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$\mathbb{NP}$-Hardness of Largest Contained and Smallest Containing Simplices for *V*- and *H*-Polytopes

Asa Packer

Department of Mathematics, Box 354350, University of Washington,
Seattle, WA 98195-4350, USA
packer@math.washington.edu

Abstract. The problem of finding a d-simplex of maximum volume in an arbitrary d-dimensional V-polytope, for arbitrary d, was shown by Gritzmann et al. [GKL] in 1995 to be $\mathbb{NP}$-hard. They conjectured that the corresponding problem for H-polytopes is also $\mathbb{NP}$-hard. This paper presents a unified way of proving the $\mathbb{NP}$-hardness of both these problems. The approach also yields $\mathbb{NP}$-hardness proofs for the problems of finding d-simplices of minimum volume containing d-dimensional V- or H-polytopes. The polytopes that play the key role in the hardness proofs are truncations of simplices. A construction is presented which associates a truncated simplex to a given directed graph, and the hardness results follow from the hardness of detecting whether a directed graph has a partition into directed triangles.

1. Introduction

This paper examines the computational difficulty of finding a simplex of maximum volume contained in a given convex polytope or finding a simplex of minimum volume containing a given polytope. These are basic problems in *computational convexity*, an area of mathematics that is concerned with the algorithmic aspects of polytopes and more general convex bodies. Because of various applications to the sciences and other areas of mathematics, the emphasis here is on the case where the dimension is variable. The paper [GKL] contains many results about largest simplices in polytopes. In particular, it includes an $\mathbb{NP}$-hardness result for one of the versions of the largest contained simplex problem considered here, and a conjecture of $\mathbb{NP}$-hardness for another version. The present paper presents a unified way of proving $\mathbb{NP}$-hardness for both versions of the largest contained simplex problem, as well as the analogous results for the smallest containing simplex problem. For a survey of known results about these and related *containment problems* in computational convexity, including some applications, see [GK2].

Another application of smallest containing simplices, which is not mentioned in [GK2], is the so-called *unmixing problem*. In this situation, one has a collection of data points in a high-dimensional space, and there is reason to think that, except for noise, the data points are all convex combinations of some unknown number m of unknown points, called *endmembers*. Thus the cloud of data points should ideally have the shape of an $(m-1)$-dimensional simplex, with the endmembers being the vertices of the simplex. The unmixing problem is to determine the endmembers from the data points. The solution process involves fitting an $(m-1)$-dimensional subspace to the data points, projecting the data points to that subspace, and then finding an $(m-1)$-dimensional simplex in the subspace that somehow approximates the shape of the projection of the data cloud. There are statistical tests one can apply to determine which guess for the value of m produces the most plausible results for a given data set. The simplex one chooses should in some sense be the "best" approximation of the cloud of projected data points, and a reasonable choice is a simplex of minimum volume containing the points. Since this is difficult to find in general, heuristic procedures are used to find simplices whose volume is close to minimal. The unmixing problem occurs in various fields. For example, the data points can represent the concentrations of various substances in seafloor sediment [Re] or the atmosphere [Wo], or the reflectances in various wavelengths of points on the surface of the earth [BG], [Go].

This paper is organized as follows. Section 2 presents some definitions and gives a precise statement of the main results of the paper. Section 3 gives an informal description of the construction used in the proofs, and some of the main steps of the arguments. Section 4 collects some background results, most of which have been proved elsewhere, that are used throughout the paper. The polytopes that play the key role in the hardness proofs are truncated simplices. We define truncated simplices in Section 5 and prove some basic facts about largest full-dimensional simplices in truncated simplices. Section 6 precisely describes the main construction, which, given a directed graph, produces a truncated simplex T such that the volume of the largest full-dimensional simplices in T reflects some aspects of the structure of the given graph. Section 7 contains some results that relate the largest contained simplices for a polytope with the smallest containing "simplicial cylinders" for the polar polytope. In Section 8 we use all these ingredients to prove the main $\mathbb{NP}$-hardness results.

2. Definitions and Statements of Main Results

Everything in this paper takes place in real Euclidean space $\mathbb{R}^d$ for some d in the set $\mathbb{N}$ of positive integers. We think of elements of $\mathbb{R}^d$ as column vectors and denote them by bold letters, e.g., $\mathbf{x} = (x_1, x_2, \ldots, x_d)^{\mathrm{T}} \in \mathbb{R}^d$. For $1 \le i \le d$, let $\mathbf{e}_i$ denote the vector whose ith entry is 1 and all of whose other entries are 0 (the dimension of the ambient space will always be clear from context). The Euclidean norm of a vector $\mathbf{x}$ will be denoted $\|\mathbf{x}\|$. Let $\mathbb{B}_d := \{\mathbf{x} \in \mathbb{R}^d\colon \|\mathbf{x}\| \le 1\}$ denote the closed unit ball of $\mathbb{R}^d$. The subscript will be omitted if there is no risk of confusion. For $n \in \mathbb{N}$, define $[n] := \{1, \ldots, n\}$. For $a, b \in \mathbb{N}$, let $K^{a\times b}$ denote the set of $a \times b$ matrices with entries in a set K. For a matrix M, let M^{T} denote the transpose of M and $M^{-\mathrm{T}}$ denote the transpose of M^{-1}. For $n \in \mathbb{N}$, let I_n denote the $n \times n$ identity matrix and J_n denote the $n \times n$ matrix all of whose entries are 1. The subscripts will be omitted if there is no risk of confusion.

For $X \subset \mathbb{R}^d$, the standard notations $\mathrm{conv}(X)$, $\mathrm{aff}(X)$, and $\mathrm{lin}(X)$ will be used to represent, respectively, the convex hull, affine hull, and linear hull of X. If $V \in \mathbb{R}^{d\times n}$, and $Y \subset \mathbb{R}^d$ is the set of points that appear as columns of V, then define $\mathrm{conv}(V) := \mathrm{conv}(Y)$, $\mathrm{aff}(V) := \mathrm{aff}(Y)$, and $\mathrm{lin}(V) = \mathrm{lin}(Y)$. A *$j$-flat* in $\mathbb{R}^d$ is the affine hull of $j+1$ affinely independent points of $\mathbb{R}^d$. If $X \subset \mathbb{R}^d$ is a compact convex set and $\mathrm{aff}(X)$ is a j-flat, then the set X is said to be j-dimensional and we write $\dim(X) = j$. Define $\dim(\emptyset) = -1$. For a j-dimensional compact convex set X, let $\mathrm{vol}(X)$ denote the j-dimensional Lebesgue measure of X. Let $\mathrm{relint}(X)$ denote the relative interior of X, i.e., the interior of X considered as a subset of the topological space $\mathrm{aff}(X)$. For $X, Y \in \mathbb{R}^d$, define

$$\mathrm{dist}(X, Y) := \inf_{\mathbf{x}\in X,\mathbf{y}\in Y} \|\mathbf{x}-\mathbf{y}\|.$$

For $X, Y \subset \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, define

$$X + Y := \{\mathbf{x}+\mathbf{y}\colon\ \mathbf{x} \in X, \mathbf{y} \in Y\}$$

and

$$\alpha X := \{\alpha\mathbf{x}\colon\ \mathbf{x} \in X\}.$$

For any $X \subset \mathbb{R}^d$, X^Δ denotes the polar set of X, i.e.,

$$X^\Delta := \{\mathbf{y} \in \mathbb{R}^d\colon\ \mathbf{y}^{\mathrm{T}}\mathbf{x} \le 1 \text{ for all } \mathbf{x} \in X\}.$$

If $j, d \in \mathbb{N}$ with $j \le d$, a *j-simplex* in $\mathbb{R}^d$ is the convex hull of $j+1$ affinely independent points of $\mathbb{R}^d$. Relative to some fixed polytope P, a j-simplex $S \subset P$ is *largest* if it has maximum volume among all j-simplices contained in P. The simplex S is *bound* to P if all of the vertices of S are also vertices of P.

If $j, d \in \mathbb{N}$ with $j \le d$, a *j-simplicial cylinder* C in $\mathbb{R}^d$ is a set of the form $C = S+L$, where S is a j-simplex with $\mathbf{0} \in \mathrm{aff}(S)$ and $L = \mathrm{aff}(S)^\perp$. The *cross-sectional volume* of C, denoted $\mathrm{vol}(C)$, is defined to be the j-dimensional volume of S. Relative to some fixed polytope P, a j-simplicial cylinder $C \supset P$ is *smallest* if it has minimum cross-sectional volume among all j-simplicial cylinders containing P. Note that a d-simplicial cylinder in $\mathbb{R}^d$ is just a d-simplex.

We assume all the standard notions having to do with $\mathbb{NP}$-hardness (see, for example, [GJ]). Our model of computation is the standard binary Turing machine. The containment problems we consider are naturally thought of as optimization problems, but we use the standard technique of phrasing them as decision problems when considering $\mathbb{NP}$-hardness questions. For example, instead of asking for a largest simplex in a polytope, we append a number λ to the input and ask if there is a simplex of volume at least λ.

Since the problems considered here involve polytopes of arbitrary dimension, it is necessary to distinguish two different ways of encoding a polytope as the input to a Turing machine. A *V-polytope* is a polytope given by a list of its vertices, and an *H-polytope* is one given by a list of its facet-defining linear inequalities. For a particular polytope the size of its encodings as a V-polytope and as an H-polytope can be quite different. For example, the d-dimensional cube has 2^d vertices and only $2d$ facets. This difference in size causes some problems to be $\mathbb{NP}$-hard for V-polytopes but not for H-polytopes, or vice versa. We consider only *rational* V-polytopes and H-polytopes, that is, all the coordinates of the vertices or coefficients of the inequalities must be rational numbers. This is because rational numbers are easy to encode as input to a Turing machine. We also work with squared volumes instead of volumes, since the squared volume of a rational simplex is rational, and with squared cross-sectional volumes of simplicial cylinders. Our restriction to rational polytopes does not allow us to include all possible types of polytopes (see, for example, [Zi] for "nonrational" polytopes), but is sufficient for showing that problems are hard.

If $f, g\colon \mathbb{N} \to \mathbb{N}$, we say that $f(d) = \Omega(g(d))$ if there is some real number $c_1 > 0$ such that $f(k) \geq c_1 g(k)$ for all $k \in \mathbb{N}$. We say that $f(d) = O(g(d))$ if there is some real number $c_2 > 0$ such that $f(k) \leq c_2 g(k)$ for all $k \in \mathbb{N}$. If $f(d) = \Omega(g(d))$ and $f(d) = O(g(d))$, then we say that $f(d) = \Theta(g(d))$.

The main result of this paper is that the following four decision problems are all $\mathbb{NP}$-hard when the function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies the following conditions: $f(d) \leq d$ for all $d \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $f(d) = \Omega(d^{1/k})$, and $f(d)$ is computable in time polynomial in d.

HLGSTSIMPLEX$_f$

Instance: $d \in \mathbb{N}$, a d-dimensional rational H-polytope $P \subset \mathbb{R}^d$, and $\lambda \in \mathbb{Q}$ with $\lambda > 0$.
Question: Is there an $f(d)$-simplex $S \subset P$ with $\mathrm{vol}^2(S) \geq \lambda$?

VLGSTSIMPLEX$_f$

Instance: $d \in \mathbb{N}$, a d-dimensional rational V-polytope $P \subset \mathbb{R}^d$, and $\lambda \in \mathbb{Q}$ with $\lambda > 0$.
Question: Is there an $f(d)$-simplex $S \subset P$ with $\mathrm{vol}^2(S) \geq \lambda$?

HSMLSTSIMPCYL$_f$

Instance: $d \in \mathbb{N}$, a d-dimensional rational H-polytope $P \subset \mathbb{R}^d$, and $\lambda \in \mathbb{Q}$ with $\lambda > 0$.
Question: Is there an $f(d)$-simplicial cylinder $C \supset P$ with $\mathrm{vol}^2(C) \leq \lambda$?

VSMLSTSIMPCYL$_f$

Instance: $d \in \mathbb{N}$, a d-dimensional rational V-polytope $P \subset \mathbb{R}^d$, and $\lambda \in \mathbb{Q}$ with $\lambda > 0$.
Question: Is there an $f(d)$-simplicial cylinder $C \supset P$ with $\mathrm{vol}^2(C) \leq \lambda$?

Our assumption that $f(d) = \Omega(d^{1/k})$ for some k is the most general under which we have been able to prove $\mathbb{NP}$-hardness for these problems. The most important case where these results apply is when $f(d) = d$. At the other extreme, when $f(d)$ is bounded by a constant, the problem VLGSTSIMPLEX$_f$ can actually be solved in polynomial time (by evaluating the volumes of all possible bound simplices). In contrast, even in the case where $f(d) = 1$ for all d and the only polytopes considered are parallelotopes, the problem HLGSTSIMPLEX$_f$ is $\mathbb{NP}$-hard—this was shown in [GK1]. We would like to know how fast $f(d)$ must grow in order for VLGSTSIMPLEX$_f$ to be $\mathbb{NP}$-hard. For example, if $f(d) = \Theta(\log(d))$, then there is no obvious polynomial-time algorithm, but we also have no proof of $\mathbb{NP}$-hardness.

The problems VLgstSimplex$_f$ and HLgstSimplex$_f$ are known to be in $\mathbb{NP}$. The problems HSmlstSimpCyl$_f$ and VSmlstSimpCyl$_f$, however, are not known to be in $\mathbb{NP}$.

The hardness result for VLgstSimplex$_f$ for the above class of functions f was first proven in [GKL]. The authors conjectured the hardness of HLgstSimplex$_f$, in particular in the case $f(d) = d$. Another proof of the hardness of VLgstSimplex$_f$, under the additional assumption that $f(d) < d$ for all $d \in \mathbb{N}$, can be found in [Pa2]. In that paper it is shown that with that additional assumption on f, the problem VLgstSimplex$_f$ remains $\mathbb{NP}$-hard even if the input V-polytope is required to be an affine crosspolytope.

In light of the above hardness results, it is natural to study the possible accuracy of polynomial-time approximation algorithms for finding largest contained simplices or smallest containing simplicial cylinders. The paper [BGK2] studies polynomial-time approximation of largest simplices in general convex bodies, not just polytopes. The authors use a more general model of computation where the convex bodies are specified by certain oracles. Their inapproximability results do not apply to the problems considered here, since presenting a polytope as a V- or H-polytope gives more information about it than presenting it via the oracles they use. However, the approximation algorithm in their paper does apply. It gives a polynomial-time algorithm for finding largest j-simplices in d-polytopes that finds simplices whose squared volume is within a factor of $O((cd)^{2j})$ of optimal, for some constant c. The paper [BGK1] applies similar methods to the problem of finding smallest simplicial cylinders containing general convex bodies. On the negative side, the paper [Pa2] shows that, if $f(d) < d$ for all d, no polynomial-time algorithm can approximately solve the optimization version of VLgstSimplex$_f$, even just for cross-polytopes, within a factor of less than 1.09. It also contains a simple polynomial-time algorithm for finding approximately largest j-simplices in d-dimensional V-polytopes. The algorithm finds simplices whose squared volume is within a factor of $O((cj)^j)$ of optimal, for some constant c.

We note that finding largest d-dimensional simplices in d-cubes subsumes the famous problem on the existence of Hadamard matrices [HKL]. However, the problem of finding smallest d-simplices containing d-parallelotopes has been completely solved (see [La] and [LZM]). There has also been work done on finding largest contained simplices and smallest containing simplices for polytopes of fixed low dimension. Klee [Kl] showed for any d that if S is a d-simplex of minimum volume enclosing a d-dimensional convex body C, then the centroid of each facet of S belongs to C. In [KL] Klee and Laskowski used this to devise an $O(n \log^2(n))$-time algorithm for finding the triangles of minimum area containing a given convex n-gon. This was improved to $O(n)$ by O'Rourke et al. [OAMB]. Chandran and Mount [CM] found a connection between the largest triangles contained in a convex n-gon and the smallest triangles containing the n-gon. They used this to come up with an $O(n)$-time algorithm that finds both largest contained triangles and smallest containing triangles for n-gons. Vegter and Yap [VY] found some conditions that must be satisfied by minimum volume tetrahedra containing 3-polytopes. Zhou and Suri [ZS] have a $\Theta(n^4)$-time algorithm for finding a tetrahedron of minimum volume enclosing a polytope $P \subset \mathbb{R}^3$ with n vertices. They also have an $O(n + 1/\varepsilon^6)$-time algorithm that produces a tetrahedron whose volume is within a factor of $1 + \varepsilon$ of optimal, for any $\varepsilon > 0$.

3. Description of Construction

In this section we give an informal description of the construction used in the rest of the paper, and some of the main steps in proving the hardness results.

The polytopes that play the pivotal role in the construction are *truncated d-simplices*. A truncated d-simplex is formed from a d-simplex by cutting along a hyperplane near each vertex. For any distinct vertices $\mathbf{v}_i, \mathbf{v}_j$ of the original simplex, there is a unique vertex $\mathbf{w}_{ij}$ of the truncated simplex that is near $\mathbf{v}_i$ and on the line segment between $\mathbf{v}_i$ and $\mathbf{v}_j$ (the notation $\mathbf{w}_{ij}$ is used in this section only).

The construction starts with a directed graph G on the vertex set $\{1, \dots, n\}$ for some n, and an $(n-1)$-simplex S. The choice of S is a technical detail—it ensures that certain quantities that must be input to a Turing machine are rational, and that we can easily compute the size of a ball around the origin that will fit inside the truncated simplex we end up with. The construction produces a truncated $(n-1)$-simplex P that is a truncation of S. Let ε be some small positive number. The idea is that if (i, j) is an edge of G, then the vertex $\mathbf{w}_{ij}$ is defined by

$$\frac{\|\mathbf{v}_i - \mathbf{w}_{ij}\|}{\|\mathbf{v}_i - \mathbf{v}_j\|} = \varepsilon,$$

and if (i, j) is not an edge of G, $\mathbf{w}_{ij}$ is defined by

$$\frac{\|\mathbf{v}_i - \mathbf{w}_{ij}\|}{\|\mathbf{v}_i - \mathbf{v}_j\|} = 2\varepsilon.$$

See Fig. 1 for an example with $n = 3$.

Let T be a bound largest $(n-1)$-simplex in P, and suppose ε is "sufficiently small." Two important facts about T are intuitively fairly clear, although somewhat tedious to prove. These two facts are the content of Theorem 5.6. The first is that for each $i \in \{1, \dots, n\}$, T has exactly one vertex that is equal to $\mathbf{w}_{ij}$ for some j. If this were not true, T would have two vertices that were equal to $\mathbf{w}_{ab}$, $\mathbf{w}_{ac}$ for some a and some $b \neq c$. Since ε is small, these two vertices would be extremely close together, and the volume of T would thus be very small.

The second part of Theorem 5.6 states that for each $i \in \{1, \dots, n\}$, T has a vertex that is equal to $\mathbf{w}_{ij}$ for some j such that (i, j) is an edge of G, unless there is no such

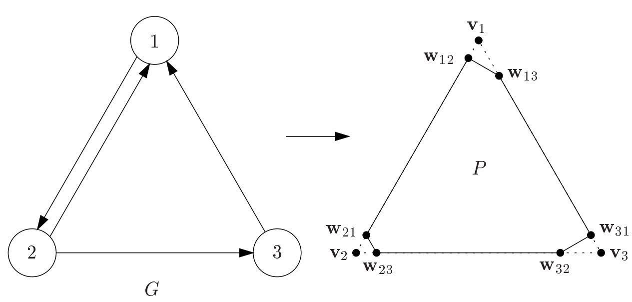

Fig. 1. An example of the construction.

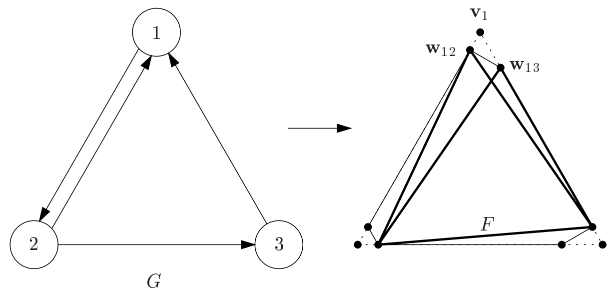

Fig. 2. The second part of Theorem 5.6.

j. The idea of the proof of this, focusing on $i = 1$, can be seen in Fig. 2. The two highlighted triangles share a facet F opposite $\mathbf{v}_1$. Because ε is small, this facet must be very close to horizontal. Since ε and 2ε have a ratio that is different enough from 1, the hyperplane defined by $\mathbf{w}_{12}$ and $\mathbf{w}_{13}$ is far from horizontal. Therefore, the vertex $\mathbf{w}_{12}$, which corresponds to an edge present in G, is further away from F than $\mathbf{w}_{13}$, which corresponds to an edge absent from G. Thus the triangle including $\mathbf{w}_{13}$ cannot be largest.

The next important fact concerns the bound largest $(n-1)$-simplices in P when G is the complete directed graph $\vec{K}_n$. In the case $n = 3$, a simple examination of all the possibilities shows that there are two largest simplices, and they correspond to the two different directed 3-cycles that exist in $\vec{K}_3$ (see Fig. 3). One might at first suspect that in the general case the largest simplices in P would correspond to directed n-cycles, but in fact that is not true. It turns out that in the case $n = 3q$, the largest simplices correspond to the ways of partitioning the graph $\vec{K}_{3q}$ into q vertex-disjoint directed 3-cycles. This is Theorem 6.2. The volume of these largest simplices is a certain easily computed number λ. This implies that, when G is an arbitrary graph with $3q$ vertices, the largest simplices in P have volume equal to λ if and only if G has a partition into q vertex-disjoint directed 3-cycles. Otherwise, the largest simplices in P have volume less than λ. This is Theorem 6.3. Determining whether a given directed graph G with $3q$ vertices has a partition into q vertex-disjoint directed 3-cycles is $\mathbb{NP}$-hard (this follows easily from the hardness of the well-known problem THREE-DIMENSIONAL MATCHING). This implies the $\mathbb{NP}$-hardness of finding largest full-dimensional simplices. The argument works for both

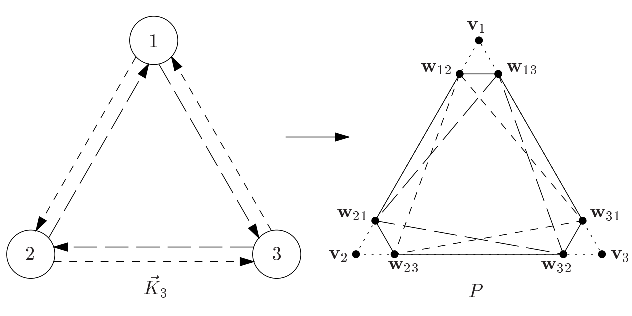

Fig. 3. The largest simplices in the truncated simplex corresponding to $\vec{K}_3$.

the H- and V-polytope problems because truncated simplices have polynomial numbers of both vertices and facets, in terms of the dimension.

To extend the hardness result to non-full-dimensional simplices, the main geometrical result required is Theorem 4.4. The idea of this theorem is to start with a j-dimensional polytope, and fatten it into a d-dimensional polytope, where $d > j$, by adding vertices very close to an interior point of the original polytope. The largest j-simplices in the resulting d-polytope are actually contained in the original j-polytope. The extension also requires some technical results about largest simplices in the truncated simplices corresponding to directed cycle graphs and to unions of directed graphs.

Theorem 7.2 is the main tool for extending hardness results to the dual problems involving smallest containing simplicial cylinders. In general there is no polarity relationship between the largest j-simplices in a d-polytope P and the smallest j-simplicial cylinders containing the polar d-polytope P^{Δ}. If $\mathbf{0} \in \operatorname{int}(P)$ and P has at least one largest j-simplex whose centroid is the origin, Theorem 7.2 shows that the smallest j-simplicial cylinders containing P^{Δ} are the polars of those largest j-simplices in P whose centroids are at the origin.

4. Background Results

In this section we collect some results, most of which have been proved elsewhere, that will be used throughout the paper. We first present the decision problem that will be the starting point of the transformations in our $\mathbb{NP}$-hardness proofs. It is closely related to the following well-known problem, which was proved $\mathbb{NP}$-complete by Karp [Ka].

THREE-DIMENSIONAL MATCHING

Instance: A set $M \subset W \times X \times Y$, where W, X, and Y are disjoint sets having the same number q of elements.
Question: Is there a subset $M' \subset M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate?

Define a *labelled directed graph* to be a pair $G = ([n], E)$, where $E \subset \{(i, j) \in [n] \times [n]\colon\ i \neq j\}$. The elements of $[n]$ are the vertices of G and the elements of E are the edges of G. Note that loops and multiple edges in the same direction are not allowed. For any $n \in \mathbb{N}$, define two particular labelled directed graphs: the complete graph $\vec{K}_n := ([n], \{(i, j) \in [n] \times [n]\colon\ i \neq j\})$ and the n-cycle $\vec{C}_n := ([n], \{(i, i+1)\colon\ i \in [n-1]\} \cup \{(n, 1)\})$. For a labelled directed graph $G = ([n], E)$, the *outdegree* of the vertex $i \in [n]$ is the number of distinct $j \in [n]$ for which $(i, j) \in E$.

Theorem 4.1. *The following decision problem is* $\mathbb{NP}$*-complete.*

PARTITION INTO DIRECTED TRIANGLES

Instance: A labelled directed graph $G = ([3q], E)$, where $q \in \mathbb{N}$.
Question: Does G contain q vertex-disjoint copies of $\vec{C}_3$?

Proof. The problem is clearly in $\mathbb{NP}$, since it is trivial to verify that a proposed partition is actually a partition. The proof of $\mathbb{NP}$-hardness is by a transformation from THREE-DIMENSIONAL MATCHING. Let $M = W \times X \times Y$ be an instance of THREE-DIMENSIONAL MATCHING, where $|W| = |X| = |Y| = q$. Identify W with the integers $1, \ldots, q$, X with the integers $q+1, \ldots, 2q$, and Y with the integers $2q+1, \ldots, 3q$. Then let G be the labelled directed graph with vertex set $[3q]$ and edges (w, x), (x, y), (y, w) for each $(w, x, y) \in M$. Then clearly G has q vertex-disjoint copies of $\vec{C}_3$ if and only if there is a subset $M' \subset M$ of size q such that no two elements of M' agree in any coordinate. □

A version of this problem for undirected graphs is listed in [GJ], credited to unpublished work of Schaefer.

We will need the following two simplex volume formulae. The first is quite well known, and the second comes from [GKL].

Theorem 4.2. *Let S be a j-simplex in $\mathbb{R}^d$. Let F be a facet of S and let $\mathbf{v}$ be the vertex of S not contained in F. Then*

$$\operatorname{vol}(S) = \frac{1}{j} \operatorname{vol}(F) \operatorname{dist}(\{\mathbf{v}\}, \operatorname{aff}(F)).$$

Theorem 4.3. *Let $V \in \mathbb{R}^{d \times (j+1)}$ have affinely independent columns, and let S be the j-simplex* $\operatorname{conv}(V)$. *If* $\mathbf{0} \in \operatorname{aff}(S)$, *then*

$$(j!)^2 \operatorname{vol}^2(S) = \det(J_{j+1} + V^{\mathrm{T}} V).$$

If $\mathbf{0}$ is a vertex of S, then

$$(j!)^2 \operatorname{vol}^2(S) = \det(W^{\mathrm{T}} W),$$

where $W \in \mathbb{R}^{d \times j}$ is formed from V by deleting the zero column.

The next result comes from [Pa1]. It is useful for extending hardness results for problems involving full-dimensional contained simplices to apply to problems involving non-full-dimensional simplices.

Theorem 4.4. *Let $m_1, m_2 \in \mathbb{N}$,*

$$A_1 = \{\mathbf{x} \in \mathbb{R}^{m_1+m_2}\colon x_{m_1+1} = \cdots = x_{m_1+m_2} = 0\}$$

and

$$A_2 = \{\mathbf{x} \in \mathbb{R}^{m_1+m_2}\colon x_1 = \cdots = x_{m_1} = 0\}.$$

For $i = 1, 2$, let $C_i \subset A_i$ be a compact convex set with $\operatorname{aff}(C_i) = A_i$ *and* $\mathbf{0} \in \operatorname{relint}(C_i)$.

Suppose $\alpha > 0$ is the largest real number for which $\alpha(\mathbb{B}_{m_1+m_2} \cap A_1) \subset C_1$, and suppose $C_2 \subset \rho(\mathbb{B}_{m_1+m_2} \cap A_2)$ for some positive $\rho < \alpha$. Let j be an integer with $1 \le j \le m_1$. Then every largest j-simplex in $\operatorname{conv}(C_1 \cup C_2)$ *is contained in C_1.*

It is shown in [GKL] that for any d-polytope P and any $j \in \mathbb{N}$ with $j \leq d$, there is a largest j-simplex in P that is bound to P. This, together with the fact that the volumes of simplices can be computed in polynomial time using Gaussian elimination to evaluate a determinant, shows that the problems HLgstSimplex$_f$ and VLgstSimplex$_f$ are in $\mathbb{NP}$. There is no analogous result for smallest containing simplicial cylinders, and we do not know if the problems HSmlstSimpCyl$_f$ and VSmlstSimpCyl$_f$ are in $\mathbb{NP}$. We require a slightly sharper version of the [GKL] result. The proof is a trivial modification of the proof in [GKL], so we omit it.

Theorem 4.5. *If P is a d-polytope and P contains a largest j-simplex which is not bound to P, then P contains two bound largest j-simplices that differ by a single vertex.*

If $X \subset \mathbb{R}^{d_1}$, $Y \subset \mathbb{R}^{d_2}$ are j-flats for some $j \geq 0$, then a surjective map $\alpha\colon X \to Y$ is a *nonsingular affine map* if for any $\mathbf{v}, \mathbf{w} \in X$ and any $c \in \mathbb{R}$,

$$\alpha(c\mathbf{v} + (1-c)\mathbf{w}) = c\alpha(\mathbf{v}) + (1-c)\alpha(\mathbf{w}).$$

The following result about the invariance of volume ratios under nonsingular affine maps is well known.

Theorem 4.6. *Let $j \geq 0$ and suppose $V_1 \in \mathbb{R}^{d_1 \times (j+1)}$, $V_2 \in \mathbb{R}^{d_2 \times (j+1)}$ each have affinely independent columns. Then there is a nonsingular affine map $\alpha\colon \mathrm{aff}(V_1) \to \mathrm{aff}(V_2)$. There is some real $c > 0$ such that $\mathrm{vol}(\alpha(C)) = c\,\mathrm{vol}(C)$ for all j-dimensional compact convex sets $C \subset \mathrm{aff}(V_1)$.*

The final result in this section allows us to make sure that the polytopes we create in our hardness proofs do not have too many vertices or facets. The proof is trivial and is omitted.

Theorem 4.7. *Suppose $P, Q \subset \mathbb{R}^d$ are polytopes of nonzero dimension a, b, respectively, where $a+b = d$. Suppose $\mathbf{0} \in \mathrm{relint}(P)$, $\mathbf{0} \in \mathrm{relint}(Q)$, and $\mathrm{lin}(P) \cap \mathrm{lin}(Q) = \{\mathbf{0}\}$. If P has v vertices and f facets, and Q has w vertices and g facets, then the d-polytope $R := \mathrm{conv}(P \cup Q)$ has $v + w$ vertices and $f \cdot g$ facets.*

5. Truncated Simplices

In this section we define truncated simplices and prove some basic facts about largest full-dimensional simplices in truncated simplices. For any $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the set of real $n \times n$ matrices M with zero diagonal such that in each row of M the off-diagonal elements are either all zero or all positive, and $M_{ij} < \frac{1}{2}$ for all $i, j \in [n]$.

Definition 5.1. If $d \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $0 \leq k \leq d+1$, a *k-truncated d-simplex* is a set of the form

$$\mathrm{Trunc}(V, M) := \mathrm{conv}(\{(1 - M_{ij})\mathbf{v}_i + M_{ij}\mathbf{v}_j\colon\ i, j \in [d+1] \text{ with } i \neq j\}),$$

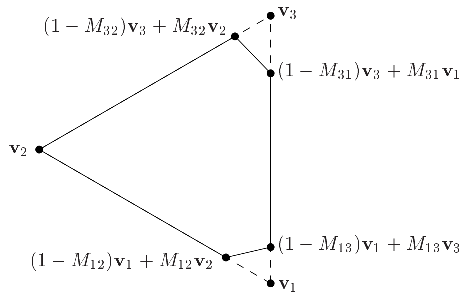

Fig. 4. The 2-truncated 2-simplex $\mathrm{Trunc}(V, M)$, where the columns of V are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and the second row of M is all zero.

where $V \in \mathbb{R}^{n\times(d+1)}$ for some n has affinely independent columns $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$, and $M \in \mathcal{M}_{d+1}$ has exactly k nonzero rows. See Fig. 4. A $(d+1)$-truncated d-simplex is also called simply a *truncated d-simplex*. For $i \in [d+1]$, the *ith truncation face* of $\mathrm{Trunc}(V, M)$ is the set $\mathrm{conv}(\{(1-M_{ij})v_i + M_{ij}v_j\colon\ j \in [d+1] \text{ with } j \neq i\})$, which is always either a vertex or a facet of $\mathrm{Trunc}(V, M)$, depending on whether the ith row of M is zero or nonzero. The *truncation set* of $\mathrm{Trunc}(V, M)$ is the set $\{i \in [d+1]\colon$ the ith row of M is nonzero$\}$.

The hardness results we prove in this paper use only $(d+1)$-truncated d-simplices, but many of the supporting results are stated and proved for general k-truncated d-simplices since it is easy to do so. For some hardness results involving k-truncated d-simplices where $k < d+1$, see [Pa1].

A k-truncated d-simplex has $d+1+k$ facets and $d+1+k(d-1)$ vertices. Thus both the number of vertices and number of facets are bounded by polynomials in d. This allows us to use truncated simplices to prove hardness of both the V-polytope versions of our problems and the H-polytope versions.

Given any two sets $X \subset Y$ and a function $g\colon X \to Y$, a finite set $C \subset X$ is said to be a *cyclic set* of g if the restriction of g to C is a cyclic permutation of C. The function g is said to be *fixed-point-free* if for all $x \in X$, $g(x) \neq x$.

Definition 5.2. If $P = \mathrm{Trunc}(V, M)$ is a k-truncated d-simplex, then a d-simplex $S \subset P$ is said to be *P-balanced* if there is a vertex of S on each truncation face of P. If $A \subset [d+1]$ is the truncation set of P, there is a natural bijection between the set of fixed-point-free functions $g\colon A \to [d+1]$ and the set of bound P-balanced d-simplices in P. We denote this bijection by Ψ_P and define it by

$$\Psi_P(g) := \mathrm{conv}(\{(1 - M_{i,g(i)})\mathbf{v}_i + M_{i,g(i)}\mathbf{v}_{g(i)}\colon\ i \in A\} \cup \{\mathbf{v}_i\colon\ i \in [d+1]\backslash A\}).$$

When calculating the volume of $\Psi_P(g)$, the cyclic sets of g play an important role, as the following result shows.

Theorem 5.3. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $P = \mathrm{Trunc}(V, M)$ be a q-truncated $(n-1)$-simplex with truncation set $A \subset [n]$, and let $g\colon A \to [n]$ be fixed-point-free. For brevity,* *we set* $x_i = M_{i,g(i)}$ *for* $i \in A$. *Define* $Q \in \mathbb{R}^{n\times n}$ *by*

$$Q_{ij} = \begin{cases} 1 & \textit{if} \quad i \notin A \quad \textit{and} \quad j = i, \\ 1 - x_i & \textit{if} \quad i \in A \quad \textit{and} \quad j = i, \\ x_i & \textit{if} \quad i \in A \quad \textit{and} \quad j = g(i), \\ 0 & \textit{otherwise}. \end{cases}$$

Let k *be the number of cyclic sets of* g. *Note that* k *can be zero if* $q < n$. *If* $k > 0$, *let* $C_1, \ldots, C_k$ *be the cyclic sets of* g. *Then*

$$\begin{aligned} \frac{\mathrm{vol}(\Psi_P(g))}{\mathrm{vol}(\mathrm{conv}(V))} &= \det(Q) \\ &= \left(\prod_{i\in A\setminus(C_1\cup\cdots\cup C_k)} (1-x_i)\right) \prod_{m=1}^{k} \left[\left(\prod_{i\in C_m} (1-x_i)\right) - \left(\prod_{i\in C_m} (-x_i)\right)\right]. \end{aligned}$$

Proof. Because of Theorem 4.6, the ratio

$$\frac{\mathrm{vol}(\Psi_P(g))}{\mathrm{vol}(\mathrm{conv}(V))}$$

is the same for all matrices V with $d+1$ affinely independent columns. In particular, we can assume that $\mathbf{0} \in \mathrm{aff}(V)$, and thus by Theorem 4.3,

$$((n-1)!)^2 \,\mathrm{vol}^2(\mathrm{conv}(V)) = \det(J + V^{\mathrm{T}}V).$$

The vertices of $\Psi_P(g)$ are the columns of the matrix VQ^{T}, and $\mathbf{0} \in \mathrm{aff}(\Psi_P(g)) = \mathrm{aff}(V)$, so

$$\begin{aligned} ((n-1)!)^2 \,\mathrm{vol}^2(\Psi_P(g)) &= \det(J + QV^{\mathrm{T}}VQ^{\mathrm{T}}) \\ &= \det(Q(J + V^{\mathrm{T}}V)Q^{\mathrm{T}}) \\ &= (\det(Q))^2 \det(J + V^{\mathrm{T}}V). \end{aligned}$$

Thus

$$\frac{\mathrm{vol}(\Psi_P(g))}{\mathrm{vol}(\mathrm{conv}(V))} = |\det(Q)|.$$

For $S \subset [n]$, we write $g^{-1}(S) = \{i \in A\colon\ g(i) \in S\}$. For any integer $r \geq 0$, let g^r denote the r-fold iteration of g. More precisely, $g^0\colon\ [n] \to [n]$ is the identity, and for $r > 0$, $g^r\colon\ g^{-1}(\mathrm{domain}(g^{r-1})) \to [n]$ is defined by $g^r(i) = g^{r-1}(g(i))$. Note that $\mathrm{image}(g^r) \subset \mathrm{image}(g^{r-1})$ and $\mathrm{domain}(g^r) \subset \mathrm{domain}(g^{r-1})$ for all $r > 0$. Our definition also allows the possibility that for some $r > 1$, $\mathrm{domain}(g^r) = \mathrm{image}(g^r) = \emptyset$.

For any $r \geq 0$, we define a matrix $Q^{(r)}$ by

$$Q^{(r)}_{ij} = \begin{cases} 1 & \text{if} \quad i \notin A \quad \text{and} \quad j = i, \\ 1 - x_i & \text{if} \quad i \in A \quad \text{and} \quad j = i, \\ x_i & \text{if} \quad i \in A, \quad j = g(i), \quad \text{and} \quad i \in \mathrm{image}(g^r), \\ 0 & \text{otherwise}. \end{cases}$$

Note that $Q^{(0)} = Q$. We claim that for all $r \geq 0$, $\det(Q^{(r+1)}) = \det(Q^{(r)})$, and therefore $\det(Q^{(r)}) = \det(Q)$ for all $r \geq 0$. Let $i \in \operatorname{image}(g^r) \setminus \operatorname{image}(g^{r+1})$, and suppose that for some $p \neq i$, $Q^{(r)}_{pi} \neq 0$. Then $p \in A$, $i = g(p)$, and $p \in \operatorname{image}(g^r)$. Therefore, $i \in \operatorname{image}(g^{r+1})$, and this contradiction shows that the only nonzero entry of the ith column of $Q^{(r)}$ is $Q^{(r)}_{ii}$. So we can zero out $Q^{(r)}_{i,g(i)}$ by adding a multiple of the ith column to the $g(i)$th column, and we get a matrix with the same determinant as $Q^{(r)}$. After doing this for all $i \in \operatorname{image}(g^r) \setminus \operatorname{image}(g^{r+1})$, we get the matrix $Q^{(r+1)}$, and therefore $\det(Q^{(r+1)}) = \det(Q^{(r)})$.

Now, since $[n]$ is a finite set, there must be some integer N such that

$$\operatorname{image}(g^r) = \operatorname{image}(g^N) \qquad \text{for all} \quad r \geq N,$$

and therefore

$$\operatorname{image}(g^N) = C_1 \cup \cdots \cup C_k.$$

If we permute the rows of $Q^{(N)}$ appropriately and permute the columns in the same way, which does not change the determinant, then $Q^{(N)}$ will be a block-diagonal matrix. There will be a 1 by 1 block equal to 1 for each $i \notin A$, and a 1 by 1 block equal to $1 - x_i$ for each $i \in A \backslash (C_1 \cup \cdots \cup C_k)$. Also, for each $m = 1, \ldots, k$, there will be a $|C_m|$ by $|C_m|$ block of the form

$$B := \begin{pmatrix} 1 - x_i & x_i & & & \\ & 1 - x_{g(i)} & x_{g(i)} & & \\ & & \ddots & \ddots & \\ & & & \ddots & x_{g^{|C_m|-2}(i)} \\ x_{g^{|C_m|-1}(i)} & & & & 1 - x_{g^{|C_m|-1}(i)} \end{pmatrix},$$

where all the blank entries are zero, and i is some element of C_m. It now suffices to show that

$$\det(B) = \left(\prod_{i \in C_m} (1 - x_i) \right) - \left(\prod_{i \in C_m} (-x_i) \right).$$

The definition of the determinant says that

$$\det(B) = \sum_{\sigma \in S_{|C_m|}} \left(\operatorname{sgn}(\sigma) \prod_{j=1}^{|C_m|} B_{j,\sigma(j)} \right).$$

However, there are only two permutations σ that can possibly make a nonzero contribution to the determinant: the identity, which gives the term

$$\prod_{i \in C_m} (1 - x_i),$$

and a certain $|C_m|$-cycle, which gives the term

$$- \prod_{i \in C_m} (-x_i).$$

Therefore, $\det(Q^{(N)}) = \det(Q)$ has the form claimed in the theorem. Since $0 < x_i < \frac{1}{2}$ for all $i \in A$, $\det(Q) > 0$, and this establishes the first equality in the theorem. $\square$

Next we show that if the entries of M are sufficiently small, every largest full-dimensional simplex in $\mathrm{Trunc}(V, M)$ must be balanced. Furthermore, if the entries of M fall into two classes, "small entries" and "large entries," we will show that a largest simplex must use small entries whenever possible. First, we need two lemmas. They are trivial calculations for which we omit the details.

Lemma 5.4. *Let $x \in \mathbb{R}$ with $0 < x < \frac{1}{2}$, and let $m \in \mathbb{N}$. Then*

$$\begin{aligned} (1-x)^2((1-x)^{m-2} + x^{m-2}) &> (1-x)^m + x^m \\ &> (1-x)^m \\ &> (1-x)^m - x^m \\ &> (1-x)^2((1-x)^{m-2} - x^{m-2}). \end{aligned}$$

Lemma 5.5. *Using the notation of Theorem* 5.3, *for any $l \in A$,*

$$\frac{\partial}{\partial x_l} \det(Q) < 0$$

whenever $0 < x_j < \frac{1}{2}$ for all $j \in A$.

Theorem 5.6. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $P = \mathrm{Trunc}(V, M)$ be a q-truncated $(n-1)$-simplex with truncation set $A \subset [n]$. Suppose $M_{ij} < 2^{-n}$ for all $i, j \in [n]$. Then every bound largest $(n-1)$-simplex in P is P-balanced.*

Suppose further that there is some $\varepsilon > 0$ such that for all $i, j \in [n]$, either $M_{ij} \leq \varepsilon$ or $M_{ij} \geq 2\varepsilon$. Then for every bound largest $(n-1)$-simplex $S = \Psi_P(g)$ in P and every $i \in A$, $M_{i,g(i)} \leq \varepsilon$ unless there is no $j \neq i$ with $M_{i,j} \leq \varepsilon$.

Proof. Let S_1 be a bound $(n-1)$-simplex in P that is not P-balanced. Then there is some $i \in [n]$ such that no vertex of S_1 is on the ith truncation face of P. Let $T = \mathrm{conv}(\{\mathbf{v}_i\} \cup \{(1-2^{-n})\mathbf{v}_j + 2^{-n}\mathbf{v}_i \colon j \in [n] \text{ with } j \neq i\})$. Then $T \subset \mathrm{conv}(V)$ is an $(n-1)$-simplex similar to $\mathrm{conv}(V)$, with similarity ratio $1 - 2^{-n}$, which does not intersect S_1. See Fig. 5. So

$$\frac{\mathrm{vol}(S_1)}{\mathrm{vol}(\mathrm{conv}(V))} \leq 1 - (1-2^{-n})^{n-1} \leq 1 - (1-2^{-n+1})^{n-1}.$$

Now let $S_2 = \Psi_P(g_2)$ be a bound P-balanced $(n-1)$-simplex in P. Let k be the number of cyclic sets of g_2, and if $k > 0$ let $C_1, \ldots, C_k$ be the cyclic sets of g_2. For

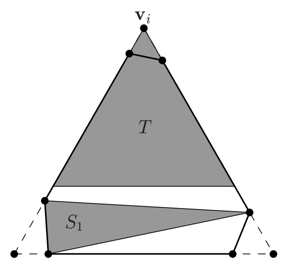

Fig. 5. The nonbalanced simplex S_1.

$i \in A$, write $x_i = M_{i,g_2(i)}$. By Theorem 5.3,

$$\frac{\mathrm{vol}(S_2)}{\mathrm{vol}(\mathrm{conv}(V))} = \left(\prod_{i\in A\setminus(C_1\cup\cdots\cup C_k)} (1-x_i)\right)\prod_{m=1}^{k}\left[\left(\prod_{i\in C_m}(1-x_i)\right)-\left(\prod_{i\in C_m}(-x_i)\right)\right],$$

so by Lemma 5.5,

$$\frac{\mathrm{vol}(S_2)}{\mathrm{vol}(\mathrm{conv}(V))} > \left(\prod_{i\in A\setminus(C_1\cup\cdots\cup C_k)} (1-2^{-n})\right)\prod_{m=1}^{k}\left[\left(\prod_{i\in C_m}(1-2^{-n})\right)-\left(\prod_{i\in C_m}(-2^{-n})\right)\right].$$

Lemma 5.4 implies that the right-hand side is minimized when all the cyclic sets are of size 2 and there are as many of them as possible. Therefore,

$$\begin{aligned}\frac{\mathrm{vol}(S_2)}{\mathrm{vol}(\mathrm{conv}(V))} &> (1-2^{-n})^{q-2\lfloor q/2\rfloor}((1-2^{-n})^2-(-2^{-n})^2)^{\lfloor q/2\rfloor}\\ &\geq ((1-2^{-n})^2-(2^{-n})^2)^{q/2}\\ &= (1-2^{-n+1})^{q/2}\\ &\geq (1-2^{-n+1})^{n-1}.\end{aligned}$$

Now we show that $\mathrm{vol}(S_2) > \mathrm{vol}(S_1)$. It suffices to show that

$$(1-2^{-n+1})^{n-1} \geq 1-(1-2^{-n+1})^{n-1},$$

or, equivalently, that

$$(1-2^{-(n-1)})^{n-1} \geq \tfrac{1}{2}.$$

This is easily checked for $n = 2$. For $n > 2$, Bernoulli's inequality implies that

$$h(n) := 1-(n-1)2^{-(n-1)} < (1-2^{-(n-1)})^{n-1}.$$

Since $h(3) = \frac{1}{2}$ and

$$\frac{d}{dn}h(n) = 2^{-(n-1)}[(n-1)\ln(2) - 1] > 0$$

if $n > 1 + 1/\ln(2) = 2.4427\ldots$, the result follows. This proves the first part of the theorem.

Now, suppose the second part of the theorem is false. Then clearly $n \geq 3$ and $A \neq \emptyset$. Because of Theorem 4.6, we may assume that the ith column of V is $\mathbf{e}_i \in \mathbb{R}^{n-1}$ for $i = 1, \ldots, n-1$, that the nth column is $\mathbf{0} \in \mathbb{R}^{n-1}$, that $n \in A$, and that there is some bound largest $(n-1)$-simplex $S = \Psi_P(g)$ in P with $M_{n,g(n)} \geq 2\varepsilon$ but there is some $j \neq n$ with $M_{n,j} \leq \varepsilon$.

We recall the definition of the infinity norm on matrices and vectors. If $A \in \mathbb{R}^{n\times n}$, we define

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |A_{ij}|,$$

and if $\mathbf{v} \in \mathbb{R}^n$, we define

$$\|\mathbf{v}\|_\infty = \max_{1\leq i\leq n} |v_i|.$$

For the properties of these norms, see, for example, [HJ]. We recall the following properties in particular. Let $A, B \in \mathbb{R}^{n\times n}$ and $\mathbf{v} \in \mathbb{R}^n$. Then

$$\|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty,$$

$$\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty,$$

and

$$\|A\mathbf{v}\|_\infty \leq \|A\|_\infty \|\mathbf{v}\|_\infty.$$

Furthermore, if $\|I - A\|_\infty < 1$, then A^{-1} exists, and $A^{-1} = \sum_{i=0}^{\infty}(I - A)^i$.

Let $R \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix whose ith column is the vertex of S on the ith truncation facet of P. Then $R^{\mathrm{T}} = I - B$ for some matrix B such that each row of B has at most two nonzero elements, each of absolute value less than 2^{-n}. So $\|I - R^{\mathrm{T}}\|_\infty = \|B\|_\infty < 2^{-n+1} < 1$. Thus $R^{-\mathrm{T}}$ exists, and $R^{-\mathrm{T}} = \sum_{i=0}^{\infty} B^i$. Therefore,

$$\begin{aligned}
\|R^{-\mathrm{T}} - I\|_\infty &= \|\sum_{i=1}^{\infty} B^i\|_\infty \\
&\leq \sum_{i=1}^{\infty} \|B^i\|_\infty \\
&\leq \sum_{i=1}^{\infty} \|B\|_\infty^i \\
&< \sum_{i=1}^{\infty} (2^{-n+1})^i \\
&= \frac{2^{-n+1}}{1 - 2^{-n+1}}.
\end{aligned}$$

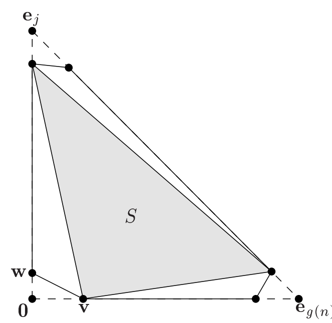

Fig. 6. The simplex S.

So

$$\|(R^{-\mathrm{T}} - I)\mathbf{1}\|_\infty \le \|R^{-\mathrm{T}} - I\|_\infty \|\mathbf{1}\|_\infty < \frac{2^{-n+1}}{1 - 2^{-n+1}} < 1.$$

This shows that every element of the vector $\mathbf{y} = R^{-\mathrm{T}}\mathbf{1}$ is positive. Let

$$\mathbf{v} = M_{n,g(n)}\mathbf{e}_{g(n)}, \qquad \mathbf{w} = M_{n,j}\mathbf{e}_j,$$

let F be the facet of S opposite $\mathbf{v}$, and let

$$S' = \mathrm{conv}(F \cup \{\mathbf{w}\}).$$

See Fig. 6. Then since S is largest,

$$0 \le \mathrm{vol}(S) - \mathrm{vol}(S') = \frac{\mathrm{vol}(F)}{n-1}(\mathrm{dist}(\{\mathbf{v}\}, \mathrm{aff}(F)) - \mathrm{dist}(\{\mathbf{w}\}, \mathrm{aff}(F))).$$

By the definition of $\mathbf{y}$, $\mathrm{aff}(F) = \{\mathbf{x}\colon\ \mathbf{x}^\mathrm{T}\mathbf{y} = 1\}$. Clearly, $\mathbf{w}$ and $\mathbf{v}$ are on the same side of $\mathrm{aff}(F)$ as $\mathbf{0}$, so $\mathbf{w}^\mathrm{T}\mathbf{y} < 1$ and $\mathbf{v}^\mathrm{T}\mathbf{y} < 1$. Therefore,

$$\mathrm{dist}(\{\mathbf{v}\}, \mathrm{aff}(F)) = \frac{1 - \mathbf{v}^\mathrm{T}\mathbf{y}}{\|\mathbf{y}\|}$$

and

$$\mathrm{dist}(\{\mathbf{w}\}, \mathrm{aff}(F)) = \frac{1 - \mathbf{w}^\mathrm{T}\mathbf{y}}{\|\mathbf{y}\|},$$

where $\|\mathbf{y}\|$ is the Euclidean norm of $\mathbf{y}$. It follows that

$$\begin{aligned} 0 \ &\le\ \|\mathbf{y}\|(\mathrm{dist}(\{\mathbf{v}\}, \mathrm{aff}(F)) - \mathrm{dist}(\{\mathbf{w}\}, \mathrm{aff}(F))) = (\mathbf{w} - \mathbf{v})^\mathrm{T}\mathbf{y} \\ &=\ M_{n,j}y_j - M_{n,g(n)}y_{g(n)} \le \varepsilon y_j - 2\varepsilon y_{g(n)}, \end{aligned}$$

and so $y_j/y_{g(n)} \ge 2$. However, since $\|\mathbf{y} - \mathbf{1}\|_\infty = \|(R^{-\mathrm{T}} - I)\mathbf{1}\|_\infty < 2^{-n+1}/(1 - 2^{-n+1})$,

$$\frac{\max_i\ y_i}{\min_i\ y_i} < \frac{1 + 2^{-n+1}/(1 - 2^{-n+1})}{1 - 2^{-n+1}/(1 - 2^{-n+1})} = \frac{1}{1 - 2^{-n+2}},$$

which is at most 2 since $n \ge 3$. This contradiction shows that the second part of the theorem is true. $\square$

6. Truncated Simplices and Directed Graphs

In this section we present a construction that associates a truncated simplex to a directed graph, and show how the structure of the graph is reflected in the volume of the largest simplices in the resulting truncated simplex.

Definition 6.1. Given a labelled directed graph $G = ([n], E)$ with $n \geq 2$, a matrix $V \in \mathbb{R}^{d\times m}$ with affinely independent columns, where $m \geq n$, and some $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \frac{1}{4}$, define $M \in \mathcal{M}_m$ by

$$M_{ij} := \begin{cases} 0 & \text{if} \quad i > n \quad \text{or} \quad j = i, \\ 2\varepsilon & \text{if} \quad i \leq n, \quad j \neq i, \quad \text{and} \quad (i, j) \notin E, \\ \varepsilon & \text{if} \quad (i, j) \in E. \end{cases}$$

Then we define $T(V, G, \varepsilon)$ to be the n-truncated $(m-1)$-simplex $\mathrm{Trunc}(V, M)$. See Fig. 7.

In the above definition, if $\varepsilon < 2^{-m-1}$, then Theorem 5.6 applies to $P = T(V, G, \varepsilon)$. In this case we get that every bound largest $(m-1)$-simplex S in P is P-balanced, and, furthermore, if $S = \Psi_P(g)$ and $i \in [n]$, then $(i, g(i))$ is an edge of G unless i has outdegree 0 in G.

Note that if $0 < \varepsilon_1 < \varepsilon_2 < \frac{1}{4}$, then $T(V, G, \varepsilon_1) \supset T(V, G, \varepsilon_2)$ for any V, G. Also, if $G_1 = ([n], E_1)$ and $G_2 = ([n], E_2)$ are two labelled directed graphs with the same number of vertices, and $E_1 \subset E_2$, then $T(V, G_1, \varepsilon) \subset T(V, G_2, \varepsilon)$ for any V, ε.

Theorem 6.2. *Let $q \in \mathbb{N}$ and $V \in \mathbb{R}^{d\times m}$ be a matrix with affinely independent columns, where $m \geq 3q$. Let $P = T(V, \vec{K}_{3q}, \varepsilon)$ for some $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2^{-m-1}$. Then the set of largest $(m-1)$-simplices in P is*

$$\{\Psi_P(g) \mid g\colon [3q] \to [m] \text{ has } q \text{ cyclic } 3\text{-sets}\}.$$

In particular, a largest $(m-1)$-simplex S in P satisfies

$$\frac{\mathrm{vol}(S)}{\mathrm{vol}(\mathrm{conv}(V))} = ((1-\varepsilon)^3 + \varepsilon^3)^q.$$

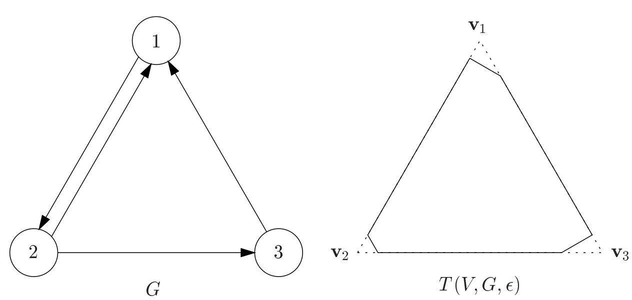

Fig. 7. The construction in Definition 6.1.

Proof. Let S be any bound largest $(m-1)$-simplex in P. Then by the first part of Theorem 5.6, S is P-balanced. Since the truncation set of P is $[3q]$, we have that $S = \Psi_P(g)$ for some fixed-point-free $g\colon [3q] \to [m]$. Since every vertex of $\vec{K}_{3q}$ has nonzero outdegree, the second part of Theorem 5.6 implies that $(i, g(i)) \in E$ for any $i \in [3q]$. So by Theorem 5.3,

$$\frac{\mathrm{vol}(S)}{\mathrm{vol}(\mathrm{conv}(V))} = (1-\varepsilon)^{3q-|C_1|-\cdots-|C_k|} \prod_{m=1}^{k} [(1-\varepsilon)^{|C_m|} - (-\varepsilon)^{|C_m|}],$$

where $C_1, \ldots, C_k$ are the cyclic sets of g. Lemma 5.4 implies that the right-hand side is maximized whenever all the cyclic sets are of size 3 and there are as many of them as possible, i.e., there are q of them. This shows that the set of *bound* largest $(m-1)$-simplices in P has the form claimed in the theorem. Now, if there were any largest $(m-1)$-simplices in P that were not bound, then by Theorem 4.5 there would be two bound largest $(m-1)$-simplices in P that differ by a single vertex. Equivalently, there would be two functions $g_1, g_2\colon [3q] \to [m]$, each with q cyclic sets of size 3, such that g_1 and g_2 agree on all but one of the elements of $[3q]$. This clearly cannot happen. □

Theorem 6.3. *Let $G = ([3q], E)$ for some $q \in \mathbb{N}$ be a labelled directed graph. Let $V \in \mathbb{R}^{d\times m}$ have affinely independent columns, where $m \geq 3q$, and let $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2^{-m-1}$. Let S be a largest $(m-1)$-simplex in $T(V, G, \varepsilon)$. Then*

$$\frac{\mathrm{vol}(S)}{\mathrm{vol}(\mathrm{conv}(V))} \leq ((1-\varepsilon)^3 + \varepsilon^3)^q,$$

with equality if and only if G has a partition into directed triangles.

Proof. Since $T(V, G, \varepsilon) \subset T(V, \vec{K}_{3q}, \varepsilon)$, S is contained in $T(V, \vec{K}_{3q}, \varepsilon)$. Therefore, the inequality follows from Theorem 6.2.

Suppose G has a partition into directed triangles. Then there is some function $g\colon [3q] \to [3q]$ with q cyclic 3-sets such that $(i, g(i)) \in E$ for all $i \in [3q]$. Let

$$S_1 := \Psi_{T(V,G,\varepsilon)}(g) \subset T(V, G, \varepsilon).$$

Since $(i, g(i)) \in E$ for all $i \in [3q]$,

$$S_1 = \Psi_{T(V,\vec{K}_{3q},\varepsilon)}(g).$$

By Theorem 6.2, S_1 is largest in $T(V, \vec{K}_{3q}, \varepsilon)$. Therefore it is largest in $T(V, G, \varepsilon)$ as well, and the equality in this theorem holds.

Now suppose equality holds, and let S_2 be any largest $(m-1)$-simplex in $T(V, G, \varepsilon)$. Then $S_2 \subset T(V, \vec{K}_{3q}, \varepsilon)$, and since it has the appropriate volume, it is a largest $(m-1)$-simplex in $T(V, \vec{K}_{3q}, \varepsilon)$. So $S_2 = \Psi_{T(V,\vec{K}_{3q},\varepsilon)}(g)$ for some $g\colon [3q] \to [3q]$ with q cyclic 3-sets. Since $S_2 \subset T(V, G, \varepsilon)$, it follows that $(i, g(i)) \in E$ for all $i \in [3q]$, and these edges form a partition of G into directed triangles. □

Theorem 6.4. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $V \in \mathbb{R}^{d\times m}$ have affinely independent columns, where $m \geq n$, and let $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2^{-m-1}$. Let S be a largest $(m-1)$-simplex in $P = T(V, \vec{C}_n, \varepsilon)$. Then*

$$\frac{\mathrm{vol}(S)}{\mathrm{vol}(\mathrm{conv}(V))} = (1-\varepsilon)^n - (-\varepsilon)^n.$$

Proof. By the first part of Theorem 5.6, S must be P-balanced, and thus $S = \Psi_P(g)$ for some fixed-point-free $g\colon [n] \to [m]$. Since every vertex of $\vec{C}_n$ has nonzero outdegree, the second part of Theorem 5.6 implies that $(i, g(i))$ must be an edge of $\vec{C}_n$ for all $i \in [n]$. Therefore g has a single cyclic set, of size n, and the result follows from Theorem 5.3. $\square$

Definition 6.5. Let $G_1 = ([n_1], E_1)$, $G_2 = ([n_2], E_2)$ be labelled directed graphs. Then we define $G_1 \cup G_2$ to be the labelled directed graph $([n_1+n_2], E_1 \cup \varphi(E_2))$, where $\varphi\colon [n_2]\times[n_2] \to [n_1+n_2]\times[n_1+n_2]$ by $\varphi((i,j)) = (i+n_1, j+n_1)$.

Theorem 6.6. *For each $i = 1, 2$, let $G_i = ([n_i], E_i)$ be a labelled directed graph such that every vertex has outdegree at least 1, and let $V_i \in \mathbb{R}^{d_i\times m_i}$ have affinely independent columns, where $m_i \geq n_i$. Let $V_3 \in \mathbb{R}^{d_3\times m_3}$ have affinely independent columns, where $m_3 \geq n_1 + n_2$. Let $r = \max(m_1, m_2, m_3)$ and let $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2^{-r-1}$.*

For each $i = 1, 2$, let S_i be a largest $(m_i - 1)$-simplex in $P_i = T(V_i, G_i, \varepsilon)$. Let S_3 be a largest (m_3-1)-simplex in $P_3 = T(V_3, G_1 \cup G_2, \varepsilon)$. Then

$$\frac{\mathrm{vol}(S_3)}{\mathrm{vol}(\mathrm{conv}(V_3))} = \frac{\mathrm{vol}(S_1)\,\mathrm{vol}(S_2)}{\mathrm{vol}(\mathrm{conv}(V_1))\,\mathrm{vol}(\mathrm{conv}(V_2))}.$$

Proof. Let Q_3 be any bound P_3-balanced (m_3-1)-simplex in P_3, so $Q_3 = \Psi_{P_3}(g)$ for some fixed-point-free $g\colon [n_1+n_2] \to [m_3]$. Since every vertex of $G_1 \cup G_2$ has outdegree at least 1, $(i, g(i))$ is an edge of $G_1 \cup G_2$ for every $i \in [n_1+n_2]$. In particular, $g([n_1]) \subset [n_1]$ and $g([n_1+n_2]\backslash[n_1]) \subset [n_1+n_2]\backslash[n_1]$, so every cyclic set of g is contained in $[n_1]$ or in $[n_1+n_2]\backslash[n_1]$. Let $C_1, \ldots, C_k$ be the cyclic sets of g that are contained in $[n_1]$, and let $C_{k+1}, \ldots, C_p$ be the cyclic sets of g that are contained in $[n_1+n_2]\backslash[n_1]$. Define $g_1\colon [n_1] \to [n_1]$ by $g_1(i) = g(i)$ and $g_2\colon [n_2] \to [n_2]$ by $g_2(i) = g(n_1+i) - n_1$. Then the cyclic sets of g_1 are $C_1, \ldots, C_k$, and the cyclic sets of g_2 are $D_1 := \{i - n_1\colon i \in C_{k+1}\}, \ldots, D_{p-k} := \{i - n_1\colon i \in C_p\}$. For $i = 1, 2$, let $Q_i = \Psi_{P_i}(g_i)$.

By Theorem 5.3,

$$\frac{\mathrm{vol}(Q_1)}{\mathrm{vol}(\mathrm{conv}(V_1))} = \left(\prod_{i\in[n_1]\backslash(C_1\cup\cdots\cup C_k)} (1-\varepsilon)\right) \prod_{m=1}^{k} \left[\left(\prod_{i\in C_m}(1-\varepsilon)\right) - \left(\prod_{i\in C_m}(-\varepsilon)\right)\right],$$

$$\frac{\operatorname{vol}(Q_2)}{\operatorname{vol}(\operatorname{conv}(V_2))}$$
$$= \left(\prod_{i\in[n_2]\setminus(D_1\cup\cdots\cup D_{p-k})}(1-\varepsilon)\right)\prod_{m=1}^{p-k}\left[\left(\prod_{i\in D_m}(1-\varepsilon)\right)-\left(\prod_{i\in D_m}(-\varepsilon)\right)\right],$$

and

$$\frac{\operatorname{vol}(Q_3)}{\operatorname{vol}(\operatorname{conv}(V_3))}$$
$$= \left(\prod_{i\in[n_1+n_2]\setminus(C_1\cup\cdots\cup C_p)}(1-\varepsilon)\right)\prod_{m=1}^{p}\left[\left(\prod_{i\in C_m}(1-\varepsilon)\right)-\left(\prod_{i\in C_m}(-\varepsilon)\right)\right].$$

So

$$\frac{\operatorname{vol}(Q_3)}{\operatorname{vol}(\operatorname{conv}(V_3))} = \frac{\operatorname{vol}(Q_1)\operatorname{vol}(Q_2)}{\operatorname{vol}(\operatorname{conv}(V_1))\operatorname{vol}(\operatorname{conv}(V_2))},$$

and clearly Q_3 is largest if and only if both Q_1 and Q_2 are largest. □

For our hardness proofs it will be necessary to know, for a given truncated simplex, the radius of a ball that fits inside it.

Definition 6.7. For $n \in \mathbb{N}$, define

$$\rho_n = \frac{1}{n}\min\left(\frac{1}{n+1}, \frac{n}{n+1} - 2^{-n-1}\right) \in \mathbb{Q},$$

and let $V_n \in \mathbb{R}^{n\times(n+1)}$ be defined by

$$(V_n)_{ij} := \begin{cases} 1 - \dfrac{1}{n+1} & \text{if} \quad j = i+1, \\ \dfrac{-1}{n+1} & \text{if} \quad j \neq i+1. \end{cases}$$

It is easy to see that $\operatorname{conv}(V_n)$ is a rational n-simplex in $\mathbb{R}^n$ whose centroid is the origin. Furthermore, $\operatorname{vol}(\operatorname{conv}(V_n)) = 1/(n!)$ and $\operatorname{conv}(V_n) \subset \mathbb{B}_n$.

Theorem 6.8. *Let $G = ([n], E)$ be a labelled directed graph with $n \geq 2$. Then for any $m \in \mathbb{N}$ with $m \geq n$ and any $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2^{-m-1}$, $\rho_{m-1}\mathbb{B} \subset T(V_{m-1}, G, \varepsilon)$.*

Proof. Let $Z = ([n], \emptyset)$ be the empty graph, and let $P = T(V_{m-1}, Z, 2^{-m-1})$. Then $T(V_{m-1}, G, \varepsilon) \supset P$, so it suffices to show that $\rho_{m-1}\mathbb{B} \subset P$. The facet-defining half-

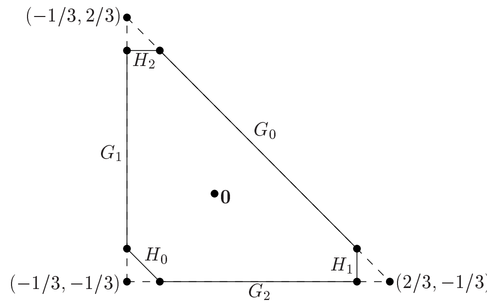

Fig. 8. The truncated simplex $T(V_{m-1}, Z, 2^{-m-1})$, for $m = 3$.

spaces of P are

$$\begin{aligned} G_0 &= \left\{ \mathbf{x}\colon\ \mathbf{1}_{m-1}^{\mathrm{T}}\mathbf{x} \le \frac{1}{m} \right\}, \\ G_i &= \left\{ \mathbf{x}\colon\ \mathbf{e}_i^{\mathrm{T}}\mathbf{x} \ge \frac{-1}{m} \right\} \qquad \text{for} \quad i \in [m-1], \\ H_0 &= \left\{ \mathbf{x}\colon\ \mathbf{1}_{m-1}^{\mathrm{T}}\mathbf{x} \ge 2^{-m} - \frac{m-1}{m} \right\}, \\ H_i &= \left\{ \mathbf{x}\colon\ \mathbf{e}_i^{\mathrm{T}}\mathbf{x} \le \frac{m-1}{m} - 2^{-m} \right\} \qquad \text{for} \quad i \in [m-1]. \end{aligned}$$

See Fig. 8. Now,

$$\begin{aligned} \operatorname{dist}(\{\mathbf{0}\}, \partial G_0) &= \left\| \left(\frac{1}{m(m-1)} \right) \mathbf{1}_{m-1} \right\| = \frac{1}{m\sqrt{m-1}}, \\ \operatorname{dist}(\{\mathbf{0}\}, \partial G_i) &= \left\| \left(-\frac{1}{m} \right) \mathbf{e}_i \right\| = \frac{1}{m} \qquad \text{for all} \quad i \in [m-1], \\ \operatorname{dist}(\{\mathbf{0}\}, \partial H_0) &= \left\| \left(\frac{2^{-m}}{m-1} - \frac{1}{m} \right) \mathbf{1}_{m-1} \right\| = \frac{1}{\sqrt{m-1}} \left(\frac{m-1}{m} - 2^{-m} \right), \end{aligned}$$

and

$$\operatorname{dist}(\{\mathbf{0}\}, \partial H_i) = \left\| \left(\frac{m-1}{m} - 2^{-m} \right) \mathbf{e}_i \right\| = \frac{m-1}{m} - 2^{-m} \qquad \text{for all} \quad i \in [m-1].$$

So the minimum distance from $\mathbf{0}$ to the boundary of P is

$$\begin{aligned} &\min\left(\frac{1}{m\sqrt{m-1}}, \frac{1}{m}, \frac{1}{\sqrt{m-1}} \left(\frac{m-1}{m} - 2^{-m} \right), \frac{m-1}{m} - 2^{-m} \right) \\ &\quad = \frac{1}{\sqrt{m-1}} \min\left(\frac{1}{m}, \frac{m-1}{m} - 2^{-m} \right) = \sqrt{m-1}\,\rho_{m-1} \ge \rho_{m-1}, \end{aligned}$$

and this completes the proof. $\square$

7. Smallest Containing Simplicial Cylinders

In this section we prove several results that relate the volume of the largest j-simplices in a polytope P with the cross-sectional volume of the smallest j-simplicial cylinders containing the polar polytope P^{Δ}.

Lemma 7.1. *Let $V \in \mathbb{R}^{m\times n}$ have affinely independent columns and suppose $\mathbf{0} \in \operatorname{aff}(V)$. Then for all $t \in \mathbb{C}$,*

$$\det(J_n + t^2 V^{\mathrm{T}} V) = t^{2n-2} \det(J_n + V^{\mathrm{T}} V).$$

If $t \neq 0$, then $J_n + t^2 V^{\mathrm{T}} V$ is invertible.

Proof. The case $t = 0$ is trivial, so suppose $t \neq 0$. Let $V_1, V_2 \in \mathbb{C}^{(m+1)\times n}$ be the matrices

$$V_1 = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ V \end{pmatrix}, \qquad V_2 = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ tV \end{pmatrix}.$$

We must have $n \leq m+1$ since the columns of V are affinely independent. If $n = m+1$, then

$$\begin{aligned} \det(J_n + t^2 V^{\mathrm{T}} V) &= \det(V_2^{\mathrm{T}} V_2) = (\det(V_2))^2 \\ &= (t^{n-1}\det(V_1))^2 = t^{2n-2}\det(V_1^{\mathrm{T}} V_1) = t^{2n-2}\det(J_n + V^{\mathrm{T}} V). \end{aligned}$$

So suppose $n < m+1$. Let $W \in \mathbb{R}^{(m+1)\times(m+1-n)}$ be a matrix whose columns form an orthonormal basis for $(\operatorname{lin}(V_1))^{\perp} \subset \mathbb{R}^{m+1}$. Since $\mathbf{0} \in \operatorname{aff}(V)$, the vector $(1, 0, \ldots, 0)^{\mathrm{T}} \in \operatorname{lin}(V_1)$, so the first row of W must be all zero. Therefore,

$$V_2^{\mathrm{T}} W = t V_1^{\mathrm{T}} W = 0.$$

Thus

$$\begin{aligned} \det(J_n + t^2 V^{\mathrm{T}} V) &= \det \begin{pmatrix} J_n + t^2 V^{\mathrm{T}} V & 0 \\ 0 & I_{m+1-n} \end{pmatrix} \\ &= \det((V_2\ W)^{\mathrm{T}}(V_2\ W)) = (\det(V_2\ W))^2. \end{aligned}$$

Since the first row of W is all zero, if we start with the matrix $(V_1\ W)$, multiply the first n columns by t, and then divide the first row by t, we get the matrix $(V_2\ W)$. Thus,

$$\begin{aligned} (\det(V_2\ W))^2 &= t^{2n-2}(\det(V_1\ W))^2 \\ &= t^{2n-2}\det \begin{pmatrix} J_n + V^{\mathrm{T}} V & 0 \\ 0 & I_{m+1-n} \end{pmatrix} = t^{2n-2}\det(J_n + V^{\mathrm{T}} V). \end{aligned}$$

By Theorem 4.3, $\det(J_n + V^{\mathrm{T}} V)$ is nonzero. Therefore, whenever $t \in \mathbb{C}$ is nonzero, $J_n + t^2 V^{\mathrm{T}} V$ is invertible. □

Theorem 7.2. *Let P be a d-polytope in $\mathbb{R}^d$ with $\mathbf{0} \in \operatorname{int}(P)$ and let $j \in \mathbb{N}$ with $j \le d$. Suppose the squared volume of a largest j-simplex in P is λ and the squared cross-sectional volume of a smallest j-simplicial cylinder containing P^Δ is μ. Then*

$$\mu \ge \frac{(j+1)^{2(j+1)}}{(j!)^4 \lambda},$$

with equality if and only if there is a largest j-simplex in P whose centroid is $\mathbf{0}$.

Proof. Let $C \supset P^\Delta$ be a j-simplicial cylinder. Then $S = C^\Delta \subset P$ is a j-simplex and $\mathbf{0} \in \operatorname{relint}(S)$. Suppose $S = \operatorname{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_{j+1}\})$. Define $\mathbf{w}_1, \dots, \mathbf{w}_{j+1} \in \operatorname{aff}(S)$ by requiring $\mathbf{w}_i^{\mathrm{T}} \mathbf{v}_k = 1$ for all $i \ne k$. Then $\mathbf{w}_1, \dots, \mathbf{w}_{j+1}$ are the vertices of the cross section of C that contains $\mathbf{0}$. Let V be the matrix whose columns are $\mathbf{v}_1, \dots, \mathbf{v}_{j+1}$ and let W be the matrix whose columns are $\mathbf{w}_1, \dots, \mathbf{w}_{j+1}$. Then $W = VM$ for some $M \in \mathbb{R}^{(j+1)\times(j+1)}$ satisfying $\mathbf{1}^{\mathrm{T}} M = \mathbf{1}^{\mathrm{T}}$, since the $\mathbf{w}_i$ are in $\operatorname{aff}(S)$. Furthermore, $W^{\mathrm{T}} V - J_{j+1}$ is diagonal. These two properties uniquely determine W.

By Lemma 7.1, the matrix $J_{j+1} - V^{\mathrm{T}} V$ is invertible, and

$$\det(J_{j+1} - V^{\mathrm{T}} V) = (-1)^j \det(J_{j+1} + V^{\mathrm{T}} V).$$

Since $\mathbf{0} \in \operatorname{aff}(S)$, there is a unique vector $\mathbf{a} = (a_1, \dots, a_{j+1})^{\mathrm{T}} \in \mathbb{R}^{j+1}$ such that $\mathbf{0} = V\mathbf{a}$ and $\mathbf{1}^{\mathrm{T}} \mathbf{a} = 1$. Since $\mathbf{0} \in \operatorname{relint}(S)$, the entries of $\mathbf{a}$ are all positive. Note that $\mathbf{a} = (J - V^{\mathrm{T}} V)^{-1} \mathbf{1}$. Let D be the diagonal matrix with diagonal entries $1/a_1, \dots, 1/a_{j+1}$. We claim that $M = (J - V^{\mathrm{T}} V)^{-1} D$. We have

$$\mathbf{1}^{\mathrm{T}} M = \mathbf{1}^{\mathrm{T}} (J - V^{\mathrm{T}} V)^{-1} D = \mathbf{a}^{\mathrm{T}} D = \mathbf{1}^{\mathrm{T}},$$

which implies that $JM = J$. Also,

$$\begin{aligned} W^{\mathrm{T}} V - J &= M^{\mathrm{T}} V^{\mathrm{T}} V - J \\ &= M^{\mathrm{T}} (V^{\mathrm{T}} V - J) \\ &= -D \end{aligned}$$

is diagonal, so we have indeed found the correct choice of M. Therefore

$$\begin{aligned} (j!)^2 \operatorname{vol}^2(C) &= \det(J + W^{\mathrm{T}} W) \\ &= \det(M^{\mathrm{T}} (J + V^{\mathrm{T}} V) M) \\ &= (\det(M))^2 \det(J + V^{\mathrm{T}} V) \\ &= \left(\frac{\det(D)}{\det(J - V^{\mathrm{T}} V)} \right)^2 \det(J + V^{\mathrm{T}} V) \\ &= \frac{(\det(D))^2}{\det(J + V^{\mathrm{T}} V)} \\ &= \frac{1}{(j!)^2 \operatorname{vol}^2(S) \prod_{i=1}^{j+1} a_i^2}. \end{aligned}$$

Now since the sum of the a_i is 1, their arithmetic mean is $1/(j+1)$. Therefore their geometric mean is at most $1/(j+1)$, with equality if and only if they are all equal, that is, if $\mathbf{0}$ is the centroid of S. So

$$\prod_{i=1}^{j+1} a_i^2 \leq \frac{1}{(j+1)^{2(j+1)}}.$$

Therefore, the smallest $\mathrm{vol}^2(C)$ can be is

$$\frac{(j+1)^{2(j+1)}}{(j!)^4\lambda^2},$$

and that is attained if and only if S is largest and has its centroid at the origin. $\square$

To apply Theorem 7.2 to truncated simplices we need the following result.

Theorem 7.3. *Let $G = ([n], E)$ be a labelled directed graph such that every vertex of G has outdegree at least 1. Let $V \in \mathbb{R}^{d\times m}$ have affinely independent columns $\mathbf{v}_1, \ldots, \mathbf{v}_m$, where $m \geq n$, and let $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 2^{-m-1}$. Let $P = T(V, G, \varepsilon)$ and let $S = \Psi_P(g)$ be a bound largest $(m-1)$-simplex in P. Then the centroid of S is the centroid of* $\mathrm{conv}(V)$ *if and only if the union of the cyclic sets of g is* $[n]$.

Proof. Let $\mathbf{c}$ be the centroid of S. Then since $(i, g(i)) \in E$ for all $i \in [n]$,

$$\mathbf{c} = \frac{1}{m}\left(\left(\sum_{i\in[n]}((1-\varepsilon)\mathbf{v}_i + \varepsilon\mathbf{v}_{g(i)})\right) + \left(\sum_{i\in[m]\setminus[n]} \mathbf{v}_i\right)\right).$$

Clearly, $\mathbf{c} = (1/m)\sum_{i\in[m]} \mathbf{v}_i$ if and only if

$$\sum_{i\in[n]} \mathbf{v}_{g(i)} = \sum_{i\in[n]} \mathbf{v}_i.$$

Because of the affine independence of the $\mathbf{v}_i$, this happens if and only if for each $j \in [n]$, $j = g(i)$ for a unique $i \in [n]$. This is equivalent to saying that the union of the cyclic sets of g is $[n]$. $\square$

8. Hardness Results

In this section we prove the $\mathbb{NP}$-hardness of the problems HLGSTSIMPLEX$_f$, VLGST-SIMPLEX$_f$, HSMLSTSIMPCYL$_f$, and VSMLSTSIMPCYL$_f$, for a certain class of functions f. The key step in proving $\mathbb{NP}$-hardness of these problems is the following construction, which relies heavily on properties of truncated simplices.

Definition 8.1. Let $G = ([3q], E)$ be a labelled directed graph for some $q \in \mathbb{N}$ and let $k, m \in \mathbb{N}$ with $3q + 1 \leq k \leq m$.

Let

$$A_1 = \{x \in \mathbb{R}^m\colon\ x_{k+1} = \cdots = x_m = 0\}$$

and

$$A_2 = \{x \in \mathbb{R}^m\colon\ x_1 = \cdots = x_k = 0\}.$$

Let $\iota_1\colon\ \mathbb{R}^k \to A_1$ by

$$\iota_1(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

and if $k < m$, let $\iota_2\colon\ \mathbb{R}^{m-k} \to A_2$ by

$$\iota_2(x_1, \ldots, x_{m-k}) = (0, \ldots, 0, x_1, \ldots, x_{m-k}).$$

Let $\varepsilon = 2^{-k-3}$, and let V_k be as in Definition 6.7. Let

$$P = T(V_k, G \cup \vec{C}_{k-3q+1}, \varepsilon) \subset \mathbb{R}^k.$$

If $k = m$, define $R_{G,k,m} = P$. If $k < m$, let V_{m-k} and ρ_k be as in Definition 6.7, and define

$$R_{G,k,m} = \operatorname{conv}(\iota_1(P) \cup \iota_2((\rho_k/2)\operatorname{conv}(V_{m-k}))) \subset \mathbb{R}^m.$$

Theorem 8.2. *Let everything be as in Definition* 8.1. *Then $R_{G,k,m}$ is a rational m-polytope in $\mathbb{R}^m$ with $\mathbf{0} \in \operatorname{int}(R_{G,k,m})$. It has $(2k+2)(m-k+1)$ facets, and if $k < m$ it has $k(k+1) + (m-k+1)$ vertices. If $k = m$ it has $k(k+1)$ vertices.*

Suppose every vertex of G has outdegree at least 1. *Let*

$$\lambda = \frac{((1-\varepsilon)^3 + \varepsilon^3)^{2q}((1-\varepsilon)^{k-3q+1} - (-\varepsilon)^{k-3q+1})^2}{(k!)^2} \in \mathbb{Q}.$$

Then there is a k-simplex $S \subset R_{G,k,m}$ with $\operatorname{vol}^2(S) \geq \lambda$ if and only if G has a partition into directed triangles. Furthermore, there is a k-simplicial cylinder $C \supset (R_{G,k,m})^\Delta$ with

$$\operatorname{vol}^2(C) \leq \frac{(k+1)^{2(k+1)}}{(k!)^4\lambda}$$

if and only if G has a partition into directed triangles.

Proof. Clearly, $R_{G,k,m}$ is a rational m-polytope in $\mathbb{R}^m$. Theorem 6.8 shows that $\mathbf{0} \in \operatorname{int}(P)$, and that $\mathbf{0} \in \operatorname{int}((\rho_k/2)\operatorname{conv}(V_{m-k}))$ if $k < m$. Therefore $\mathbf{0} \in \operatorname{int}(R_{G,k,m})$. Since P is a truncated k-simplex, P has $2k+2$ facets and $k(k+1)$ vertices. If $k < m$, $(\rho_k/2)\operatorname{conv}(V_{m-k})$ is an $(m-k)$-simplex, so it has $m-k+1$ vertices and $m-k+1$ facets. Thus, if $k < m$, Theorem 4.7 implies that $R_{G,k,m}$ has $k(k+1) + (m-k+1)$ vertices and $(2k+2)(m-k+1)$ facets.

Theorem 6.8 shows that $\rho_k\mathbb{B}_k \subset P$. If $k < m$, then since $(\rho_k/2)\operatorname{conv}(V_{m-k}) \subset (\rho_k/2)\mathbb{B}_{m-k}$ and $\rho_k/2 < \rho_k$, Theorem 4.4 implies that every largest k-simplex S in

$R_{G,k,m}$ is contained in $\iota_1(P)$. This also trivially holds when $k = m$. Since every vertex of G has outdegree at least 1, Theorem 6.6 implies that

$$\frac{\mathrm{vol}^2(S)}{\mathrm{vol}^2(\mathrm{conv}(V_k))} = \frac{\mathrm{vol}^2(S_1)}{\mathrm{vol}^2(\mathrm{conv}(V_{3q-1}))} \cdot \frac{\mathrm{vol}^2(S_2)}{\mathrm{vol}^2(\mathrm{conv}(V_{k-3q}))},$$

where S_1 is a largest $(3q-1)$-simplex in $T(V_{3q-1}, G, \varepsilon)$ and S_2 is a largest $(k-3q)$-simplex in $T(V_{k-3q}, \vec{C}_{k-3q+1}, \varepsilon)$. By Theorem 6.4,

$$\frac{\mathrm{vol}^2(S_2)}{\mathrm{vol}^2(\mathrm{conv}(V_{k-3q}))} = ((1-\varepsilon)^{k-3q+1} - (-\varepsilon)^{k-3q+1})^2.$$

By Theorem 6.3,

$$\frac{\mathrm{vol}^2(S_1)}{\mathrm{vol}^2(\mathrm{conv}(V_{3q-1}))} \le ((1-\varepsilon)^3 + \varepsilon^3)^{2q},$$

with equality if and only if G has a partition into directed triangles. So

$$\frac{\mathrm{vol}^2(S)}{\mathrm{vol}^2(\mathrm{conv}(V_k))} \le (k!)^2\lambda,$$

with equality if and only if G has a partition into directed triangles. Since

$$\mathrm{vol}(\mathrm{conv}(V_k)) = \frac{1}{k!},$$

we have proven the first part of the theorem.

Now, let α be the squared volume of a largest k-simplex in $R_{G,k,m}$ and let μ be the squared cross-sectional volume of a smallest k-simplicial cylinder containing $(R_{G,k,m})^\Delta$. If G does not have a partition into directed triangles, then $\alpha < \lambda$, so by Theorem 7.2,

$$\mu \ge \frac{(k+1)^{2(k+1)}}{(k!)^4\alpha} > \frac{(k+1)^{2(k+1)}}{(k!)^4\lambda}.$$

If G does have a partition into directed triangles, then $\alpha = \lambda$. Let S be a largest k-simplex in $R_{G,k,m}$. Then S is a largest k-simplex in $\iota_1(P)$, and $\Psi^{-1}_{\iota_1(P)}(S)$ has q cyclic 3-sets and one cyclic set of size $k-3q+1$. So the union of the cyclic sets of $\Psi^{-1}_{\iota_1(P)}(S)$ is $[k+1]$, and by Theorem 7.3, the centroid of S is the centroid of $\mathrm{conv}(V_k)$, which is the origin. So in this case, Theorem 7.2 gives

$$\mu = \frac{(k+1)^{2(k+1)}}{(k!)^4\lambda},$$

which completes the proof. □

Theorem 8.3. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that $f(d) \le d$ for all d, $f(d)$ is computable in time polynomial in d, and there is some positive integer k such that $f(d) = \Omega(d^{1/k})$. Then the problems* HLgstSimplex$_f$, VLgstSimplex$_f$, HSmlstSimpCyl$_f$, *and* VSmlstSimpCyl$_f$ *are all* $\mathbb{NP}$*-hard.*

Proof. Let $G = ([3q], E)$ be an instance of PARTITION INTO DIRECTED TRIANGLES. We can determine in polynomial time whether there is a vertex of G with outdegree 0, and if there is then G clearly cannot have a partition into directed triangles. So assume that every vertex of G has outdegree at least 1.

Find the smallest $m \in \mathbb{N}$ such that $f(m) \geq 3q + 1$. This can be done in polynomial time simply by evaluating $f(3q + 1), f(3q + 2), \ldots,$ until a suitably large value is found. By our assumptions on f, the number of evaluations required is polynomially bounded, and each evaluation takes polynomial time.

Set

$$\varepsilon = 2^{-f(m)-3},$$

$$\lambda = \frac{((1-\varepsilon)^3 + \varepsilon^3)^{2q}((1-\varepsilon)^{f(m)-3q+1} - (-\varepsilon)^{f(m)-3q+1})^2}{((f(m))!)^2} \in \mathbb{Q},$$

and

$$\mu = \frac{(f(m)+1)^{2(f(m)+1)}}{((f(m))!)^4 \lambda} \in \mathbb{Q}.$$

It is possible to construct either an H- or V-presentation of either $R_{G,f(m),m}$ or $(R_{G,f(m),m})^{\Delta}$ in polynomial time, since each polytope has a polynomial number of both vertices and facets, and we know exactly which vertices and facets are incident. By Theorem 8.2, we can decide whether G has a partition into directed triangles by asking whether $R_{G,f(m),m}$ contains an $f(m)$-simplex of squared volume at least λ, or by asking whether there is an $f(m)$-simplicial cylinder containing $(R_{G,f(m),m})^{\Delta}$ whose squared cross-sectional volume is at most μ. This shows the $\mathbb{NP}$-hardness of all four problems. □

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