Progress of Theoretical Physics, Vol. 25, No. 1, January 1961

Nuclear Forces in the Momentum Space*

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(Received August 20, 1960)

Several fundamental problems on a two-nucleon system in the momentum space are discussed with the view that they will be useful for treating the two-nucleon problem completely nonstatically, i. e. without making use of the expansion in terms of the inverse of the mass of the nucleon.

General forms for a two-nucleon potential in the momentum space are derived, and the integral equations which are the Fourier transform of the Schrödinger equation and their solutions are briefly discussed. Formulas for matrix elements of the most general types of potentials are evaluated and are applied to the nonstatic one-pion-exchange potential.

§ 1. Introduction

As has been stressed by Taketani and Machida,[1] one of the most important problems in the theory of nuclear forces will be to evaluate the nonstatic effects in a consistent way, especially avoiding the expansion with respect to the inverse of the mass of the nucleon. One way to do so will be to treat the problem in the momentum space throughout. It is the aim of this paper to give fundamental formulas to carry out such a program.

First, we will derive in § 2 the most general expression for a potential in the momentum space, which has been used by Hoshizaki and Machida[2),3)] (hereafter referred to as HMI and HMII respectively). In § 3, the integral equations in the momentum space, which are the Fourier transforms of the Schrödinger equations in the configuration space, are discussed, and after separating the angular variables, scattering amplitudes are connected with phase shifts. In § 4, formulas for matrix elements of the most general forms of a potential in the momentum space will be given, and in § 5 we will apply these formulas to the one-pion-exchange potential with full recoil in the case of the pseudoscalar coupling, and in § 6 the phase shifts in the Born approximation will be obtained. Actual solutions of the integral equations derived in § 3 will not be given in this paper, but will be reported in a forthcoming paper.

§ 2. General forms of a potential in the momentum space

We consider the nucleon-nucleon scattering as shown in Fig. 1 in the centre-

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* This work has been done as a part of the 1960~61 Annual Research Project on Nuclear Forces organized by the Research Institute for Fundamental Physics, Kyoto University.

of-mass system, and define the following quantities:

$$(1/2)(\boldsymbol{p}+\boldsymbol{p}')\equiv\boldsymbol{q},\quad \boldsymbol{p}-\boldsymbol{p}'\equiv\boldsymbol{k},$$

and

$$\boldsymbol{l}=\frac{\boldsymbol{k}}{|\boldsymbol{k}|},\quad \boldsymbol{m}=\frac{\boldsymbol{q}}{|\boldsymbol{q}|},\quad\text{and}\quad \boldsymbol{n}=\frac{\boldsymbol{k}\times\boldsymbol{q}}{|\boldsymbol{k}\times\boldsymbol{q}|}=\frac{\boldsymbol{p}\times\boldsymbol{p}'}{|\boldsymbol{p}\times\boldsymbol{p}'|}.$$

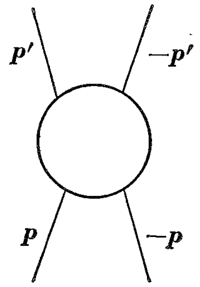

Fig. 1

Then, the most general form of the potential between two particles with spin one half in the momentum space is as follows:

$$V=V_0+i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\boldsymbol{n}\cdot V_1+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{l})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{l})\cdot V_2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{m})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{m})\cdot V_3$$
$$+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n})\cdot V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\cdot V_5, \qquad (2\cdot1)$$

where $V_0, \cdots, V_5$ are real functions of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$. In order to derive Eq. (2·1) we have assumed invariance[4),5),6)] with respect to translation, Galilei transformation, the exchange of two particles, rotation, space reflection and time reversal, and Hermiticity of the potential.

Okubo and Marshak[7)] first derived this general form which corresponds to the Fourier transform of Eq. (2·1) with respect to $\boldsymbol{k}$. Since we will need the relation between x-representation and p-representation which is not necessarily trivial (see the Appendix of HMI), it will be convenient to derive the above formula in our own way.

Consider the matrix element of a potential operator, v, from a two-nucleon state with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ to the one with momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$, which may be written as the Fourier transform of a nonlocal potential in x-space in general,

$$\langle\boldsymbol{p}_1', \boldsymbol{p}_2'|v|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle=\frac{1}{(2\pi)^6}\int d\boldsymbol{r}_1'd\boldsymbol{r}_2'd\boldsymbol{r}_1d\boldsymbol{r}_2\langle\boldsymbol{r}_1', \boldsymbol{r}_2'|v|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle$$
$$\times\exp i(\boldsymbol{p}_1'\cdot\boldsymbol{r}_1'+\boldsymbol{p}_2'\cdot\boldsymbol{r}_2'-\boldsymbol{p}_1\cdot\boldsymbol{r}_1-\boldsymbol{p}_2\cdot\boldsymbol{r}_2). \qquad (2\cdot2)$$

(I) *Translation invariance*

We have

$$\langle\boldsymbol{r}_1'+\boldsymbol{a}, \boldsymbol{r}_2'+\boldsymbol{a}|v|\boldsymbol{r}_1+\boldsymbol{a}, \boldsymbol{r}_2+\boldsymbol{a}\rangle=\langle\boldsymbol{r}_1', \boldsymbol{r}_2'|v|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle, \qquad (2\cdot3)$$

where $\boldsymbol{a}$ is an arbitrary vector. Inserting Eq. (2·3) into Eq. (2·2), we obtain

$$\langle\boldsymbol{p}_1', \boldsymbol{p}_2'|v|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle=\langle\boldsymbol{p}_1', \boldsymbol{p}_2'|v|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle\cdot\exp[-i(\boldsymbol{p}_1'+\boldsymbol{p}_2'-\boldsymbol{p}_1-\boldsymbol{p}_2)\boldsymbol{a}]. \qquad (2\cdot4)$$

Therefore, one may write

$$\langle\boldsymbol{p}_1', \boldsymbol{p}_2'|v|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle=\delta(\boldsymbol{p}_1'+\boldsymbol{p}_2'-\boldsymbol{p}_1-\boldsymbol{p}_2)\langle\boldsymbol{p}_1', \boldsymbol{p}_2'|V|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle$$
$$=\delta(\boldsymbol{Q}'-\boldsymbol{Q})V(\boldsymbol{Q}, \boldsymbol{P}', \boldsymbol{P}), \qquad (\mathrm{i})$$

where

$$\boldsymbol{Q}=\boldsymbol{p}_1+\boldsymbol{p}_2,\qquad \boldsymbol{Q}'=\boldsymbol{p}_1'+\boldsymbol{p}_2',$$
$$\boldsymbol{P}=(1/2)(\boldsymbol{p}_1-\boldsymbol{p}_2),\quad \boldsymbol{P}'=(1/2)(\boldsymbol{p}_1'-\boldsymbol{p}_2'). \qquad (2\cdot5)$$

(II) *Galilei invariance*

The condition for Galilei invariance gives

$$\langle \boldsymbol{p}_1'+\boldsymbol{q},\ \boldsymbol{p}_2'+\boldsymbol{q}\,|v|\,\boldsymbol{p}_1+\boldsymbol{q},\ \boldsymbol{p}_2+\boldsymbol{q}\rangle=\langle \boldsymbol{p}_1',\ \boldsymbol{p}_2'|v|\boldsymbol{p}_1,\ \boldsymbol{p}_2\rangle, \tag{2·6}$$

where $\boldsymbol{q}$ is an arbitrary momentum vector and masses of participating particles are assumed to be equal. Eq. (2·6) gives a condition on $V(\boldsymbol{Q}, \boldsymbol{P}', \boldsymbol{P})$,

$$V(\boldsymbol{Q}+2\boldsymbol{q}, \boldsymbol{P}', \boldsymbol{P})=V(\boldsymbol{Q}, \boldsymbol{P}', \boldsymbol{P}). \tag{2·7}$$

So we have, writing the spin dependence explicitly,

$$V=V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q}), \tag{ii}$$

where

$$\boldsymbol{k}=\boldsymbol{P}-\boldsymbol{P}'=\boldsymbol{p}_1-\boldsymbol{p}_1'=-(\boldsymbol{p}_2-\boldsymbol{p}_2'),$$
$$\boldsymbol{q}=(1/2)(\boldsymbol{P}+\boldsymbol{P}')=(1/4)(\boldsymbol{p}_1+\boldsymbol{p}_1'-\boldsymbol{p}_2-\boldsymbol{p}_2').$$

(III) *Symmetry condition*

V is invariant when particles 1 and 2 are interchanged, i. e.

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q})=V(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}^{(1)}, -\boldsymbol{k}, -\boldsymbol{q}). \tag{iii}$$

(IV) *Rotation invariance*

Invariant functions are functions of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$.

(V) *Space reflection invariance*

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\ \boldsymbol{k}, \boldsymbol{q})=V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, -\boldsymbol{k}, -\boldsymbol{q}). \tag{iv}$$

From (iii) and (iv), we have

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q})=V(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{k}, \boldsymbol{q}). \tag{v}$$

(VI) *Time reversal invariance*

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q})=V^T(-\boldsymbol{\sigma}^{(1)}, -\boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, -\boldsymbol{q}). \tag{vi}$$

V^T means to reverse the order of operators.

(VII) *Hermiticity*

$$V=V^+.$$

As we assume charge independence, $V_0, \cdots, V_5$ are divided into $\boldsymbol{1}$-terms and $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})$-terms in isotopic space. As for the $\boldsymbol{\sigma}$-dependence, they are separated to the zeroth, the first and the second order terms with respect to $\boldsymbol{\sigma}$. Then, the zeroth order term, V_0, is a function of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$ from (IV). In the first order term of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ appear always in a form $\boldsymbol{S}=(1/2)(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})$, i. e. $(\boldsymbol{S}\cdot\boldsymbol{k}\times\boldsymbol{q})$. And the second order terms of $\boldsymbol{\sigma}$ may have the following factors,

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}),$$
$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\ \text{and}\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q}).$$

Among the above five terms one may be expressed by other four types, so we take the next four terms as independent terms,

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})\ \text{and}\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q}).$$

Therefore, we have six independent types of potentials, $V_0, \cdots, V_5$, in general, as we see in Eq. $(2\cdot1)$. In the static approximation, only V_0, V_2 and V_5 survive, and if we evaluate the potential up to the first order of the inverse of the mass of the nucleon V_1 appears. V_3 and V_4 appear as the second order correction to the static potential with respect to the inverse of the mass of the nucleon. As our intention is to take the nonstatic effects fully into account, it is necessary to treat all types of potentials on the equal footing.

When we consider the matrix element of a potential, the energy conservation law does not hold in general. However, when we consider the scattering matrix for two nucleons, where the energy must be conserved, we have just five (instead of six) independent types and, furthermore, invariant functions are functions of $\boldsymbol{k}^2$ and $\boldsymbol{q}^2$ only. This is because of the identity,

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\cdot\boldsymbol{q}^2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})\cdot\boldsymbol{k}^2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})$$
$$=(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})(\boldsymbol{k}\times\boldsymbol{q})^2+(\boldsymbol{k}\cdot\boldsymbol{q})\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\},$$

and equations

$$(\boldsymbol{k}\cdot\boldsymbol{q})=0,\ (\boldsymbol{k}\times\boldsymbol{q})^2=\boldsymbol{k}^2\cdot\boldsymbol{q}^2.$$

It should be remarked that the assumption of the conservation of energy in the derivation of a non-static potential introduces serious errors in general (see HMI).

§ 3. Integral equations in the momentum space and phase shifts

In order to compare the potential in § 2 with the experiment, it is necessary to solve the integral equation which corresponds to the Schrödinger equation in the momentum space. We will discuss only the main points, which will be necessary in our later discussions, referring to a paper by Signell[8] for more detailed discussions.

The basic equation is the following,

$$(2E_{p_0}-2E_{p'})\phi(\boldsymbol{p}')=\int d\boldsymbol{p}\,V(\boldsymbol{p},\boldsymbol{p}')\phi(\boldsymbol{p}),\tag{3·1}$$

where $\boldsymbol{p}$ and $\boldsymbol{p}'$ are momenta as shown in Fig. 1, $E_{p_0}=(1/2)E$ is the incident energy of each particle in the centre-of-mass system, and $E_{p'}=(1/2)H_0$ where H_0 is the free Hamiltonian.

In order to separate the angular variables in Eq. $(3\cdot1)$, we expand the wave function and the potential into the spherical harmonics:

$$\phi(\boldsymbol{p})=\sum_l g_l(p)\cdot Y^M_{J,l,S},\tag{3·2}$$

where[9])

$$Y^M_{J,l,S}=\sum_{m_l,\,m_s} C_{lS}(J,M\,;\,m_l,\,m_s)\,Y^{m_l}_l(\Omega_p)\chi^{m_s}_S.$$

For simplicity we will treat the uncoupled states and give the relation between scattering amplitudes and phase shifts. The coupled states may also be treated by a slight generalization as discussed by Signell[8]. Using the following matrix elements of the potential,

$$V_l(p,\,p')\equiv\int Y^{M\,*}_{J,l,S}(\Omega_{p'})\,V(\mathbf{p},\,\mathbf{p}')\,Y^M_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p, \tag{3·3}$$

we can rewrite Eq. (3·1) in one-dimensional equation,

$$(2E_{p_0}-2E_{p'})g_l(p')=\int p^2\,dp\,V_l(p,\,p')g_l(p). \tag{3·4}$$

Thus the scattering solution is given by[10])

$$g_l(p')=\delta(2E_{p_0}-2E_{p'})+\frac{1}{2E_{p_0}-2E_{p'}}\int p^2\,dp\,V_l(p,\,p')g_l(p),$$

and may also be written in the following way:

$$g_l(p')=\delta(2E_{p_0}-2E_{p'})+\mathrm{P}\frac{1}{2E_{p_0}-2E_{p'}}f_l(p'), \tag{3·5}$$

where

$$f_l(p')=\int_0^\infty p^2\,dp\,V_l(p,\,p')g_l(p).$$

Using Eq. (3·5) and

$$\delta(2E_{p_0}-2E_{p'})=\left(\frac{E_{p_0}}{2p_0}\right)\delta(p_0-p');$$

we obtain an equation satisfied by $f_l(p')$,

$$f_l(p')=f_{l,B}(p')+\mathrm{P}\int_0^\infty\frac{p^2\,dp\,V_l(p,\,p')f_l(p)}{2E_{p_0}-2E_p}, \tag{3·6}$$

where $f_{l,B}(p')$ is the amplitude in Born approximation,

$$f_{l,B}(p')\equiv\frac{p_0\cdot E_{p_0}}{2}V_l(p_0,\,p'). \tag{3·7}$$

Now phase shifts, δ_l, are defined by the asymptotic wave function in the configuration space,

$$\phi_l(r)\sim\sin\left(p_0r-\frac{l\pi}{2}\right)+\tan\delta_l\cdot\cos\left(p_0r-\frac{l\pi}{2}\right). \tag{3·8}$$

Transforming this equation into the momentum space in the vicinity of the singularity, we have

$$\phi_l(p') \sim \delta(p_0 - p') - \frac{\tan\delta_l}{\pi} \mathrm{P} \cdot \frac{1}{p_0 - p'} + R(p'), \tag{3·9}$$

where $R(p')$ is a non-singular function at p_0.

It is seen that Eq. (3·5) is written as follows:

$$g_l(p') \sim \delta(p_0 - p') + \mathrm{P} \cdot \frac{1}{p_0 - p'} f_l(p_0) + R(p'). \tag{3·10}$$

Comparing Eq. (3·9) with Eq. (3·10), the relation between the scattering solution and phase shifts is given by

$$\tan\delta_l = -\pi f_l(p_0). \tag{3·11}$$

In the Born approximation, we have

$$\tan\delta_{l,B} = -\frac{\pi p_0 E_{p0}}{2} V_l(p_0, p_0). \tag{3·12}$$

So we must solve Eq. (3·6) to obtain scattering phase shifts. Using

$$\frac{1}{2E_{p_0} - 2E_p} = \begin{cases} \dfrac{M}{p_0^2 - p^2} & \text{for the nonrelativistic case,} \\ \dfrac{E_{p_0} + E_p}{2(p_0^2 - p^2)} & \text{for the relativistic case,} \end{cases} \tag{3·13}$$

we obtain

$$f_l(p') = f_{l,B}(p') + \mathrm{P}\int_0^\infty \frac{p^2 dp\, U(p', p) f_l(p)}{p_0 - p}, \tag{3·14}$$

where $U(p', p)$ is not singular anywhere (we assume $U(p', p)$ is bounded when $p' \to \infty$ or $p \to \infty$), and is defined by

$$\frac{U(p', p)}{p_0 - p} = \frac{V(p', p)}{2E_{p_0} - 2E_p}. \tag{3·15}$$

Eq. (3·14) is a singular integral equation, the kernel of which has a pole of the first order at a fixed point $p = p_0$ in the range of the integration. The integral equation of this type is generally known to have a solution.[11]

Eq. (3·14) is also easily transformed into a Fredholm integral equation of the third kind with a symmetrical kernel,

$$I(p') F_l(p') = f_{l,B}(p') + \int_0^\infty V_l(p', p) F_l(p) dp, \tag{3·16}$$

where

$$I(p) = \frac{2(E_{p_0} - E_p)}{p^2},$$

$$F_l(p)=\frac{f_l(p)}{I(p)}.$$

Eq. (3·16) may be solved by standard methods used to solve the Fredholm integral equation of the second kind. Evaluation of phase shifts will be treated in a separate paper making use of these integral equations.

§ 4. General formulas for matrix elements of potentials in the momentum space

In this section we calculate the matrix elements of the potentials by operating the most general potentials in § 2 to the eigenfunctions of a two-nucleon system.

If we write

$$V_{l,l'}=\int Y^{M*}_{J,l',S}(\Omega_{p'})\,V\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p,$$

there are four diagonal elements $V_{l,l}$, one for $S=0$ and three for $S=1$ with $l=J$, $J\pm1$. Also, there are two non-diagonal elements $V_{l,l'}$, for $S=1$ between $l'=J\mp1$ and $l=J\pm1$. There are, in all, five independent matrix elements, because $V_{l,l'}=V_{l',l}$. $V_{l,l}$ and $V_{l,l'}$ are real functions and they do not depend on M, the magnetic quantum number, owing to the rotation invariance and hermiticity of the potential. S is also conserved between two states before and after the potential operator because of parity conservation and charge independence. So we may use the following notation,

$$V_{l,l}(p,p')\equiv\int Y^{M*}_{J,l,S}(\Omega_{p'})\,V(\mathbf{p},\mathbf{p}')\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p, \tag{4·1}$$

$$V_{l,l'}(p,p')\equiv\int Y^{M*}_{J,l',S}(\Omega_{p'})\,V(\mathbf{p},\mathbf{p}')\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p. \tag{4·2}$$

Inserting Eq. (2·1) into Eqs. (4·1) and (4·2), we have carried out the integral of angular variables for each type of V_0, $\cdots$, V_5, and the results are given in the following. Since V_i,s in Eq. (2·1) contain not only $\mathbf{k}^2$ but also $\mathbf{q}^2$ and $(\mathbf{k}\times\mathbf{q})^2$ (which correspond to the most general nonlocal and angular momentum dependent potentials in x space), their calculations are very troublesome though straightforward.

Two states before and after the potential operator may be designated by l, since J and S are conserved between an initial and a final state. For example, a matrix element for $l=J$ in a spin singlet state of V_0 type is written as $\langle j|V_0|j\rangle$.

(1) For spin triplet states:

$$\langle J-1|V_0|J-1\rangle=2\pi A^0_{J-1},$$

$$\langle J|V_0|J\rangle=2\pi A^0_J,$$

$$\langle J+1|V_0|J+1\rangle=2\pi A^0_{J+1},$$

$$\langle J+1|V_0|J-1\rangle=\langle J-1|V_0|J+1\rangle=0.$$

For a spin singlet state:

$$\langle j|V_0|j\rangle = 2\pi A_j^0,$$

where

$$A_n^0 = \int V_0 P_n(z)\, dz,$$

and V_0 is the same one as V_0 in Eq. (2·1), $P_n(z)$'s are Legendre functions of the first kind, and $z \equiv (\boldsymbol{p}\cdot\boldsymbol{p}')/|\boldsymbol{p}|\cdot|\boldsymbol{p}'|$.

(2) For spin triplet states:

$$\langle J-1|V_1|J-1\rangle = 4\pi \frac{(J-1)}{(2J-1)}[A_J^1 - A_{J-2}^1],$$

$$\langle J|V_1|J\rangle = 4\pi\left[-\frac{(J+1)}{(2J+1)^2}A_{J+1}^1 + \frac{1}{(2J+1)}A_{J-1}^1\right],$$

$$\langle J+1|V_1|J+1\rangle = 4\pi\left[-\frac{(J+2)}{(2J+3)}A_{J+2}^1 + \frac{(J+2)}{(2J+5)}A_J^1\right],$$

$$\langle J+1|V_1|J-1\rangle \equiv \langle J-1|V_1|J+1\rangle = 0.$$

For a spin singlet state:

$$\langle j|V_1|j\rangle = 0,$$

where

$$A_n^1 = \int \frac{V_1 P_n(z)}{\sin(\boldsymbol{p}\cdot\boldsymbol{p}')}\, dz.$$

(3) For spin triplet states

$$\langle J-1|V_2|J-1\rangle = 2\pi\left[(p^2+p'^2)\frac{A_{J-1}^{2(0)}}{(2J+1)} - 2pp'\frac{A_J^{2(0)}}{(2J+1)}\right],$$

$$\langle J|V_2|J\rangle = 2\pi\left[(p^2+p'^2)A_J^{2(0)} - \frac{2pp'}{(2J+1)}\{J\cdot A_{J+1}^{2(0)} + (J+1)A_{J-1}^{2(0)}\}\right],$$

$$\langle J+1|V_2|J+1\rangle = 2\pi\left[-(p^2+p'^2)\frac{A_{J+1}^{2(0)}}{(2J+1)} + 2pp'\frac{A_J^{2(0)}}{(2J+1)}\right],$$

$$\langle J+1|V_2|J-1\rangle \equiv \langle J-1|V_2|J+1\rangle$$

$$= \frac{4\pi\sqrt{J(J+1)}}{(2J+1)}[p'^2\cdot A_{J-1}^{2(0)} + p^2 A_{J+1}^{2(0)} - 2pp'\cdot A_J^{2(0)}].$$

For a spin singlet state:

$$\langle j|V_2|j\rangle = 2\pi[-(p^2+p'^2)A_j^{2(0)} + 2pp' A_j^{2(1)}],$$

where

$$A_n^{2(l)} = \int \frac{V_2}{\boldsymbol{k}^2} P_n(z) z^l\, dz.$$

(4) For spin triplet states:

$$\langle J-1|V_3|J-1\rangle = 2\pi\left[(p^2+p'^2)\frac{A^{3(0)}_{J-1}}{(2J+1)}+2pp'\frac{A^{3(0)}_{J}}{(2J+1)}\right],$$

$$\langle J|V_3|J\rangle = 2\pi\left[(p^2+p'^2)A^{3(0)}_J+\frac{2pp'}{(2J+1)}\{J\cdot A^{3(0)}_{J+1}+(J+1)\cdot A^{3(0)}_{J-1}\}\right],$$

$$\langle J+1|V_3|J+1\rangle = -2\pi\left[(p^2+p'^2)\frac{A^{3(0)}_{J+1}}{(2J+1)}+2pp'\frac{A^{3(0)}_{J}}{(2J+1)}\right],$$

$$\langle J+1|V_3|J-1\rangle \equiv \langle J-1|V_3|J+1\rangle$$

$$=\frac{4\pi\sqrt{J(J+1)}}{(2J+1)}[p'^2\cdot A^{3(0)}_{J-1}+p^2A^{3(0)}_{J+1}+2pp'\cdot A^{3(0)}_J].$$

For a spin singlet state:

$$\langle j|V_3|j\rangle = -2\pi[(p^2+p'^2)A^{3(0)}_j+2pp'\cdot A^{3(1)}_j],$$

where

$$A^{3(l)}_n = \int\frac{V_3}{\boldsymbol{q}^2}P_n(z)z^l\,dz.$$

(5) For spin triplet states:

$$\langle J-1|V_4|J-1\rangle = 2\pi p^2p'^2\left[\frac{2}{(2J+1)}A^{4(1)}_J+\frac{(2J-1)}{(2J+1)}A^{4(0)}_{J-1}-A^{4(2)}_{J-1}\right],$$

$$\langle J|V_4|J\rangle = 2\pi p^2p'^2\left[-A^{4(0)}_J+\frac{(J+2)}{(2J+1)}A^{4(1)}_{J-1}+\frac{(J-1)}{(2J+1)}A^{4(1)}_{J+1}\right],$$

$$\langle J+1|V_4|J+1\rangle = 2\pi p^2p'^2\left[-\frac{2}{(2J+1)}A^{4(1)}_J-A^{4(2)}_{J+1}+\frac{(2J+3)}{(2J+1)}A^{4(0)}_{J+1}\right],$$

$$\langle J+1|V_4|J-1\rangle \equiv \langle J-1|V_4|J+1\rangle = \frac{4\pi p^2p'^2\sqrt{J(J+1)}}{(2J+1)^2}[A^{4(0)}_{J+1}-A^{4(0)}_{J-1}].$$

For a spin singlet state:

$$\langle j|V_4|j\rangle = 2\pi p^2\cdot p'^2[-A^{4(0)}_j+A^{4(2)}_j],$$

where

$$A^{4(l)}_n = \int\frac{V_4\ P_n(z)\cdot z^l}{\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}dz.$$

(6) For spin triplet states:

$$\langle J-1|V_5|J-1\rangle = 2\pi A^5_{J-1},$$

$$\langle J|V_5|J\rangle = 2\pi A_J^5,$$

$$\langle J+1|V_5|J+1\rangle = 2\pi A^5_{J+1},$$

$$\langle J+1|V_5|J-1\rangle \equiv \langle J-1|V_5|J+1\rangle = 0.$$

For a spin singlet state:

$$\langle j|V_5|j\rangle=-6\pi A_j^5,$$

where

$$A_n^5=\int V_5 P_n(z)\,dz.$$

§ 5. The one-pion-exchange potential (OPEP)

In this paragraph we will consider the one-pion-exchange potential, $V^{(2)}$, assuming the pseudoscalar coupling.* The types V_2, V_3, V_4 and V_5 appear of the six possible types of the potentials in Eq. (2·1),

$$\begin{aligned}V^{(2)}&=(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{l})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{l})V_2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{m})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{m})V_3\\&\quad+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n})V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\cdot V_5\\&=\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{|\boldsymbol{k}^2|}V_2+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{|\boldsymbol{q}^2|}V_3\\&\quad+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})}{|\boldsymbol{k}^2|\cdot|\boldsymbol{q}^2|\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\cdot V_5,\end{aligned}\qquad(5\cdot1)$$

where

$$\begin{aligned}V_2&=\frac{G}{A+Bz}\boldsymbol{k}^2\left\{\beta+\frac{2\boldsymbol{q}^2}{(\boldsymbol{k}\cdot\boldsymbol{q})}\alpha\right\},\\V_3&=\frac{G}{A+Bz}\boldsymbol{q}^2\left\{\gamma+\frac{\boldsymbol{k}^2\cdot\alpha}{(\boldsymbol{k}\cdot\boldsymbol{q})}\right\},\\V_4&=\frac{G}{A+Bz}\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})\frac{2\alpha}{(\boldsymbol{k}\cdot\boldsymbol{q})},\\V_5&=\frac{G}{A+Bz}\cdot\frac{-2\alpha}{(\boldsymbol{k}\cdot\boldsymbol{q})}\{\boldsymbol{k}^2\cdot\boldsymbol{q}^2-(\boldsymbol{k}\cdot\boldsymbol{q})^2\},\end{aligned}\qquad(5\cdot2)$$

$$G=-\frac{1}{16}g^2\cdot(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{1}{M^2}\frac{(E_p+M)(E_{p'}+M)}{E_p\cdot E_{p'}},$$

$$A=p^2+p'^2+\mu^2-(E_p-E_{p'})^2,$$

$$B=-2pp',$$

$$\alpha=\frac{1}{(1+(E_p/M))^2}-\frac{1}{(1+(E_{p'}/M))^2},$$

$$\beta=\left(\frac{1}{1+(E_p/M)}+\frac{1}{1+(E_{p'}/M)}\right)^2,$$

* Properties of the one-pion-exchange potential, both in the cases of the pseudoscalar and pseudovector couplings, have been investigated in detail by HMT.

$$\gamma=\left(\frac{1}{1+(E_p/M)}-\frac{1}{1+(E_{p'}/M)}\right)^2. \tag{5·3}$$

g is the pseudoscalar coupling constant and μ is the mass of the pion. These potentials contain nonstatic effects completely, i. e. they are not expanded with respect to the inverse of the mass of the nucleon. Only V_2 term remains in the static approximation:

$$V^{(2)}_{static}=-\left(\frac{g}{2M}\right)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{(\boldsymbol{k}^2+\mu^2)}\cdot\frac{1}{(2\pi)^3}$$
$$=-\left(\frac{f}{\mu}\right)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{(\boldsymbol{k}^2+\mu^2)}\frac{1}{(2\pi)^3} \tag{5·4}$$

When we apply the general formulas in § 4 for the one-pion-exchange potential, we obtain the following matrix elements. From equations given in § 4, we get, for spin singlet states,

$$\langle j|V^{(2)}(p,p')|j\rangle=\int(2\pi)\left[-(p^2+p'^2)\frac{V_2}{\boldsymbol{k}^2}P_J(z)+2pp'\frac{V_2}{\boldsymbol{k}^2}P_J(z)\cdot z\right]dz$$
$$-\int(2\pi)\left[(p^2+p'^2)\frac{V_3}{\boldsymbol{q}^2}P_J(z)+2pp'\frac{V_3}{\boldsymbol{q}^2}P_J(z)\,z\right]dz$$
$$+\int(2\pi)p^2p'^2\left[-\frac{V_4\cdot P_J(z)}{\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}+\frac{V_4\cdot P_J(z)}{\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}z^2\right]dz$$
$$-\int(6\pi)V_5\cdot P_J(z)\,dz. \tag{5·5}$$

Inserting Eq. (5·2) into Eq. (5·5), and carrying out the integral with respect to z. we obtain

$$\langle j|V^{(2)}(p,p')|j\rangle=\langle j|V_2^{(2)}|j\rangle+\langle j|V_3^{(2)}|j\rangle+\langle j|V_4^{(2)}|j\rangle+\langle j|V_5^{(2)}|j\rangle,$$

where

$$\langle j|V_2^{(2)}|j\rangle$$
$$=4\pi G\left[\left\{\frac{(p^2+p'^2)\cdot\beta}{B}+\frac{(p^2+p'^2)^2\alpha}{(p^2-p'^2)B}-\frac{2pp'\beta}{B}x_0-\frac{4p^2p'^2\alpha}{(p^2-p'^2)B}x_0^2\right\}Q_J(x_0)\right.$$
$$\left.+\left\{\frac{2pp'\beta+(4p^2p'^2\alpha/(p^2-p'^2))x_0}{B}\right\}\delta_{J,0}+\frac{1}{3B}\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\delta_{J,1}\right].$$

$$\langle j|V_3^{(2)}|j\rangle$$
$$=4\pi G\left[\left\{\frac{(p^2+p'^2)\gamma}{B}+\frac{(p^2+p'^2)^2\alpha}{(p^2-p'^2)B}+\frac{2pp'\gamma}{B}x_0-\frac{4p^2p'^2\alpha}{(p^2-p'^2)B}x_0^2\right\}Q_J(x_0)\right.$$
$$\left.+\left\{\frac{-2pp'\gamma+(4p^2p'^2\alpha/(p^2-p'^2))x_0}{B}\right\}\delta_{J,0}+\frac{1}{3B}\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\delta_{J,1}\right],$$

$$\langle j|V_4^{(2)}|j\rangle$$
$$=16G\frac{\alpha}{B}\Big[\frac{p^2p'^2}{(p^2-p'^2)}(1-x_0^2)Q_J(x_0)+\frac{p^2p'^2}{(p^2-p'^2)}\Big(x_0\delta_{J,0}+\frac{1}{3}\delta_{J,1}\Big)\Big],$$
$$\langle j|V_5^{(2)}|j\rangle$$
$$=12\pi G\frac{\alpha}{B}\Big[\Big\{-\frac{(p^2+p'^2)^2}{(p^2-p'^2)}+(p^2-p'^2)+\frac{4p^2p'^2}{(p^2-p'^2)}x_0^2\Big\}Q_J(x_0)$$
$$-\frac{4p^2p'^2}{(p^2-p'^2)}\Big(x_0\delta_{J,0}+\frac{1}{3}\delta_{J,1}\Big)\Big], \tag{5·6}$$

where $Q_J(x_0)$'s are Legendre functions of the second kind. Then we find the result which agrees with Signell's results from Eq. (5·5) and Eq. (5·6), i.e.

$$\langle j|V^{(2)}(p,p')|j\rangle=-\frac{g^2}{4\pi}\frac{[(x_0-\eta)Q_J(x_0)-\delta_{J,0}]}{2\pi E_p\cdot E_{p'}}, \tag{5·7}$$

where

$$x_0\equiv\frac{\mu^2+2(E_p\cdot E_{p'}-M^2)}{2pp'},$$
$$\eta\equiv\frac{p(E_{p'}+M)}{2p'(E_p+M)}+\frac{p'(E_p+M)}{2p(E_{p'}+M)}.$$

Other matrix elements for spin triplet states are also evaluated in the same way, and the results are given by

$$\langle J-1|V^{(2)}(p\cdot p')|J-1\rangle$$
$$=\langle J-1|V_2^{(2)}|J-1\rangle+\langle J-1|V_3^{(2)}|J-1\rangle+\langle J-1|V_4^{(2)}|J-1\rangle+\langle J-1|V_5^{(2)}|J-1\rangle,$$

where

$$\langle J-1|V_2^{(2)}|J-1\rangle$$
$$=2\pi G\Big[\frac{(p^2+p'^2)}{(2J+1)}\Big\{\Big(\beta+\frac{(p^2+p'^2)}{(p^2-p'^2)}\alpha+\frac{2pp'\alpha}{(p^2-p'^2)}x_0\Big)\frac{(-2)}{B}Q_{J-1}(x_0)$$
$$+\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,1}\Big\}-\frac{2pp'}{(2J+1)}\Big\{\Big(\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}+\frac{2pp'\alpha}{(p^2-p'^2)}x_0\Big)$$
$$\times\frac{(-2)}{B}Q_J(x_0)+\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,0}\Big\}\Big],$$

$$\langle J-1|V_3^{(2)}|J-1\rangle$$
$$=2\pi G\Big[\frac{(p^2+p'^2)}{(2J+1)}\Big\{\Big(\gamma+\frac{(p^2+p'^2)}{(p^2-p'^2)}\alpha-\frac{2pp'\alpha}{(p^2-p'^2)}x_0\Big)\frac{(-2)}{B}Q_{J-1}(x_0)$$
$$-\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,1}\Big\}+\frac{2pp'}{(2J+1)}\Big\{\Big(\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}-\frac{2pp'\alpha}{(p^2-p'^2)}x_0\Big)$$

$$\times\frac{(-2)}{B}Q_J(x_0)-\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,0}\Big\}\Big],$$

$\langle J-1|V_4^{(2)}|J-1\rangle$

$$=2\pi G\Big[\frac{8p^2p'^2\alpha}{(p^2-p'^2)(2J+1)}\Big(\frac{2A}{B^2}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)+\frac{4p^2p'^2\alpha(2J-1)}{(p^2-p'^2)(2J+1)}$$

$$\times\Big(-\frac{2}{B}Q_{J-1}(x_0)\Big)-\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2x_0^2}{B}Q_{J-1}(x_0)+\frac{2x_0}{B}\delta_{J,1}+\frac{2}{3B}\delta_{J,2}\Big)\Big],$$

$\langle J-1|V_5^{(2)}|J-1\rangle$

$$=2\pi G\Big[\frac{8p^2p'^2\alpha}{(p^2-p'^2)B}Q_{J-1}(x_0)+\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2}{B}Q_{J-1}(x_0)\cdot x_0^2$$

$$+\frac{2x_0}{B}\delta_{J,1}+\frac{2}{3B}\delta_{J,2}\Big)\Big].$$

From this we obtain

$$\langle J-1|V^{(2)}|J-1\rangle=\Big(\frac{g^2}{4\pi}\Big)\frac{[Q_J(x_0)-\eta Q_{J-1}(x_0)]}{2\pi E_pE_{p'}(2J+1)}.$$

Also, for $L=J+1$, we obtain

$\langle J+1|V^{(2)}(p,p')|J+1\rangle$

$$=\langle J+1|V_2^{(2)}|J+1\rangle+\langle J+1|V_3^{(2)}|J+1\rangle+\langle J+1|V_4^{(2)}|J+1\rangle+\langle J+1|V_5^{(2)}|J+1\rangle,$$

where

$\langle J+1|V_2^{(2)}|J+1\rangle$

$$=2\pi G\Big[\frac{-(p^2+p'^2)}{(2J+1)}\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_{J+1}(x_0)-\frac{(p^2+p'^2)}{(2J+1)}\frac{2pp'\alpha}{(p^2-p'^2)}$$

$$\times\Big(\frac{-2x_0}{B}Q_{J+1}(x_0)\Big)+\frac{2pp'}{(2J+1)}\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_J(x_0)$$

$$+\frac{4p^2p'^2\alpha}{(2J+1)(p^2-p'^2)}\Big(\frac{-2x_0}{B}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)\Big],$$

$\langle J+1|V_3^{(2)}|J+1\rangle$

$$=2\pi G\Big[-\frac{(p^2+p'^2)}{(2J+1)}\Big\{\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_{J+1}(x_0)+\frac{(p^2+p'^2)}{(p^2-p'^2)}\frac{2pp'\alpha}{(2J+1)}$$

$$\times\Big(\frac{-2x_0}{B}Q_{J+1}(x_0)\Big)-\frac{2pp'}{(2J+1)}\Big\{\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_J(x_0)$$

$$+\frac{4p^2p'^2\alpha}{(2J+1)(p^2-p'^2)}\Big(\frac{-2x_0}{B}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)\Big],$$

$$\langle J+1|V_4^{(2)}|J+1\rangle$$
$$=2\pi G\Big[\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big\{-\frac{2}{(2J+1)}\Big(-\frac{2x_0}{B}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)$$
$$-\Big(-\frac{2x_0^2}{B}Q_{J+1}(x_0)+\frac{2x_0}{B}\delta_{J,0}+\frac{2}{3B}\delta_{J,1}\Big)+\frac{(2J+3)}{(2J+1)}\Big(-\frac{2}{B}Q_{J+1}(x_0)\Big)\Big\}\Big],$$

$$\langle J+1|V_5^{(2)}|J+1\rangle$$
$$=2\pi G\Big[-\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2}{B}\Big)Q_{J+1}(x_0)+\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2x_0^2}{B}Q_{J+1}(x_0)$$
$$+\frac{2x_0}{B}\delta_{J,0}+\frac{2}{3B}\delta_{J,1}\Big)\Big],$$

and

$$\langle J+1|V^{(2)}(p,p')|J+1\rangle=\Big(\frac{g^2}{4\pi}\Big)\frac{[-Q_J(x_0)+\eta Q_{J+1}(x_0)]}{2\pi E_pE_{p'}(2J+1)}.$$

And, for $L=J$, we obtain

$$\langle J|V^{(2)}(p,p')|J\rangle=\langle J|V_2^{(2)}|J\rangle+\langle J|V_3^{(2)}|J\rangle+\langle J|V_4^{(2)}|J\rangle+\langle J|V_5^{(2)}|J\rangle,$$

where

$$\langle J|V_2^{(2)}|J\rangle$$
$$=2\pi G\Big[(p^2+p'^2)\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(-\frac{2}{B}Q_J(x_0)\Big)+\frac{2pp'(p^2+p'^2)\alpha}{(p^2-p'^2)}$$
$$\times\Big(-\frac{2}{B}x_0Q_J(x_0)\Big)-\frac{(2pp')}{(2J+1)}\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}$$
$$\times\Big\{J\cdot\Big(-\frac{2}{B}\Big)Q_{J+1}(x_0)+(J+1)\Big(-\frac{2}{B}\Big)Q_{J-1}(x_0)\Big\}$$
$$-\frac{(2pp')}{(2J+1)}\frac{(2pp'\alpha)}{(p^2-p'^2)}\Big\{J\Big(-\frac{2}{B}x_0Q_{J+1}(x_0)\Big)$$
$$+(J+1)\Big(-\frac{2}{B}x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\Big)\Big\}\Big],$$

$$\langle J|V_3^{(2)}|J\rangle$$
$$=2\pi G\Big[(p^2+p'^2)\Big(\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big)\Big(-\frac{2}{B}Q_J(x_0)\Big)-\frac{2pp'(p^2+p'^2)\alpha}{(p^2-p'^2)}$$
$$\times\Big(-\frac{2}{B}Q_J(x_0)x_0\Big)+\Big(\frac{2pp'}{2J+1}\Big)\Big(\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big)$$
$$\times\Big\{J\Big(-\frac{2}{B}\Big)Q_{J+1}(x_0)+(J+1)\Big(-\frac{2}{B}\Big)Q_{J-1}(x_0)\Big\}-\frac{(2pp')}{(2J+1)}$$

$$\times\frac{(2pp'\alpha)}{(p^2-p'^2)}\left\{J\left(-\frac{2}{B}\right)x_0Q_{J+1}(x_0)+(J+1)\left(-\frac{2}{B}x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right)\right\}\Bigg],$$

$\langle J|V_4^{(2)}|J\rangle$

$$=2\pi G\Bigg[\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left\{-\left(-\frac{2}{B}\right)Q_J(x_0)+\frac{(J+2)}{(2J+1)}\left(-\frac{2x_0}{B}Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right)\right.$$

$$\left.+\frac{(J-1)}{(2J+1)}\left(-\frac{2}{B}x_0Q_{J+1}(x_0)\right)\right\}\Bigg],$$

$\langle J|V_5^{(2)}|J\rangle$

$$=2\pi G\Bigg[-\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left(-\frac{2}{B}Q_J(x_0)\right)+\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left(-\frac{2}{B}x_0^2Q_J(x_0)+\frac{2}{3B}\delta_{J,1}\right)\Bigg],$$

so we obtain

$\langle J|V^{(2)}(p,p')|J\rangle$

$$=-\left(\frac{g^2}{4\pi}\right)\frac{[(2J+1)\eta Q_J(x_0)-JQ_{J+1}(x_0)-(J+1)Q_{J-1}(x_0)]}{2\pi E_pE_{p'}(2J+1)}.$$

For a non-diagonal element, we have

$$\langle J+1|V^{(2)}(p,p')|J-1\rangle\equiv\langle J-1|V^{(2)}(p,p')|J+1\rangle$$

$$=\langle J+1|V_2^{(2)}|J-1\rangle+\langle J+1|V_3^{(2)}|J-1\rangle+\langle J+1|V_4^{(2)}|J-1\rangle,$$

where

$\langle J+1|V_2^{(2)}|J-1\rangle$

$$=4\pi G\frac{\sqrt{J(J+1)}}{(2J+1)}\Bigg[\left\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\right\}\left\{p'^2\left(-\frac{2}{B}\right)Q_{J-1}(x_0)+p^2\left(-\frac{2}{B}\right)Q_{J+1}(x_0)\right.$$

$$\left.-2pp'\left(-\frac{2}{B}\right)Q_J(x_0)\right\}+\frac{2pp'\alpha}{(p^2-p'^2)}\left\{p'^2\left(-\frac{2}{B}\right)x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right.$$

$$\left.+p^2\left(-\frac{2}{B}\right)x_0Q_{J+1}(x_0)-2pp'\left(\left(-\frac{2}{B}\right)x_0Q_J(x_0)+\frac{2}{B}\delta_{J,0}\right)\right\}\Bigg],$$

$\langle J+1|V_3^{(2)}|J-1\rangle$

$$=4\pi G\frac{\sqrt{J(J+1)}}{(2J+1)}\Bigg[\left\{\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\right\}\left\{p'^2\left(-\frac{2}{B}\right)Q_{J-1}(x_0)+p^2\left(-\frac{2}{B}\right)Q_{J+1}(x_0)\right.$$

$$\left.+2pp'\left(-\frac{2}{B}\right)Q_J(x_0)\right\}-\frac{2pp'\alpha}{(p^2-p'^2)}\left\{p'^2\left(-\frac{2}{B}x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right)\right.$$

$$\left.+p^2\left(-\frac{2}{B}x_0Q_{J+1}(x_0)\right)-2pp'\left(-\frac{2}{B}x_0Q_J(x_0)+\frac{2}{B}\delta_{J,0}\right)\right\}\Bigg],$$

$$\langle J+1|V_4^{(2)}|J-1\rangle$$
$$=4\pi G\frac{\sqrt{J(J+1)}}{(2J+1)^2}\left[\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left(-\frac{2}{B}Q_{J+1}(x_0)-\left(-\frac{2}{B}\right)Q_{J-1}(x_0)\right)\right],$$

so we obtain

$$\langle J+1|V^{(2)}(p,p')|J-1\rangle=-\left(\frac{g^2}{4\pi}\right)\frac{\sqrt{J(J+1)}}{2\pi E_pE_{p'}(2J+1)}$$
$$\times\left[\frac{p(E_{p'}+M)}{p'(E_p+M)}Q_{J+1}(x_0)+\frac{p'(E_p+M)}{p(E_{p'}+M)}Q_{J-1}(x_0)-2Q_J(x_0)\right].$$

§ 6. Phase shifts in the Born approximation

Phase shifts for the most general potential in the Born approximation are given without solving the integral equations. Using Eq. (3·12) and Blatt-Biedenharn phase parameters,[12] we obtain the following results, where $|p|=|p'|\equiv p_0$;
for a spin singlet state

$$\tan{}^1\delta_{j,B}=-\frac{\pi p_0E_{p0}}{2}\langle j|V(p_0,p_0)|j\rangle;$$

for spin triplet states

$$\tan{}^3\delta_{J,B}=-\frac{\pi p_0E_{p0}}{2}\langle J|V(p_0,p_0)|J\rangle,$$

$$x_{J-1,B}=-\frac{\pi p_0E_{p0}}{2}\langle J-1|V(p_0,p_0)|J-1\rangle,$$

$$x_{J+1,B}=-\frac{\pi p_0E_{p0}}{2}\langle J+1|V(p_0,p_0)|J+1\rangle,$$

$$y_{J-1,B}=+\frac{\pi p_0E_{p0}}{2}\langle J+1|V(p_0,p_0)|J-1\rangle,$$

$$y_{J+1,B}=+\frac{\pi p_0E_{p0}}{2}\langle J-1|V(p_0,p_0)|J+1\rangle,$$

so we have

$$y_{J-1,B}=y_{J+1,B}\equiv y_B.$$

The results may be written in the following way:

(1) V_0 *type*

$$\tan{}^1\delta_{j,B}=-\pi^2p_0E_{p0}A_j^0,$$
$$\tan{}^3\delta_{J,B}=-\pi^2p_0E_{p0}A_J^0,$$
$$x_{J-1,B}=-\pi^2p_0E_{p0}A_{J-1}^0,$$
$$x_{J+1,B}=-\pi^2p_0E_{p0}A_{J+1}^0,$$

$$y_B=0,$$

where

$$A_n^0=\int V_0(p_0,z)P_n(z)dz.$$

(2) V_1 *type*

$$\tan^1\delta_{J,B}=0,$$

$$\tan^3\delta_{J,B}=-2\pi^2 p_0 E_{p0}\left[-\frac{(J+1)}{(2J+1)^2}A^1_{J+1}+\frac{1}{(2J+1)}A^1_{J-1}\right],$$

$$x_{J-1,B}=-2\pi^2 p_0 E_{p0}\frac{(J-1)}{(2J-1)}[A_J^1-A^1_{J-2}],$$

$$x_{J+1,B}=-2\pi^2 p_0 E_{p0}\left[-\frac{(J+2)}{(2J+3)}A^1_{J+2}+\frac{(J+2)}{(2J+5)}A_J^1\right],$$

$$y_B=0,$$

where

$$A_n^1=\int\frac{V_1(p_0,z)P_n(z)}{\sin(\boldsymbol{p}\cdot\boldsymbol{p}')}dz$$

provided that $|p|=|p'|\equiv p_0$.

(3) V_2 *type*

$$\tan^1\delta_{J,B}=-\pi^2 p_0 E_{p0}[-A_J^{2(0)}+A_J^{2(1)}],$$

$$\tan^3\delta_{J,B}=-\pi^2 p_0 E_{p0}\left[A_J^{2(0)}-\frac{J}{(2J+1)}A^{2(0)}_{J+1}-\frac{(J+1)}{(2J+1)}A^{2(0)}_{J-1}\right],$$

$$x_{J-1,B}=-\pi^2 p_0 E_{p0}\left[\frac{1}{(2J+1)}A^{2(0)}_{J-1}-\frac{1}{(2J+1)}A_J^{2(0)}\right],$$

$$x_{J+1,B}=-\pi^2 p_0 E_{p0}\left[-\frac{1}{(2J+1)}A^{2(0)}_{J+1}+\frac{1}{(2J+1)}A_J^{2(0)}\right],$$

$$y_B=\pi^2 p_0 E_{p0}\frac{\sqrt{J(J+1)}}{(2J+1)}[A^{2(0)}_{J-1}+A^{2(0)}_{J+1}-2A_J^{2(0)}],$$

where

$$A_n^{2(l)}=\int\frac{V_2(p_0,z)P_n(z)z^l}{(1-z)}dz.$$

(4) V_3 *type*

$$\tan^1\delta_{J,B}=4\pi^2 p_0 E_{p0}[A_J^{3(0)}+A_J^{3(1)}],$$

$$\tan^3\delta_{J,B}=-4\pi^2 p_0 E_{p0}\left[A_J^{3(0)}+\frac{J}{(2J+1)}A^{3(0)}_{J+1}+\frac{(J-1)}{(2J+1)}A^{3(0)}_{J-1}\right],$$

$$x_{J-1,B}=-4\pi^2 p_0 E_{p0}\left[\frac{1}{(2J+1)}A^{3(0)}_{J-1}+\frac{1}{(2J+1)}A^{3(0)}_{J}\right],$$

$$x_{J+1,B}=4\pi^2 p_0 E_{p0}\left[\frac{1}{(2J+1)}A^{3(0)}_{J+1}+\frac{1}{(2J+1)}A^{3(0)}_{J}\right],$$

$$y_B=4\pi^2 p_0 E_{p0}\frac{\sqrt{J(J+1)}}{(2J+1)}[A^{3(0)}_{J-1}+A^{3(0)}_{J+1}+2A^{3(0)}_{J}],$$

where

$$A^{3(l)}_n=\int\frac{V_3(p_0,z)P_n(z)z^l}{(1+z)}dz.$$

(5) V_4 *type*

$$\tan^1\delta_{J,B}=-\pi^2 p_0 E_{p0}[-A^{4(0)}_J+A^{4(2)}_J],$$

$$\tan^3\delta_{J,B}=-\pi^2 p_0 E_{p0}\left[-A^{4(0)}_J+\frac{(J+2)}{(2J+1)}A^{4(1)}_{J-1}+\frac{(J-1)}{(2J+1)}A^{4(1)}_{J+1}\right],$$

$$x_{J-1,B}=-\pi^2 p_0 E_{p0}\left[\frac{2}{(2J+1)}A^{4(1)}_J+\frac{(2J-1)}{(2J+1)}A^{4(0)}_{J-1}-A^{4(2)}_{J-1}\right],$$

$$x_{J+1,B}=-\pi^2 p_0 E_{p0}\left[-\frac{2}{(2J+1)}A^{4(1)}_J-A^{4(2)}_{J+1}+\frac{(2J+3)}{(2J+1)}A^{4(0)}_{J+1}\right],$$

$$y_B=2\pi^2 p_0 E_{p0}\frac{\sqrt{J(J+1)}}{(2J+1)^2}[A^{4(0)}_{J+1}-A^{4(0)}_{J-1}],$$

where

$$A^{4(l)}_n=\int\frac{V_4(p_0,z)P_n(z)z^l}{\sin^2(k\cdot q)(1-z^2)}dz$$

provided that

$$|p|=|p'|=p_0.$$

(6) V_5 *type*

$$\tan^1\delta_{J,B}=3\pi^2 p_0 E_{p0}A_J{}^5,$$

$$\tan^3\delta_{J,B}=-\pi^2 p_0 E_{p0}A_J{}^5,$$

$$x_{J-1,B}=-\pi^2 p_0 E_{p0}A^5_{J-1},$$

$$x_{J+1,B}=-\pi^2 p_0 E_{p0}A^5_{J+1},$$

$$y_B=0,$$

where

$$A_n{}^5=\int V_5(p_0,z)p_n(z)dz.$$

The authors would like to thank Professor M. Taketani for his stimulating discussions. One of the authors (S. M.) takes this opportunity to recall with pleasure several interesting discussions with Professor R. E. Marshak and Dr. P. S. Signell while he was staying in U.S.A.

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