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Nuclear Level Density and Inertia Parameters

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The formulae of calculation of the nuclear level density based on the theory of superconductivity are presented. Special attention is paid to the dependence of the level density on the angular momentum. The moment of inertia is evaluated using the cranking model. For a deformed nucleus, the Coriolis anti-pairing effect is taken into account. The obtained formulae do not contain any open parameters. The numerical calculations are made for the single-particle spectrum of the Nilsson model without the use of asymptotic integral expressions. The results of calculations are compared with experimental data for ^{68}Ge and ^{242}Pu. The agreement is satisfactory.

§ 1. Introduction

Recently the successful method of the theory of superconductivity[1] has been employed in the study of nuclear structure. Studies in this field are likewise focussed mainly upon the ground states or the low-lying excited states. The authors have considered it worth while applying the method to higher states, and to discuss its physical consequences. In order to make the investigation of the over all character of nuclear sepectra, it is necessary at least to make use of known properties of nuclei at low excitation, and to extend the low energy picture to higher excitation.

In a previous paper,[2] we employed a statistical approach to examine the effect of the pairing interaction on the gross structure of nuclear spectrum. The main conclusion is the existence of the transition energy of about 5 to 10 MeV below which the Fermi gas model is invalidated. In fact, in this superconducting phase, the energy-temperature relation is much different from that expected for the latter model, and the level density is much smaller than is expected by the extrapolation from the normal phase. Afterwards, such an investigation was performed by many authors,[3] and it was shown that the agreement between experiments and theory is better than when the Fermi-gas model is applied. However, we made no reference to the angular momentum in the previous paper. As the angular momentum is a good quantum number for atomic nuclei, the statistical considerations should be applied also to statistical ensemble with a definite value of angular momentum. The angular momentum plays an essential role in the analysis of compound nuclear reactions such as the isomeric cross section ratio and the angular anisotropics of evaporated particles and fission fragments. Therefore it is important to determine the density of nuclear levels with a given angular momentum.

In § 2 we give a formulation of the statistical dynamics of the system taking into account both the pairing interaction and the angular momentum. This is applied in § 3 where the thermodynamical properties of nuclei are studied. For a deformed nucleus, if we assume an axially symmetric shape, the theory contains two inertia parameters; one is a moment of inertia about an axis perpendicular to the symmetry axis and the other is a moment of inertia about the symmetry axis. For spherically symmetric nucleus we can describe the distribution of angular momentum with only one inertia parameter corresponding to rotation about the z-axis. The moment of inertia in the Fermi-gas model or non-interacting particle model coincides with the moment of inertia of the rigid rotor. However, the moments of inertia which can be defined from the rotational sequences of the low-lying energy levels are smaller than those of the rigid rotor. It is seen that the observed values are from 0.2 to 0.5 of the rigid-body values. If the effect of the pairing correlation is taken into account, the values of moment of inertia are in a satisfactory agreement with experiments due to the existence of the energy gap. In our treatment, the energy gap parameter Δ is a function of both temperature and angular velocity, i.e., the excitation energy and the angular momentum. It decreases with increasing the excitation energy and also the angular momentnm. The reduction of the energy gap Δ results in an increase in the moment of inertia. At the limit of $\Delta=0$, the moment of inertia becomes a rigid rotor value. The critical temperature or the transition energy corresponding to disappearance of the energy gap is also a function of the angular momentum. A relationship between the transition energy and the angular momentum is derived. In § 4, we give the derivation of level density formula and discuss the effects of the pairing interaction and the angular momentum on the gross structure of nuclear spectrum.

In the numerical calculation, we try to employ the single particle levels of the Nilsson model as in the calculations related to the ground states or the low-lying excited states. At the present time the analysis of the density of excited states of nuclei is usually carried out on the basis of simple analytic expressions in the Fermi-gas model. The influence of the discrete structure of single-particle spectra on the behavior of the thermodynamic functions was considered by a few authors.[4] In the continuous-spectrum approximation, the sum over the single-particle states is substituted into an integral on the single particle energy. The derivation of the thermodynamic functions in analytical form always involves numerous approximations. A series of these approximations may be avoided by using the Nilsson levels. The results obtained are given in § 5.

§ 2. Pairing interaction and thermodynamical functions

We consider a system of nucleons interacting with the pairing force. The

Hamiltonian of this system is written as follows:

$$H=\sum_{\alpha}(\varepsilon_\alpha-\lambda)c_\alpha{}^+c_\alpha+\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}c_\alpha{}^+c_\beta{}^+c_\delta c_\gamma\,, \tag{1}$$

where $c_\alpha{}^+$ and c_α are the creation and annihilation operators of a nucleon in the state α, ε_α is the energy of this single particle state, λ the chemical potential of the system and $V_{\alpha\beta\gamma\delta}$ the matrix elements of the interaction.

The introduction of angular momentum is made by the technique of a Lagrange-multiplier. For spherically symmetric nuclei, in addition to being characterized by the energy, the nuclear states are also characterized by the projection of the angular momentum on the z-axis. For deformed nuclei, in addition to the rotational motion about the symmetry axis z, we must also consider the rotational motion about the x-axis perpendicular to the symmetry axis. For simplicity, we consider the z-component of angular momentum and the x-component independently. This approximation corresponds to the case such that we assume the angular momenta to be classical vectors. The terms which must add to the Hamiltonian (1) are

$$H_{\omega_\perp}=-\omega_\perp J_x=-\omega_\perp\sum_{\alpha\beta}J_{\alpha\beta}c_\alpha{}^+c_\beta\,, \tag{2}$$

$$H_{\omega_\parallel}=-\omega_\parallel J_z=-\omega_\parallel\sum_{\alpha}K_\alpha c_\alpha{}^+c_\alpha\,, \tag{3}$$

where $\omega_\perp$ and $\omega_\parallel$ are components of the angular velocity on the x- and z-axes, $J_{\alpha\beta}$ and K_α are matrix elements of the angular momentum operator.

$$J_{\alpha\beta}=\langle\alpha|j_x|\beta\rangle, \tag{4a}$$

$$K_\alpha=\langle\alpha|j_z|\alpha\rangle. \tag{4b}$$

Thermodynamical behavior of a quantum mechanical system can be described by the statistical operator $e^{-\beta\mathcal{H}}$ of the ensemble. Here β is the inverse of temperature of the system and $\mathcal{H}$ is of the form

$$\mathcal{H}=H-\omega_\perp J_x-\omega_\parallel J_z\,. \tag{5}$$

For spherically symmetric nuclei, however, it is not necessary to consider the x-component of the angular momentum. The term corresponding to the x-component of the angular momentum should be equal to zero.

The approximation method developed by BCS, Bogoliubov, and Valatin[1)] consists, at non-zero temperature, of a self-consistent linearization of the statistical operator on such a way that the operator represents an ensemble of independent quasi-particles. The previous expression is then replaced by $e^{-\beta\mathcal{H}_0}$, where $\mathcal{H}_0$ is the linearized Hamiltonian. The normalized statistical operator is then

$$U_0=\frac{e^{-\beta\mathcal{H}_0}}{\mathrm{Tr}\,e^{-\beta\mathcal{H}_0}}\,, \tag{6}$$

and the ensemble average of any quantity A is given by

$$\langle A\rangle=\mathrm{Tr}(AU_0). \tag{7}$$

In order to obtain the linearized Hamiltonian $\mathcal{H}_0$, we perform the generalized Bogoliubov transformation[5])

$$c_\alpha=\sum_i(A_\alpha^i a_i+B_\alpha^{i*}a_i^+),\tag{8}$$

where a_i is an annihilation operator of a quasi-particle. The transformation coefficients A and B obey the orthogonality conditions

$$\sum_\alpha(A_\alpha^{i*}A_\alpha^j+B_\alpha^{i*}B_\alpha^j)=\delta_{ij},\qquad \sum_\alpha(A_\alpha^i B_\alpha^j+B_\alpha^i A_\alpha^j)=0,\tag{9a}$$

$$\sum_i(A_\alpha^i A_\beta^{i*}+B_\alpha^{i*}B_\beta^i)=\delta_{\alpha\beta},\qquad \sum_i(A_\alpha^i B_\beta^{i*}+B_\alpha^{i*}A_\beta^i)=0.\tag{9b}$$

We assume $\mathcal{H}_0$ to have the form

$$\mathcal{H}_0=\text{const}+\sum_i \hat{E}_i^{(T)}a_i^+a_i.\tag{10}$$

For the expectation value of an operator

$$\hat{n}_i=a_i^+a_i,\tag{11}$$

we obtain

$$\langle\hat{n}_i\rangle=f_i=\frac{1}{1+\exp[\beta\hat{E}_i^{(T)}]},\tag{12}$$

that is, a Fermi distribution of independent quasi-particles. The ensemble average of the Hamiltonian $\mathcal{H}$ is given by

$$\begin{aligned}W^{(T)}&=\langle\mathcal{H}\rangle\\&=\sum_\alpha(\varepsilon_\alpha-\lambda)\rho_{\alpha\alpha}^{(T)}+\tfrac{1}{2}\sum_{\alpha\gamma}\Gamma_{\alpha\gamma}^{(T)}\rho_{\alpha\gamma}^{(T)}+\tfrac{1}{2}\sum_{\alpha\beta}\Delta_{\alpha\beta}^{(T)}\kappa_{\alpha\beta}^{(T)*}\\&\qquad-\omega_\perp\sum_{\alpha\gamma}J_{\alpha\gamma}\rho^{(T)}-\omega_\parallel\sum_\alpha K_\alpha\rho_{\alpha\alpha}^{(T)}.\end{aligned}\tag{13}$$

The quantities $\Gamma^{(T)}$ and $\Delta^{(T)}$ are defined by

$$\Delta_{\alpha\beta}^{(T)}=-\Delta_{\beta\alpha}^{(T)}=2\sum_{\gamma\delta}V_{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}^{(T)},\tag{14a}$$

$$\Gamma_{\alpha\gamma}^{(T)}=\Gamma_{\gamma\alpha}^{(T)*}=2\sum_{\beta\delta}(V_{\alpha\beta\gamma\delta}-V_{\alpha\beta\delta\gamma})\rho_{\beta\delta}^{(T)},\tag{14b}$$

where

$$\kappa_{\gamma\delta}^{(T)}=-\kappa_{\delta\gamma}^{(T)}=\langle c_\delta c_\gamma\rangle=\sum_i(A_\delta^i B_\gamma^{i*}(1-f_i)+A_\gamma^i B_\delta^{i*}f_i),\tag{15a}$$

$$\rho_{\beta\delta}^{(T)}=\rho_{\delta\beta}^{(T)*}=\langle c_\beta^+c_\delta\rangle=\sum_i(A_\beta^{i*}A_\delta^i f_i+B_\beta^i B_\delta^{i*}(1-f_i)).\tag{15b}$$

The free energy of the system is obtained from $W^{(T)}$ by adding an entropy term:

$$F=W^{(T)}-TS.\tag{16}$$

The entropy expression for the system of independent quasi-particles is given by

$$TS=-\beta^{-1}\sum_i\{f_i\log f_i+(1-f_i)\log(1-f_i)\}.\tag{17}$$

This depends only on the Fermi distribution f_i. One can minimize the free energy independently with respect to the functions f_i, $\rho^{(T)}$ and $\kappa^{(T)}$. The minimization with respect to f_i for given $\rho^{(T)}$ and $\kappa^{(T)}$ leads to the Fermion distribution (12). If we vary the free energy with respect to $\rho^{(T)}$ and $\kappa^{(T)}$ for f_i fixed, this amounts to the same as varying $W^{(T)}$. We then obtain the Hartree-Fock-Bogoliubov equation:

$$\hat{E}_i^{(T)}A_\alpha{}^i=(\varepsilon_\alpha-\lambda)A_\alpha{}^i+\sum_\gamma \Gamma_{\alpha\gamma}^{(T)}A_\gamma{}^i+\sum_\beta \Delta_{\alpha\beta}^{(T)}B_\beta{}^i-\omega_\perp\sum_\gamma J_{\alpha\gamma}A_\gamma{}^i-\omega_\parallel K_\alpha A_\alpha{}^i,$$

$$\hat{E}_i^{(T)}B_\alpha{}^i=-(\varepsilon_\alpha-\lambda)B_\alpha{}^i-\sum_\gamma \Gamma_{\alpha\gamma}^{(T)*}B_\gamma{}^i-\sum_\beta \Delta_{\alpha\beta}^{(T)*}A_\beta{}^i+\omega_\perp\sum_\gamma J_{\alpha\gamma}^*B_\gamma{}^i+\omega_\parallel K_\alpha B_\alpha{}^i. \tag{18}$$

The complex conjugate of these equations becomes

$$-\hat{E}_i^{(T)}B_\alpha{}^{i*}=(\varepsilon_\alpha-\lambda)B_\alpha{}^{i*}+\sum_\gamma \Gamma_{\alpha\gamma}^{(T)}B_\gamma{}^{i*}+\sum_\beta \Delta_{\alpha\beta}^{(T)}A_\beta{}^{i*}-\omega_\perp\sum_\gamma J_{\alpha\gamma}B_\gamma{}^{i*}-\omega_\parallel K_\alpha B_\alpha{}^{i*},$$

$$-\hat{E}_i^{(T)}A_\alpha{}^{i*}=-(\varepsilon_\alpha-\lambda)A_\alpha{}^{i*}-\sum_\gamma \Gamma_{\alpha\gamma}^{(T)}A_\gamma{}^{i*}-\sum_\beta \Delta_{\alpha\beta}^{(T)}B_\beta{}^{i*}+\omega_\perp\sum_\gamma J_{\alpha\gamma}^*A_\gamma{}^{i*}+\omega_\parallel K_\alpha A_\alpha{}^{i*}. \tag{19}$$

The chemical potential λ is determined by fixing the average number of particles

$$N=\langle\hat{N}\rangle=\langle\sum_\alpha c_\alpha{}^+c_\alpha\rangle=\sum_\alpha \rho_{\alpha\alpha}^{(T)}. \tag{20}$$

The angular velocity ω must be determined from the following conditions:

$$\sqrt{I(I+1)-K^2}\equiv\langle J_x\rangle=\sum_{\alpha\beta}J_{\alpha\beta}\rho_{\alpha\beta}^{(T)}=\omega_\perp\mathcal{J}_\perp{}^{(T)}, \tag{21}$$

$$K\equiv\langle J_z\rangle=\sum_\alpha K_\alpha\rho_{\alpha\alpha}^{(T)}=\omega_\parallel\mathcal{J}_\parallel{}^{(T)}, \tag{22}$$

were I is total angular momentum and K is its projection on the symmetry axis z. These equations also give the definition of the moments of inertia $\mathcal{J}_\perp$ and $\mathcal{J}_\parallel$.

By using Eqs. (18), (19), (21) and (22) we obtain the energy of the system

$$E(T)=\langle\mathcal{H}+\omega_\perp J_x+\omega_\parallel J_z\rangle$$

$$=U(T)+\frac{I(I+1)-K^2}{2\mathcal{J}_\perp{}^{(T)}}+\frac{K^2}{2\mathcal{J}_\parallel{}^{(T)}}, \tag{23a}$$

where

$$U(T)=\tfrac{1}{2}\sum_{\alpha,i}(\varepsilon_\alpha-\lambda)(B_\alpha{}^iB_\alpha{}^{i*}(1-f_i)+A_\alpha{}^iA_\alpha{}^{i*}f_i)$$

$$-\tfrac{1}{2}\sum_{\alpha,i}\hat{E}_i^{(T)}(B_\alpha{}^iB_\alpha{}^{i*}(1-f_i)-A_\alpha{}^{i*}A_\alpha{}^if_i). \tag{23b}$$

Thermodynamical functions of our interest, such as the entropy S and the specific heat C, can be derived from Eqs. (12) and (17) as follows:

$$S=\sum_i\log(1+\exp(-\beta\hat{E}_i^{(T)}))+\beta\sum_i\frac{\hat{E}_i^{(T)}}{1+\exp(\beta\hat{E}_i{}^T)}, \tag{24}$$

$$C=-\beta\frac{dS}{d\beta}=\frac{1}{4}\sum_i\frac{\widehat{E}_i{}^{(T)}}{\cosh^2(\beta\widehat{E}_i{}^{(T)}/2)}\left[\frac{\widehat{E}_i{}^{(T)}}{T^2}-\frac{1}{T}\frac{\partial\widehat{E}_i{}^{(T)}}{\partial T}\right]. \tag{25}$$

§ 3. Thermodynamical properties of atomic nuclei

In this section we discuss some characteristic features of the thermodynamical functions with the aid of simplifying approximations.

First of all we assume that the strengh of the pairing interaction is a constant. Then the pairing potential can be written as

$$\Delta_{\alpha\beta}^{(T)}=\delta_{\alpha,-\beta}s_\beta\Delta(T) \tag{26}$$

with

$$\Delta(T)=-\frac{G}{2}\sum_\gamma s_\gamma\kappa_{\gamma,-\gamma}^{(T)}{}_{-\gamma}\,. \tag{27}$$

Here $\Delta(T)$ is the so-called the energy gap parameter, s_α is a phase factor related to the time reversal of the state α, and $s_\alpha{}^2=1$, $s_\alpha s_{-\alpha}=-1$. With the further substitution

$$C_\alpha{}^i=s_\alpha B_{-\alpha}^i\,, \tag{28}$$

the Hartree-Fock-Bogoliubov equation (18) becomes

$$\begin{aligned}\widehat{E}_i{}^{(T)}A_\alpha{}^i&=(\varepsilon_\alpha-\lambda)A_\alpha{}^i+\sum_\gamma\Gamma_{\alpha\gamma}^{(T)}A_\gamma{}^i+\Delta(T)C_\alpha{}^i-\omega_\perp\sum_\gamma J_{\alpha\gamma}A_\gamma{}^i-\omega_\parallel K_\alpha A_\alpha{}^i,\\ \widehat{E}_i{}^{(T)}C_\alpha{}^i&=-(\varepsilon_\alpha-\lambda)C_\alpha{}^i-\sum_\gamma\Gamma_{\bar\alpha\bar\gamma}^{(T)*}C_\gamma{}^i+\Delta(T)A_\alpha{}^i-\omega_\perp\sum_\gamma J_{\alpha\gamma}C_\gamma{}^i-\omega_\parallel K_\alpha C_\alpha{}^i.\end{aligned} \tag{29}$$

3-1 *Perturbation approach*

In order to investigate thermodynamical properties of nuclei we want to solve Eq. (29) by using a simple perturbation calculation. The matrix element of the angular momentum about the axis of rotation parallel to the symmetry axis of the nucleus is diagonal. Then we have not any difficulty in solving the Hartree-Fock-Bogoliubov equation.

We treat $\omega_\perp J_x$ as a small perturbation. A and C are expanded in powers of $\omega_\perp J_x$. The zeroth-order amplitudes $A^{(0)}$ and $C^{(0)}$ obey the following equations:

$$\begin{aligned}E_i{}^{(T)}A_\alpha{}^{(0)i}&=(\varepsilon_\alpha-\lambda)A_\alpha{}^{(0)i}+\sum_\gamma\Gamma_{\alpha\gamma}^{(T)}A_\gamma{}^{(0)i}-\Delta(T)C_\alpha{}^{(0)i},\\ E_i{}^{(T)}C_\alpha{}^{(0)i}&=-(\varepsilon_\alpha-\lambda)C_\alpha{}^{(0)i}-\sum_\gamma\Gamma_{\alpha\gamma}^{(T)*}C_\gamma{}^{(0)i}-\Delta(T)A_\alpha{}^{(0)i},\end{aligned} \tag{30}$$

where

$$E_i{}^{(T)}=\widehat{E}_i{}^{(T)}+\omega_\parallel K_i\,.$$

It is possible to find the representation diagonalizing these equations. However, for the sake of simplicity we assume here as follows:

$$A_\alpha^{(0)i}=u_i W_\alpha^{\ i}, \qquad C_\alpha^{(0)i}=v_i W_\alpha^{\ i}. \tag{31}$$

Under these assumptions we have

$$u_i^2=\frac{1}{2}\left[1+\frac{\varepsilon_i-\lambda}{E_i^{(T)}}\right], \qquad v_i^2=\frac{1}{2}\left[1-\frac{\varepsilon_i-\lambda}{E_i^{(T)}}\right], \tag{32a}$$

$$E_i^{(T)}=\sqrt{(\varepsilon_i-\lambda)^2+\Delta(T)^2}, \tag{32b}$$

where ε_i and $W_\alpha^{\ i}$ are the solutions of the eigenvalue equation

$$\varepsilon_i W_\alpha^{\ i}=\varepsilon_\alpha W_\alpha^{\ i}+\sum_\gamma \Gamma^{(0)}_{\alpha\gamma} W_\gamma^{\ i}. \tag{33}$$

This equation involves only the Hartree potential $\Gamma^{(0)}$ and determines the energies ε_i of odd-particle states. Higher order terms in $\omega_\perp$ of A and C can be calculated from Eq. (29) and these complex conjugate equations. We take into account to the second order terms of $\omega_\perp J_x$. The results are given as follows:
first-order terms

$$A_\alpha^{(1)i}=\sum_j \frac{-\omega_\perp J_{ji}}{E_i^{(T)2}-E_j^{(T)2}}\{v_i\Delta(T)+E_i^{(T)}u_i+(\varepsilon_j-\lambda)u_i\}W_\alpha^{\ j}, \tag{34a}$$

$$C_\alpha^{(1)i}=\sum_j \frac{-\omega_\perp J_{ji}}{E_i^{(T)2}-E_j^{(T)2}}\{u_i\Delta(T)+E_i^{(T)}v_i-(\varepsilon_j-\lambda)v_i\}W_\alpha^{\ j}, \tag{34b}$$

second-order terms

$$\begin{aligned}A_\alpha^{(2)i}=&\sum_{j,k}\frac{\omega_\perp^2 J_{kj}J_{ji}}{(E_i^{(T)2}-E_j^{(T)2})(E_i^{(T)2}-E_k^{(T)2})}[u_i\{E_i^{(T)2}+E_i^{(T)}(\varepsilon_j-\lambda)+E_i^{(T)}(\varepsilon_k-\lambda)\\&+\Delta(T)^2+(\varepsilon_j-\lambda)(\varepsilon_k-\lambda)\}+v_i\{2E_i^{(T)}\Delta(T)\\&\qquad+\Delta(T)(\varepsilon_k-\lambda)-\Delta(T)(\varepsilon_j-\lambda)\}]W_\alpha^{\ k}\\&-\sum_j\frac{\omega_\perp^2 J_{ji}^2 E_j^{(T)}}{(E_i^{(T)2}-E_j^{(T)2})E_i^{(T)}}\left\{v_i\frac{\Delta(T)(\varepsilon_i-\varepsilon_j)}{2E_i^{(T)}E_j^{(T)}}\right\}W_\alpha^{\ i}\\&-\frac{1}{2}\sum_j\frac{\omega_\perp^2 J_{ji}^2 u_i}{(E_i^{(T)2}-E_j^{(T)2})^2}\Big\{(E_i^{(T)}+E_j^{(T)})^2-4E_i^{(T)}E_j^{(T)}\\&\times\frac{1}{2}\left(1-\frac{(\varepsilon_i-\lambda)(\varepsilon_j-\lambda)+\Delta(T)^2}{E_i^{(T)}E_j^{(T)}}\right)\Big\}W_\alpha^{\ i},\end{aligned} \tag{34c}$$

$$\begin{aligned}C_\alpha^{(2)i}=&\sum_{j,k}\frac{\omega_\perp^2 J_{ji}^2 E_j^{(T)}}{(E_i^{(T)2}-E_j^{(T)2})(E_i^{(T)2}-E_k^{(T)2})}[v_i\{E_i^{(T)2}-E_i^{(T)}(\varepsilon_j-\lambda)-E_i^{(T)}(\varepsilon_k-\lambda)\\&+\Delta(T)^2+(\varepsilon_k-\lambda)(\varepsilon_j-\lambda)\}+u_i\{2E_i^{(T)}\Delta(T)-\Delta(T)(\varepsilon_k-\lambda)+\Delta(T)(\varepsilon_j-\lambda)\}]W_\alpha^{\ k}\\&+\sum_j\frac{\omega_\perp^2 J_{ji}^2 E_j^{(T)}}{(E_i^{(T)2}-E_j^{(T)2})E_i^{(T)}}\left\{u_i\frac{\Delta(T)(\varepsilon_i-\varepsilon_j)}{2E_i^{(T)}E_j^{(T)}}W_\alpha^{\ i}\right.\\&-\frac{1}{2}\sum_j\frac{\omega_\perp^2 J_{ji}^2 v_i}{(E_i^{(T)2}-E_j^{(T)2})^2}\Big\{(E_i^{(T)}+E_j^{(T)})^2-4E_i^{(T)}E_j^{(T)}\end{aligned}$$

$$\times \frac{1}{2}\left(1-\frac{(\varepsilon_i-\lambda)(\varepsilon_j-\lambda)+\Delta(T)^2}{E_i^{(T)}E_j^{(T)}}\right)\Bigg\} W_a{}^i. \tag{34d}$$

3-2 *Energy gap and critical temperature*

As a result of the perturbation calculation, the gap equation (27) becomes

$$\frac{4}{G}=\sum_i \frac{(1-2f_i)}{E_i^{(T)}}[1-\omega_\perp{}^2 \sum_j X_{ij}^{(T)}], \tag{35}$$

where

$$X_{ij}^{(T)}=\frac{J_{ij}^2}{(E_i^{(T)}+E_j^{(T)})^2}\left(1-\frac{(\varepsilon_i-\lambda)(\varepsilon_j-\lambda)+\Delta(T)^2}{E_i^{(T)}E_j^{(T)}}\right)+\frac{J_{ij}^2}{E_i^{(T)}+E_j^{(T)}}\frac{(\varepsilon_i-\lambda)(\varepsilon_i-\varepsilon_j)}{E_i^{(T)2}E_j^{(T)}}.$$

This gives a dependence of Δ on the angular velocity in addition to the temperature. The energy gap equation in the ground state rotational band[6] is given by the zero-temperature limit of this equation. For sufficiently large values of the angular velocity, the Coriolis force prevents the formation of the coupled pair. Thus it is expected that there is a critical value of the angular velocity or the angular momentum. The relationship between the transition temperature and the angular velocity is given by the following equation:

$$\frac{4}{G}=\sum_i \frac{\tanh\{(|\varepsilon_i-\lambda|-\omega_\parallel K_i)/2T_c\}}{|\varepsilon_i-\lambda|}[1-\omega_\perp{}^2\sum_j X_{ij}^{(T)}(\Delta=0)]. \tag{36}$$

If Eqs. (35) and (36) are expanded in powers of $\omega_\parallel K_i$, it may be approximated that the coupling constant G of the pairing interaction is replaced by an effective constant $\overline{G}^{(T)}$,

$$\overline{G}^{(T)}\simeq G\{1-\omega_\parallel{}^2\langle Z_{ii}^{(T)}\rangle_{\mathrm{av}}-\omega_\perp{}^2\langle \sum_j X_{ij}^{(T)}\rangle_{\mathrm{av}}\}, \tag{37}$$

where $Z_{ii}^{(T)}$ is given by

$$Z_{ii}^{(T)}=\frac{\beta^2 K_i{}^2 \exp(\beta E_i^{(T)})}{(1+\exp(\beta E_i^{(T)}))^2}.$$

The average $\langle\ \rangle_{\mathrm{av}}$ is taken over the states i. The effect of the angular velocity is to reduce the value of the effective coupling constant $\overline{G}^{(T)}$. We shall discuss some qualitative features of the pairing system with the aid of the following rather crude assumptions.

a) The strength of the pairing interaction is a constant G with an energy interval δ around the Fermi surface

$$|\varepsilon_i-\lambda|\lesssim\delta \tag{38}$$

and is equal to zero otherwise.

b) The sums over i can be approximated by integrals with the single particle level density set equal to a constant g. In every case of our interest the assumption $gG\ll 1$ is justified.

The energy gap $\varDelta(\omega)$ at the zero temperature is determined from Eqs. (35) and (37) with the aid of above assumptions.

$$\frac{2}{g\overline{G}^{(0)}}=\int_{-\delta}^{\delta}\frac{d\xi}{\sqrt{\xi^2+\varDelta(\omega)^2}},\qquad \xi=\varepsilon-\lambda\,. \tag{39}$$

This equation gives

$$\varDelta(\omega)=\frac{\delta}{\sinh(1/g\overline{G}^{(0)})}, \tag{40}$$

and as $g\overline{G}^{(0)}\ll 1$ is valid in our case, we have the approximate equality

$$\varDelta(\omega)=2\delta\exp(-1/g\overline{G}^{(0)}). \tag{41}$$

Equation (36) determining the critical temperature similarly becomes

$$\frac{2}{g\overline{G}^{(T_c)}}=\int_{-\delta}^{\delta}\frac{d\xi}{\xi}\tanh\frac{\xi}{2T_c}, \tag{42}$$

from which we obtain

$$T_c=1.14\,\delta\exp(-1/g\overline{G}^{(T_c)}). \tag{43}$$

From Eqs. (41) and (43) follows the relation between the transition temperature and energy gap at the zero temperature,

$$2\varDelta(\omega)/T_c=3.50\times\exp\left[-\frac{1}{g}\left(\frac{1}{\overline{G}^{(0)}}-\frac{1}{\overline{G}^{(T_c)}}\right)\right]. \tag{44}$$

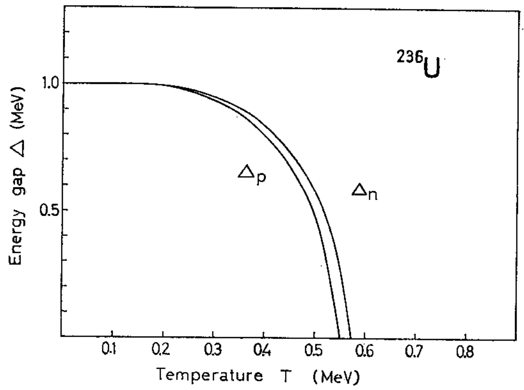

Fig. 1. Temperature dependence of the neutron and proton energy gap parameters for ^{236}U.

3-3 *Moments of inertia*

The moment of inertia which can be defined from the rotational sequences of low-lying energy levels has been calculated by means of the cranking model.[7] If the effect of the pairing correlation is taken into account, the values of moments of inertia are in satisfactory agreement with experiment.

At a finite temperature, the moment of inertia about the axis perpendicular to the symmetry axis of deformed nucleus can be calculated from Eq. (21) and (34). The lowest order term is given by the following equation:

$$\mathcal{J}_\perp^{(T)}=\frac{1}{2}\sum_{ij}\frac{J_{ji}^2(1-f_i-f_j)}{E_i^{(T)}+E_j^{(T)}}\left(1-\frac{(\varepsilon_i-\lambda)(\varepsilon_j-\lambda)+\varDelta(T)^2}{E_i^{(T)}E_j^{(T)}}\right)+\frac{1}{2}\sum_{ij}\frac{J_{ji}^2(f_j-f_i)}{E_i^{(T)}-E_j^{(T)}}\left(1+\frac{(\varepsilon_i-\lambda)(\varepsilon_j-\lambda)+\varDelta(T)^2}{E_i^{(T)}E_j^{(T)}}\right). \tag{45}$$

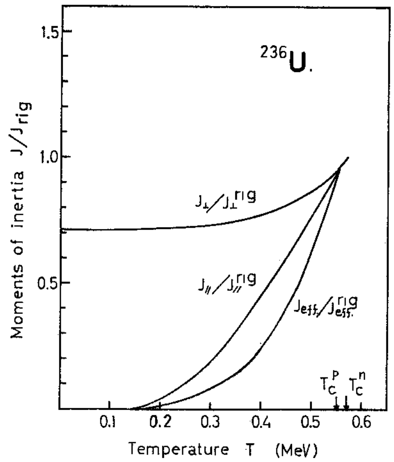

Fig. 2. The moments of inertia for ^{236}U relative to the rigid rotor values, $\mathcal{I}_\parallel{}^{\rm rig}=$ 64.8 MeV^{-1}, $\mathcal{I}_\perp{}^{\rm rig}=$259.6 MeV^{-1} and $\mathcal{I}_{\rm eff}^{\rm rig}=$86.4 MeV versus the nuclear temperature for the deformation $\delta=0.6$.

The moment of inertia at the zero temperature is obtained by putting $T=0$ into the above equation.

$$\mathcal{I}_\perp{}^{(0)}=\frac{1}{2}\sum_{ij}\frac{J_{ji}^2}{E_i{}^{(0)}+E_j{}^{(0)}}\times\left(1-\frac{(\varepsilon_i-\lambda)(\varepsilon_j-\lambda)+\Delta(0)^2}{E_i{}^{(0)}E_j{}^{(0)}}\right). \tag{46}$$

The expression reduces for $\Delta=0$ to the cranking model formula.

$$\mathcal{I}_\perp{}^{\rm rig}=\sum_{ij}\frac{J_{ji}^2}{\varepsilon_i-\varepsilon_j},\quad (\varepsilon_i>\varepsilon_F>\varepsilon_j),\quad \omega_\perp\geqq\omega_{\perp c}, \tag{47}$$

where ε_F means the Fermi energy. This also gives an expression for the moment of inertia associated with the rotation of a rigid body.

The moment of inertia about the symmetry axis of deformed nucleus or about the z-axis of spherical nucleus can be derived from Eq. (22) as follows:

$$\mathcal{I}_\parallel{}^{(T)}=\frac{1}{\omega_\parallel}\sum_i K_i f_i\,. \tag{48}$$

If we expand f_i in powers of $\omega_\parallel K_i$ and take it into account to the first order contribution, then the moment of inertia becomes

$$\mathcal{I}_\parallel{}^{(T)}=\beta\sum_i K_i^2\frac{\exp(\beta E_i{}^{(T)})}{(1+\exp(\beta E_i{}^{(T)}))^2}\,. \tag{49}$$

Equation (49) may be simplified by making use of some approximations. With the aid of assumptions (a) and (b) we replace the sums over i by integrals. Since the statistical factor $e^{\beta E_i}/(1+e^{\beta E_i})^2$ is peaked at $E_i=0$, the integration reduces practically to an integration in a shell near the Fermi surface. If K_i^2 is replaced by the average $\langle K_i^2\rangle_{\rm av}$ taken over the states i near the Fermi surface, Eq. (49) becomes

$$\mathcal{I}_\parallel{}^{(T)}=2g\langle K_i^2\rangle_{\rm av}\int_0^\infty d\xi\,\frac{\beta\exp(\beta E^{(T)})}{(1+\exp(\beta E^{(T)}))^2}\,. \tag{50}$$

For temperature sufficiently low as to satisfy $\Delta(0)/T\gg1$ we have

$$\mathcal{I}_\parallel{}^{(T)}\simeq\sqrt{\frac{2\pi\Delta(0)}{T}}\,e^{-\Delta(0)/T}\mathcal{I}_\parallel{}^{(0)}, \tag{51}$$

where

$$\mathcal{J}_{/\!/}^{(0)}=2g\langle K_i^2\rangle_{av}=\mathcal{J}_{/\!/}^{rig}.$$

For a normal state, we find the expected result

$$\mathcal{J}_{/\!/}^{(T)}\simeq 2g\langle K_i^2\rangle_{av}\int_0^\infty \frac{\beta e^{\beta\varepsilon}}{(1+e^{\beta\varepsilon})^2}d\varepsilon$$
$$=2g\langle K_i^2\rangle_{av}\,. \qquad (52)$$

3-4 *Excitation energy and temperature*

The intrinsic energy of the system with temperature T can be written as

$$E_{int}(T)\simeq U(T)+\lambda\langle\hat{N}\rangle$$
$$=\frac{1}{2}\sum_i\Big[\varepsilon_i\Big\{1-\frac{\varepsilon_i-\lambda}{E_i^{(T)}}(1-2f_i)\Big\}-\frac{\Delta(T)^2}{2E_i^{(T)}}(1-2f_i)\Big]-\frac{1}{2}\sum_i\omega_{/\!/}K_if_i\,. \qquad (53)$$

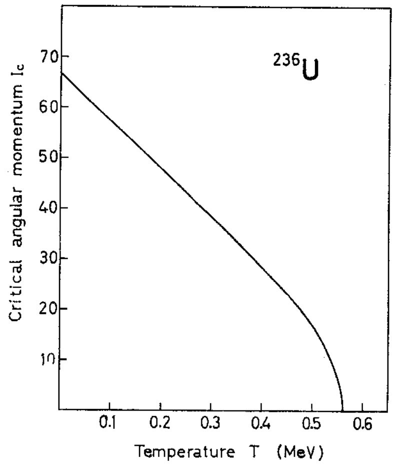

Fig. 3. The values of the critical angular momentum for ^{236}U plotted as a function of the temperature.

The intrinsic energy of the ground state is obtained by putting $T=0$ into the above equation. Under the assumptions (a) and (b) one has

$$E_{int}(0)=\frac{1}{2}\sum_i\varepsilon_i\Big(1-\frac{\varepsilon_i-\lambda}{\sqrt{(\varepsilon_i-\lambda)^2+\Delta(0)^2}}\Big)-\frac{1}{4}\sum_i\frac{\Delta(0)^2}{\sqrt{(\varepsilon_i-\lambda)^2+\Delta(0)^2}}$$
$$\simeq E_N(0)-\frac{1}{2}g\Delta(0)^2, \qquad (54)$$

where $E_N(0)$ is the normal ground state energy. Thus the intrinsic excitation energy of the nucleus corresponding to the temperature T is given by

$$\mathcal{E}_{int}=E_{int}(T)-E_{int}(0)$$
$$=\frac{1}{2}\sum_i\varepsilon_i\Big\{1-\frac{\varepsilon_i-\lambda}{E_i^{(T)}}(1-2f_i)\Big\}-\frac{1}{4}\sum_i\frac{\Delta(T)^2}{E_i^{(T)}}(1-2f_i)-\frac{1}{2}\sum_i\omega_{/\!/}K_if_i$$
$$-E_N(0)+\frac{1}{2}g\Delta(0)^2. \qquad (55)$$

Below the critical temperature the functional relationship of the excitation energy and the temperature is not so simple. Assuming $\beta\omega_{/\!/}K_i$ as a small perturbation and taking it into account to the second order terms, Eq. (55) can be expanded as

$$\mathcal{E}_{int}\simeq\frac{1}{2}\sum_i\Big[\varepsilon_i\Big\{1-\frac{\varepsilon_i-\lambda}{E_i^{(T)}}\tanh\frac{\beta E_i^{(T)}}{2}\Big\}-\frac{\Delta(T)^2}{2E_i^{(T)}}\tanh\frac{\beta E_i^{(T)}}{2}\Big]-\frac{K^2}{2\mathcal{J}_{/\!/}^{(T)}}$$
$$-\frac{1}{2}\beta^2\sum_i\omega_{/\!/}^2K_i^2\frac{2\varepsilon_i(\varepsilon_i-\lambda)+\Delta(T)^2}{2E_i^{(T)}}\frac{\exp(\beta E_i^{(T)})(1-\exp(\beta E_i^{(T)}))}{(1+\exp(\beta E_i^{(T)}))^3}\,. \qquad (56)$$

For low energies, however, we have

$$\mathcal{E}_{\text{int}} \cong 2g\Delta(0)^2[K_0(\Delta(0)/T) + K_2(\Delta(0)/T)] + \frac{K^2}{2\mathcal{I}_{\parallel}(T)}(\Delta(0)/T - 1), \quad (57)$$

where $\mathcal{I}_{\parallel}$ is given by Eq. (51) and the function K_n is the modified Bessel function of the second kind.

Above the critical temperature T_c, the energy expression is much simplified. In fact, in the normal phase one has, putting $\Delta=0$ into Eq. (53),

$$E_{\text{int}}(T) = \sum_i \frac{\varepsilon_i}{1+\exp[\{(\varepsilon_i - \lambda) - \omega_{\parallel} K_i\}/T]} - \frac{K^2}{2\mathcal{I}_{\parallel}^{(T)}} . \quad (58)$$

Similarly the occupation number of the single particle state i becomes

$$n_i = \frac{1}{1+\exp[\{(\varepsilon_i - \lambda) - \omega_{\parallel} K_i\}/T]} . \quad (59)$$

In this case the occupation number n_i includes the term $\omega_{\parallel} K_i$. This has been neglected in the calculations based on the Fermi gas model employed by many authors. Equation (58) allows the usual low temperature expansion in terms of T, and then the intrinsic excitation energy is given by

$$\mathcal{E}_{\text{int}} \cong aT^2 + \tfrac{1}{2} g\Delta(0)^2, \quad (60)$$

where $a = (\pi^2/6)g$. This equation not only shows the quadratic dependence of the energy on temperature but also reflects the lowering of the ground state energy due to the pairing interaction.

From Eq. (23a), the total excitation energy becomes

$$\mathcal{E} = \mathcal{E}_{\text{int}}(T) + \frac{I(I+1) - K^2}{2\mathcal{I}_{\perp}^{(T)}} + \frac{K^2}{2\mathcal{I}_{\parallel}^{(T)}} . \quad (61)$$

Here, we have measured the excitation energy from the ground state with the angular momentum $I=0$. For odd or odd-odd nuclei, we must further take into account the energy correction for the angular momentum of the ground state and the pairing energy. For spherical nuclei, the second term in the right-hand side of Eq. (61) must be disregarded because the rotational motion about the axis perpendicular to the symmetry is physically meaningless.

The transition energy $\mathcal{E}_c$, corresponding to the critical temperature T_c, is given by

$$\mathcal{E}_c = \mathcal{E}_{\text{int}}(T_c) + \frac{I_c(I_c+1) - K_c^2}{2\mathcal{I}_{\perp}^{(T_c)}} + \frac{K_c^2}{2\mathcal{I}_{\parallel}^{(T_c)}} . \quad (62)$$

The critical temperature T_c is determined by Eq. (36). The critical temperature T_c has different values corresponding to the angular momenta I and K, and then the transition energy $\mathcal{E}_c$ is a function of the angular momenta.

3-5 *Entropy and specific heat*

Taking into account to the second order terms with respect to $\omega_{\parallel}$, the entropy S can be expanded as follows:

$$S \cong \sum_i \log(1+\exp(-\beta E_i^{(T)})) + \beta \sum_i \frac{E_i^{(T)}}{1+\exp(\beta E_i^{(T)})} - \frac{\beta K^2}{2\mathcal{I}_{\parallel}^{(T)}}$$
$$-\frac{1}{2}\beta^3 \sum_i \omega_{\parallel}^2 K_i^2 \frac{E_i^{(T)} \exp(\beta E_i^{(T)})(1-\exp(\beta E_i^{(T)}))}{(1+\exp(\beta E_i^{(T)}))^3}. \qquad (63)$$

For temperatures sufficiently low as to satisfy $\Delta(0)/T \gg 1$ we have similarly to Eq. (57)

$$S = S_s \cong 2g\beta\Delta(0)^2[K_0(\Delta(0)/T) + K_2(\Delta(0)/T)] + \frac{\beta K^2}{2\mathcal{I}_{\parallel}^{(T)}}(\Delta(0)/T - 1). \qquad (64)$$

In the superconducting phase the entropy is a much more complicated function of the energy. Combining Eqs. (57), (61) and (64), we have

$$S = S_s \simeq \beta\mathcal{E} - \frac{\beta[I(I+1)-K^2]}{2\mathcal{I}_{\perp}^{(T)}} - \frac{\beta K^2}{2\mathcal{I}_{\parallel}^{(T)}}. \qquad (65)$$

In the normal phase the entropy is much simplified as

$$S = S_n \simeq 2aT. \qquad (66)$$

Combining Eqs. (60), (61) and (66) one has

$$S_n \cong 2\sqrt{a\left[\mathcal{E} - \frac{1}{2}g\Delta(0)^2 - \frac{I(I+1)-K^2}{2\mathcal{I}_{\perp}^{(T)}(\Delta=0)} - \frac{K^2}{2\mathcal{I}_{\parallel}^{(T)}(\Delta=0)}\right]}. \qquad (67)$$

As long as the rotational energy $\{(I(I+1)-K^2)/2\mathcal{I}_{\perp}^{(T)}(\Delta=0) + K^2/2\mathcal{I}_{\parallel}^{(T)}(\Delta=0)\}$ is small, we can expand Eq. (67) as

$$S_n \cong 2\sqrt{a\left(\mathcal{E} - \frac{1}{2}g\Delta(0)^2\right)} - \frac{I(I+1)}{2T\mathcal{I}_{\perp}^{(T)}(\Delta=0)} - \frac{1}{2T}\left(\frac{1}{\mathcal{I}_{\parallel}^{(T)}(\Delta=0)} - \frac{1}{\mathcal{I}_{\perp}^{(T)}(\Delta=0)}\right)K^2, \qquad (68)$$

where the nuclear temperature T is defined by the equation

$$\mathcal{E} - \tfrac{1}{2}g\Delta(0)^2 = aT^2. \qquad (69)$$

The specific heat (25) can be rewritten as

$$C \cong \frac{1}{4}\sum_i \frac{E_i^{(T)}}{\cosh^2(\beta E_i^{(T)}/2)}\left[\frac{E_i^{(T)}}{T^2} - \frac{1}{T}\frac{\partial E_i^{(T)}}{\partial T}\right] - \frac{1}{4}\sum_i \frac{\omega_{\parallel}^2 K_i^2}{\cosh^2(\beta E_i^{(T)}/2)}$$
$$\times\left[\left(\frac{E_i^{(T)}}{T^2} - \frac{1}{T}\frac{\partial E_i^{(T)}}{\partial T}\right)\left\{\frac{\beta^2 E_i^{(T)}}{4}\left(1 - 3\tanh^2\left(\frac{\beta E_i^{(T)}}{2}\right)\right)\right.\right.$$
$$\left.\left.+\beta\tanh\left(\frac{\beta E_i^{(T)}}{2}\right)\right\} + \beta^3 E_i^{(T)}\tanh\left(\frac{\beta E_i^{(T)}}{2}\right) - \beta^2\right]. \qquad (70)$$

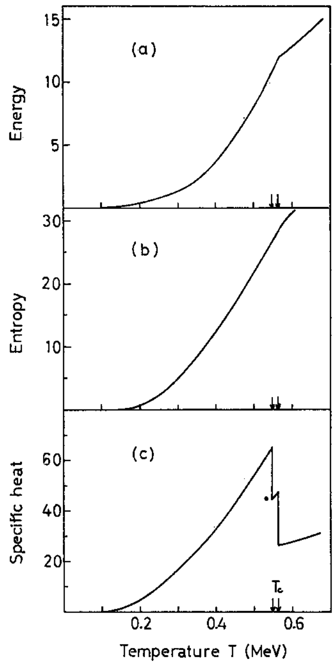

Fig. 4. Intrinsic excitation energy (a), entropy (b) and specific heat (c) for ^{236}U plotted as functions of the temperature. The arrows indicate the energies of the phase transition from the superconducting state to the normal state.

The second term in the right-hand side is a rather complicated function of the temperature. If we neglect this term, below the critical temperature the same approximation as used for Eqs. (57) and (64) gives

$$C=C_s\cong g\beta^2\Delta(0)^3[3K_1(\Delta(0)/T)+K_3(\Delta(0)/T)]. \tag{71}$$

Above the critical temperature the specific heat becomes

$$C=C_n\sim 2aT. \tag{72}$$

The discontinuity of the specific heat at the transition temperature is given by

$$(C_s-C_n)_{T=T_c}\cong -\frac{1}{4}\sum \mathrm{sech}^2\frac{\varepsilon_i-\lambda}{2T_c}\cdot\left(\frac{d\Delta(T)^2}{dT}\right)_{T=T_c}$$

$$\approx -g\left(\frac{d\Delta(T)^2}{dT}\right)_{T=T_c}. \tag{73}$$

The energy gap Δ is a function of the angular velocity, that is, the angular momentum. Therefore, the value of $(d\Delta^2/dT)_{T=T_c}$ depends on the angular momentum. If we neglect the contribution from the angular momentum, we obtain the result derived by BCS,

$$\frac{1}{T_c}\left(\frac{d\Delta(T)^2}{dT}\right)_{T=T_c}\cong -9.4. \tag{74}$$

§4. Nuclear level density

We have discussed in the preceding section the thermodynamical properties of atomic nuclei. In our case many of the thermodynamical quantities are not measured directly in contrast with the case of the superconducting metals. These properties, however, are expected to manifest themselves in various aspects of nuclear reactions. The nuclear level density is the most characteristic quantity in this respect as it can be inferred from nuclear reaction experiments. In the following we will first mention the level density of spherically symmetric nuclei. The case of deformed nuclei will be discussed later.

4-1 *Level density for spherically symmetric nuclei*

In addition to the energy ε_i, the single Fermion states are also characterized by the projection of the angular momentum on the z-axis, m_i. In the superconducting model the nucleons having angular momentum $(m_i, -m_i)$ couple so as to form a quasi-bound pair state. Therefore, the projection of the total angular momentum J, M, on the z-axis of the system is the sum of the projections of the excited particles. The level density $W(E, M)$ of the N nucleon system having energy E and angular momentum M is related to the thermodynamical functions by the following relation:

$$e^{-\beta F} = \iint W(E, M) e^{-\beta(E-\omega M)} dE dM . \tag{75}$$

The inverse Laplace transformation of Eq. (75) gives

$$W(E, M) = \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} d\beta \int_{-i\infty}^{+i\infty} d\alpha e^{\beta(E-F)-\alpha M} . \qquad (\alpha=\beta\omega, \gamma>0) \tag{76}$$

These integrals can be evaluated by the method of steepest descent at the saddle point given by

$$E = \frac{\partial}{\partial\beta}(\beta F)_{N,\omega} , \qquad M = -\frac{\partial}{\partial\alpha}(\beta F)_{N,\beta} . \tag{77}$$

Thus we have

$$W(E, M) = \frac{e^{\beta(E-F-\omega M)}}{2\pi\sqrt{\det A}} = \frac{e^S}{2\pi\sqrt{\det A}} , \tag{78}$$

where

$$\det A = \begin{vmatrix} -\dfrac{\partial^2}{\partial\beta^2}(\beta F), & -\dfrac{\partial^2}{\partial\beta\partial\alpha}(\beta F) \\ -\dfrac{\partial^2}{\partial\alpha\partial\beta}(\beta F), & -\dfrac{\partial^2}{\partial\alpha^2}(\beta F) \end{vmatrix} . \tag{79}$$

It is allowable to neglect higher than first-order angular velocity terms in the denominator in comparison to the much more rapid variation of the exponent. Then, we can approximately obtain the following relations:

$$-\frac{\partial^2}{\partial\beta^2}(\beta F) \simeq T^2 C ,$$

$$-\frac{\partial^2}{\partial\alpha^2}(\beta F) \simeq T \mathcal{I}_{\parallel}^{(T)} ,$$

$$-\frac{\partial^2}{\partial\alpha\partial\beta}(\beta F) \simeq 0 .$$

Thus we have

$$W(E, M) \cong \frac{e^S}{2\pi\sqrt{\mathcal{I}_{\parallel}^{(T)} C T^3}} . \tag{80}$$

In accordance with the usual convention we consider the level density as a function of the excitation energy and write as $W(\mathcal{E}, M)$. As we discuss in the following, the level density $W(\mathcal{E}, M)$ differs in many respects from that expected on the ground of the Fermi-gas model.

Above the transition energy, the situation is very much alike to the case where the pairing interaction is absent. In fact thermodynamical functions are in accordance with the prediction of the Fermi gas model with only an exception that the relation between the excitation energy and temperature is different from that predicted by this model. As we discussed already in § 3, this relation is given by

$$\mathcal{E} \cong aT^2 + \frac{1}{2} g\Delta^2(0) + \frac{M^2}{2\mathcal{I}_{\parallel}^{(T)}}. \tag{81}$$

The second term in the right-hand side of this equation represents the pairing-interaction energy in the ground state. In view of the importance of the pairing energy in nuclear spectroscopy, phenomenological attempts[8] have been made to introduce the even-odd difference in the nuclear level density. The ground states of the even and odd mass nuclei have been depressed by an amount similar to the corresponding odd-even shift in the semi-empirical mass formula. The even-odd difference in this case was attributed to the pairing interaction which was supposed to affect only the ground state of even nucleus. Accordingly, the energy-temperature relation for even nucleus obtained was written as $\mathcal{E} = aT^2 + P + M^2/2\mathcal{I}_{\parallel}$, $P \simeq 2\Delta(0)$. Contrary to this approach, the second term in Eq. (81) is quadratic in the pairing energy. The even-odd difference should be attributed to the fine structure of the second term in the right-hand side of Eq. (81). When a level near the Fermi surface is occupied by an odd particle, the effect of the pairing correlations is reduced. The reduction necessarily depends on what level is occupied. The blocking effect can be taken into account by subtracting the occupied odd particle state from the sum in the right-hand side of Eq. (27) which determines the energy gap. The change in Δ between the even and the odd case due to the blocking of one level by the odd particle is estimated as[9]

$$\Delta^{\text{odd}}(0) \simeq \Delta^{\text{even}}(0) - \frac{1}{(\Delta^{\text{even}}(0))^2} \left(\sum_{i \neq i'} \frac{1}{E_i^{(0)3}} \right)^{-1}, \tag{82}$$

where i' indicates a state occupied by the odd particle. The actual calculations, in which the blocking effect has been included exactly, indicate a difference in Δ between even- and odd-system of the order of 20%. These results are roughly in agreement with Eq. (82). Presumably $\Delta^{\text{odd}}(0)$ will be slightly smaller. It should be noted, however, that the pairing qualitatively produces an odd-even effect of the magnitude observed although the change in the energy gap parameter is relatively small. In the superconducting model, the energy correction is quadratic in $\Delta(0)$ rather than linear as is usually assumed.

Above the transition energy, the entropy, specific heat and excitation energy are much simplified, and are given by Eqs. (67), (72) and (82). However, in these equations we should disregard the terms with the component of the angular momentum perpendicular to the z-axis. Below the transition energy the thermodynamical functions are rather complicated functions of the temperature. At low temperature, however, Eqs. (57), (65) and (71) are available. In the two extreme cases, the level density can be written as

$$W(\mathcal{E}, M) = W(\mathcal{E}) \frac{\exp(-M^2/2\mathcal{J}_{\parallel}^{(T)}T)}{\sqrt{2\pi T \mathcal{J}_{\parallel}^{(T)}}}, \tag{83a}$$

where

$$W(\mathcal{E}) = \frac{\exp(\beta\mathcal{E})}{\sqrt{2\pi T^2 C_s}}, \quad (\Delta(0)/T \gg 1) \tag{83b}$$

$$W(\mathcal{E}) \simeq \frac{2\sqrt{a(\mathcal{E}-\frac{1}{2}g\Delta(0)^2)}}{\sqrt{2\pi T^2 C_n}} \cdot (\mathcal{E} \geq \mathcal{E}_c) \tag{83c}$$

Making use of the fact that each nuclear level consists of $(2I+1)$ magnetic substates corresponding to the values of $M=0, \pm 1, \pm 2, \cdots, \pm I$, the level density with angular momentum I is given by the difference between the level density of angular momenta $M=I$ and $M=I+1$.

$$\begin{aligned} W(\mathcal{E}, J) &= W(\mathcal{E}, M=I) - W(\mathcal{E}, M=I+1) \\ &\simeq W(\mathcal{E}) \frac{(2I+1)}{\sqrt{8\pi(\mathcal{J}_{\parallel}^{(T)}T)^3}} \exp(-I(I+1)/2\mathcal{J}_{\parallel}^{(T)}T). \end{aligned} \tag{84}$$

4-2 *Level density for deformed nuclei*

The single particle states arise from the motion of a nucleon in the deformed average potential. They are characterized by the projection Ω of the angular momentum on the nuclear symmetry axis. The projection of the total angular momentum, K, on the nuclear symmetry axis is the sum of the projections of the excited particles. The distribution of the angular momentum K is associated with a rotational energy about the symmetry axis. In addition to the rotational motion about the symmetry axis, we must also consider the rotational motion about the axis perpendicular to the symmetry axis.

The level density $W(E, J)$ of the N nucleon system becomes

$$W(E, J) = \left(\frac{1}{2\pi i}\right)^4 \int \cdots \int \exp(\beta(E - F - \boldsymbol{\omega}\cdot\boldsymbol{J}))\, d\beta d^3(\beta\omega), \tag{85}$$

where

$$\boldsymbol{\omega}\cdot\boldsymbol{J} = \omega_x J_x + \omega_y J_y + \omega_z J_z \,.$$

The integrals can be evaluated by the method of the steepest descent at the saddle point given by

$$E=\frac{\partial}{\partial\beta}(\beta F)_{N,\omega}\,, \qquad \boldsymbol{J}=-\frac{1}{\beta}\frac{\partial}{\partial\boldsymbol{\omega}}(\beta F)_{N,\beta}\,. \tag{86}$$

Thus we obtain

$$W(E,J)=\frac{1}{(2\pi)^2}\frac{\exp(\beta(E-F-\boldsymbol{\omega}\cdot\boldsymbol{J}))}{\sqrt{\det A}}=\frac{1}{(2\pi)^2}\frac{e^S}{\sqrt{\det A}}\,, \tag{87}$$

where

$$\det A=\left|-\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}(\beta F)_N\right|, \qquad \alpha=(\beta,\beta\boldsymbol{\omega}). \tag{88}$$

By using the transformation of the coordinate system by the rotation about the z-axis, we can write

$$\begin{aligned}\omega_x J_x+\omega_y J_y&=\omega_\perp J_{x'}\,,\\ \omega_\perp&=\sqrt{\omega_x^2+\omega_y^2}\,.\end{aligned} \tag{89}$$

The x'-axis is perpendicular to the z-axis and its direction is the same as that of the vector $(\omega_x, \omega_y, 0)$. By using Eq. (89), in the last step of Eq. (87) the exponent was identified with the entropy (17). Neglecting the higher-order terms than ω, we get

$$\det A\cong T^5 C\mathcal{J}_\perp^{(T)2}\mathcal{J}_\parallel^{(T)}. \tag{90}$$

The level density of the N nucleon system having excitation energy $\mathcal{E}$, total angular momentum I and its projection K can be written as follows:

$$W(\mathcal{E},I,K)=\frac{e^S}{(2\pi)^2\sqrt{T^5C\mathcal{J}_\perp^{(T)2}\mathcal{J}_\parallel^{(T)}}}\,. \tag{91}$$

Now we will consider the level density in the two extreme cases, i. e., the low and high excitation energy corresponding to superconducting and normal state, respectively. We will not reproduce the expression for the level density in the intermediate region which is somewhat complicated. The level density is thus given by

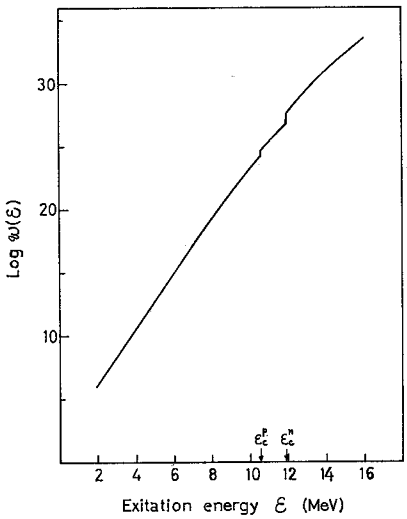

Fig. 5. Level density for ^{236}U calculated on the basis of the Nilsson diagram.

$$W(\mathcal{E}, I, K) \cong \frac{W(\mathcal{E})}{\sqrt{2\pi T^3 \mathcal{J}_\perp^{(T)2} \mathcal{J}_\parallel^{(T)}}} \exp\left[-\frac{I(I+1)}{2\mathcal{J}_\perp^{(T)} T} - \frac{K^2}{2\mathcal{J}_{\text{eff}} T}\right], \tag{92}$$

where $W(\mathcal{E})$ is given by Eqs. (83b) and (83c) and $\mathcal{J}_{\text{eff}}$ is defined as

$$\mathcal{J}_{\text{eff}} = \left(\frac{1}{\mathcal{J}_\parallel^{(T)}} - \frac{1}{\mathcal{J}_\perp^{(T)}}\right)^{-1}. \tag{93}$$

The reduction of the energy gap Δ results in an increase in the moment of inertia. The nuclear phase transition will certainly not have the sharpness of the transition in a superconductor. However, it may manifest itself by a more rapid increase of the level density in the transition region and by a sudden increase in the moment of inertia associated with the angular momentum distributions.

§ 5. Numerical calculations

There are several parameters entering in the calculation, such as the single-particle energies ε_i, the strength G of the pairing interaction and the deformation parameter δ, etc. The calculations are made for ^{68}Ge and actinide nuclei by using the Nilsson levels employed by Gustafson et al.[10] without the use of asymptotic integral expressions. We included all states of the $N=2$, 3, 4, 5 shells for protons and neutrons in the former case, and also all states of the $N=4$, 5, 6, 7, 8 shells for protons and all states of the $N=5$, 6, 7, 8, 9 shells for neutrons in the latter case. The interaction between the neutron and proton is neglected. In so far as the neutron-proton correlation can be neglected, the neutron and proton superfluids are independent. Then the values of the thermodynamic functions are the sum of those for neutrons and protons. For example, the intrinsic excitation energy corresponding to a given temperature is

$$\mathcal{E}_{\text{int}} = \mathcal{E}_{\text{int}}^{(p)} + \mathcal{E}_{\text{int}}^{(n)}.$$

We must know the strength of the pairing interaction. It can be determined by relating the energy gap parameter with the empirical odd-even mass difference P. If one assumes the same quasi-particle vacuum for the odd and even nucleus, this leads to $P \cong 2\Delta$. The energy gap Δ_n at the fission barrier can be determined from the difference between the fission barrier heights B for odd and even nucleus.

$$B(\text{odd}) - B(\text{even}) \cong \Delta_n - \Delta_n(0).$$

The results obtained in this way is $\Delta_n \cong 1.0$ MeV for ^{236}U. The behaviour of energy gap parameters for protons and neutrons is shown as functions of the temperature in Fig. 1 for ^{236}U. There are two critical temperatures corresponding to disappearance of the neutron and proton energy gaps. Here, we assumed that the energy gap $\Delta_p(0)$ is equal to $\Delta_n(0)$ at the zero temperature, the nuclear shape at the fission barrier is of the quadrupole deformation with $\delta=0.6$ and the Coriolis anti-pairing effect is neglected.

Since the interaction between the proton and neutron is neglected, the value of the moment of inertia is the sum of proton and neutron moments of inertia.

$$\mathcal{I}=\mathcal{I}_p+\mathcal{I}_n\,.$$

Figure 2 shows the moment of inertia for ^{236}U plotted as a function of the nuclear temperature. The quantities $\mathcal{I}_{\parallel}$, $\mathcal{I}_{\perp}$ and $\mathcal{I}_{\text{eff}}$ were evaluated at the top of the fission barrier. In fact, the energy gap parameter and the moment of inertia are functions of both the temperature and the angular momentum, and then there is a critical angular momentum for a fixed temperature. The values of the critical angular momentum obtained for ^{236}U are shown as a function of the temperature in Fig. 3.

It is known that the energy levels in the ground state rotational band are well reproduced by taking account of the Coriolis anti-pairing effect. On the other hand, the necessity of taking account of the Coriolis anti-pairing effect for the inertia parameters in the nuclear level density is not yet found out.

In Figs. 4(a) and 4(b), the excitation energy as well as the entropy are plotted as functions of the temperature. The specific heat is given in Fig. 4(c). The discontinuity of the specific heat is seen at the critical temperature. In Fig. 5 the calculated level density $W(\mathcal{E})$ is presented. The discontinuity of the level density at the transition energy is due to the denominator of Eq. (83).

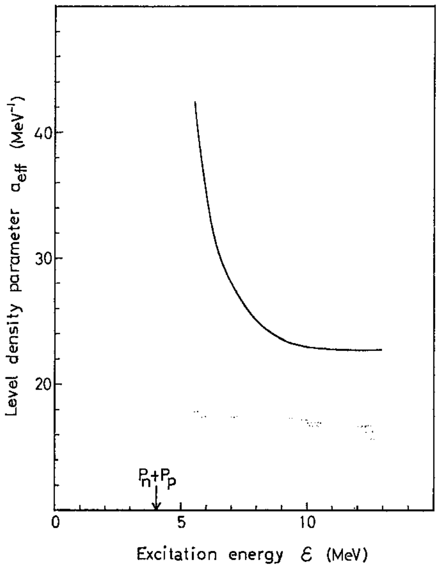

Fig. 6. Effective level density parameter, a_{eff}, for ^{236}U.

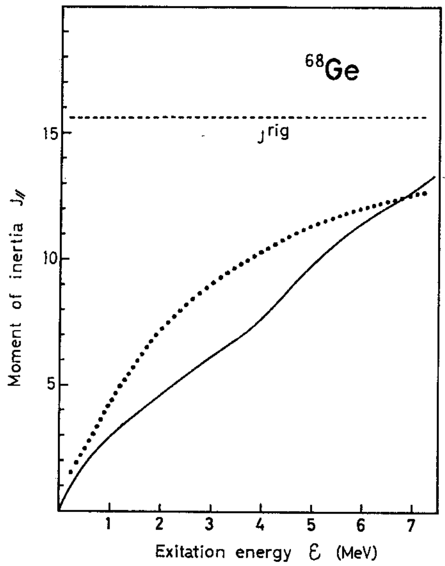

Fig. 7. The moment of inertia with excitation energy for ^{68}Ge. The closed circles denote the observed values and the solid line shows the results calculated with the parameters defining the single-particle level spectrum employed by Gustafson et al.[10] The strength of the pairing interaction is adjusted to give the empirical values, $\Delta_n(0)=1.6$ MeV and $\Delta_p(0)=1.4$ MeV for neutrons and protons, respectively.

Above the transition energy, the situation is very much alike to the case where the pairing interaction is absent and the relation between the excitation energy and temperature is given by Eq. (60). Below the transition energy, as we have discussed in the preceding section, thermodynamic functions are rather complicated functions of the temperature. It may, however, be convenient to define the relations like those for the Fermi gas.

$$\mathcal{E}_{\rm int}=a'T^2+\tfrac{1}{2}g\Delta(0)^2, \tag{94}$$

$$S=2a''T. \tag{95}$$

Here, a' and a'' are functions of the temperature and do not agree with each other below the critical temperature. We can then define a parameter $a_{\rm eff}$ which is an analogue of the level-density parameter a in the Fermi-gas model.

$$a_{\rm eff}=\frac{a''^2}{a'}=\frac{S^2}{4(\mathcal{E}_{\rm int}-\frac{1}{2}g\Delta(0)^2)}. \tag{96}$$

The energy dependence of $a_{\rm eff}$ for ^{236}U is plotted in Fig. 6. Tne effective level-density parameter $a_{\rm eff}$ becomes approximately independent of the excitation energy above the critical temperature.

The moment of inertia is expected to be less than the rigid-body value below the critical temperature. In Fig. 7, the theoretical prediction for the moment of inertia $\mathcal{J}_{\parallel}$ is compared with the result obtained by Vogt et al.[11] for ^{68}Ge. We assumed the spherical shape for ^{68}Ge. The calculated values are smaller than the experimental values.

The temperature dependence of $\mathcal{J}$[12] is especially seen in data on the angular distribution of fission fragments. Such angular distributions depend on the quantity K_0^2 which is the mean square value of the projection of angular momentum on the nuclear symmetry axis. The quantity K_0^2 is given by

$$K_0^2=T\mathcal{J}_{\rm eff}=T\left(\frac{1}{\mathcal{J}_{\parallel}^{(T)}}-\frac{1}{\mathcal{J}_{\perp}^{(T)}}\right)^{-1}. \tag{97}$$

The dependence of K_0^2 upon excita-

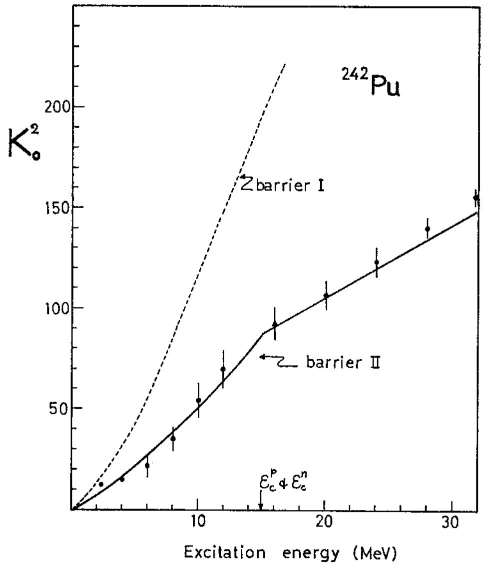

Fig. 8. Variation of K_0^2 with the excitation energy of the nucleus ^{242}Pu. The experimental points are taken from Ref. 12). The calculated values of K_0^2 corresponding to the barrier I ($\delta=0.37$) and barrier II ($\delta=0.63$) are shown by dashed line and the solid line, respectively. The calculation was made with the energy gap parameters $\Delta_n(0)=\Delta_p(0)=1.2$ MeV.

tion energy is therefore a good test of the persistence of superconducting effects to finite excitation energies. Figure 8 shows the experimental values of K_0^2 versus the excitation energy for a typical case of alpha-induced fission of ^{238}U, together with the theoretical curves calculated for the shapes corresponding to barrier I($\delta=0.37$) and barrier II($\delta=0.63$). It is seen that the agreement between the calculation for the shape corresponding to barrier II and the experiment is good. In the calculation, however, the Coriolis anti-pairing effect was not taken into account. The evidence for the Coriolis anti-pairing effect is not seen in the analysis of the level-density.

We hope that in time the model of nuclear level density presented in this paper will serve as means of calculating the total cross sections, the cross section ratios for isomeric state formation in reactions involving the intermediated formation of compound nuclei and the angular distribution of fission fragments.

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