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Nuclear Coulomb Effects on the Induced Pseudoscalar Interaction in the Theory of Muon Capture by Nuclei

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Variations of the strength of the induced pseudoscalar coupling C_P in muon capture by nuclei due to nuclear Coulomb effects on both pion propagator and muon wave function are calculated by assuming point nuclei with the effective charge $Z_{\rm eff}e$. Those Coulomb effects depend not only on $Z_{\rm eff}$ but also on the energy of the neutrino emitted in the process in consideration. In the low-energy region of the neutrino the Coulomb effects on both quantities are nearly cancelled out by each other, while in the high-energy region (corresponding to the case of normal muon capture) a considerable net effect remains and C_P is reduced by several percent (5~9% for $Z_{\rm eff}=7.47\sim16.17$, i. e. for $Z=8\sim20$) compared with its magnitude when no Coulomb effects are taken into account.

§ 1. Introduction

The theoretical treatments of the "induced pseudoscalar interaction" (hereafter abbreviated to IPSI) in muon capture by a proton inside a nucleus have hitherto been carried out[1] under the following two assumptions: (1) IPSI is induced by a virtual free-pion which is exchanged between nucleon current and lepton $(\mu^- - \nu_\mu)$ current,*) where "free" means that use has been made of pion propagator "in vacuo"; (2) the muon may be considered to be captured locally by each proton just at its position through the (effective) direct pseudoscalar interaction with the coupling constant C_P which is derived from one-pion-exchange picture. This means that the calculations of the muon-capture matrix elements have been carried out in terms of the muon wave function evaluated at the position of each proton by which the muon is captured.

First the assumption (1) is incorrect, since the virtual charged pion propagates under the influence of the nuclear Coulomb field.

Next the assumption (2) is also invalid, since the muon does not annihilate at the position of the capturing proton but just at the position where the muon encounters the virtual charged pion.

The aim of this paper is to evaluate the change of C_P resulting from improving those two defects of the usual treatments along the abovementioned

*) Of course, processes other than the one-pion-exchange process (hereafter abbreviated to OPEP) also contribute to IPSI. However, since OPEP seems to be the most dominant,[2] we shall confine all our arguments to OPEP in this paper.

criticism. Although we carry out this evaluation while adopting both a crude model of nuclei and perturbative treatment, the present task must serve as a basis for more detailed future analysis of the same problem.

In § 2, following Glauber-Martin's procedure for electron Green's function under nuclear Coulomb field,[3] we derive a modified pion-propagator*) under nuclear Coulomb field**) as a solution of Whittaker's confluent hypergeometric equation. Variations of C_P due to the use of the modified pion-propagator are numerically evaluated as functions of both ν (the energy of the neutrino emitted) and Z_{eff}, where Z_{eff}—the effective nuclear charge divided by the elementary charge e representing the charge of a proton***)—is used everywhere in this paper in order to take into account the fact that the orbit of the muon is partly embedded inside the nucleus. Of course $Z_{\text{eff}}e$ for the muon is smaller than the actual nuclear charge Ze. Those facts concerning Z_{eff} are well known since Wheeler's original work.[6] In this paper we adopt Sens's results[7] for Z_{eff}. The effect treated in § 2 is called the Coulomb effect on pion propagator. The numerical results show that this effect increases the magnitude of C_P especially for both low ν and high Z_{eff}.

In § 3 the second assumption is improved by taking into account the non-locality of our capture process commented above. The effect thus obtained is called the Coulomb effect on muon wave function, since it depends on Z_{eff} (the meaning of this nomenclature for the effect shall later be clarified in § 5). In order to evaluate this effect apart from the Coulomb effect on pion propagator, in § 3 we use intentionally pion propagator in vacuo. Thus we obtain another variation of C_P which is reduced in contrast with the increasing effect in § 2.

In § 4 the combined Coulomb effect on both pion propagator and muon wave function is calculated in a compact form. As it can already be foreseen from the results in § 2 and § 3, and also as can be confirmed from the results in § 4, both Coulomb effects nearly cancel out each other in the region of the low ν which corresponds to a part of the high-energy region of the γ-ray spectrum of radiative muon capture.****) On the contrary, considerable net Coulomb effects which reduce C_P appreciably remain in the region of the high ν which corresponds mainly to normal muon capture.

For simplicity the same value of Z_{eff} is assumed for the pion by neglecting mass- and spin-differences. Implication of this assumption is briefly stated in § 5 together with other discussions and summary for the preceding sections.

Finally we will emphasize the fact that the Coulomb effects treated in this

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*) It was pointed out by Chu et al. concerning radiative muon capture by nuclei that we shall need a somewhat modified pion-propagator.[4] See also the fifth footnote in § 5.

**) A modified "muon"-propagator under nuclear Coulomb field has already been used by Lobov for the virtual muon after the emission of γ-ray in the problem of radiative muon capture by a point nucleus,[5] but, of course, it has no direct relation with IPSI.

***) This definition of e (i. e. $e>0$) is used throughout this paper.

****) Concerning a relation to radiative muon capture, refer to the brief discussion in § 5.

paper (especially the Coulomb effect on pion propagator) are different from those studied by Tadić,[8] and Blokhintsev and Dolinsky.[9),*)] Originally, Tadić considered IPSI to be proportional to $m_e + ieA_4\gamma_4$ [the electron mass plus Coulomb potential]. Later, Blokhintsev and Dolinsky showed that Tadić's Coulomb potential term is exactly cancelled out by another term appearing by the gauge invariance requirement, and there remains only an "ordinary" IPSI term which is proportional to m_e. This is, of course, correct. However our Coulomb effect on pion propagator modifies the "ordinary" IPSI term itself as well as both Tadić's term and its counter term. The Coulomb effect on muon wave function is also irrelevant to the criticism made in the reference 9). The detailed discussions are given in Appendix.

§ 2. Coulomb effect on pion propagator

Our derivation is based on the close analogy to the procedure of Glauber-Martin's theory of radiative electron capture[3] with a correspondence**) shown in the combination of Figs. 1a and 1b: That is, the virtual electron-, initial electron- and photon-lines in Fig. 1a correspond to the pion-, muon- and neutrino***)-lines respectively in Fig. 1b.

After elementary calculations closely analogous to Glauber-Martin's ones, we obtain the following S-matrix element for the process of Fig. 1b:

$$\langle S(\infty)\rangle = (2\pi)^4 i\delta(M_p + E_{\mu 1S} - E_n - \nu)\,\delta(\boldsymbol{p}_n + \boldsymbol{\nu})\,C\sqrt{\frac{M_n}{2\nu E_n}}\{\bar{u}_n(\boldsymbol{p}_n)\gamma_5 u_p(0)\}$$

$$\times \int \mathcal{G}_{E_{\mu 1S}-\nu}(0, \boldsymbol{r})\,\bar{u}_\nu(\boldsymbol{\nu})\exp(-i\boldsymbol{\nu}\cdot\boldsymbol{r})\,(1-\gamma_5)\,\psi_{\mu 1S}(\boldsymbol{r})\,d\boldsymbol{r} \qquad (2\cdot 1)$$

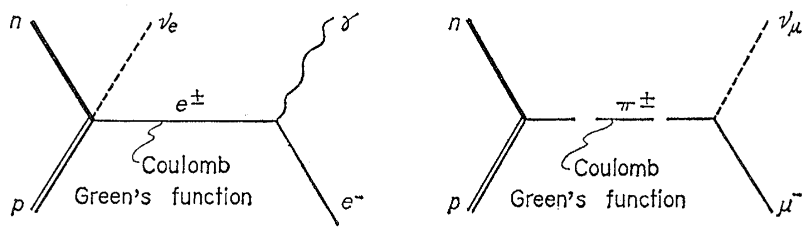

Fig. 1a. Fig. 1b.

Fig. 1. Diagrams showing the correspondence of the present consideration to Glauber-Martin's theory; Fig. 1a: Radiative electron capture; Fig. 1b: Muon capture through one pion exchange.

*) Although they treated mainly the case of β-decay, the essence of the matter is the same for the case of μ-capture.

**) Another correspondence to Glauber-Martin's theory, in which merely all electron lines in Fig. 1a are replaced by muon lines, was adopted by Lobov[5] (see the third footnote in § 1).

***) Since there is no need to say that the neutrinos in Figs. 1a and 1b are the electron- and muon-neutrino respectively, we omit the distinctive words for neutrinos—"electron-" and "muon-"—throughout this paper.

with

$$C = \sqrt{2}g_\pi (2\pi)^{-9/2} m_\mu f_V / m_\pi \,.$$

Here, besides the obvious notations, $\psi_{\mu 1S}(\boldsymbol{r})$ denotes the $1S$-state muon wave function with the total muon energy $E_{\mu 1S}$; g_π is the coupling constant for the strong interaction between charged pions and nucleons; f_V represents the coupling constant for the phenomenological $\pi \to \mu\nu$ decay interaction [see Eq. (A·4b)]; M_p, M_n, m_μ and m_π are the proton-, neutron-, muon- and pion-masses respectively; $\mathcal{G}_E(0, \boldsymbol{r})$, which satisfies the Green's function equation

$$\{\nabla^2 - m_\pi{}^2 + (E + eA_0)^2\}\mathcal{G}_E(0, \boldsymbol{r}) = -\delta(\boldsymbol{r}) \tag{2·2}$$

or in an approximation up to the order of the fine structure constant α

$$(\nabla^2 - m_\pi{}^2 + E^2 + 2EeA_0)\mathcal{G}_E(0, \boldsymbol{r}) = -\delta(\boldsymbol{r}), \tag{2·2'}$$

is a pion Green's function corresponding to the propagation of π^- with energy E from an arbitrary point $\boldsymbol{r}$ to an origin 0 (and the propagation of π^+ with energy $-E$ from 0 to $\boldsymbol{r}$) under the Coulomb field of nuclear charge $Z_{\text{eff}}e$ located at the origin; in Eqs. (2·2) and (2·2′) A_0 is given by

$$A_0 = Z_{\text{eff}}e/r^{*),**)} \tag{2·3}$$

with $r \equiv |\boldsymbol{r}|$.

These Green's function equations (2·2) and (2·2′) are indeed the same as those for spinless electron (Klein-Gordon particle) which were derived by Glauber and Martin,[3] where the atomic number Z was used in lieu of Z_{eff}. Thus, following Glauber and Martin, we find a solution for Eq. (2·2′) by virtue of Whittaker's confluent hypergeometric equation as below:

$$\mathcal{G}_E(0, \boldsymbol{r}) \equiv \mathcal{G}_E(0, r) = \frac{\mu}{2\pi} e^{-\mu r} \int_0^\infty e^{-2\mu r s} s^{-\eta} (1+s)^\eta ds\,, \tag{2·4}$$

where

$$\mu \equiv \sqrt{m_\pi{}^2 - E^2}\,, \qquad \eta \equiv \frac{Z_{\text{eff}} e^2 E}{\mu} = \frac{Z_{\text{eff}}\alpha E}{\mu}\,.$$

Of course the function $\mathcal{G}_E(0, r)$ given by Eq. (2·4) is reduced to the free-pion Green's function $e^{-\mu r}/4\pi r$ when the parameter Z_{eff} (or consequently η) tends to zero.

Substituting the solution (2·4) into the expression (2·1), we have the integral part of $\langle S(\infty)\rangle$ including the retardation factor of the neutrino as follows:

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*) In this paper we use the Gaussian system of units for the elementary charge e, and also use is made of the convention $c = \hbar = 1$, so that the fine structure constant α equals e^2.

**) As stated in § 1, here the same value of Z_{eff} is assumed for the pion; implication of this assumption is stated in § 5.

$$
\begin{aligned}
\mathcal{J}_{E_{\mu1S}-\nu} &\equiv \int \mathcal{G}_{E_{\mu1S}-\nu}(0, r) \exp(-i\boldsymbol{\nu}\cdot\boldsymbol{r}) \varphi_{\mu1S}(\boldsymbol{r}) d\boldsymbol{r} \\
&\cong \varphi_{\mu1S}(0) \int \mathcal{G}_{E_{\mu1S}-\nu}(0, r) \exp(-i\boldsymbol{\nu}\cdot\boldsymbol{r}) d\boldsymbol{r} \\
&= \varphi_{\mu1S}(0) \frac{4\mu^2}{(\mu^2+\nu^2)^2} \int_0^\infty \{(1+s)/s\}^\eta (1+2s) \Big/ \left\{1+\frac{4\mu^2 s(1+s)}{\mu^2+\nu^2}\right\}^2 ds \\
&= \varphi_{\mu1S}(0) \frac{A}{\mu^2+\nu^2} \int_0^1 \frac{v^{-\eta}(1-v^2)}{\{1+(A-2)v+v^2\}^2} dv
\end{aligned}
\tag{2·5}
$$

with

$$
A = 4\mu^2/(\mu^2+\nu^2).
$$

The last integral can be obtained by a change of variable $v=s/(1+s)$ from the preceding integral. $\varphi_{\mu1S}(\boldsymbol{r})$ is the spatial part of muon wave function $\psi_{\mu1S}(\boldsymbol{r})$ which includes the spinor part. In deriving this result (2·5) we have taken the muon wave function $\varphi_{\mu1S}$ out from the integral as if $\varphi_{\mu1S}(\boldsymbol{r})$ took a constant value equalling its value at the origin, or in other words, as if the muon were captured at the origin where the capturing proton lies. This crude assumption shall be improved in the following sections. We would, however, like to use the result (2·5) in this section in order to see separately how much influence the Coulomb effect on virtual pion only exerts upon C_P.

If we put $Z_{\text{eff}}=0$, i.e. $\eta=0$ in the expression (2·5), $\mathcal{J}_{E_{\mu1S}-\nu}$ is reduced to the following form as can easily be confirmed:

$$
\begin{aligned}
\mathcal{J}_{E_{\mu1S}-\nu} &\longrightarrow \varphi_{\mu1S}(0) \frac{1}{\mu^2+\nu^2} \\
&= \varphi_{\mu1S}(0) \frac{1}{m_\pi{}^2+q^2},
\end{aligned}
\tag{2·6}
$$

where q is the 4-momentum transferred at the $\pi\mu\nu$-vertex in Fig. 1b. The factor $1/(m_\pi{}^2+q^2)$ in the result (2·6) is just the pion propagator in vacuo which contributes as a factor to the value of C_P. Therefore, by comparing the expression (2·5) with the expression (2·6), we can obtain the aimed ratio Q of C_P' to C_P as follows, where C_P' is the strength of IPSI including the effect of nuclear Coulomb field on pion propagator, while C_P is the one resulting from pion propagator in vacuo:

$$
\begin{aligned}
Q(\nu, Z_{\text{eff}}) &\equiv \frac{C_P'}{C_P} = \frac{\{A/(\mu^2+\nu^2)\}\int_0^1 [v^{-\eta}(1-v^2)/\{1+(A-2)v+v^2\}^2] dv}{1/(m_\pi{}^2+q^2)} \\
&= A\int_0^1 \frac{v^{-\eta}(1-v^2)}{\{1+(A-2)v+v^2\}^2} dv\,.
\end{aligned}
\tag{2·7}
$$

The integration in the expression (2·7) has been carried out numerically

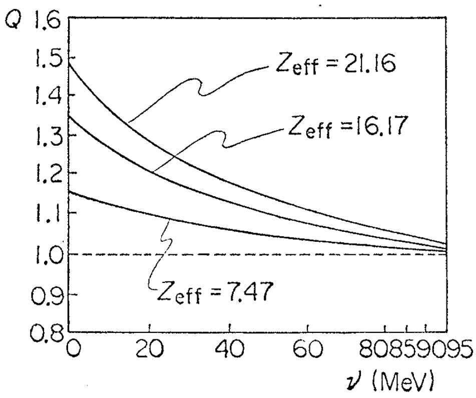

Fig. 2. The ratios Q of C_P' to C_P as functions of the neutrino energy ν for some values of Z_{eff}.

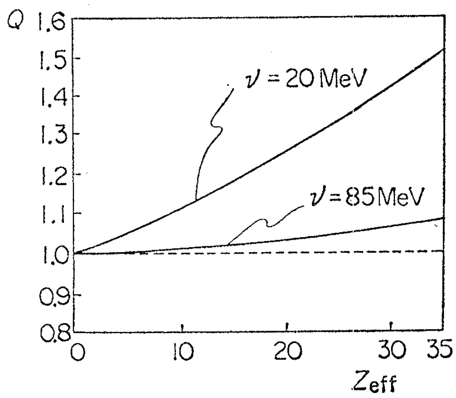

Fig. 3. The ratios Q of C_P' to C_P as functions of Z_{eff} for fixed values of the neutrino energy ν.

using FACOM-270/20 EDPS after some analytic manipulation concerning the singularity of the integrand at the origin. The numerical results for Q are shown in Figs. 2 and 3 both graphically, and also are given in Table I.*)

Since, in normal muon capture, ν is comparatively high ($\nu = 80 \sim 90$ MeV), the variation of C_P is considerably small even for high Z_{eff} as can be seen from Fig. 2 and also from Table I. For instance the rate of increase of C_P amounts to only about 2.4% for $\nu = 90$ MeV in the case of $Z_{eff} = 16.17$.

Table I. [a] The ratios Q of C_p' to C_p for several values of both Z_{eff} and the neutrino energy ν in units of MeV.

ν \ Z_{eff}	3.97 (Be)	5.75 (C)	7.47 (O)	10.72 (Mg)	12.27 (Si)	16.17 (Ca)	19.63 (Fe)	21.16 (Cu)
0	1.070	1.104	1.138	1.207	1.242	1.335	1.427	1.471
20	1.046	1.068	1.089	1.131	1.152	1.207	1.259	1.283
40	1.031	1.045	1.059	1.086	1.099	1.133	1.165	1.179
60	1.019	1.028	1.036	1.053	1.061	1.081	1.100	1.108
80	1.010	1.014	1.019	1.027	1.031	1.041	1.050	1.054
85	1.008	1.011	1.015	1.021	1.025	1.032	1.040	1.043
90	1.006	1.009	1.011	1.016	1.018	1.024	1.029	1.032
95	1.004	1.006	1.007	1.011	1.012	1.016	1.020	1.021

a) In the derivation of this results we have employed the physical constants which were used by Ohtsubo and Fujii:[10] $m_\mu = 105.660 \pm 0.003$, $m_\pi = 139.58 \pm 0.05$ and $\alpha = (7.297203 \pm 0.000101) \times 10^{-3}$, where the masses are given in units of MeV.

*) All the numerical results as functions of ν varying one by one MeV from 0 to 100 MeV for all values of Z_{eff} are now under the present author's hand together with the corresponding results concerning §§ 3 and 4.

§ 3. Coulomb effect on muon wave function
——Effect of initial muon momentum——

In this section we improve the crude assumption adopted in § 2 that we have taken the muon wave function $\varphi_{\mu 1S}$ out of the integration on derivation of the results of Eq. (2·5). Thus the integration of the first line of Eq. (2·5) with respect to the spatial variable $\boldsymbol{r}$ must now be carried out while keeping $\varphi_{\mu 1S}(\boldsymbol{r})$ under the sign of integration. But, in the integration, use will be made of the free-pion Green's function for $\mathcal{G}_{E_{\mu 1S}-\nu}(0, r)$ instead of the Coulomb Green's function (2·4) in order to inspect separately how much influence the Coulomb effect on muon wave function only exerts upon C_P.

Substituting the $1S$-state Coulomb wave function

$$\varphi_{\mu 1S}(\boldsymbol{r}) = \sqrt{m_\mu{}^3\alpha^3 Z_{\rm eff}^3/\pi}\,\exp(-m_\mu \alpha Z_{\rm eff} r)$$
$$= \varphi_{\mu 1S}(0)\,\exp(-m_\mu \alpha Z_{\rm eff} r) \tag{3·1}$$

and the free-pion Green's function

$$\mathcal{G}_{E_{\mu 1S}-\nu}(0, r) = e^{-\mu r}/4\pi r \tag{3·2}$$

into the integral expression of the first line of Eq. (2·5), we have $\mathcal{J}_{E_{\mu 1S}-\nu}$ as follows:

$$\mathcal{J}_{E_{\mu 1S}-\nu} \to \varphi_{\mu 1S}(0)/\{\nu^2 + (\mu + m_\mu \alpha Z_{\rm eff})^2\}$$
$$\simeq \varphi_{\mu 1S}(0)/(\nu^2 + \mu^2 + 2\mu m_\mu \alpha Z_{\rm eff}). \tag{3·3}$$

In the second line of Eq. (3·3) we have neglected the α^2-order term as in Eq. (2·2′).

Thus the last result of Eq. (3·3) divided by the expression (2·6) gives the ratio K of C_P'' to C_P as follows, where C_P'' denotes the strength of IPSI including the Coulomb effect on muon wave function only:

$$K(\nu, Z_{\rm eff}) = \frac{C_P''}{C_P} = \frac{1}{1 + 2\mu m_\mu \alpha Z_{\rm eff}/(\nu^2 + \mu^2)}. \tag{3·4}$$

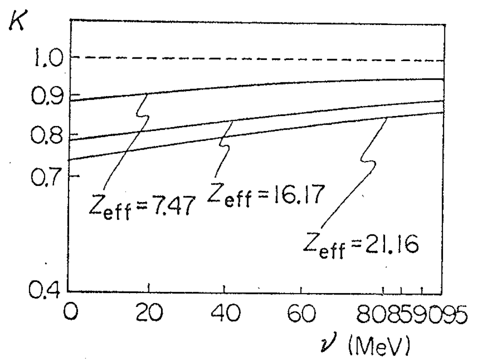

Fig. 4. The ratios K of C_P'' to C_P as functions of ν.

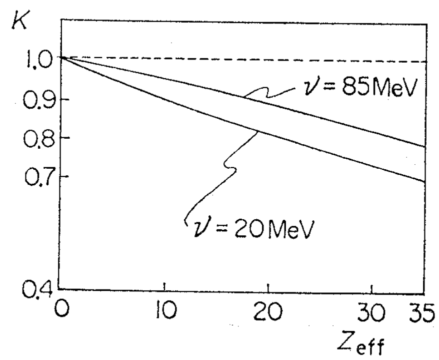

Fig. 5. The ratios K of C_P'' to C_P as functions of $Z_{\rm eff}$.

Table II. The ratios K of C_P'' to C_P for several values of both $Z_{\rm eff}$ and the neutrino energy ν in units of MeV.

ν \ $Z_{\rm eff}$	3.97 (Be)	5.75 (C)	7.47 (O)	10.72 (Mg)	12.27 (Si)	16.17 (Ca)	19.63 (Fe)	21.16 (Cu)
0	0.937	0.911	0.888	0.847	0.828	0.785	0.751	0.737
20	0.949	0.928	0.908	0.873	0.858	0.820	0.790	0.777
40	0.957	0.939	0.922	0.892	0.878	0.845	0.818	0.807
60	0.963	0.947	0.933	0.906	0.894	0.865	0.840	0.830
80	0.968	0.954	0.941	0.918	0.907	0.881	0.859	0.849
85	0.969	0.956	0.943	0.920	0.910	0.884	0.863	0.854
90	0.970	0.957	0.945	0.923	0.912	0.888	0.867	0.858
95	0.971	0.958	0.947	0.925	0.915	0.891	0.871	0.862

The numerical results for the expression (3·4) are given in Figs. 4, 5 and Table II in the same manner as in the previous section.

As can be seen from Fig. 4, the ratios K increase quite gradually with increasing ν under the line of $K=1$ for each $Z_{\rm eff}$. Thus they hold still appreciable deviations from unity even in the high-energy region of the neutrino in which we take our main interest. On the other hand, in the same region of ν, the ratios Q for each $Z_{\rm eff}$ in the previous section are considerably small although they are slightly larger than unity. Hence we may expect that an appreciable net effect will remain in the high-energy region of the neutrino for each $Z_{\rm eff}$ if we combine these effects in both § 2 and this section. Against this situation it is not the case in the low-energy region of the neutrino as can be confirmed in the next section.

Finally we would like to mention another physical meaning of the effect mentioned in this section besides those in § 1. As shown by the subtitle of this section, the effect concerned is also called the effect of initial muon momentum following Glauber-Martin's nomenclature,*) since keeping $\varphi_{\mu 1S}$ under the sign of integration means that the gradient of the initial muon wave function $\varphi_{\mu 1S}$ is correctly taken into account and the non-vanishing gradient corresponds to just the non-vanishing initial momentum, while taking $\varphi_{\mu 1S}$ out from the integration as a constant means the neglect of initial muon momentum.

§ 4. Net Coulomb effect

Now we calculate a combined influence upon C_P from the two kinds of Coulomb effects considered above in a compact form.

We return to the integral expression of the first line of Eq. (2·5). Substituting the Coulomb Green's function for pion (2·4) and the $1S$-state Coulomb wave function of muon (3·1) into $\mathcal{G}_{E_{\mu 1S}-\nu}(0,r)$ and $\varphi_{\mu 1S}(\boldsymbol{r})$ respectively, we obtain

*) See especially the latter half of § 5 of reference 3).

$$\mathcal{G}_{E_{\mu 1S}-\nu}=\varphi_{\mu 1S}(0)\frac{\Lambda}{\mu^2+\nu^2}\int_0^1\frac{v^{-\eta}\{(1-v^2)+2M_\mu(1-v)^2\}}{\{\Lambda v+\Lambda M_\mu(1-v^2)+(\Lambda M_\mu^{\ 2}+1)(1-v)^2\}^2}dv$$

or up to the order of α (i.e. neglecting the terms including $M_\mu^{\ 2}$)

$$\mathcal{G}_{E_{\mu 1S}-\nu}=\varphi_{\mu 1S}(0)\frac{1}{\Lambda(\mu^2+\nu^2)}\int_0^1\frac{v^{-\eta}\{(1-v^2)+2M_\mu(1-v)^2\}}{\{v+M_\mu(1-v^2)+(1-v)^2/\Lambda\}^2}dv \qquad (4\cdot 1)$$

with

$$M_\mu\equiv\frac{m_\mu\alpha Z_{\text{eff}}}{2\mu}.$$

Thus, as before, the result (4·1) divided by the expression (2·6) gives the following ratio R of C_P''' to C_P as a net Coulomb effect, where C_P''' denotes the strength of IPSI due to the combined effect:

$$R(\nu, Z_{\text{eff}})\equiv\frac{C_P'''}{C_P}=\frac{1}{\Lambda}\int_0^1\frac{v^{-\eta}\{(1-v^2)+2M_\mu(1-v)^2\}}{\{v+M_\mu(1-v^2)+(1-v)^2/\Lambda\}^2}dv\,. \qquad (4\cdot 2)$$

The numerical results for the expression (4·2) are given in Figs. 6, 7 and Table III again in the same manner as in the preceding sections.

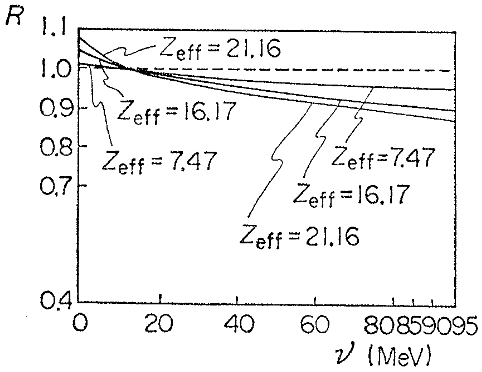

Fig. 6. The ratios R of C_P''' to C_P as functions of ν.

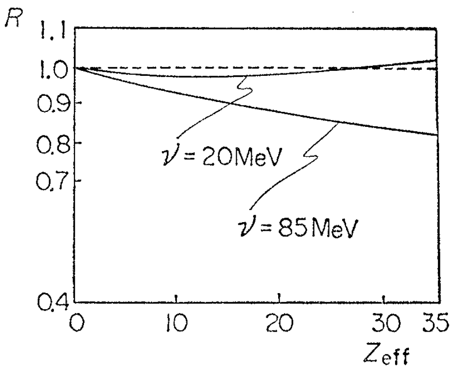

Fig. 7. The ratios R of C_P''' to C_P as functions of Z_{eff}.

Table III. The ratios R of C_P''' to C_P for several values of both Z_{eff} and the neutrino energy ν in units of MeV.

ν \ Z_{eff}	3.97 (Be)	5.75 (C)	7.47 (O)	10.72 (Mg)	12.27 (Si)	16.17 (Ca)	19.63 (Fe)	21.16 (Cu)
0	1.002	1.005	1.008	1.016	1.021	1.036	1.052	1.061
20	0.992	0.990	0.987	0.985	0.984	0.984	0.985	0.987
40	0.986	0.980	0.975	0.966	0.963	0.954	0.948	0.945
60	0.981	0.973	0.966	0.953	0.947	0.933	0.921	0.917
80	0.977	0.968	0.959	0.942	0.934	0.916	0.901	0.894
85	0.976	0.966	0.957	0.940	0.932	0.912	0.896	0.889
90	0.976	0.965	0.955	0.937	0.929	0.909	0.892	0.885
95	0.975	0.964	0.954	0.935	0.926	0.905	0.888	0.880

The results of Fig. 6 or Table III show just the behavior expected in the previous section, i.e. the deviations of R from unity amount to considerable percentages for both high-energy region of the neutrino and intermediately or further high values of Z_{eff}.

§ 5. Summary and discussion

We have calculated the matrix element for muon capture by a proton which is a member of a point-like nucleus bearing positive charge $Z_{\text{eff}}e$. In the calculations we have taken account of both the effect of the nuclear Coulomb field on pion propagator and the effect resulting from the overlap of the spatial variation of the $1S$-state muon wave function on the spatial variation of the pion Green's function respectively up to the order of α. By comparing this matrix element with the one in which none of those effects are taken into account, we have obtained the ratio of the modified C_P to the original C_P.

The numerical results given in Table III show that the effects which we considered lie yet within the range of experimental error at the present stage of experiments. Namely the magnitudes of C_P, or strictly speaking, the ratios C_P/C_A*) deduced from theoretical analyses of experimental data for muon capture by nuclei lie more or less in rather broad regions**) as follows:

The case of C^{12}:

$C_P/C_A = 5\sim30$ by M. Hirooka, T. Konishi, R. Morita, H. Narumi, M. Soga and M. Morita,[13]

$= 5\sim28$ by L. L. Foldy and J. D. Walecka.[14]

The case of O^{16}:

$C_P/C_A = 5\sim20$ by V. Gillet and D. A. Jenkins,[15]

$= 6\sim16$ by M. Morita, M. Hirooka and H. Narumi,[16]

$= 5\sim14$ by A. Fujii, M. Morita and H. Ohtsubo.[17]

The values $7\sim8$ for the ratio C_P/C_A predicted theoretically by Wolfenstein,[18] Goldberger-Treiman[19] and Primakoff[20] lie within all the above-listed "experimental" regions, in which our Coulomb effects are also embedded. Since our effects, however, are not too small, they should be considered in the analysis of muon capture by nuclei in the future when the experimental accuracy shall more increase in muon capture by nuclei and also the magnitude of C_P/C_A for the cases of capture by isolated protons (i.e. capture from μ-mesic hydrogen atom[21]

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*) C_A is the axial vector coupling constant.

**) For the cases corresponding to low values of Z_{eff}, as were clarified by Fujii-Yamaguchi,[11] M. Rho,[12] and Ohtsubo-Fujii,[10] the ratio C_P/C_A is not so ambiguous, but in such cases our Coulomb effects are almost negligible.

or μ-mesic hydrogen molecular ion[22),10)]) shall be established with high reliability.*)

In the actual determination of C_P, however, it is also necessary to pay attention to the exchange current effects which have some influence on the Fermi-coupling constant[23),24)] and/or the Gamow-Teller one,[25)] and thus indirectly on C_P through the evaluation of the capture rate.

As it is clearly seen from the mass-difference between pion and muon, Z_{eff} must be smaller for pion than for muon. Thus, if we use the "true Z_{eff} for pion" in § 2 as we should do so, the "true" values of Q must be always smaller than those given in Table I. On the contrary the "true" values of K remain as they stand in Table II, since we should use "Z_{eff} for muon" from the first in § 3. Therefore the deviation of the "true" net Coulomb effect R from unity must be always larger than those given in Table III. In other words our results are rather underestimated.

Semi-classically speaking, after the muon is transmuted into a neutrino and a negatively charged pion, the latter must be attracted towards the nucleus according to Coulomb's attraction law and shall then be absorbed by a proton inside the nucleus through the strong interaction. A similar consideration is possible for the case of positively charged pion. Thus in either case the muon capture by nuclei through OPEP may be more facilitated by nuclear Coulomb field than without it so long as we consider the effect of the field on virtual pion; therefore Q becomes larger than unity. On the other hand our procedure in § 3 means evidently that we have taken into account the influence of nuclear Coulomb field on the muon by keeping the muon wave function under the sign of integration in the calculation of $\mathcal{G}_{E_{\mu 1S}-\nu}$ defined by the first line of Eq. (2·5). The reason why K is smaller than unity is evident from the behavior of the $1S$-state muon wave function (decrease with increasing r). Those which are mentioned here are also the origin of our nomenclature—"Coulomb effect on muon wave function"—for the effect treated in § 3.

In some processes of radiative muon capture through OPEP in which the high-energy γ-ray is emitted from, for instance, proton- or neutron-lines of the corresponding diagram, the energy parameter $E=E_{\mu 1S}-\nu$ of pion Green's function is large owing to low ν in that case because of energy conservation. Therefore the modified C_P becomes large owing to a mathematical character of the Coulomb Green's function for pion as the case that the low ν of neutrino were emitted in normal muon capture.**) This seems to be accordant with the experimental tendency that C_P in radiative muon capture is larger than C_P in normal muon capture—which has not yet been successfully explained.[26)] However we shall not enter any detailed discussion of the case of radiative muon capture

*) The capture rate is also a function of the induced tensor coupling constant for a fixed magnitude of C_P/C_A, for example $C_P/C_A \sim 8$(or 7) given by Goldberger and Treiman. Ohtsubo and Fujii[10)] performed an analysis of the data on muon capture in liquid hydrogen under the above assumption.

**) See Figs. 2 and 3.

which shall need a separate investigation because of its great complexity.

Our results may comparatively be correct for low values of Z_{eff}, but the accuracy of them may decrease in spite of the use of Z_{eff} instead of Z as Z_{eff} becomes higher. The reason is as follows: Against the former cases (low Z_{eff}), in the latter ones (high Z_{eff}) the nuclei may no longer be considered to be point-like for both muon and pion, and thus not only the electrostatic field acting on the virtual charged pion may considerably deviate from the literal Coulomb field but also the ground state muon wave function must not be the familiar exponential type of the $1S$-state wave function.

Another ambiguity involved in our results may be introduced from the fact that we have not considered the effect of the nuclear potential (due to the strong interaction) on the virtual pion. For instance, we may adopt the so-called optical potential as such an interaction,*) and those potentials for pion have been studied by several authors.[29] We have, however, not taken account of the modification of the pion propagator due to the optical potential.

In order to obtain more precise results, we should give up the use of Z_{eff}; instead, we have to find the "true" muon wave functions by solving numerically the Dirac equation for muon while treating Z protons as they stand in the nucleus. [This was, in fact, done in the cases of C^{12} and O^{16} in references 30) and 16), respectively, and for the both cases in the reference 17).] Then, according to our physical points of view in the present work, we should integrate the "true" muon wave function together with the "rigorously" modified pion propagator as well as other functional factors with respect to each spatial variable which indicates the position where the virtual pion associated with each proton encounters the muon. Here the "rigorously" modified pion propagator means the one including the effects of both the actual electrostatic field inside (and/or near) the nucleus and the nuclear potential (due to the strong interaction) on pion. Of course, all these calculations must also be performed numerically. This has to be performed in the future work. As a first step, we have in the present work treated the problem by slightly simplifying it in order to inspect how much influence our Coulomb effects exert upon C_P.

The formulae for Q, K and R in previous sections are also applicable to the case of K-electron capture by nuclei after necessary replacements of parameters. However the variations of C_P in this case are negligibly small. For example, $Q=1.00141\sim1.00004$[i.e. increase by $0.141\sim0.004\%$], $K=0.99606$ (nearly constant for ν) [i.e. decrease by 0.394%] and $R=0.99747\sim0.99610$ [i.e. decrease by $0.253\sim0.390\%$] for both $Z=74$**) [i.e. $Z_{\text{eff}}=32.77$ (tungsten)] and $\nu=0.1\sim0.5\,\text{MeV}$.

*) This idea was suggested for the present author by T. Miyazima.[27] Independently of this suggestion, however, it is reported that M. Rho has attempted to take into account the optical potential effect on the pion propagator.[28]

**) Now it is sufficient to use still Z_{eff} in order to obtain the ratio Q by Eq. (2·7), but we should use Z in lieu of Z_{eff} in Eq. (3·4) to compute the ratio K. On the other hand, in Eq. (4·2), it is valid to use Z_{eff} itself in the definition of η while replacing Z_{eff} in the definition of M_μ by Z in order to obtain the resultant ratio R reasonably. Of course we must use always m_e in lieu of m_μ.

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Appendix

Proof of the irrelevance of the Coulomb effects treated in this paper to the Coulomb effects studied by Tadić, Blokhintsev and Dolinsky

We start from the total Hamiltonian in Schrödinger representation $\mathcal{H}_{\text{total}}$:

$$\mathcal{H}_{\text{total}} = \mathcal{H}^{\text{free}} + \mathcal{H}^{\text{int}}_{(1)} + \mathcal{H}^{\text{int}}_{(2)}$$
$$= \mathcal{H}^{(0)} + \mathcal{H}^{\text{int}}_{(2)},$$

where

$$\mathcal{H}^{\text{free}} \equiv \mathcal{H}_N{}^{\text{free}} + \mathcal{H}_\pi{}^{\text{free}} + \mathcal{H}_\mu{}^{\text{free}} + \mathcal{H}_\nu{}^{\text{free}}, \tag{A·1a}$$

$$\mathcal{H}^{\text{int}}_{(1)} \equiv \mathcal{H}_{\pi A} + \mathcal{H}_{\mu A}, \tag{A·1b}$$

$$\mathcal{H}^{\text{int}}_{(2)} \equiv \mathcal{H}_{N\pi} + \mathcal{H}_{\pi\mu\nu} + \mathcal{H}_{NA} + \mathcal{H}_{A\pi\mu\nu} \tag{A·1c}$$

and

$$\mathcal{H}^{(0)} \equiv \mathcal{H}^{\text{free}} + \mathcal{H}^{\text{int}}_{(1)}. \tag{A·1d}$$

In Eqs. (A·1) the subscripts N, π, μ, ν and A denote nucleon-, (charged) pion-, muon- and neutrino-fields, and an external electromagnetic field respectively.

Now we go over a new representation, so to speak, a semi-Heisenberg (hereafter abbreviated to S-H) representation (not the so-called interaction representation) through a unitary transformation caused by a unitary operator

$$U \equiv e^{-i\mathcal{H}^{(0)}t}.$$

Since $\mathcal{H}^{(0)}$ includes $\mathcal{H}_{\pi A}$ and $\mathcal{H}_{\mu A}$ as well as $\mathcal{H}^{\text{free}}$, the field operators for charged pions φ (and its Hermite conjugate $\varphi^\dagger$) and for muon ψ_μ in S-H representation respectively satisfy the following equations of motion:*),**)

$$\left\{\left(\frac{\partial}{\partial x_\lambda} - ieA_\lambda\right)^2 - m_\pi{}^2\right\}\varphi = 0\,, \tag{A·2a}$$

*) Note that our e is positive in all formulae according to the fourth footnote in § 1.

**) Actually we substitute "Coulomb potential of a nucleus" $A_\lambda = (0, 0, 0, iA_0)$ for A_λ, where A_0 is given by Eq. (2·3).

$$\left\{\left(\frac{\partial}{\partial x_\lambda}+ieA_\lambda\right)^2-m_\pi{}^2\right\}\varphi^\dagger=0 \tag{A·2b}$$

and

$$\left\{\gamma_\lambda\left(\frac{\partial}{\partial x_\lambda}+ieA_\lambda\right)+m_\mu\right\}\psi_\mu=0\,, \tag{A·3}$$

whereas other field operators satisfy their own free-field equations. Of course, if necessary, we can proceed with other formulation in which field operators other than those for charged pions and muon also satisfy non-free field equations by modifying the definition of $\mathcal{H}^{(0)}$ in Eq. (A·1d). It is, however, sufficient to choose $\mathcal{H}^{(0)}$ as in Eq. (A·1d) in order to clarify the essence of the problem with which we are concerned in this Appendix.

Schrödinger functional Ψ in S-H representation obeys the equation of motion

$$i\frac{\partial\Psi}{\partial t}=\mathcal{H}_{(2)}^{\rm int}\Psi\,.$$

From this one can construct the S-matrix in the usual way. The lowest order S-matrix element $\langle S_{2,\mu-{\rm cap}}\rangle$ for muon capture thus obtained is as follows (see Figs. A.1 and A.2):

$$\langle S_{2,\mu-{\rm cap}}\rangle=M_1+M_2\,,$$

where

$$M_1\equiv-\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\langle P(H_{N\pi}(x_1)H_{\pi\mu\nu}(x_2))\rangle_0\,,$$

$$M_2\equiv-\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\langle P(H_{N\pi}(x_1)H_{A\pi\mu\nu}(x_2))\rangle_0\,,$$

$$H_{N\pi}(x)=i\sqrt{2}g_\pi\bar{\psi}_n(x)\gamma_5\psi_p(x)\varphi^\dagger(x)\,, \tag{A·4a}$$

$$H_{\pi\mu\nu}(x)=\frac{f_V}{m_\pi}\bar{\psi}_\nu(x)(1-\gamma_5)i\gamma_5\gamma_\lambda\psi_\mu(x)\frac{\partial\varphi(x)}{\partial x_\lambda} \tag{A·4b}$$

and

$$H_{A\pi\mu\nu}(x)=-\frac{f_V}{m_\pi}\bar{\psi}_\nu(x)(1-\gamma_5)i\gamma_5\gamma_\lambda ieA_\lambda(x)\psi_\mu(x)\varphi(x)\,. \tag{A·4c}$$

$H_{A\pi\mu\nu}(x)$ is an additional interaction Hamiltonian density obtained from (A·4b) by the well-known replacement

$$\frac{\partial\varphi}{\partial x_\lambda}\longrightarrow\frac{\partial\varphi}{\partial x_\lambda}-ieA_\lambda\varphi$$

which is required from gauge invariance. P denotes Dyson's time-ordering P-symbol, and $\langle\ \rangle_0$ means the vacuum expectation value for pion-field operators.

After integration by parts with respect to the variable x_2, we have

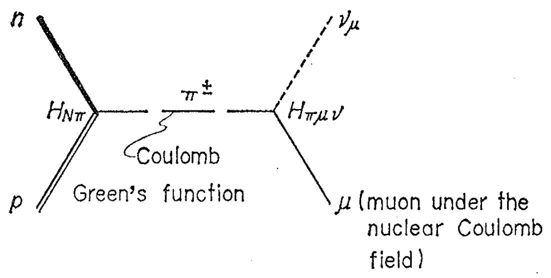

Fig. A.1. Diagram corresponding to M_1.

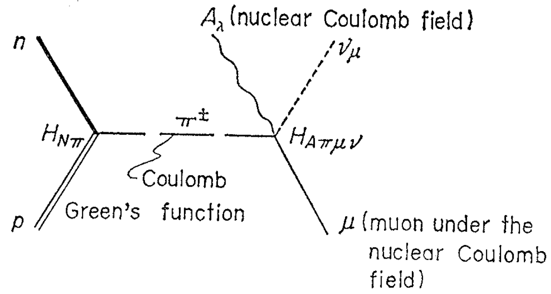

Fig. A.2. Diagram corresponding to M_2.

$$M_1 = i\sqrt{2}g_\pi \frac{f_V}{m_\pi}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\{\bar{\psi}_n(x_1)\gamma_5\psi_p(x_1)\}\left\{\bar{\psi}_\nu(x_2)(1-\gamma_5)i\gamma_5\gamma_\lambda \times\frac{\partial\psi_\mu(x_2)}{\partial x_{2\lambda}}\right\}\langle P(\varphi^\dagger(x_1)\varphi(x_2))\rangle_0 . \tag{A·5}$$

Inserting the relation

$$\gamma_\lambda\frac{\partial\psi_\mu(x_2)}{\partial x_{2\lambda}} = -m_\mu\psi_\mu(x_2) - ieA_\lambda(x_2)\gamma_\lambda\psi_\mu(x_2)$$

coming from the equation of motion (A·3) into Eq. (A·5), we find that M_1 is separated into two parts as follows:

$$M_1 = M_{1a} + M_{1b},$$

where

$$M_{1a} \equiv -i\sqrt{2}g_\pi\frac{f_V}{m_\pi}m_\mu\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\{\bar{\psi}_n(x_1)\gamma_5\psi_p(x_1)\}\{\bar{\psi}_\nu(x_2)(1-\gamma_5) \times i\gamma_5\psi_\mu(x_2)\}\langle P(\varphi^\dagger(x_1)\varphi(x_2))\rangle_0 \tag{A·6}$$

and

$$M_{1b} \equiv -i\sqrt{2}g_\pi\frac{f_V}{m_\pi}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\{\bar{\psi}_n(x_1)\gamma_5\psi_p(x_1)\} \times\{\bar{\psi}_\nu(x_2)(1-\gamma_5)i\gamma_5\gamma_\lambda ieA_\lambda(x_2)\psi_\mu(x_2)\}\langle P(\varphi^\dagger(x_1)\varphi(x_2))\rangle_0 .$$

M_{1b} corresponds just to Tadić's Coulomb term.[8),*)] On the other hand M_2 reads from Eqs. (A·4a) and (A·4c)

$$M_2 = i\sqrt{2}g_\pi\frac{f_V}{m_\pi}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\{\bar{\psi}_n(x_1)\gamma_5\psi_p(x_1)\} \times\{\bar{\psi}_\nu(x_2)(1-\gamma_5)i\gamma_5\gamma_\lambda ieA_\lambda(x_2)\psi_\mu(x_2)\}\langle P(\varphi^\dagger(x_1)\varphi(x_2))\rangle_0 ,$$

*) In Tadić's article, as well as in the article by Blokhintsev and Dolinsky, there appear no pion-field operators explicitly. But the essence of the matter is the same as in their investigations.

which is found at a glance to be cancelled by M_{1b}. The fact of this cancellation is the essence of the criticism by Blokhintsev and Dolinsky.[9] As pointed out by them, there remains only M_{1a} which does not include A_λ field explicitly.*) But $\varphi^\dagger$ and φ in Eq. (A·6) now satisfy the equations of motion under the external (Coulomb) field A_λ (A·2a) and (A·2b). The effect of A_λ in such a meaning is just the Coulomb effect on pion propagator evaluated in §2 of the present text. Indeed the matrix element $\langle S(\infty)\rangle$ given by Eq. (2·1) is nothing but M_{1a}.

The irrelevance of our Coulomb effect on muon wave function to the Coulomb effects argued by Tadić, Blokhintsev and Dolinsky is now no need to say.

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*) When we adopt the pseudovector (PV) coupling instead of the pseudoscalar (PS) one $H_{N\pi}$ [(A·4a)], a similar conclusion can be drawn by using the Dirac equations for the nucleons under the Coulomb potential: The two redundant terms including A_λ explicitly (one of which comes from the Dirac equation for the proton, and another one does from the N-π-A_λ vertex appearing due to the gauge invariance requirement) cancel out each other in a very similar way as in this Appendix, and there remains only a matrix element including no A_λ explicitly which turns out equal to the same expression as M_{1a} [(A·6)] by virtue of the equivalence theorem. If, however, we further introduce in the equations of motion of nucleons the interaction between A_λ and the anomalous magnetic moment (AMM) of the nucleons in the phenomenological form of the Pauli-term [a type of $(\partial A_\nu/\partial x_\mu-\partial A_\mu/\partial x_\nu)\gamma_\mu\gamma_\nu$], there remains another non-vanishing term including AMM besides the above result, since the Pauli-term is gauge invariant by itself and so it has no counter term which cancels it. But, in our case, there is a doubt concerning such an introduction of AMM from the field-theoretic point of view. Thus the problem is open in this meaning.

On the other hand, in the case of the PS coupling $H_{N\pi}$, there also appears a matrix element in the diagram of which the A_λ-line attaches to the nucleon-lines, which reduces in non-relativistic limit to the N-π-A_λ vertex type of matrix element in the case of the PV coupling. This "Coulomb effect on nucleon wave function" has been omitted in this paper, since it may presumably be smaller than the Coulomb effect on pion propagator (and/or on muon wave function) for the reason of the small mass ratios m_π/M and m_μ/M, where M is the nucleon mass. Of course the calculation of this effect has to be done in a more detailed investigation.

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