its followers have used integral equations with a variable boundary. In this paper, the quadrature formula is applied to calculate the definite integral which has been encountered in the integral equation, and the methods are constructed for solving of the initial-value problem for ODE (such as Adams methods) which can be written as follows:

$$y' = f(x,y), \quad y(x_0) = y_0, \quad x_0 \le x \le X. \qquad (1)$$

Suppose that a sufficiently smooth function: $f(x,y)$ is defined in some close domain. If we integrate the equality obtained from (1) by taking into our account that its solution has been found by some methods, then the following can be written:

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s)), \quad x \in [x_0, X]. \qquad (2)$$

To determine the numerical solution of the initial-value problem (1), let us define the mesh points in the form: $x_i = x_0 + ih$, $(i = 0,1,...,N)$ here h step-size, but the values of the solution $y(x)$ and its first derivatives at the points x_i $(i \ge 0)$, denoted by the $y(x_i)$ and $y'(x_i)$ $(i = 0,1,...,N)$, respectively.

By choosing the values of x as the x_i $(i = 1,2,...,N)$, one can find the values y_i $(i = 1,2,...,N)$ of the solution $y(x)$ of the problem (1), by using some numerical methods.

It is evident that the problem (1) on the segment $[x_i, x_{i+1}]$ can be written as the follows:

$$y' = f(x,y), \quad y(x_i) = y_i, \quad x \in [x_i, x_{i+1}]. \qquad (3)$$

Taking this into account in the equation (2) and applying some quadrature formula to calculate the definite integral, one can construct the following method:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i}. \qquad (4)$$

It is not difficult to prove that Adams methods that can be received from the method (4) as the partial cases. It follows that by choosing the coefficients α_i, β_i $(i = 0,1,...,k)$ in the method (4), one can receive many methods as the Trapezoidal rule, Simpson`s method, and Midpoint rule. Method (4) is fundamentally investigated by many authors (see for example [8]-[21]).

In the work, [11] defined the sources of the errors arising in the application of the method (4). The conception for convergence is also defined. This conception in the works [12] and [13] named as the stability of the method (4). Bakhvalov proved that, if the method (4) is stable in the case $\alpha_k \neq 0,\ \beta_k = 0$, then $p \leq k$ for $k \leq 10$. Here, p is the degree and k is the order for the method (4). Dahlquist proved that if the method (4) is stable and has the degree p, then there are methods with the degree $P_{\max} = 2[k/2]+2$ for all the values of k if $\alpha_k \neq 0$ and $\beta_k \neq 0$, if $\beta_k = 0$, then there is a stable method of the type (4) with the degree $P_{\max} \leq k$. In addition, Dahlquist proved that for the degree of the stable methods of the type (4) the condition $p > k$ is hold. By Dahlquist rule, we obtain that if $p > k$, then α_k and β_k have the same sin, it is obvious that if $\alpha_k > 0$, then $\beta_k > 0$ holds.

The conception of stability and degree for the method (4) are determined as the follows:

Definition 1. Method (4) is stable if the roots of the polynomial:

$$\rho(\lambda) = \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1\lambda + \alpha_0$$

lie in the unit circle, on the boundaries of which there are no multiply roots.

Definition 2. The integer p is called as the degree of the method (4) if the following asymptotic equality holds:

$$\sum_{i=0}^{k}\left(\alpha_i y(x+ih) - h\beta_i y'(x+ih)\right) = O(h^{p+1}), h \to 0.$$

By Dahlquist results, the degree for the stable methods of the type (4) is bounded. Therefore, the scientists have proposed to use the following method to construct stable methods with the degree $p > k+2$:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{j=1}^{s}h^j\sum_{i=0}^{k}\beta_i^{(j)} y_{n+i}^{(j)}. \qquad (5)$$

This method is called as the multistep multiderivative method (MMM), which for the case $s = 1$ and $s = 2$ has been investigated by some authors (see for example [19] - [26], [31] - [33]). It is known that depending on the values of the coefficients $\beta_i^{(j)}$ $(i = 0,1,...,k;\ j = 0,1,...,s)$, the properties of the method (5) are changed. In even case, the properties of stability can be defined in a different form. In the work [26], the method (5) was fundamentally investigated and defined the maximal value of the degree for stable methods which have been constructed for different values of the coefficients:

$$\beta_i^{(j)}\ (i = 0,1,...,k;\ j = 0,1,...,s),\ \beta_i^{(2)} = 0$$
$$(i = 0,1,...,k).$$

In addition, the method (5) for $s = 3$ has been investigated in the cases $\beta_i^{(1)} = 0$ and $\left|\beta_0^{(1)}\right| + \left|\beta_1^{(1)}\right| + ... + \left|\beta_k^{(1)}\right| \neq 0$, and the method (5) in this case $\beta_i^{(1)} = 0$ $(i = 0,1,...,k)$ cannot be stable.

One of classical methods of this type is Stërmer method, which is not stable (see definition 1). If we consider the case: $\alpha_k = 0,\ \alpha_{k-1} \neq 0$ and $\left|\beta_k^{(1)}\right| + \left|\beta_k^{(2)}\right| + ... + \left|\beta_k^{(s)}\right| \neq 0$, then we receive the multistep method with new properties which are usually called as the forward-jumping (advanced) methods. The stable forward-jumping methods are more exact than the implicit methods.

Note that the maximal value of the degree for the method (5) can be presented as $p \leq s(k+1)+1$ for $s = 2m-1$ and $k = 2l$ and also $p \leq s(k+1)$ for $s = 2m-1,\ k = 2l-1$ but if $s = 2m$, then $p \leq s(k+1)$ for $k = 2l$ and $k = 2l+1$ (all the values of k).

Let us note that the investigated method (5) is very difficult than others. The properties of this method depend on the values of coefficients that can be changed a lot. For example, if $\beta_i^{(1)} = 0$ $(i = 0,1,...,k)$ then in the class of methods (5) is not stable method. If $\beta_i^{(3)} = 0$ $(i = 0,1,...,k)$ and if $\left|\beta_k^{(j)}\right| + \left|\beta_{k-1}^{(j)}\right| + \cdots + \left|\beta_0^{(j)}\right| \neq 0$ for $j = 1,2$, then the domain of application of the method of (5) is narrow. If method (5) is stable and has the degree p which receives the maximal value, then arises necessity to define the sign of the coefficients $\beta_k^{(j)}$. This is usually used in the construction of two sided or bilateral methods. As a result, the class of (5) is wider than the others. As was noted above, defining the sign for some coefficients presents some interests for the specialists studies application of the different variants of method (5).

From the above described way, we note that by using the following quadrature formula:

$$\int_{x_0}^{x_k} f(s)ds = h\sum_{i=0}^{k}A_i f_i + R(x), \qquad (6)$$

one can solve the problem (1). Here, A_i are the coefficients of the quadrature formula.

It is not difficult to understand that by using the formula (6), one can construct a method for solving problem (1), which can be presented as follows:

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$$y_k = y_0 + h\sum_{i=0}^{k} \beta_i f_i \tag{7}$$

It is clear that for the fixed point $x_n = x_0 + nh$, method (7) can be written in the following form:

$$y_{n+k} = y_n + h\sum_{i=0}^{k} \beta_i f_{n+i} \tag{8}$$

It is known that multistep method with constant coefficients, which has applied to solving of the problem (1), can be written as the formula (4). Thus, we note that the formula (8) can be applied to solve problem (1) and to the calculation of following defined integral:

$$\int_a^b f(s)\,ds.$$

Let's consider that the following indefinite function $y(x) = \int_{x_0}^{x} f(s)ds$, then from here we get: $y' = f(x),\ y(x_0) = 0$, which is the partial case of the problem (3). Let us now compare the methods (8) with the method (4). The method of (8) is also one of the partial case of the method (4), which has mostly been applied to calculate definite integrals. It follows that method (8) is stable and has the degree $p \le k+2$. It is evident that the class methods of (4) is wider than class of method (8).

If we apply the method (4) or method (5) to the calculation of the definite integral participated in the equality (6), we obtain a wide class of methods to calculate the definite integrals. It has also been shown that by using the named way, we have constructed a new way to calculate the double integral, which can be written as:

$$I = \int_a^b \int_c^d f(s,t)dsdt. \tag{9}$$

Hence, we note that investigating the class methods of (4) is perspective. By proving the advantages of proposed methods, we consider the application of the method (4) and its modification to solve Volterra integral and Volterra integro-differential equations. It is not difficult to understand that if equation (2) is generalized in the following form:

$$y(x) = y(x_0) + \int_{x_0}^{x} K(x,s,y(s))ds,$$
$$x_0 \le s \le x \le X, \tag{10}$$

then we have the nonlinear Volterra integral equation. By differentiation of the equality (10) (here suppose that by some way the solution of the Volterra integral equation, it has been found after using of which in equation (10) the equality is obtained), we have the Volterra integro-differential equation which can be written as follows:

$$y'(x) = \varphi(x,y) + v(x), \quad y(x_0) = y_0. \tag{11}$$

Here, the functions: $\varphi(x,y)$ and $v(x)$ are known, and the function $v(x)$ is defined as:

$$v(x) = \int_{x_0}^{x} K(x,s,y(s))ds, \ \ x_0 \le s \le x. \tag{12}$$

However, the function $\varphi(x,y)$ is defined in the form: $\varphi(x,y) = K(x,x,y)$. One can take the expression of (12) Volterra integral equation of the first kind. By this description, the above-mentioned problems: computation of definite integrals (including the double integrals), solving of initial-value problem for ODE and the Volterra integro-differential equation and also solving of Volterra integral equations have the direct connection. By the comparison of the numerical methods applied to solve the named problems, there is some region in which all problems can be solved in the same methods. For example, by using the method of (4), we can define the region in which all above-mentioned problems can be solved by the same method.

II. The Determination of the Set in Which All Above-Mentioned Problems Are Equivalent.

Let us determine the region in which all above mentioned problems can be solved by the same method (in our case the method (4) and its modification). For this aim, let us begin from the investigation of the problem (1). As noted above, this problem was fundamentally investigated by many authors. Therefore, let us define the problem (1) as the basic problem and consider the construction of the way by which solving of other problems can be reduced to solve of the problem (1). In this case, by using the same method, one can solve all above-mentioned problems. If we consider the case: $K(x,s,y(s)) \equiv f(s,y(s))$, then by differentiation of the equality (10) gets the initial-value problem for the ODE (here we suppose that by some way, it has been found the solution of the problem (1) and by using that in (1), we obtain the equality which is differentiable).

Let us put $f(s,y(s)) = F(s)$ and $y(x_0) = 0$. In this case, from the equality (2), it follows that:

$$y(x) = \int_{x_0}^{x} F(s)ds.$$

For the values $x_0 = a,\ x = b,$ we get:

$y(b) = \int_a^b F(s)ds,$ which is a definite integral. From here, the calculation of the definite integrals and the problem (1) in the case $f(s,y(s)) = F(s)$ is equivalent, and all definite integrals have the corresponding initial-value problem for ODEs of the first order and vise versa. However, the problem (1) and the integral equation (10) can be equivalent in the case, when $K(x,s,y) = f(s,y)$ holds. Thus, if $K(x,s,y) = f(s,y)$ and $f(s,y) = F(s)$ hold, then by using one and the same quadrature formula, one can solve the problem (1) and the equation (10) by computing the definite integral. Let us note that computing of the double integrals is more complex than the computing of single definite integral. For this aim, let us consider the following double integral and the function $u(x,y)$:

$$DI = \int_a^b \int_c^d f(s,t)dsdt;$$
$$u(x,y) = \int_a^x \int_c^y f(s,t)dsdt. \tag{13}$$

It is evident that $DI = u(b,d)$ and function $u(x,y)$ can be found by solving of the hyperbolic equation: $\partial^2 u(x,y)/\partial x \partial y = f(x,y)$ (see for example [29] p. 148-149). This problem can be solved by using the problem (1). By using the theory of interpolation polynomials, one can construct the method to compute the values of the double integral. In this case, constructed methods will have the fixed degree. To construct more exact method for computation of double integral (13), one can propose to use the Hermit or Gauss interpolation polynomials (see for example [28], [29]). In this case, we also receive the quadrature formula with the fixed degrees. Here, to solve the proposed problem, we use the finite-difference method by the modification of which one can be constructed more exact methods to compute the double integrals. Let us now continue our discussion by considering the case when the function $K(x,s,y)$ is degenerate. In this case, that can be presented as the follows:

$$K(x,s,y) = \sum_{j=1}^{m} a_j(x) b_j(s,y). \tag{14}$$

By taking this into the general form of the equation (10), we receive:

$$y(x) = g(x) + \sum_{j=1}^{m} a_j(x) \int_{x_0}^{x} b_j(s,y)ds, \tag{15}$$

Where the continuous function $g(x)$ is known. This equation can be written as follows:

$$y(x) = g(x) + \sum_{j=1}^{m} a_j(x) v_j(x), \tag{16}$$

$$v_j'(x) = b_j(x,y),\ v_j(x_0) = 0,\ j = 1,2,...,m. \tag{17}$$

Thus, the problem (1) and Volterra integral equations can be solved in the case when the function of $K(x,s,y)$ that can be presented as the (14) will be equivalent. Namely, by using these properties, the intersection domain of equivalence for the equation of (10) and problem (1) can be extended.

Let us change on the interval $[x_0, x_{n+k}]$ the function $K(x,s,y)$ by the Lagrange polynomial. In this case, we receive the following:

$$K(x,s,y) = \sum_{j=0}^{k} l_j(x) K(x_j,s,y) + R_n(x), \tag{18}$$

where $R_n(x)$ is the remainder term, and $l_j(x)$ $(j = 0,1,...k)$ is aLagrange basic function (see for example [26, p.121]). By taking into account that in the Volterra integral equation, we receive:

$$y(x) = g(x) + \sum_{j=0}^{k} l_j(x) \int_{x_0}^{x} K(x_j,s,y(s))ds$$
$$+ \int_{x_0}^{x} R_n(s)ds, \quad x \in [x_0, x_{n+k}]. \tag{19}$$

If we change the functions $K_j(x_j,x,y)$ $(j = 0,1,..,k)$ by the functions $b_j(x,y)$ and discard the remainder term $R_n(x)$, then we receive the problem similar to the problem (16) and (17), which can be solved by using the methods that are usually applied to solve of the initial-value problem of the ODEs. Thus, we prove that by using the above described way, one can solve the problem (1) and the following problem by using the same method:

$$y(x) = g(x) + \sum_{j=0}^{k} l_j(x) v_j(x), \tag{20}$$

$$v_j'(x) = K(x_j,x,y(x)),\ v_j(x_0) = 0,\ (j = 0,1,..,k). \tag{21}$$

The exactness of the described method can be estimated by the value of the remainder term $R_n(x)$.

Thus, we prove that the ODE and definite integral with the variable bounders are equivalent. All solutions of the ODE can be written by the definite integrals with the variable bounders and vise versa. We also prove that there exists a set in which the numerical solution of the initial-value problem for ODE and Volterra integral equation can be found by one and the same formula. It follows that these problems are equivalent. This set can be extended by decreasing the corresponding remainder term. For this aim, as noted above, one can use more exact interpolation polynomials. In the next section, the finite-difference methods are constructed with high degree. Let us now consider the determined set in which the above-mentioned problems and initial-value problem for Volterra integro-differential equations are equivalent. For this aim, let us compare the initial-value problem for the both ODE and Volterra integro-differential equations. It is evident that if the function $v(x)$ is defined by the equality of (12) and satisfies the condition $v(x) = 0$, then we receive that the problems (1) and (21) is one and the same problem. Therefore, we suppose that $v(x) \neq 0$ and $K(x,s,y) = \varphi(s,y)$. In this case, the problem (11) can be written as:

$$y'(x) = \varphi(x,y) + v(x),\ y(x_0) = y_0,\ x \in [x_0, X], \tag{22}$$

$$v'(x) = \psi(x,y(x)),\ v(x_0) = 0, x \in [x_0, X]. \tag{23}$$

It is clear that the function $\varphi(x, y(x))$ can be defined by the function $K_x^{\prime}(x, x, y(x))$, taking into account the equality of (12). As it follows from here the problems (1) and (11) are equivalent. It is not difficult to understand that the problems (22) and (23) can be taken as the initial-value problem for the ODE of the second order in the following form:

$$y''(x) = \varphi_x'(x, y) + \varphi_y'(x, y)y' + \psi(x, y), \quad (24)$$
$$y(x_0) = y_0, \ y'(x_0) = \varphi(x_0, y_0).$$

It follows from here that the problem (1) and (11) in the case $K(x, s, y) = \varphi(s, y)$ are equivalent. Now, let us suppose that function $K(x, s, y)$ is degenerate which can be presented as (14). In this case, the problems (22) and (23) can be written as follows:

$$y'(x) = \varphi(x, y) + \sum_{j=1}^{m} a_i(x) v_j(x),$$
$$y(x_0) = y_0, \ x \in [x_0, X] \quad (25)$$
$$v_j'(x) = b_j(x, y), \ v_j(x_0) = 0, \ j = 1,2,\ldots,m. \quad (26)$$

The problems (25) and (26) are the initial-value problem for the ODE of the first order. The following finite-difference method can be applied to solve of above-mentioned problems:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i}. \quad (27)$$

Let us note that in the case $y' = f(x, y)$ from the method (27), it implies method (4).

It is noted that the properties of the numerical methods depend on the values of its coefficients. Therefore, the methods depend on the way in which they have been applied to determine the values of the coefficients $\alpha_i, \beta_i \ (i = 0,1,\ldots,k)$ that can be called in different form. If the construction of the method of the type (27) uses the interpolation polynomials, then the received method will be partial case of the method (27). For example, if the construction of the method (27) uses the Lagrange polynomial, then we receive one of the Adams methods. The coefficients of Adams methods are calculated by some definite integral. From here, we receive that for the construction the methods of the type (27), one can use the different polynomials. For every polynomial, one can construct separate methods. But here we propose to use methods of the unknown coefficients and by choosing the coefficients, one can be construct new methods. Therefore, for the determined values of the coefficients $\alpha_i, \beta_i \ (i = 0,1,\ldots,k)$ proposed here to use the methods of unknowns' coefficients.

By the above-described way, we receive that if the kernel of the integral is degenerate function, then the problems (1) and (11) are equivalent. It is not difficult to understand that the function $K(x, s, y)$ can be approached by the following formula:

$$K(x, s, y) = \sum_{j=1}^{m} a_j(x) b_j(s, y) + R_n(x). \quad (28)$$

This similar presentation of the kernel can be used one of the above-mentioned methods. It follows from here that extending of the set of equivalency for the named problems depends on the values of $R_n(x)$. Therefore, in the construction of the approximation function for the kernel of $K(x, s, y)$, one can use the Gauss interpolation polynomial. But in this case, we receive that the application of the new function is more complex than the formula (28). It is noted that for the construction stable multistep methods with the high degree, one can use hybrid methods.

By taking into account the above mentioned disadvantages that have been proposed for the methods which are more general than above-mentioned.

Let us note that to extend the set of the above-mentioned equivalent problems, we receive that it is necessary to construct the function which approaches the kernel $K(x, s, y)$ with high rate. Otherwise, the remainder term must be sufficiently small. Here, for this aim, we use more exact methods constructed at the junction of the multistep and hybrid methods. It is evident, that if the remainder term $R_n(x)$ satisfies the conditions $R_n(x) \equiv 0$, then application of all numerical methods to solve above-mentioned problems gives some errors. These errors are related to the truncation errors of the used methods and computational technologies. Therefore, let us consider the construction of numerical methods and their application to solve above-mentioned problems.

III. Construction of the Methods with the Best Properties with Applications

From all above-described discussion problems, they can be reduced to solve the initial-value problem for ODE or to calculate the definite integrals. By taking this into account the construction of quadrature formula is considered with the best properties. For this aim, let us consider the following formula:

$$\int_{x_n}^{x_{n+k}} F(s)ds = h\sum_{i=0}^{k} \beta_i F_{n+i} + h\sum_{i=0}^{k} \beta_i F_{n+i+v_i} + R_n, \quad (29)$$
$$(|v_i| < 1; i = 0,1,\ldots,k).$$

This formula can be received by using the following way. One of the classical quadrature formulas for the computing of definite integrals can be written as:

$$\alpha \int_{x_n}^{x_{n+k}} F(s)ds = h\sum_{i=0}^{k} \bar{\beta}_i F_{n+i} + \alpha R_n^{(1)}. \quad (30)$$

It is not difficult to show that the Gauss quadrature formula applied to computing of definite integrals can be written as follows:

$$(1-\alpha)\int_{x_n}^{x_{n+k}} F(s)ds = h\sum_{i=0}^{k} \bar{\gamma}_i F_{n+l_i} + (1-\alpha)R_n^{(2)}, \quad (31)$$

here $l_i \ (i = 0,1,\ldots,k)$ are the Gauss nodes which can be written as $l_i = i + \nu_i \ (j = 0,1,...,k)$.

By summing the equalities (30) and (31), we receive the equality (29). If in the equality (2) put $y(x_0) = 0$ and $f(s, y) = F(s)$, then we get:

$$y(x) = \int_{x_0}^{x} F(s)ds. \tag{32}$$

By differentiation from this equality, we receive:

$$y'(x) = F(x), \ \ y(x_0) = 0, \ \ x \in [x_0, b]. \tag{33}$$

If apply the method (29) to solve the problem (33), then we obtain:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i F_{n+i} + h\sum_{i=0}^{k} \gamma_i F_{n+i+\nu_i} + R_n, \quad (|\nu_i| < 1; \quad i = 0,1,...,k). \tag{34}$$

As known by the order of the remainder term R_n, one can define the values of the degree for the method (34). But from the above described way, the order of accuracy for the R_n can be defined by the order of accuracy $R_n^{(1)}$ and $R_n^{(2)}$ so as $R_n = \overline{\alpha} R_n^{(1)} + (1-\overline{\alpha}) R_n^{(2)}$. It is not difficult to show that by choosing α the values of the order of accuracy for the R_n can be defined and corrected by the order of accuracy $R_n^{(1)}$ or $R_n^{(2)}$ (usually, it should be chosen with the least order of accuracy among these remaining terms).

Let us note that to define the values of R_n by the proposed method is not always correct. By using disadvantages of the above-mentioned methods, here to define the values of the coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,...,k)$ proposed to use the following way. To present this method, it is needed to use the following Teylor series:

$$\begin{aligned}
&y(x+ih) = y(x) + ihy'(x) + \\
&+\frac{(ih)^2}{2!} y''(x) + + ... + \frac{(ih)^p}{p!} y^{(p)}(x) + O(h^{p+1}) \\[2ex]
&y'(x+ih) = y'(x) + ihy''(x) + \\
&+\frac{(ih)^2}{2!} y'''(x) + ... + \frac{(ih)^{p-1}}{(p-1)!} y^{(p)}(x) + O(h^p), \\[2ex]
&y'(x+m_i h) = y'(x) + m_i hy''(x) + \\
&+\frac{(m_i h)^2}{2!} y'''(x) + .. + \frac{(m_i h)^p}{(p-1)!} y^{(p)}(x) + O(h^p), \\
&m_i = i + \nu_i (i = 0,1,...,k).
\end{aligned} \tag{35}$$

Let us suppose that method (34) has the degree of p. The degree for the method (34) defines by the analogic way applied to the method (16). Then by using the equalities (35) in the asymptotic equality for the degree of method (34), one can write:

$$\begin{aligned}
&\sum_{i=0}^{k} \alpha_i y(x) - h\sum_{i=0}^{k} (i\alpha_i + \beta_i + \gamma_i) y'(x) + \\
&- h^2 \sum_{i=0}^{k} (\frac{i^2}{2!}\alpha_i + i\beta_i + m_i\gamma_i) y''(x) + ... + \\
&- h^p \sum_{i=0}^{k} (\frac{i^p}{p!}\alpha_i + \frac{i^{p-1}}{(p-1)!}\beta_i + \frac{m_i^{p-1}}{(p-1)!}\gamma_i) y^{(p)}(x) = \\
&= O(h^{p+1}), \ h \to 0
\end{aligned} \tag{36}$$

By using linear independence of the system $1, h, h^2, ..., h^p$ or $y(x), y'(x), y''(x), ..., y^{(p)}(x)$ (in assumption that $y^{(j)}(x) \neq 0$ $(i = 0, k$, in . the asymptotic equality (36), we receive that the following must be hold:

$$\begin{aligned}
&\sum_{i=0}^{k} \alpha_i = 0; \ \sum_{i=0}^{k} (\beta_i + \gamma_i) = \sum_{i=0}^{k} i\alpha_i; \\
&\sum_{i=0}^{k} (\frac{i^2}{2!}\alpha_i + i\beta_i + m_i\gamma_i) = \sum_{i=0}^{k} \frac{i^2}{2!}\alpha_i; ... \ ; \\
&\sum_{i=0}^{k} (\frac{i^{p-1}}{(p-1)!}\beta_i + \frac{m_i^{p-1}}{(p-1)!}\gamma_i) = \sum_{i=0}^{k} \frac{i^p}{p!}\alpha_i.
\end{aligned} \tag{37}$$

By the above described way proved the following lemma:

Lemma: In order to the method of (34) has the degree of p, a necessary and sufficient condition satisfies its coefficients system of (37).

Thus receive the nonlinear homogeneous system of algebraic equations. In this system the amount of equations equal to $p+1$ but the amount of the unknowns equal to $4k+4$. If in this system put $\gamma_i = 0 \ (i = 0,1,...,k)$, then nonlinear system (37) transfer to the linear system of algebraic equations. As is known the received system for the value $p = 2k$ has the unique solution and there are methods with the degree $p < 2k$. Therefore some authors suppose that the system (37) has the unique solution in the case $p = 4k + 2$. This condition is hold for the value $k = 1$ and $k = 2$. Some time ago to solve the system (37) have been used the program MathCard-2015 and receive that the system (35) will be able to have the solution in the case when $p > 4k + 2$. This is available because the system (37) is nonlinear, and by the program MathCard-215 receive the approximately solution, but not exact solution. From the received results it follows that method (34) is more accurate than the following multistep second derivative methods:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}. \tag{38}$$

As is known if this method has the degree p, then

$p \le 3k+1$ (see for example [17]-[18],[22]-[26]).

Let us note that in the method (38) the coefficients are determined by the linear system. It is known that the implicit method presents both theoretical and practical interest. As is known if the method (38) is stable then there exist the methods with the degree $p = 3k+1$. If the method (29) is stable and has the degree p for the case when $\beta_i = 0\ (i = 0,1,...,k)$, then there are methods with the degree $p = 2k+2$. By simple comparison of the named methods receive that the method (29) is more perspective. Therefore let us consider the application of the method (29) to compute definite integrals and solving of the initial-value problem for the Volterra integro-differential equations. For this aim, let us write the method (34) in the following form:

$$y_{n+k} = \sum_{i=0}^{k-1} \bar{\alpha}_i y_{n+i} + h\sum_{i=0}^{k} \bar{\beta}_i y'_{n+i} + +h\sum_{i=0}^{k} \bar{\gamma}_i y'_{n+i+\nu_i}, \quad (|\nu_i| < 1;\ \ i = 0,1,...,k), \tag{39}$$

here $\bar{\alpha}_i = \alpha_i / \alpha_k \quad (i = 0,1,...,k-1)$ $\bar{\beta}_i = \beta_i / \alpha_k$, $\bar{\gamma}_i = \gamma_i / \alpha_k \quad (i = 0,1,...,k)$.

In the application of the method (39) to solve of the problem (33) are not arise any difficulties, so as $y'(x) = F(x)$, and $F(x)$ is known function. Note that these properties satisfy and to compute the values $F(x_n + (i+\nu_i)h)$.

As noted above, most difficulties arise in the calculation the values $\bar{\bar{y}}_{n+i+\nu_i}\ (0 \le i \le k)$. If the function $F(x)$ presented in the form $F(x,y)$, then for the calculation of the values $F_{n+i+\nu_i}$ it is necessary to find the values $y_{n+i+\nu_i}\ (i = 0,1,\ldots,k)$, which are not easy. Therefore application of the hybrid methods to solve initial value problem for ODE is more difficult.

As it follows from here, these difficulties can be arisen and also in solving of other problems. In addition, let us consider the application of method (39) to solve the problem (11). For this aim, we consider the application of method (39) to solve of the equation (12) which can be considered as the partial case of the Volterra integral equations. As a result, the application of the method (39) to solve of the problem (11) obtains (see for example [30], [35]):

$$y_{n+k} = \sum_{i=0}^{k-1} \bar{\alpha}_i y_{n+i} + h\sum_{i=0}^{k} \bar{\beta}_i (\varphi_{n+i} + v_{n+i}) + +h\sum_{i=0}^{k} \bar{\gamma}_i (\varphi_{n+i+\nu_i} + v_{n+i+\nu_i})\ , \tag{40}$$

$$v_{n+k} = \sum_{i=0}^{k-1} \bar{\alpha}_i v_{n+i} + h\sum_{i=0}^{k}\sum_{j=i}^{k} \bar{\beta}_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) + +h\sum_{i=0}^{k}\sum_{j=i}^{k} \bar{\gamma}_i^{(j)} K(x_{n+j+\nu_j}, x_{n+i+\nu_i}, y_{n+i+\nu_i}). \tag{41}$$

It is not difficult to verify that if the values $y_l, v_l\ (l = 0,1,...,k-1)$ are known, then by using methods (40) and (41) one can find the values $y_{k+l}, v_{k+l}\ (l = 0,1,...,N-k)$. It follows that to use the above-mentioned methods it is needed to use some methods for calculation of the values $y_{n+i+\nu_i}$ and $v_{n+i+\nu_i}\ (i \ge 0)$. To be freed from these difficulties, one can choose the values of $\nu_i\ (0 \le i \le k)$ in the form $\nu_i = m_i / t_i\ (0 \le i \le k)$. Here m and t are integer values, therefore ν_i will be functional. But in this case, the order of accuracy (values for the degree) for the given method will be decreasing.

If compare the above-mentioned methods then receive that all methods which have been applied to solve of above considered problems are one and the same without the method (41). In the construction of the method (41) one can take $j = k$ and in this case receive that the method (41) is the same with the method (40). It follows from here that the method of type (40) can be constructed by the same way which has used for the construction of methods of the type (39).

And now note that for fundamental investigation of above-mentioned problems it is need to impose some conditions on the coefficients of the used methods. Similar conditions for the coefficients of the method (4) have been received in the [14] and that called as the A,B,C conditions. These conditions for the method (39) can be presented as the following:

The coefficients $\alpha_i, \bar{\beta}_i, \bar{\gamma}_i, \nu_i\ (i = 0,1,...,k)$ are some real numbers and $\alpha_k \ne 0$.

A. Characteristic polynomials: $\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i$,

$$\sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i; \qquad \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}$$

have no common factor different from the constant.

B. The conditions $\sigma(1) + \gamma(1) \ne 0$ and $p \ge 1$ are hold.

Let us note that all properties of the methods (34) and (39) are the same. Therefore, to determine the values of its coefficients, let us use the same way. In our case for construction of the methods of the types (34) and (41), one can use the system (37) and the following systems:

$$\sum_{j=i}^{k} \beta_i^{(j)} = \beta_i; \sum_{j=i}^{k} \gamma_i^{(j)} = \gamma_i \quad (i = 0,1,...,k). \tag{42}$$

By using the above described way, let us construct the stable methods to solve the above-mentioned problems. And let us began to construct those methods by using the problems (1) and (11).

As noted above, the domain in which all above-mentioned problems are equivalent and has been shown that in outside of that domain the problems are equivalent with some errors which can be estimated. By using the last properties, the constructed method by which can be solved all above-mentioned problems by some errors. For the simplicity, let us put $k=1$. In this case, by using the solution of the system (37), one can construct the following methods:

$$y_{n+1} = y_n + h(y'_{n+\alpha} + y'_{n+1-\alpha})/2;\ \alpha = 1/2 - \sqrt{3}/6, \qquad (43)$$

$$\begin{aligned} y_{n+1} &= y_n + h(y'_n + y'_{n+1})/12 + \\ &+5h(y'_{n+\beta} + y'_{n+1-\beta})/12, \beta = 1/2 - \sqrt{5}/10. \end{aligned} \qquad (44)$$

It is not difficult to show that there exists a stable method of the type (34) with the degree $p_{\max} = 6$, for the value $k=1$ which can be defined by the formula (44). Now, let us construct more exact stable method. For this aim, we put $k=2$. As was noted above the system (35) is nonlinear, therefore the finding its exact solutions is difficult. For the simplicity here assume that $\gamma_1 = 0$ and $\nu_1 = 1$. In this case, the stable method with the degree $p=8$ can be written as follows:

$$\begin{aligned} y_{i+2} &= y_i + h(9y'_{i+2} + 64y'_{i+1} + 9y'_i) \\ &/90 + 49h(y'_{i+1+\alpha} + y'_{i+1-\alpha})/90, 7\alpha = \sqrt{21}/7. \end{aligned} \qquad (45)$$

It can be argued that the method of (45) is symmetric, but someone could argue that it can not be symmetric. If the method of (45) can be written in the following form:

$$y_{i+2} = y_i + \frac{h(9y'_{i+2} + 32y'_{i+1} + 9y'_i)}{90} + \\ + \frac{h(49y'_{i+1+a} + 32y'_{i+1} + 49y'_{i+1-a})}{90}$$

,then we receive that this method can be taken as symmetric.

This method is two-step method which can be written as the one-step method in the following form:

$$\begin{aligned} y_{i+1} &= y_i + 9h(y'_{i+1} + y'_i)/90 + \\ &+h(49y'_{i+1/2+\beta} + 64y'_{i+1/2} + 49y'_{i+1/2-\beta})/90, \\ \beta &= \sqrt{21}/14. \end{aligned} \qquad (46)$$

This is one-step method with the degree $p=8$. In the class of one-step methods, that can be taken as the one of the best methods.

Let us note that in the application of the constructed here methods of type (39) do not arise some difficulties. Let us consider application of method of (43) to solve problem (11). In this case, we receive:

$$y_{n+1} = y_n + \frac{h(\varphi_{n+a} + \varphi_{n+1-a})}{2} + \frac{h(\vartheta_{n+a} + \vartheta_{n+1-a})}{2}.$$

Here, $\varphi_m = \varphi(x_m, y_m),\ \nu_m = \nu(x_m),\ (m \ge 0)$.

By applying this method to solve problem (11) that arises some difficulties related with the calculation of values $\bar{y}(x_{n+i+v_i})\ \ (i = 0,1,..,k;\ |v_i| < 1)$ for which one can use any methods constructed by the help of the known methods as the trapezoidal rule, Simpson method, and others.

It is not difficult to prove that methods of (43)-(46) are the symmetric. It is noted that method (43) remains the corresponding Gauss method. The other coefficients of methods also satisfy the Gauss conditions. Now, let us consider calculation of double integral.

Methods (43)-(46) can be taken as the symmetric methods and the known Gauss and Chebishev methods are also can be taken as symmetric. Method (45) is symmetric priory of Dahlquist.

IV. ON THE CALCULATION OF DOUBLE INTEGRALS

It is known that with the calculation of the double integral (9) encounter that to find the volume of some geometric figures (see [27 ,p.22]). For this aim, we use the function (13). The investigation of the function (13), bind with the following hyperbolic equation (see [28, p.147]):

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} = f(x,y),\ a \le x \le b,\ c \le y \le d. \qquad (47)$$

Thus, we receive that calculation of the double integral (9) is equivalent to solve the equation of (47).

For the investigation of the double integral (9), one can use the solution of the following problem:

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} = f(x,y),\ U(a,y) = U(x,c) = 0$$

$$a \le x \le b,\ c \le y \le d.$$

As known that the double definite integral can be calculated by the following formula (see [5, p.147]):

$$\begin{aligned} \int_a^b \int_c^d f(s,t)\, ds dt &= U(b,d) - \\ &-U(a,d) - U(b,c) + U(a,c). \end{aligned} \qquad (48)$$

One of the popular method for solving hyperbolic equation is the finite-difference method. If we appy simple finite difference method to solve equation (47), then we receive:

$$\left.\frac{\partial^2 U}{\partial x \partial y}\right|_{\substack{x=x_i \\ y=y_j}} = \frac{U_{i,j} - U_{i-1,j} + U_{i-1,j-1}}{h\tau}. \qquad (49)$$

If in the construction of finite-difference method suppose that $h = \tau = 1$, and $x_{i-1} = a, b = x_i,\ \ c = y_{i-1}$ and $d = y_j$, then receive that method (48) and (49) are the same. But if there for approximation of the equation (47) used more exact formula, then the above properties would not take a place and in the results of which we would receive some system of algebraic equation. It should be noted that to determine the solution of such systems of algebraic equation perhaps but not always. Therefore here proposed very simple method for calculation the values of double integral (9). For this aim, let us consider the following integral:

$$F(x,y) = \int_c^y f(x,t)\, dt. \qquad (50)$$

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The function $F_y^{'}(x,y)$ by using equality (50) can be written as:

$$F_y^{'}(x,y)\Big|_{x=x_1} = f(x_i, y),\ F(x_i,c)=0,\ i=0,1,2,...,n\,, \tag{51}$$

$$U_x^{'}(x,d) = F(x,d),\ U(a,d)=0,\ x_0 = a\,. \tag{52}$$

By solving these systems one can be find the value $U(b,d)$. Noted that all the equations of these systems are separated. Therefore we can investigate the system of (51) and (52) for the fixed value of the variable i. For example, let us consider the following method:

$$\sum_{i=0}^{k}\alpha_i F(x_j, y_{n+i}) = h\sum_{i=0}^{k}\beta_i f(x_j, y_{n+i}) + +h\sum_{i=0}^{k}\gamma_i f(x_j, y_{n+i+\nu_i}). \tag{53}$$

This method is the same as the method (39) and one can be taken that as the generalization of the method (27).

Let's note that, Gauss, Chebyshev, and many other methods can be received from the method (53), but it's impossible to receive some of them from the method (27) as a partial case. Therefore, numerical methods of type (53) can be taken as a perspective.

By comparison, the conception of stability and degree for the linear multistep methods with constant coefficients and for the method of (53) receive that the above mentioned conceptions are defined by one and the same way. By using this approach, one can prove that the conception of stability is necessary and sufficient condition for its convergence. One of the main problems of Modern Computational Mathematics is the construction of the stable methods with the higher order of accuracy. By taking into account this, many specialists have proposed to construct methods of hybrid type. It is clear that like other methods the hybrid methods also have some advantages and disadvantages. There are different way for solving named problem. One of them is the constructed methods for the calculation values of the type $y_{n+i+\nu_i}$ $(0 \le i \le k)$. If κ- satisfies the condition $\kappa \ge 3$, then in construction of such formulas, some difficulties are arisen.

Therefore, we use some known methods having the type (4). For example, let us construct method for calculation of the value $y_{n+1/2}$ with the rate of approach $O(h^2)$, one can be used the explicit Euler or trapezoidal method. If applied proposed here way to calculation value $y_{n+1/2}$, for the calculation that receive following formulas:

$$y_{n+\frac{1}{2}} = y_n + \frac{hy_n^{'}}{2};\ \ y_{n+\frac{1}{2}} = y_n + \frac{h\left(y_n^{'} + y_{n+\frac{1}{2}}^{'}\right)}{2}\,. \tag{54}$$

This way has used for calculation y_{n+1} by application of Simpson's method, but trapezoidal method had used in application of method (43) to solve some concrete example. Let us note that the Simpson method in this case can be written in the following form:

$$y_{n+1} = y_n + \frac{h\left(y_n^{'} + 4y_{n+\frac{1}{2}}^{'} + y_{n+1}^{'}\right)}{2}.$$

For application of this method to solve some problems, one can recommend method (54).

V. Numerical results

For the illustration of our obtained numerical results, let us consider following example:

$$y' = \lambda y(x) - m(1-\exp(-\lambda x)) + \lambda m\int_0^x y(-s)ds\,.$$

The exact solution of this problem can be written as:

$$y(x) = \exp(\lambda x)\,.$$

Suppose that solution of initial value problem (55) defined on the interval [1,2], but results obtained on the segment [0,1] are not present an interest. Therefore, we determine the numerical solution of the named problem on the interval [1,2] Here, we also have considered to calculate the following integrals:

$\int_0^1 \exp(s)ds$ and $\int_0^1\int_0^1 \exp(s+t)dsdt$ and also solving the following integral equation

$$y(x) = 1 + \lambda\int_0^1 y(s)ds,\ \ 0 \le x \le 1\,.$$

It is not difficult to understand that in the case $m=0$ from the example, we receive the initial-value problem for the ODE of the first order. The results have been obtained from the application of the method (43) to solving of our problem that have been tabulated in Table 1 and Table 2:

Table 1. For the case $h=0.1$ and $\lambda = \pm 1$.

x_i	$\lambda=1$, $m=1$	$\lambda=1$, $m=0$	$\lambda=-1$, $m=1$	$\lambda=-1$, $m=0$
1.1	1.51 E-7	1.58E-7	7.36E-8	2.06E-8
1.4	8.95E-8	8.90E-7	5.70E-6	5.86E-8
1.7	7.40E-7	2.11E-6	2.009E-5	7.55E-8
2.0	2.26E-6	4.08E-6	4.45E-5	7.97E-8

From the above table content, our obtained results for ODE are better than the results obtained for the Volterra Integro-differential equations, which are related to the calculation of the integral participated in the problem of (55).

Table 2. For the case $h=0.1$ and $\lambda = \pm 5$.

x_i	$\lambda=5$, $m=1$	$\lambda=5$, $m=0$	$\lambda=-5$, $m=1$	$\lambda=-5$, $m=0$
1.1	4.30E-6	4.30E-6	1.45E-5	7.42E-11
1.4	7.73E-5	7.73E-5	3,17E-4	6.61E-11
1.7	6,06E-4	6,06E-4	1,92E-3	2.58E-11
2.0	3,88E-3	3,88E-3	9,05E-3	8.22E-12

This example has been solved in the work [28] for the case $m=0$ and $m \neq 0$ by using the forward-jumping method with the degree $p=5$, which is asymmetric To illustrate our results received here by the implicit method (43) that has been applied to solve the following Volterra integral equations:

$$y(x)=1+\lambda \int_0^x y(s)ds, \ 0 \leq x \leq 1.$$

Let us note that the method (43) is one-step and has the degree $p=4$. The results received here have been tabulated in Table 3.

Table 3. For the cases $h=0.1$ and $h=0.01$

x_i	$h=0.1$, $m=1$	$h=0.1$, $m=-1$	$h=0.01$, $m=5$	$h=0.01$, $m=-5$
1.1	1.2E-4	1.65E-5	6,12E-4	1.10E-8
1.4	2.18E-4	1.55E-5	1,09E-2	9.88E-9
1.7	3.57E-4	1.40E-5	8.60E-2	3.86E-9
2.0	5.68E-4	1.22E-5	5.51E-1	1.23E-9

For comparison of all results received by using the method (43), let us apply the method (43) to compute the following definite integral:

$$I=\int_0^1 \exp(s)ds.$$

The received results in the calculation of definite integral for $h=0{,}01$ can be written as follows: $1.40521816405226\text{E-}10$.

Let us now consider calculation of following definite integral:

$$\int_0^1\int_0^1 \exp(s+t)dsdt$$

by using method (44), for the step-size $h=0{,}05$. The error can be presented as $\varepsilon=6{,}17E-14$ for the $h=0{,}05$, but for the step-size $h=0{,}1$ the error can be presented as $1{,}4E-10$.

Noted that symmetric method can be constructed and also by using the forward-jumping (advanced) methods. For example the following method can be included to class of symmetric methods:

$$y_{n+1}=y_n+\frac{h\left(8y'_{n+1}+5y'_n-y'_{n+2}\right)}{12}.$$

For the determined the value y_{n+1}, the necessity of using the values y_n and y_{n+2} are arisen which are located symmetrically to y_{n+1}, therefore methods can be taken as symmetric.

Noted that for the application of these methods to solve fructional derivatives and integral equations that can be modified. In this case, there are known works of different authors (see for example [37-39]). In the recent time, one of the basic question related to this study, the construction of regression algorithm and investigation risk factor of sustainability (see for example [40,41]).

VI. Conclusion

Here have been proven that the domain of the solution of the initial-value problem for ODE and Volterra integro-differential equation, and the domain of the solution of Volterra integral equation has an intersection different from zero. Have shown that here are indefinite functions and the domain of existence of which has also been included in the above-described intersection of domains. It is not difficult to obtain from here, that there exist some domains in which the above-mentioned four problems can be solved by the same method. These methods can be taken as the class of multistep methods which have been applied to solve ODEs. In order to solve the integral equations with the help of stable multistep methods with an extended region of stability, some modification of the above-mentioned methods can be applied (by using the system (40)). The maximal value of the degrees has been defined for the stable and unstable multistep methods which are recommended in order to solve the above-mentioned problems. By taking into account that the hybrid methods are more exact, here have been constructed the methods on the intersection of the multistep and hybrid methods. Have been defined the advantages of these methods. Identified some connection between Gauss and constructed here methods. By the development of the theory of Dahlquist here have been received some connections between the degree and the order for the proposed methods. The constructed concrete algorithm, which has been applied to solve some model problems, has the same solution. The obtained results are in agreement with the theoretical ones. It should be noted that the proposed methods in this work are interesting for a wide class of specialists, and hence, they are perspective.

Note that in [36], an interesting result has been obtained for solving the initial-value problem for differential equation [37], which has investigated the calculation of definite integrals in comparison with the known quadrature methods. And in [30], important research has been done for the calculation and application of definite integrals for computation of the geometric figures and many applied foundations.

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