

Nuclear Fluid Dynamics at a Finite Temperature

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(Received May 12, 1985)

An irrotational nuclear fluid dynamical scheme, on the basis of the quantal action principle, is presented, where the effect of temperature is taken into account. A low temperature expansion, retaining terms up to T^4 , is considered. The temperature dependence of the energies of isoscalar giant resonances is calculated. The nuclear radius, chemical potential, excitation energy and incompressibility of hot nuclei are investigated as functions of the temperature.

§ 1. Introduction

It has been shown that selfconsistent HF plus RPA is a powerful tool in the study of giant multipole states. However when these calculations are made for a finite temperature the numerical situation turns out to be of a higher degree of complexity because the configuration space of possible transitions grows drastically when mixed states take the place of pure states (zero temperature). One thus expects the methods of fluid mechanics to be an appropriate tool to describe such phenomena.

Macroscopic models¹⁾⁻¹¹⁾ have contributed enormously to our understanding of the nucleus. Obviously one should try to justify successful macroscopic models in microscopic terms as for instance TDHF. We expect that the semiclassical methods, which already have given good results for zero temperature, are also appropriate for describing thermally excited systems.

In Ref. 12) a variational derivation of the Liouville-von Neumann equation of statistical mechanics was presented in order to formulate a mean field approximation appropriate to mixed states. A mixed state of equilibrium is described by a density matrix D_0 satisfying the so-called stability condition

$$\text{tr}(D_0 H) \leq \text{tr}(U D_0 U^\dagger H), \quad (1)$$

for all unitary operators U . In the sequel, Eq. (1) is implemented in the independent particle approximation, which means that U and D_0 are exponentials of one body operators. For physical situations requiring the use of mixed states the time evolution of the density matrix may be derived from the lagrangian

$$L = i\hbar \text{tr}(\dot{U} D_0 U^\dagger) - \text{tr}(U D_0 U^\dagger H), \quad (2)$$

where D_0 is a fixed (time independent) density matrix satisfying the condition (1), $U = U(t)$ is a time dependent unitary operator and the dot over U means partial time derivative. We write

$$U = \exp\left(\frac{i}{\hbar} S\right), \quad (3)$$

where S is a time dependent hermitian operator. If a stationary state is slightly perturbed, the system is afterwards described by a density matrix which at all instants differs only

infinitesimally from the stationary one. We neglect therefore in (2) terms of higher order in S than the second obtaining then the lagrangian

$$L^{(2)} = \text{tr} \left(D_o \left(\frac{i}{2\hbar} [S, \dot{S}] - \frac{1}{2\hbar^2} [S, [H, S]] \right) \right), \quad (4)$$

which describes the dynamics of states in the neighborhood of a state of equilibrium. The density matrix is

$$D = D_o - \frac{i}{\hbar} [D_o, S] - \frac{1}{2\hbar^2} [[D_o, S], S] + \dots, \quad (5)$$

where S is the generator of deviations from equilibrium.

In this note we will consider the classical limit⁵⁾ of the lagrangian (4), retaining only the leading order terms in a Wigner-Kirkwood expansion. Presently we restrict the generator S to the first two terms in an expansion in powers of the momentum

$$S = \sum_{i=1}^A S(\mathbf{x}_i, \mathbf{p}_i, t),$$

$$S(\mathbf{x}, \mathbf{p}, t) = \chi(\mathbf{x}, t) + \mathbf{p} \cdot \mathbf{s}(\mathbf{x}, t). \quad (6)$$

The fields χ and \mathbf{s} will be taken as variational fields. Different truncation schemes of the generator S , such as those presented in Refs. 6)~11) provide reliable approximation schemes to the exact dynamics of the atomic nucleus.

§ 2. The static case

It is well-known that the density matrix D_o , exponential of a one-body operator, describes a state of equilibrium if we have

$$\text{tr}(\delta S [H, D_o]) = 0, \quad (7)$$

for all one body operators δS . Equation (7) alone does not ensure stable equilibrium. The stability condition (1) leads to short term equilibrium and should not be confused with the condition for statistical equilibrium which is obtained minimizing the free energy and should be understood as a long term equilibrium.

From Eq. (7) the well-known Hartree-Fock equation for the reduced density matrix $\hat{\rho}_o$ may be derived

$$[h, \hat{\rho}_o] = 0, \quad (8)$$

where h is the selfconsistent single particle hamiltonian. The classical version of this equation is the static form of the Vlasov equation

$$\{h[f_o], f_o\} = 0, \quad (9)$$

where f_o is the equilibrium distribution function (Wigner transform of $\hat{\rho}_o$) and

$$h[f] = \frac{p^2}{2m} + \int d\Gamma_2 v_{12} f(2) + \frac{1}{2!} \iint d\Gamma_2 d\Gamma_3 v_{123} f(2) f(3) + \dots \quad (10)$$

Here the following simplifying notation is adopted: $f(i)$ denotes a function of $\mathbf{x}_i, \mathbf{p}_i$ and

t ; the quantities v_{12}, v_{123}, \dots , stand respectively for the two body, three body, \dots , interactions. The volume integral element is $d\Gamma = g d^3x d^3p / (2\pi\hbar)^3$.

Let W denote the energy functional associated to a distribution function f

$$W[f] = \int d\Gamma_1 \frac{p_1^2}{2m} f(1) + \frac{1}{2!} \iint d\Gamma_1 d\Gamma_2 v_{12} f(1) f(2) \\ + \frac{1}{3!} \iiint d\Gamma_1 d\Gamma_2 d\Gamma_3 v_{123} f(1) f(2) f(3) + \dots \quad (11)$$

Let $f(\mathbf{x}, \mathbf{p})$ be a fixed function of \mathbf{x} and \mathbf{p} and Λ_f the set of all distribution functions f' which are obtained from f by a canonical transformation,

$$\Lambda_f = \{f'(\mathbf{x}, \mathbf{p}): f'(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}, \mathbf{p}) + \{f, S\} \\ + \frac{1}{2} \{\{f, S\}, S\} + \dots, S = S(\mathbf{x}, \mathbf{p}) \text{ real}\}. \quad (12)$$

Let $f_o \in \Lambda_f$ be such that

$$W[f_o] \leq W[f'], \quad (13)$$

for all $f' \in \Lambda_f$. Clearly f_o represents a state of equilibrium satisfying Eq. (9).

In this note we will consider the classical limit of the quantal lagrangian and we will consider density dependent δ forces. In this case f_o may be conveniently written

$$f_o = \frac{\theta(R-r)}{1 + \exp\left[\left(\frac{p^2}{2m} - \zeta\right)/T\right]}, \quad (14)$$

where $\zeta = \zeta(T)$ and $R = R(T)$ are functions of T determined by the condition (13).

The integration in momentum space can be performed immediately

$$E^{(o)} \equiv W[f_o] = \int d^3x \left(\tau_o + \sum_{\sigma} a_{\sigma} \rho_o^{\sigma} \right), \quad (15)$$

where

$$\rho_o = g \int \frac{d^3p}{(2\pi\hbar)^3} f_o \quad (16)$$

and

$$\tau_o = g \int \frac{d^3p}{(2\pi\hbar)^3} f_o \frac{p^2}{2m}. \quad (17)$$

As we are not taking into account the so-called gradient corrections, which give rise to surface contributions, the equilibrium density consistent with our model is a spherical square density

$$\rho_o(\mathbf{x}) = \rho_o(0) \theta(R-r), \quad (18)$$

where $\rho_o(0)$ and R are determined by minimizing the energy, taking into account the conservation of the particle number. The nuclear radius R is fixed by $\rho_o(0)$ and by the particle number A .

Since the interaction through δ function forces leads to a constant selfconsistent potential we chose to consider, due to its simplicity, the following trial distribution function:

$$f = \frac{\theta(R' - r)}{1 + \exp\left[\left(\frac{p^2}{2m} - \zeta'\right)/T'\right]}, \quad (19)$$

where R' , ζ' and T' are only parameters. In particular, the parameter T' should not be interpreted as the temperature.

The distribution function f_0 , characterizing the state of equilibrium, is determined by requiring that the energy is stationary with respect to a canonical transformation. A canonical transformation is a coordinate transformation that preserves Poisson bracket relations. We consider the following coordinate transformation:

$$\mathbf{p} \longrightarrow \frac{\mathbf{p}}{\eta}, \quad \mathbf{x} \longrightarrow \eta \mathbf{x}, \quad (20)$$

where η is an arbitrary quantity. Under this transformation the distribution function becomes replaced by

$$f = \frac{\theta\left(\frac{R'}{\eta} - r\right)}{1 + \exp\left[\left(\frac{p^2}{2m} - \zeta'\eta^2\right)/T'\eta^2\right]}. \quad (21)$$

Thus the canonical transformation (20) leads to the following replacements:

$$T' \longrightarrow T = \eta^2 T', \quad R' \longrightarrow R = \frac{R'}{\eta}, \quad \zeta' \longrightarrow \zeta = \eta^2 \zeta'. \quad (22)$$

The quantities ρ_0 and τ_0 are easily obtained replacing the expression (21) in the definitions (16) and (17). The integration in momentum space is done with the well-known formula, which applies to degenerate Fermi gases,¹³⁾

$$\int_0^\infty d\varepsilon \frac{g(\varepsilon)}{1 + \exp[(\varepsilon - \zeta)/T]} = \int_0^\zeta g(\varepsilon) d\varepsilon + \frac{\pi^2}{6} T^2 g'(\zeta) + \frac{7\pi^4}{360} T^4 g'''(\zeta) + \dots \quad (23)$$

Since we are only interested in low temperatures we will keep only the terms up to the order T^4 in the expansion of thermodynamic quantities. Taking into account (23), we obtain

$$\rho_0 = \alpha \theta\left(\frac{R'}{\eta} - r\right) \left[\frac{2}{3} (\eta^2 \zeta')^{3/2} + \frac{\pi^2}{12} (\eta^2 T')^2 (\eta^2 \zeta')^{-1/2} + \frac{7\pi^4}{960} (\eta^2 T')^4 (\eta^2 \zeta')^{-5/2} \right], \quad (24)$$

$$\tau_0 = \alpha \theta\left(\frac{R'}{\eta} - r\right) \left[\frac{2}{5} (\eta^2 \zeta')^{5/2} + \frac{\pi^2}{4} (\eta^2 T')^2 (\eta^2 \zeta')^{1/2} - \frac{7\pi^4}{960} (\eta^2 T')^4 (\eta^2 \zeta')^{-3/2} \right], \quad (25)$$

where

$$\alpha = 4\pi\sqrt{2}g\left(\frac{\sqrt{m}}{2\pi\hbar}\right)^3. \quad (26)$$

Let \mathcal{Q} be the volume of the nucleus,

$$\Omega = \frac{4\pi}{3} R^3 = \frac{4\pi}{3} \frac{R'^3}{\eta^3}. \quad (27)$$

The particle number A remains constant when we change η . Thus we have

$$-\frac{\delta\Omega}{\Omega} = \frac{\delta\rho}{\rho} = 3\frac{\delta\eta}{\eta}. \quad (28)$$

We will now determine the canonical transformation which minimizes E by considering arbitrary variations with respect to η

$$\begin{aligned} \delta E &= \delta\Omega(\tau + \sum_{\sigma} a_{\sigma}\rho^{\sigma}) + \Omega(\delta\tau + \delta\rho \sum_{\sigma} a_{\sigma}\rho^{\sigma-1}) \\ &= -\Omega \frac{3\delta\eta}{\eta} (\tau + \sum_{\sigma} a_{\sigma}\rho^{\sigma}) + \Omega \frac{3\delta\eta}{\eta} \left(\frac{5}{3}\tau + \sum_{\sigma} a_{\sigma}\rho^{\sigma} \right) \\ &= \Omega \frac{3\delta\eta}{\eta} \left(\frac{2}{3}\tau + \sum_{\sigma} a_{\sigma}(\sigma-1)\rho^{\sigma} \right). \end{aligned} \quad (29)$$

Thus for a given temperature $T(=\eta^2 T')$ the parameter $\zeta(=\eta^2 \zeta')$ appearing in the equilibrium distribution function f_o (Eq. (14)) is determined by introducing ρ_o and τ_o , as given by Eqs. (16) and (17), in the equilibrium condition

$$\frac{2}{3}\tau_o + \sum_{\sigma} a_{\sigma}(\sigma-1)\rho_o^{\sigma} = 0. \quad (30)$$

Once we know the quantity ζ , for a given temperature T , it is straightforward to obtain the density ρ_o and the kinetic energy density τ_o corresponding to the equilibrium state. These results are the generalization of previous results derived in Ref. 9) for $T=0$. Equation (30) insures the stability of the thermal Hartree-Fock state. At a finite temperature we find from Eq. (30) the following expression for ζ :

$$\zeta(T) = \zeta(0) + \lambda T^2 + \gamma T^4 + \dots, \quad (31)$$

where

$$\lambda = B/D, \quad \gamma = C/D,$$

$$B = -\frac{\pi^2}{6\zeta(0)} \left[\zeta^{1/2}(0) + \zeta^{-1/2}(0) \sum_{\sigma} a_{\sigma} \frac{\sigma(\sigma-1)}{2} \left(\alpha \frac{2}{3} \zeta^{3/2}(0) \right)^{\sigma-1} \right], \quad (32)$$

$$\begin{aligned} C = & -\frac{\pi^4}{\zeta(0)^3} \left\{ \zeta^{1/2}(0) \left[\frac{1}{2} \left(\zeta(0) \frac{\lambda}{\pi^2} \right)^2 + \frac{1}{12} \zeta(0) \frac{\lambda}{\pi^2} - \frac{7}{1440} \right] \right. \\ & + \zeta^{-1/2}(0) \sum_{\sigma} a_{\sigma} \sigma(\sigma-1) \left(\alpha \frac{2}{3} \zeta^{3/2}(0) \right)^{\sigma-1} \\ & \left. \times \left[\frac{1}{4} \left(\zeta(0) \frac{\lambda}{\pi^2} \right)^2 - \frac{1}{24} \zeta(0) \frac{\lambda}{\pi^2} + \frac{7}{960} + \frac{3}{4} (\sigma-1) \left(\zeta(0) \frac{\lambda}{\pi^2} + \frac{1}{12} \right)^2 \right] \right\}, \end{aligned} \quad (33)$$

$$D = \frac{2}{3} \zeta^{1/2}(0) + \zeta^{-1/2}(0) \sum_{\sigma} a_{\sigma} \sigma(\sigma-1) \left(\alpha \frac{2}{3} \zeta^{3/2}(0) \right)^{\sigma-1}. \quad (34)$$

Since the interaction is constructed in terms of δ forces, it follows from (10) that the single particle hamiltonian has the following expression:

$$h = \frac{p^2}{2m} + \sum_{\sigma} a_{\sigma} \rho \rho^{\sigma-1}. \quad (35)$$

Taking into account the definition of ζ , and the expression (35) for the single particle hamiltonian, we see that the chemical potential $\mu(T)$ is obtained adding to $\zeta(T)$ the selfconsistent potential

$$\mu = \zeta + \sum_{\sigma} a_{\sigma} \rho \rho^{\sigma-1}. \quad (36)$$

§ 3. Lagrangian, equations of motion and boundary conditions

We shall now be concerned with the time dependent problem, namely we will be interested in small amplitude oscillations around the equilibrium state due to a small external perturbation. For studying the detailed response of the system to a given external field, it is desirable to derive and solve dynamical equations of motion, rather than imposing an apriori specified form of motion. The linear response relies on a cutoff of the expansion in powers of the generator S at the leading order. Taking the classical limit of (4), with the generator S given by (6), we obtain, in an analogous way to Refs. 6) ~ 11), the nuclear fluid dynamical lagrangian

$$L^{(2)} = \int d^3x \left(\rho_o \mathbf{s} \cdot \nabla \dot{\chi} - \frac{\rho_o}{2m} (\nabla \chi) \cdot (\nabla \chi) \right) - E^{(2)}[\mathbf{s}]. \quad (37)$$

The intrinsic energy functional $E[\mathbf{s}]$ contains all \mathbf{s} dependent terms of $\text{tr}(DH)$,

$$E^{(2)}[\mathbf{s}] = \int d^3x \left[\frac{\tau_o}{3} ((\nabla \cdot \mathbf{s})^2 + \frac{1}{2} (\partial_{\alpha} s_{\beta} + \partial_{\beta} s_{\alpha})^2) + \sum_{\sigma} a_{\sigma} \frac{\sigma(\sigma-1)}{2} \rho_o^{\sigma} (\nabla \cdot \mathbf{s})^2 \right]. \quad (38)$$

Here and hereafter we employ the summation convention over repeated vector indices. The linear terms in the energy functional $E[\mathbf{s}]$ vanish in view of Eq. (30). This equation also enables us to eliminate the surface term, involving $d\theta/dr$,

$$\frac{1}{2} \int d^3x (\nabla \cdot \mathbf{s}) \mathbf{s} \cdot \nabla \left[\frac{2\tau_o}{3} + \sum_{\sigma} a_{\sigma} (\sigma-1) \rho_o^{\sigma} \right], \quad (39)$$

in the energy functional $E^{(2)}[\mathbf{s}]$. We see that, for finite temperatures, the structure of the fluid dynamical lagrangian is the same as in the zero temperature limit.⁶⁾ The essential difference to this limit lies in the fact that ρ_o and τ_o depend on the temperature T . At this point we introduce a slight generalization of the model of Ref. 6) by allowing for an arbitrary field \mathbf{s} , instead of restricting \mathbf{s} to be irrotational. We will find out the macroscopic fields which satisfy the equations of motion and boundary conditions obtained from arbitrary variations of the dynamical fields which in this case are χ and \mathbf{s} . In fact nothing implies that the field \mathbf{s} should be taken as irrotational. Allowing for arbitrary variations of \mathbf{s} we obtain a fluid dynamical description which is more general than the scheme proposed in Ref. 6) where the scaling field is restricted to the form $\mathbf{s} = \nabla F$. Arbitrary variation of χ leads to the continuity equation

$$\nabla \cdot \left(\rho_o \dot{\mathbf{s}} + \frac{\rho_o}{m} \nabla \chi \right) = 0, \quad (40)$$

and arbitrary variation of \mathbf{s} leads to the Euler type equation

$$\rho_o \partial_\alpha \dot{\chi} = \partial_\beta P_{\alpha\beta} \left(\equiv \frac{\delta E^{(2)}}{\delta s_\alpha} \right), \quad (41)$$

where $P_{\alpha\beta}$ is the pressure tensor and has the following expression:

$$P_{\alpha\beta} = -\frac{2}{3} \tau_o (\delta_{\alpha\beta} \nabla \cdot \mathbf{s} + \partial_\alpha s_\beta + \partial_\beta s_\alpha) - \delta_{\alpha\beta} \sum_\sigma a_\sigma \sigma (\sigma - 1) \rho_o^\sigma \nabla \cdot \mathbf{s}. \quad (42)$$

Equations (40) and (41), together with the analytical form of the quantities ρ_o and τ_o , are sufficient to determine the eigenfrequencies ω_k as well as the corresponding eigenfields $\chi^{(k)}$ and $\mathbf{s}^{(k)}$. All other quantities with physical meaning, such as the transition density $\delta\rho$ and the current \mathbf{j} are determined by χ and \mathbf{s} (see Ref. 6)). If we had taken into consideration a force with a finite range, or if we had taken into account the gradient corrections, then the expressions for $E^{(0)}$, $E^{(2)}$ and $L^{(2)}$ would be different, and in particular the selfconsistent density ρ_o would have a smooth profile. Still we would obtain the equations of motion from the variation of χ and \mathbf{s} and as long as the equilibrium state is known these equations are sufficient to determine the normal modes.

Due to the special form of ρ_o and τ_o , which in our approach are constant for $r \leq R$ and zero for $r > R$, it is clear that the following boundary conditions are contained in the equations of motion (40) and (41)

$$\mathbf{x} \cdot \left(\rho_o \dot{\mathbf{s}} + \frac{\rho_o}{m} \nabla \chi \right) \Big|_{r=R} = 0, \quad (43)$$

$$x_\alpha P_{\alpha\beta} \Big|_{r=R} = 0. \quad (44)$$

The analytical expression of \mathbf{s} , for an arbitrary eigenmode, may be written in general as

$$-\rho_o(0) \dot{\mathbf{s}} = \dot{\beta}(t) \{ \nabla [(\gamma_1 j_l(K_L r) + \gamma_2 r^l) Y_{lm}] + \gamma_3 \nabla \times (r^{l+2} \mathbf{r} \times \nabla Y_{lm}) \}, \quad (45)$$

where we have assumed an harmonic time dependence which is factorized out in the time dependent function $\beta(t)$ ($\ddot{\beta}(t) = -\omega^2 \beta(t)$).

From Eqs. (40) and (41) we can write

$$\rho_o \nabla \cdot \dot{\mathbf{s}} + \frac{\rho_o}{m} \Delta \chi = 0 \quad (46)$$

and

$$\rho_o \nabla \dot{\chi} = - \left(\frac{4}{3} \tau_o + \sum_\sigma a_\sigma \sigma (\sigma - 1) \rho_o^\sigma \right) \nabla (\nabla \cdot \mathbf{s}) - \frac{2}{3} \tau_o \Delta \mathbf{s}. \quad (47)$$

Taking the time derivative of Eq. (47) and inserting the analytical expression (45) for \mathbf{s} we obtain

$$-\omega^2 \rho_o \nabla \chi = -\frac{\dot{\beta}(t)}{\rho_o(0)} \left\{ (2\tau_o + \sum_{\sigma} a_{\sigma} \sigma (\sigma-1) \rho_o^{\sigma}) \nabla (\gamma_1 K_L^2 j_l(K_L r) Y_{lm}) + \frac{2}{3} \tau_o \nabla (\gamma_3 2(2l+3)(l+1) r^l Y_{lm}) \right\}. \quad (48)$$

We now consider the divergence of Eq. (48) and we substitute $\rho_o \Delta \chi$ by $-m \rho_o \nabla \cdot \dot{\mathbf{s}}$ (see Eq. (46)) obtaining

$$\begin{aligned} \omega^2 m \dot{\beta}(t) (\gamma_1 K_L^2 j_l(K_L r) Y_{lm}) \\ = \frac{\dot{\beta}(t)}{\rho_o(0)} (2\tau_o(0) + \sum_{\sigma} a_{\sigma} \sigma (\sigma-1) \rho_o^{\sigma}(0)) \gamma_1 K_L^4 j_l(K_L r) Y_{lm}. \end{aligned} \quad (49)$$

We observe that $\tau_o(0) = \tau_o(r)|_{r=0}$. From Eq. (49) we obtain

$$\omega^2 = K_L^2 \frac{2\tau_o(0) + \sum_{\sigma} a_{\sigma} \sigma (\sigma-1) \rho_o^{\sigma}(0)}{m \rho_o(0)} \quad (50)$$

and therefore the sound velocity is

$$C_L = \left(\frac{2\tau_o(0) + \sum_{\sigma} a_{\sigma} \sigma (\sigma-1) \rho_o^{\sigma}(0)}{m \rho_o(0)} \right)^{1/2}. \quad (51)$$

From Eq. (48), taking into account that $\omega^2 = K_L^2 C_L^2$, we obtain that

$$\frac{\rho_o(0)}{m} \nabla \chi = \dot{\beta}(t) \nabla [(\gamma_1 j_l(K_L r) + \gamma_4 r^l) Y_{lm}], \quad (52)$$

where, for $\omega^2 \neq 0$,

$$\gamma_4 = \frac{4}{3} \frac{\tau_o(0)}{m \rho_o(0) \omega^2} (2l+3)(l+1) \gamma_3. \quad (53)$$

Inserting the expressions (45) and (52) in the boundary condition (43) we obtain

$$l(\gamma_4 - \gamma_2) + l(l+1) \gamma_3 R^2 = 0. \quad (54)$$

Finally, we insert the analytical expression for \mathbf{s} , given by Eq. (45), in the boundary condition (44), $x_{\alpha} P_{\alpha\beta}|_{r=R} = 0 (= P_1(R) \partial_{\beta} Y_{lm} + P_2(R) x_{\beta} Y_{lm})$. For $l \geq 1$ we must have $P_1(R) = 0$ and $P_2(R) = 0$ which reduce respectively to the following equations:

$$\gamma_1 (r \partial_r - 1) j_l(K_L r) + \gamma_2 (l-1) r^l - \gamma_3 l(l+2) r^{l+2} |_{r=R} = 0 \quad (55)$$

and

$$\begin{aligned} \gamma_1 \left[\frac{3\omega^2 m \rho_o(0)}{2\tau_o(0)} - \frac{2l(l+1)}{r^2} + \frac{4}{r} \partial_r \right] j_l(K_L r) \\ - 2\gamma_2 l(l-1) r^{l-2} + 2\gamma_3 l(l+1)^2 r^l |_{r=R} = 0, \end{aligned} \quad (56)$$

where obviously Eq. (55) (derived from the requirement $P_1(R) = 0$) does not apply for $l=0$.

For $l=0$, the terms in γ_2 and γ_3 , appearing in Eq. (45), vanish. Then the values of K_L , corresponding to different eigenmodes, are fixed by the boundary condition (56) which reduces to

$$\left[\frac{3}{2} K_L^2 \left(2 + \frac{\sum_{\sigma} a_{\sigma} \sigma (\sigma - 1) \rho_o^{\sigma}(0)}{\tau_o(0)} \right) + \frac{4}{r} \partial_r \right] j_l(K_L r) |_{r=R} = 0. \quad (57)$$

This boundary condition was derived for $l=0$ and for $T=0$ in Refs. 15), 16) and 6). For $l>0$ the frequency ω and the constants $\gamma_1, \gamma_2, \gamma_3$ and γ_4 , corresponding to each eigenmode, are fixed by Eqs. (53) ~ (56) and by the orthogonality relation¹⁷⁾

$$- \int d^3x \rho_o \mathbf{s}^{(i)} \cdot \nabla \chi^{(k)} = \delta_{ik}, \quad (58)$$

where $\mathbf{s}^{(i)}$ and $\chi^{(k)}$ are time independent eigenfields corresponding to the eigenmodes with frequencies ω_i and ω_k .

It is instructive to examine the possibility of solutions with $\omega=0$ for $l \geq 1$. In this case Eq. (53) does not apply and we may consider $\gamma_1=0$. Then we see from Eqs. (55) and (56) that, except for $l=1$, γ_2 and γ_3 must be zero and from Eq. (54) it follows also that $\gamma_4=0$. Therefore for $l>1$ there is no nontrivial solution with $\omega=0$. However for $l=1$ we obtain that $\gamma_3=0$ and from Eq. (54) we have that $\gamma_4=\gamma_2$. Thus for $l=1$ there exists a $\omega=0$ solution, namely

$$\frac{\rho_o(0)}{m} \nabla \chi = -\rho_o(0) \dot{\mathbf{s}} = \gamma_4 \nabla (r Y_{10}). \quad (59)$$

This solution should appear and obviously corresponds to a uniform translation in space.

At this point it is convenient to recall the expression for the current in order to allow for comparison with other nuclear fluid dynamical schemes. The current is obtained integrating in momentum space the distribution function multiplied with \mathbf{p}/m

$$\mathbf{j} = g \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} f_o = \frac{\rho_o}{m} \nabla \chi. \quad (60)$$

Several authors have considered nuclear fluid dynamical schemes where the relation $\mathbf{j} = -\rho_o \dot{\mathbf{s}}$ is assumed and in particular, in Ref. 6) this relation is directly derived from an action principle. It is however clear from Eqs. (45), (52), (53) and (60) that such a relation is not valid in this simplified scheme, except for $l=0$ and also for $l=1$ when $\omega=0$. In fact, since we have allowed for arbitrary variations of \mathbf{s} we expect that \mathbf{s} will acquire rotational components which cannot contribute to \mathbf{j} since the current is only determined by the scalar field χ .

§ 4. Numerical results

The numerical results presented in this section are for a spherical nucleus with $A=208$. As a model for the collective dynamics we consider the system described by a purely zero range density dependent interaction previously considered in other fluid dynamical calculations^{6)~11)} and which implies the following potential energy:

$$\sum_{\sigma} a_{\sigma} \rho^{\sigma} = a_2 \rho^2 + a_{2+1/6} \rho^{2+1/6}, \quad (61)$$

where

$$a_2 = -\frac{3}{8} \times 3075.8 \text{ MeV fm}^3,$$

$$a_{2+1/6} = \frac{1}{16} \times 2016.4 \text{ MeV fm}^{3+1/2}. \quad (62)$$

Hereby we assume that the parameters a_2 and $a_{2+1/6}$ do not depend on T .

In the present section results are presented in the temperature range from 0 to 6 MeV. As the temperature increases, there is a slight decrease of the equilibrium density. Since the particle number A remains constant there is also a slight growth of the nuclear radius. In Fig. 1 we show the nuclear radius as function of the temperature. Naturally, as the temperature increases there is also an increase of the energy E (see Eq. (15)) corresponding to the equilibrium state. In Fig. 2 we plot the dependence of the excitation energy E^* on the temperature, where E^* is defined as

$$E^*(T) = E(T) - E(0). \quad (63)$$

For a finite temperature the nuclear incompressibility can be defined by

$$K = \frac{9}{A} \rho^2 \frac{d^2 E}{d\rho^2}. \quad (64)$$

By similar arguments to the ones used in § 2, we find

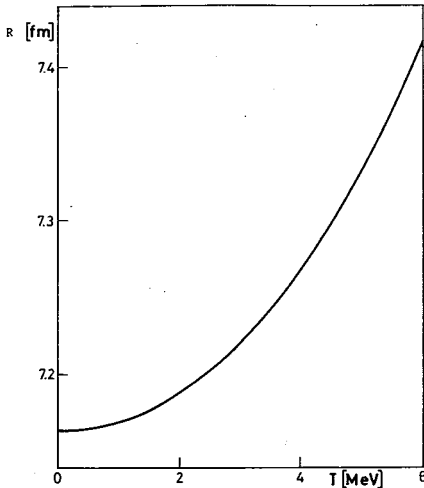


Fig. 1. Nuclear radius ($A=208$) as function of the temperature.

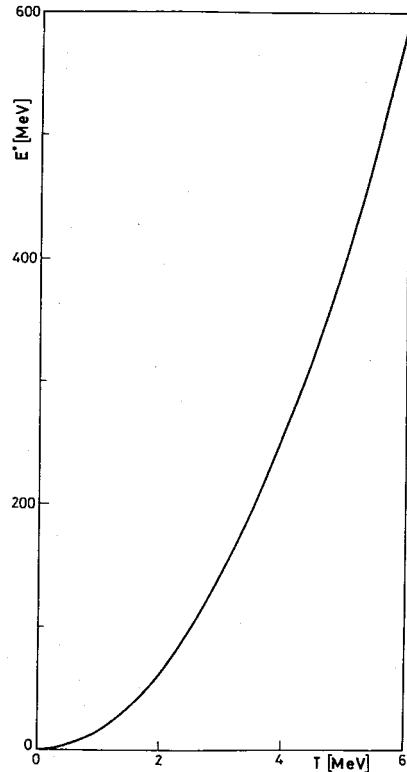


Fig. 2. Excitation energy of the state of equilibrium (see Eq. (63)) as function of the temperature ($A=208$).

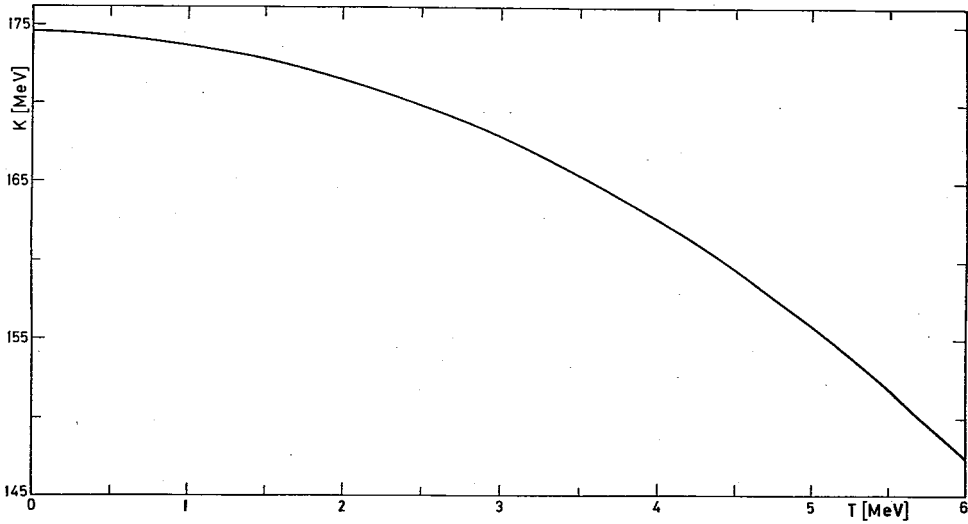


Fig. 3. Nuclear incompressibility (Eq. (65)) as function of the temperature.

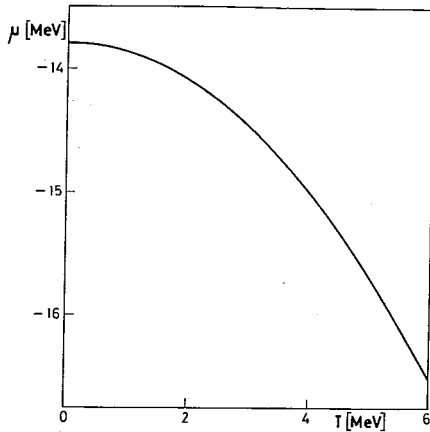


Fig. 4. Chemical potential, defined by Eq. (36), as function of the temperature.

$$\delta^2 E = \mathcal{Q} \left(\frac{3\delta\eta}{\eta} \right)^2 \left(\frac{10}{9} \tau_o(0) + \sum_{\sigma} a_{\sigma} \sigma(\sigma-1) \rho_o^{\sigma}(0) \right). \quad (65)$$

Then, for a finite temperature, the nuclear incompressibility may still be written as

$$K = \frac{9}{\rho_o(0)} \left(\frac{10}{9} \tau_o(0) + \sum_{\sigma} a_{\sigma} \sigma(\sigma-1) \rho_o^{\sigma}(0) \right). \quad (66)$$

In Figs. 3 and 4 we plot respectively K and the chemical potential μ (see Eq. (36)) as functions of the temperature.

Reduced $B(E1)$ values are calculated⁹⁾

according to

$$B(E1) = \frac{\hbar}{2\omega} \left(\frac{Ze}{A} \right)^2 \frac{\left[\int d^3x g_l(x) \nabla \cdot (\rho_o \mathbf{s}) \right]^2}{\int d^3x (-m \dot{\mathbf{s}} \cdot \mathbf{j})}, \quad (67)$$

for local excitations

$$g_0 = r^2 Y_{00}, \quad g_1 = j_1(qr) Y_{10} \text{ with } qR = 4.49 \quad \text{and} \quad g_l = r^l Y_{l0} \text{ for } l \geq 2. \quad (68)$$

For comparison we give in Table I the energy and percentage of the energy weighted sum (EWS) corresponding to different eigenmodes for the temperatures $T=0$, $T=3$ MeV and $T=5$ MeV. We further give in Table II, for the same temperatures, model independent quantities $(m_3/m_1)^{1/2}$ which indicate the position of a resonance that would exhaust both

Table I. For the states listed in the first column, energies (in MeV) and percentages of the EWS, for the excitation operators (68), are given. The second and third columns show the results obtained for $T=0$, the fourth and fifth columns show the results obtained for $T=3$ MeV and the sixth and seventh columns show the results for $T=5$ MeV. A nucleus with $A=208$ has been considered.

l^π	$T=0$		$T=3$ MeV		$T=5$ MeV	
	$\hbar\omega$	EWS	$\hbar\omega$	EWS	$\hbar\omega$	EWS
0_1^+	13.70	0.974	13.38	0.976	12.76	0.978
0_2^+	33.02	19.8×10^{-3}	32.54	18.8×10^{-3}	31.62	17.1×10^{-3}
1_1^-	0	0	0	0	0	0
1_2^-	20.20	0.974	19.73	0.972	18.86	0.968
1_3^-	40.37	21.5×10^{-3}	39.76	23.1×10^{-3}	38.63	26.1×10^{-3}
2_1^+	8.98	0.999	8.99	0.999	8.98	0.999
2_2^+	26.89	0.35×10^{-3}	26.32	0.39×10^{-3}	25.25	0.50×10^{-3}
3_1^-	14.14	0.999	14.15	0.999	14.15	0.999
3_2^-	33.39	0.56×10^{-3}	32.72	0.64×10^{-3}	31.46	0.81×10^{-3}
4_1^+	18.99	0.999	19.01	0.999	19.01	0.999
4_2^+	39.75	0.61×10^{-3}	38.99	0.69×10^{-3}	37.54	0.89×10^{-3}

Table II. For the multiplicities listed in the first column, the quantities $(m_3/m_1)^{1/2}$ are calculated for $T=0$, 3 MeV and 5 MeV respectively in the second, third and fourth columns, where m_1 and m_3 are respectively the energy-weighted and the cubic energy-weighted sums calculated for the equilibrium distribution function (14) and the excitation operators (68).

l	$T=0$	$T=3$ MeV	$T=5$ MeV
	$\sqrt{m_3/m_1}$	$\sqrt{m_3/m_1}$	$\sqrt{m_3/m_1}$
0	15.33	14.92	14.16
1	25.98	25.68	25.11
2	10.32	10.33	10.33
3	17.27	17.29	17.29
4	23.98	24.01	24.00

the m_1 (EWS) and the m_3 (cubic weighted) sums. A main feature of Table I is that for each multipolarity there is a state (for $l \neq 1$ it is the one with lowest energy) which exhausts almost completely the EWS. Due to the severe constraint imposed on the generator S , implied by Eq. (6), we obtain a small number of eigenmodes in comparison with more general schemes.^{(10),(11)} Due to the small splitting of states obtained, most of the strength associated with each multipolarity l and for the excitation operators (68) is concentrated in a single state. Only for the cases where the RPA reproduces eigenmodes

which exhaust a large fraction of the EWS, such as the isoscalar giant monopole and quadrupole resonances⁽⁶⁾ we can expect a good agreement when we compare the RPA results with the results of the present fluid dynamical approach. We will now proceed to a short discussion of the results obtained for different multiplicities.

$l=0^+$: Monopole modes are purely longitudinal. In this case the results obtained for $T=0$ are identical to previous results.^{(6),(15),(16)} We see in Fig. 5 that, as temperature increases, the energy of the giant monopole resonance lowers from 13.70 MeV at $T=0$ to 12.76 MeV for $T=5$ MeV. For a local excitation operator $g(x)$ the quantity m_3/m_1 is equal to (see Ref. 17))

$$\frac{m_3}{m_1} = \frac{\frac{\hbar^4}{m^2} E^{(2)}[\nabla g]}{\hbar^2 \int d^3x \frac{\rho_0}{2m} (\nabla g) \cdot (\nabla g)}, \quad (69)$$

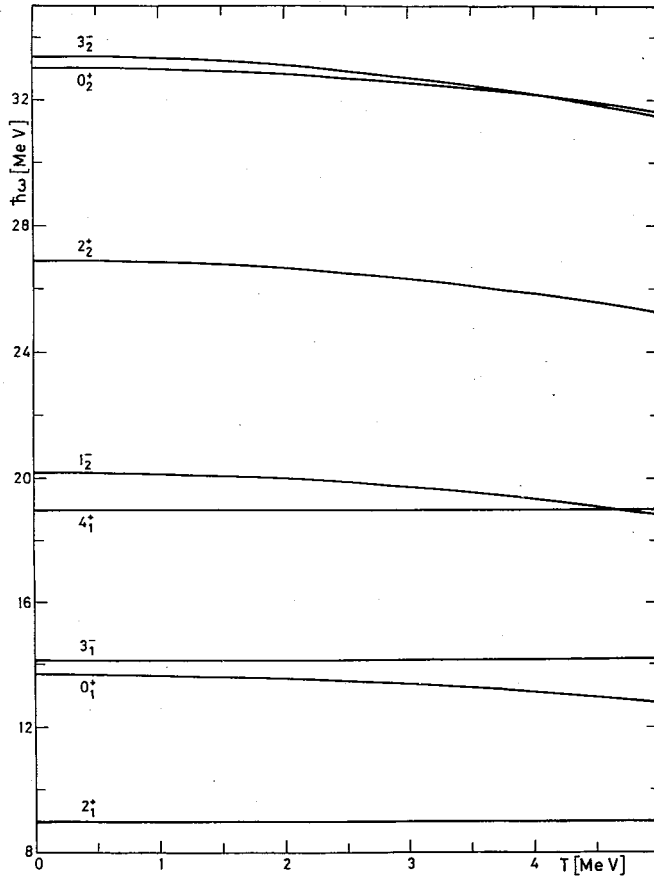


Fig. 5. Energies of the isoscalar giant resonances as function of the temperature ($A=208$).

where $E^{(2)}$ is defined by Eq. (38). For $l=0$ we consider $g(\mathbf{x})=r^2 Y_{00}$ and therefore Eq. (69) reduces to

$$\frac{m_3}{m_1} = \frac{5}{3} \frac{\hbar^2}{m} \frac{\{10\tau_o(0) + 9\sum_{\sigma} a_{\sigma}\sigma(\sigma-1)\rho_o^{\sigma}(0)\}}{\rho_o(0)R^2} \left(\equiv \frac{\hbar^2 K}{m\langle r^2 \rangle} \right). \quad (70)$$

We note that the curly bracket in Eq. (70) is equal to $\rho_o(0)K$. It may be seen from Tables I and II that the energy of the giant monopole resonance shows a dependence on the temperature analogous to the one exhibited by $(\hbar^2 K/(m\langle r^2 \rangle))^{1/2}$. The quantity $\langle r^2 \rangle$ denotes the mean square radius and, for the distribution function (14), $\langle r^2 \rangle$ is equal to $3R^2/5$.

For electric modes the flow field may always be decomposed in the following way in terms of vector spherical harmonics:

$$\mathbf{j}(\mathbf{x}) = j_+(r) \mathbf{Y}_{l,l+1,0} + j_-(r) \mathbf{Y}_{l,l-1,0}. \quad (71)$$

For $l=0$ only the radial function $j_+(r)$ is different from zero. In Fig. 6 we compare the field $j_+(r)$ corresponding to the giant monopole resonance for the temperatures $T=0$ and $T=5$ MeV. In both cases the fields have been normalized according to the orthogonality relation (58). As it is clear from Fig. 6, the flow field maintains the same shape but its

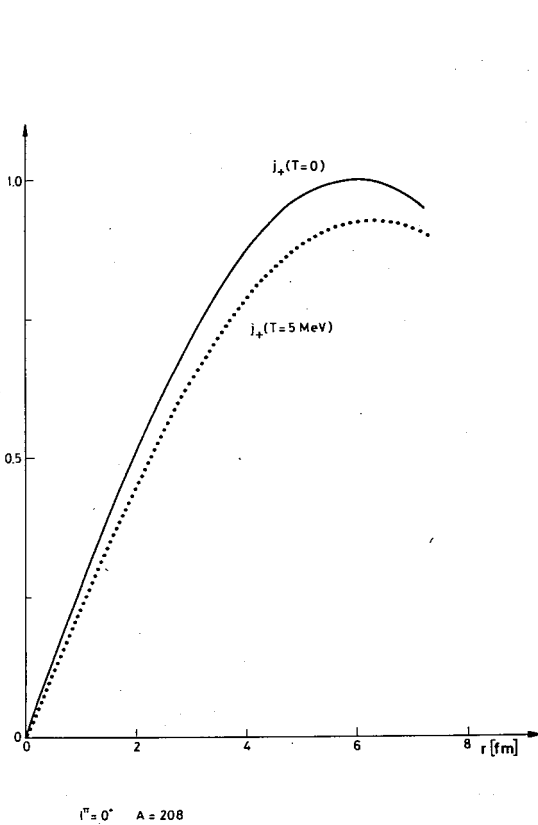


Fig. 6. The radial function of j , according to (71), of the isoscalar giant monopole resonance for a system with $A=208$. The full line ($\hbar\omega=13.7$ MeV) refers to the temperature $T=0$ and the dotted line ($\hbar\omega=12.76$ MeV) refers to $T=5$ MeV. The curves are normalized according to Eq. (58).

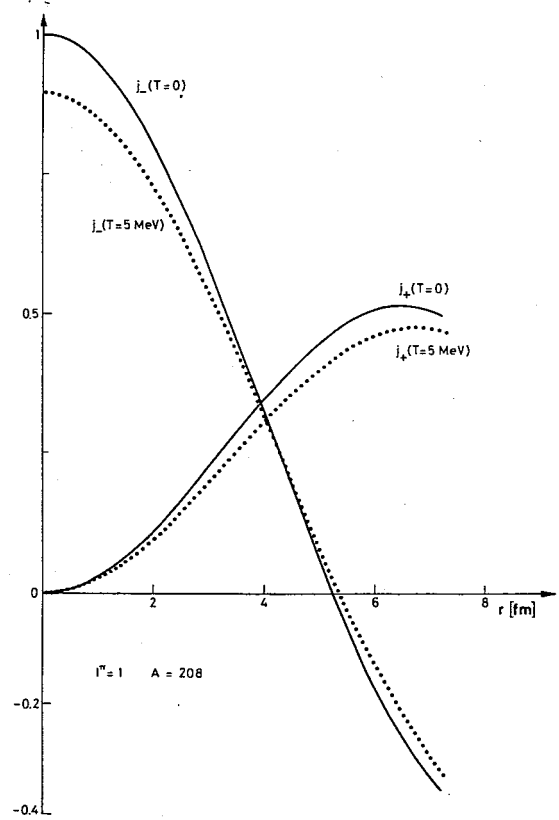


Fig. 7. The radial functions of j , according to (71), of the giant squeezing dipole for $A=208$: The full lines ($\hbar\omega=20.2$ MeV) refer to $T=0$ and the dotted lines ($\hbar\omega=18.86$ MeV) refer to $T=5$ MeV. The curves are normalized according to Eq. (58).

amplitude decreases by a small amount as the temperature increases from $T=0$ to $T=5$ MeV.

$l^\pi=1^-$: The lowest 1^- mode is the uniform translation which occurs at $\omega=0$. The giant squeezing dipole occurs at 20.20 MeV for $T=0$. Again its energy decreases as the temperature increases reaching the value of 18.86 MeV for $T=5$ MeV (see Fig. 5). With respect to the velocity field it is clear from Fig. 7 that there are same changes similar to the ones exhibited by the giant monopole resonance.

$l^\pi=2^+, 3^-, 4^+$: In Figs. 8 and 9 the flow fields for the lowest excited states with $l^\pi=2^+, 3^-, 4^+$ are presented. It is most remarkable to which extent the Tassie relation,¹⁴⁾ $\mathbf{v} \propto \nabla r^l Y_{lm}$, is recovered (we note that in a square density model the flow field \mathbf{j} is proportional to the velocity field \mathbf{v}). In fact we conclude from the well-known formula¹⁸⁾

$$\nabla(\phi(r) Y_{lm}) = -\sqrt{\frac{l+1}{2l+1}} \left(\frac{d}{dr} - \frac{l}{r} \right) \phi Y_{l+1,m}$$

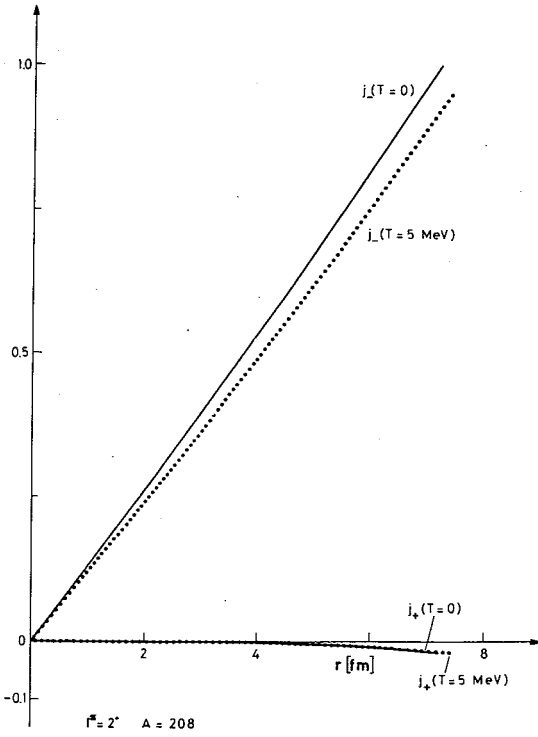


Fig. 8. The radial function of j for the giant quadrupole resonance ($A=208$): The full lines ($\hbar\omega=8.98$ MeV) refer to $T=0$ and the dotted lines ($\hbar\omega=8.98$ MeV) refer to $T=5$ MeV.

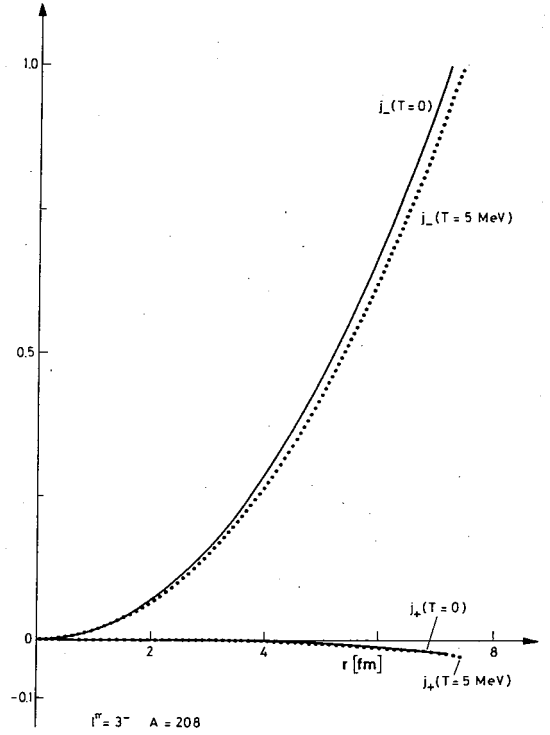


Fig. 9. The radial function of j for the giant octupole resonance ($A=208$): The full lines ($\hbar\omega=14.14$ MeV) refer to $T=0$ and the dotted lines ($\hbar\omega=14.15$ MeV) refer to $T=5$ MeV.

$$+\sqrt{\frac{l}{2l+1}}\left(\frac{d}{dr}+\frac{l+1}{r}\right)\phi Y_{l,l-1,m}, \quad (72)$$

that the velocity potential of the giant resonances with $l \geq 2$ is approximately $r^l Y_{lm}$ since, as it is evident from Figs. 8 and 9, the field j_+ remains always very close to zero and j_- is approximately proportional to r^{l-1} . Such velocity fields have been imposed in nuclear fluid dynamics with good results. However, in this model, they appear in an approximate way as solutions of the equations of motion and boundary conditions. In view of the well-known relation $\Delta r^l Y_{lm} = 0$ it is also clear that these modes may be interpreted as surface modes (characterized by a velocity field \mathbf{v} satisfying $\nabla \cdot \mathbf{v} = 0$) since in the interior of the nucleus $\delta\rho \approx 0$ and at the surface $\delta\rho$ is proportional to a δ function, as may be easily seen if we introduce the velocity field $(1/m)\nabla\chi \propto \nabla r^l Y_{lm}$ in Eq. (40). This is in good agreement with the statement that, if there exists a single collective state which exhausts the EWS for the excitation operator $r^l Y_{lm}$, the velocity field for that state should be irrotational and incompressible.¹⁹⁾ In fact if the velocity field for a given normal mode was exactly proportional to $\nabla r^l Y_{lm}$, then due to the orthogonality relation (58), all the other normal modes would carry no strength for the excitation operator $r^l Y_{lm}$. Since this scheme fulfils the sum rules m_1 and m_3 (see Ref. 17)) such a state would exhaust completely these two sums and therefore its energy would be exactly equal to $(m_3/m_1)^{1/2}$.

Although this scheme preserves the energy-weighted and cubic energy-weighted sum rules,¹⁷⁾ it may be seen from Table II that the quantities $(m_3/m_1)^{1/2}$ have in general

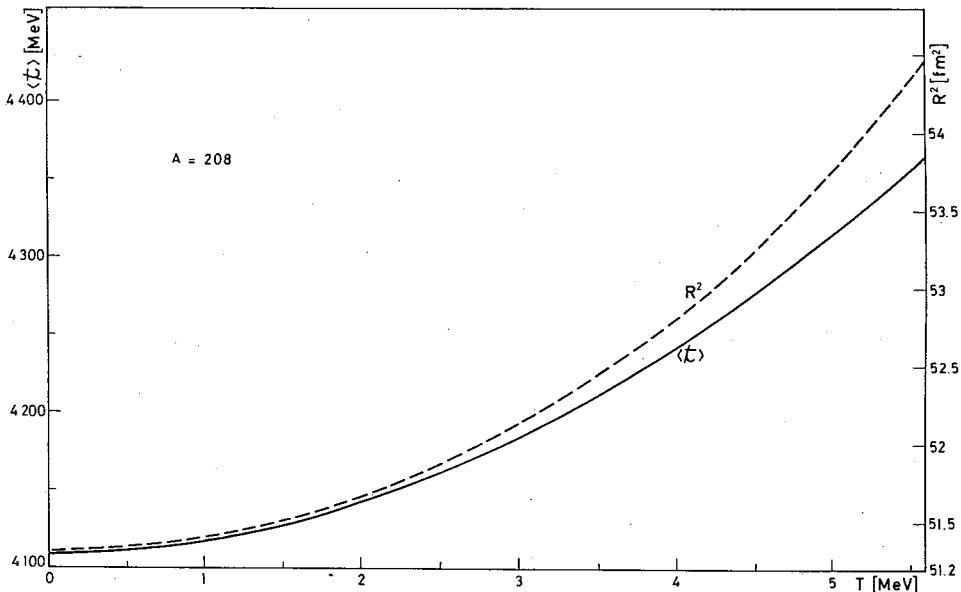


Fig. 10. For comparison the total kinetic energy $\langle T \rangle$ (solid curve) and the square of the nuclear radius R^2 (dashed curve) are plotted as functions of the temperature ($A=208$). The scale on the right refers to the square of the nuclear radius and the scale on the left refers to the total kinetic energy.

different values from the lowest energy levels for each multipolarity. The reason why the energies of the giant resonances do not coincide with $(m_3/m_1)^{1/2}$ is due to the distribution of strength over several levels. It is interesting to note that for $l \geq 2$ the energies of the giant resonances (see Table I) and the quantities $(m_3/m_1)^{1/2}$ do not seem to be affected in a significant way by the temperature. If we consider an excitation operator $g(\mathbf{x}) = r^l Y_{lm}$, we obtain⁴⁾ from Eq. (69)

$$\frac{m_3}{m_1} = \frac{4}{3}(l-1)(2l+1) \frac{\hbar^2 \tau_o(0)}{m \rho_o(0) R^2} = \frac{4}{3}(l-1)(2l+1) \frac{\hbar^2 \langle T \rangle}{m R^2 A}, \quad (73)$$

where $\langle T \rangle = \Omega \tau_o(0)$ is the total kinetic energy of the nucleus. Therefore, the fact that the quantity $(m_3/m_1)^{1/2}$, evaluated for an excitation operator of the form $g(\mathbf{x}) = r^l Y_{lm}$, does not depend on temperature in a significant way, is related to the fact that the equilibrium condition (30) and the zero range force here considered imply in an approximate way that the total kinetic energy $\langle T \rangle$ is proportional to the square of the nuclear radius (see Fig. 10).

Finally we would like to note that fluid dynamical calculations as well as RPA calculations show that, for $l=1$ and $l \geq 3$ the strength is distributed covering a wide energy region.¹¹⁾ Our present results mean that the parametrization of S , through a scalar field $\chi(\mathbf{x}, t)$ and the generalized scaling field $\mathbf{s}(\mathbf{x}, t)$, is not adequate to describe the fragmentation of strength, especially when a low lying state exists. In particular, we observe that, for $T=0$, a variational scheme,^{9)~11),20),21)} based on a more general choice of S , namely $S = \chi + \mathbf{s} \cdot \mathbf{p} + (1/2) p_\alpha p_\beta \varphi_{\alpha\beta}$, provides a good description of the case $l=1$, since it succeeds in reproducing a low-lying mode and the giant squeezing dipole in agreement with the RPA (see table 1, p. 483, in Ref. 11) and Refs. 20) and 21)).

§ 5. Conclusion

A variational formulation of nuclear fluid dynamics has been presented which is valid both for zero and finite temperatures. The equations of motion and boundary conditions for small amplitude vibrations around a stationary state were derived from an appropriate lagrangian. This formulation can be regarded as an approximation to a quantal treatment of the equations of motion as described for instance in the time dependent Hartree-Fock Theory or equivalently in the RPA. In fact in order to account for all the high multipolarity distortions of the Fermi surface predicted for example by Landau's zero sound theory we should not impose the severe ansatz (6) for the generator S and instead we would have to allow for a completely general generator. Fluid dynamical schemes involving larger variational spaces, than the one implied by (6), have already been considered for $T=0$ in Refs. 9) ~ 11).

We have obtained the dependence on the temperature of the energy levels predicted by this model. We also obtained the dependence of the strength function on the temperature. We have therefore achieved a variational formulation of nuclear fluid dynamics where temperature effects are most easily incorporated. This preliminary calculation aims only at illustrating the power of the method proposed. The natural way to improve the description of the strength distribution is to generalize the formulation presented here by including in the generator S the next term in an expansion in the momentum coordinate, namely $(1/2)p_{\alpha\beta}\varphi_{\alpha\beta}$ (see Refs. 20) and 21)). In fact the truncation of an expansion of S in momentum space seems to be an adequate method of extracting important degrees of freedom for the collective motions, from the complicated nuclear system. A main feature of the results presented in this note and in Refs. 20) and 21) is that the energy of surface modes remains rather stable, when the temperature increases from zero to 5 MeV, in contrast with the behaviour of compression modes which show a significant dependence of the energy on the temperature.

Nuclear fluid dynamics has the advantage of being a simple and physically transparent formulation which is entirely selfconsistent and does not require complicated numerical calculations. We believe that nuclear fluid dynamics leads to a better understanding of nuclear collective motion which is one of the main themes of nuclear physics.

Acknowledgements

The author is grateful to Professor J. da Providência for suggesting the extension of Nuclear Fluid Dynamics to a finite temperature and also for useful discussions. The author also acknowledges valid suggestions of Dr. S. Nishizaki. The author was supported by the Instituto Nacional de Investigação Científica, Lisboa and by the Calouste Gulbenkian Foundation.

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