SIAM J. APPL. MATH.
Vol. 26, No. 3, May 1974

NUCLEOLUSES OF COMPOUND SIMPLE GAMES*

NIMROD MEGIDDO†

Abstract. The nucleolus is a unique solution concept for a cooperative game. It belongs to the kernel of the game. Given m games over disjoint sets of players and an m-player game, one defines a compound game over the union of the m disjoint sets. These m games are the components and the above m-player game is called the quotient.

The nucleolus of the compound game is a combination of the nucleoluses of the components, i.e., the payoffs in the components are proportional to the payoffs in the compound. The combination depends on the quotient game and on the two highest levels of excesses with respect to the nucleoluses of the components. This combination is the solution of a problem of the same nature as that of the nucleolus.

1. Introduction. The nucleolus of a cooperative game was defined by D. Schmeidler in [10]. It is a solution concept for a characteristic function game which consists of a unique point. As a point of the kernel of the game (see [2]) the nucleolus reflects strength relations between players and symmetry properties of the characteristic function.

Compound simple games were defined by L. S. Shapley in [11]. Shapley proved in [12], [13], [14] (also proved by G. Owen in [9]) that von Neumann–Morgenstern solutions of the component games compose in a natural manner which results in a solution of the compound game. Kernels have a very similar property (see [5]).

The nucleolus of the product of simple games was characterized in [4]. In most of the cases (when the maximum excesses with respect to the nucleoluses of the component games are distinct) the nucleolus of the product coincides with the nucleolus of the component the maximum excess of which is the least. In this case the players in the other components get zero. When the maximum excesses in the components are not distinct the nucleolus of the product is a convex combination of the components' nucleoluses.

In this paper we generalize the results of [4]. A compound simple game is a tensor composition (see [9]) with simple components. In view of the results presented here, the computation of the nucleolus of a decomposable game (with simple components) is easier than that of a prime one. It is equivalent to the computation of the components' nucleoluses plus a similar computation in the quotient game.

2. Definitions. A *characteristic function game* is a pair $\Gamma = (N; v)$ where $N = \{1, \cdots, n\}$ is a nonempty finite set (whose elements are the *players*) and v is a real-valued function defined over the subsets of N (the *coalitions* in the game).

A *simple game* is a characteristic function game $\Gamma = (N; v)$ where for every $S \subset N$ either $v(S) = 1$ or $v(S) = 0$. Those coalitions that have a unit value are called *winning coalitions*. The set of the winning coalitions is denoted by $\mathscr{W}$ and the game is represented also by $\Gamma = (N; \mathscr{W})$. We always assume $N \in \mathscr{W}$. A game

* Received by the editors September 6, 1972, and in revised form February 23, 1973.
† Department of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel.

is said to be *monotonic* if the function v is monotonic with respect to the inclusion ordering. A player i in a simple game $\Gamma = (N; \mathscr{W})$ is called *veto player* if $i \in S$ for every $S \in \mathscr{W}$.

A compound game is defined as follows. Let $\Gamma_i = (N_i; \mathscr{W}^i)$, $i = 1, \cdots, m$, be m simple games over disjoint sets of players, N_i, $i = 1, \cdots, m$. Let $\Gamma_0 = (M; u)$ be an m-player characteristic function game ($M = \{1, \cdots, m\}$). The *compound* game $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$ is defined over the set $N = N_1 \cup \cdots \cup N_m$, and its characteristic function v is defined by

$$v(S) = u(\{i \in M : S \cap N_i \in \mathscr{W}^i\}), \qquad S \subset N. \tag{2.1}$$

Γ_0 is the *quotient* and $\Gamma_1, \cdots, \Gamma_m$ are the *components*.

A *simple committee* in a game $\Gamma = (N; v)$ is a coalition $C \subset N$ such that for every $S \subset N$,

$$v(S) = \begin{cases} v(S \cup C) & \text{if } S \cap C \in \mathscr{W}_C, \\ v(S \setminus C) & \text{if } S \cap C \notin \mathscr{W}_C, \end{cases} \tag{2.2}$$

where $\mathscr{W}_C$ is a set of subsets of C. $\Gamma_C = (C; \mathscr{W}_C)$ is a *simple committee-game* of Γ. A simple committee-game is equivalent to a simple component in a compound game (see [6, Lemma 2.4]).

An *imputation* in an n-player game is an n-tuple $x = (x_1, \cdots, x_n)$ of non-negative numbers such that $\sum_{i=1}^{n} x_i = v(N)$. The set of all the imputations in a game Γ is denoted by $X(\Gamma)$. For $\emptyset \neq S \subset N$ and $x \in X(\Gamma)$ we denote $x(S) = \sum_{i \in S} x_i$ and define $x(\emptyset)$ to be equal to zero. The *excess* of S with respect to x is defined by

$$e(S, x) = v(S) - x(S). \tag{2.3}$$

For every $x \in X(\Gamma)$ we denote by $\theta(x)$ the 2^n-tuple whose components are the numbers $e(S, x)$, $S \subset N$, arranged according to their magnitude, i.e.,

$$\theta_i(x) \geqq \theta_j(x) \quad \text{for } 1 \leqq i \leqq j \leqq 2^n.$$

The *nucleolus* of a game Γ is the imputation $v \in X(\Gamma)$ such that $\theta(v)$ is a lexicographical minimum in the set $\{\theta(x) : x \in X(\Gamma)\}$. The existence of a unique v such that $\theta(v)$ precedes every $\theta(x)$, $v \neq x \in X(\Gamma)$, in the lexicographical order on R^{2^n} was proved in [10].

The *barycentric projection* of an imputation x on a coalition S such that $x(S) \neq 0$ is defined to be the imputation $B_S x$ where $(B_S x)_i = x_i / x(S)$ for $i \in S$ and $(B_S x)_i = 0$ for $i \notin S$ (see [14, p. 237]).

3. Some general properties of the nucleolus.

LEMMA 3.1. *Let $y \in X(\Gamma)$ be any imputation different from the nucleolus, v, of a game $\Gamma = (N; v)$. In this case*

$$\max\{e(S, v) : e(S, v) \neq e(S, y)\} < \max\{e(S, y) : e(S, v) \neq e(S, y)\}. \tag{3.1}$$

Proof. Suppose

$$\theta(v) = (e(S_1, v), \cdots, e(S_{2^n}, v)), \tag{3.2}$$

$$\theta(y) = (e(T_1, y), \cdots, e(T_{2^n}, y)). \tag{3.3}$$

That means

$$e(S_1, v) \geqq \cdots \geqq e(S_{2^n}, v), \tag{3.4}$$

$$e(T_1, y) \geqq \cdots \geqq e(T_{2^n}, y). \tag{3.5}$$

Let i_0 $(1 \leqq i_0 \leqq 2^n)$ be the first index of unequal excesses:

$$e(S_{i_0}, v) < e(T_{i_0}, y), \tag{3.6}$$

$$i < i_0 \Rightarrow e(S_i, v) = e(T_i, y). \tag{3.7}$$

We shall prove that for every $i < i_0$,

$$e(S_i, v) = e(S_i, y). \tag{3.8}$$

First,

$$e(S_{i_0 - 1}, v) > e(S_{i_0}, v) \tag{3.9}$$

(otherwise (3.5) and (3.7) contradict (3.6)). Let i^* $(1 \leqq i^* \leqq 2^n)$ be defined by

$$e(S_{i^*}, v) \neq e(S_i, y) \tag{3.10}$$

and

$$i < i^* \Rightarrow e(S_i, v) = e(S_i, y). \tag{3.11}$$

Without loss of generality we assume that for every $i > i^*$,

$$e(S_{i^*}, v) = e(S_i, v) \Rightarrow e(S_{i^*}, y) \geqq e(S_i, y) \tag{3.12}$$

and for every $i < i^*$, $S_i = T_i$. It can be easily seen that $i^* \leqq i_0$. We shall prove that $i^* = i_0$. Suppose

$$i^* < i_0. \tag{3.13}$$

Statements (3.7), (3.10) and (3.13) imply

$$e(S_{i^*}, v) > e(S_{i^*}, y). \tag{3.14}$$

Statements (3.9) and (3.13) imply

$$e(S_{i^*}, v) > e(S_{i_0}, v). \tag{3.15}$$

There exists $\varepsilon > 0$ such that the imputation $z = (1 - \varepsilon)v + \varepsilon y$ satisfies for every $i > i^*$ (see (3.12))

$$\begin{aligned} e(S_{i^*}, z) &= (1 - \varepsilon)e(S_{i^*}, v) + \varepsilon e(S_{i^*}, y) \\ &\geqq (1 - \varepsilon)e(S_i, v) + \varepsilon e(S_i, y) \\ &= e(S_i, z). \end{aligned} \tag{3.16}$$

Also, (3.11) implies

$$i < i^* \Rightarrow e(S_i, v) = e(S_i, z) \tag{3.17}$$

and (3.14) implies

$$e(S_{i^*}, z) < e(S_{i^*}, v). \tag{3.18}$$

Statements (3.16)–(3.18) imply that $\theta(z)$ precedes $\theta(v)$ in the lexicographical order on R^{2^n} in contradiction to the assumption that v is the nucleolus. Thus, $i^* = i_0$ and (3.8) is proved. Relation (3.1) follows from (3.6) and (3.8).

The nucleolus was defined to be an imputation whose excesses vector is a lexicographical minimum. In order to characterize the nucleolus of a compound game we would like to define here a general lexicographical problem.

PROBLEM 3.2. Given a set of m affine functionals over R^n,

$$a_i(x) = \sum_{j=1}^{n} a_{i_j} x_j + v_i, \qquad i = 1, \cdots, m, \quad x \in R^n, \tag{3.19}$$

we denote by $a(x)$ an m-tuple whose components are the numbers $a_i(x)$, $i = 1, \cdots, m$, arranged in decreasing order. Given a convex polytope $X \subset R^n$, our problem is to find $x_0 \in X$ such that $a(x_0)$ is minimal in the lexicographical order on R^m in the set $\{a(x): x \in X\}$.

Remark 3.3. One possible way to solve an n-dimensional lexicographical problem with m functionals is by solving at most $\min(m, n)$ linear problems (see [1]). The first one is: Minimize t subject to the constraints

$$a_i(x) \leqq t, \qquad i \in A_0 = \{1, \cdots, m\}, \tag{3.20}$$

$$x \in X_0 = X. \tag{3.21}$$

Suppose X_k is the set of the solutions to the kth problem and A_k is the set of indices i such that $a_i(x)$ is not constant in X_k. The $(k + 1)$th problem is: Minimize t subject to

$$a_i(x) \leqq t, \qquad i \in A_k, \tag{3.22}$$

$$x \in X_k. \tag{3.23}$$

In view of this method, a lexicographical problem may be simplified as follows. Denote

$$\bar{a}_i(x) = a_i(x) - v_i, \qquad i = 1, \cdots, m. \tag{3.24}$$

Suppose $\bar{a}_{i_0}$ is a linear combination of $\bar{a}_{i_1}, \cdots, \bar{a}_{i_p}$ and

$$a_{i_0}(x) \leqq \min\{a_{i_j}(x): j = 1, \cdots, p\} \tag{3.25}$$

for every $x \in X$. Then a_{i_0} may be omitted without affecting the solution. Our last claim follows from the fact that, throughout the computation, in every problem, k, either $a_{i_0}(x)$ is constant in X_k or the constraint $a_{i_0}(x) \leqq t$ may be omitted (see (3.25)).

DEFINITION 3.4. Let $\mathcal{S} \subset 2^N$ be any collection of coalitions in a game $\Gamma = (N; v)$.

(i) The *nucleolus with respect to* $\mathcal{S}$ is defined to be the set of the solutions of the lexicographical problem over $X(\Gamma)$ with the functionals $e(S, x)$, $S \in \mathcal{S}$. We denote this set by $\mathcal{N}_{\mathcal{S}}(\Gamma)$.

(ii) $\mathcal{S}$ is called a *nucleo-sufficient* collection if $\mathcal{N}_{\mathcal{S}}(\Gamma)$ coincides with the nucleolus $\mathcal{N}(\Gamma) = \mathcal{N}_{2^N}(\Gamma)$ of Γ. That means that throughout the computation of the nucleolus, one may be restricted to constraints corresponding to the coalitions in $\mathcal{S}$.

LEMMA 3.5. *Let* $\Gamma = (N; v)$ *be a monotonic game and denote*

$$\mathscr{R}_1(\Gamma) = \{S \subset N : (\exists i \in S)(v(S \setminus \{i\}) < v(S))\}, \tag{3.26}$$

$$\mathscr{R}_2(\Gamma) = \{S \subset N : (\forall T \subset S)(|S \setminus T| \geqq 2 \Rightarrow v(T) < v(S))\}, \tag{3.27}$$

$$\binom{N}{1} = \{\{i\} : i \in N\}, \tag{3.28}$$

$$\mathscr{R}(\Gamma) = \binom{N}{1} \cup (\mathscr{R}_1(\Gamma) \cap \mathscr{R}_2(\Gamma)). \tag{3.29}$$

Then $\mathscr{R}(\Gamma)$ *is a nucleo-sufficient collection in* Γ.

Proof. A coalition S has *property* I if there are $S_1, \cdots, S_k \in \mathscr{R}(\Gamma)$ such that $S_i \subset S, v(S_i) = v(S), i = 1, \cdots, k$, and the characteristic vector of S, χ^S, is a linear combination of the characteristic vectors χ^{S_i} of $S_i, i = 1, \cdots, k$. We shall prove that every coalition S has property I. This is certainly true for the 1-player coalitions (see (3.29)). Let S consist of more than one player and assume, by induction, that every $T \subsetneqq S$ has property I. If $S \in \mathscr{R}(\Gamma)$ then S has property I, so let us assume that $S \notin \mathscr{R}(\Gamma)$. If $S \notin \mathscr{R}_1(\Gamma)$, then for every $i \in S$,

$$v(S \setminus \{i\}) = v(S) \tag{3.30}$$

and χ^S is a linear combination of the vectors $\chi^{S \setminus \{i\}}, i \in S$ (recall that there are at least two players in S).

If $S \notin \mathscr{R}_2(\Gamma)$ then there exist $i, j \in S$ $(i \neq j)$ such that

$$v(S \setminus \{i, j\}) = v(S \setminus \{i\}) = v(S \setminus \{j\}) = v(S). \tag{3.31}$$

Also, χ^S is a linear combination of $\chi^{S \setminus \{i\}}, \chi^{S \setminus \{j\}}$ and $\chi^{S \setminus \{i,j\}}$. Thus, there always exist coalitions $S_1, \cdots, S_k$ such that $S_i \subset S, v(S_i) = v(S), S_i$ has property I, $i = 1, \cdots, k$, and χ^S is a linear combination of $\chi^{S_1}, \cdots, \chi^{S_k}$. It follows that S has property I. In view of Remark 3.3, since every coalition S has property I, it follows that $\mathscr{R}(\Gamma)$ is a nucleo-sufficient collection in Γ.

LEMMA 3.6. *Let* $\Gamma = (N; \mathscr{W})$ *be a monotonic simple game with no veto players and denote*

$$\mathscr{W}^* = \{S \in \mathscr{W} : (\forall T \subset S)(|S \setminus T| \geqq 2 \Rightarrow T \notin \mathscr{W}) \wedge (\exists i \in S)(S \setminus \{i\} \notin \mathscr{W})\}. \tag{3.32}$$

Then $\mathscr{W}^*$ *is a nucleo-sufficient collection.*

Proof. It is easily verified that

$$\mathscr{R}(\Gamma) = \binom{N}{1} \cup \mathscr{W}^*. \tag{3.33}$$

Using arguments similar to those of Lemma 3.5, we can prove that

$$\mathscr{N}_{\mathscr{W}}(\Gamma) = \mathscr{N}_{\mathscr{W}^*}(\Gamma) \tag{3.34}$$

(see Definition 3.4). Since $\mathscr{R}(\Gamma) = \binom{N}{1} \cup \mathscr{W}^*$, it is sufficient to prove that $\mathscr{W}$ is a nucleo-sufficient collection, i.e.,

$$\mathcal{N}_{\mathscr{W}}(\Gamma) = \mathcal{N}\binom{N}{1} \cup \mathscr{W}(\Gamma). \tag{3.35}$$

For every player i there is a winning coalition S such that $i \notin S$. Also, $S \cup \{i\}$ is winning because of the monotonicity. If $\{i\}$ is not winning, then for every $x \in X(\Gamma)$,

$$e(S, x) \geqq e(S \cup \{i\}, x) \geqq e(\{i\}, x). \tag{3.36}$$

Moreover, $\chi^{\{i\}}$ is a linear combination of χ^S and $\chi^{S \cup \{i\}}$ and therefore $e(\{i\}, x)$ may be omitted throughout the computation of the nucleolus (see Remark 3.3). Thus, (3.35) is proved.

4. The nucleolus of a simple committee-game. In this section we prove that the nucleolus of a simple committee-game is the barycentric projection of the nucleolus of the grand game on this committee. Thus, strength relations between the members of the committee, as reflected in the nucleolus, are the same in the committee-game and in the grand game.

We shall use the notation

$$\mathscr{D}(x) = \{a_i : a_i(x) \geqq a_j(x), j = 1, \cdots, m\}, \qquad x \in X, \tag{4.1}$$

when dealing with Problem 3.2.

LEMMA 4.1. *If* $x \in X_1(\Gamma)$ (using the notation of Remark 3.3 in the problem of computing the nucleolus), *where* $\Gamma = (N; v)$ *is a monotonic game, then for every* $i \in N$ *there is* $S \in \mathscr{D}(x)$ *such that* $i \in S$.

Proof. The proof is immediate in case $x_i = 0$. If $x_i > 0$ and[1] $i \notin \bigcup\{S : S \in \mathscr{D}(x)\}$, then $\varnothing \notin \mathscr{D}(x) (N \notin \mathscr{D}(x), e(\varnothing, x) = e(N, x))$. For $\varepsilon > 0$ small enough we can define an imputation y,

$$y_k = \begin{cases} x_i - (n-1)\varepsilon & \text{if } k = i, \\ x_k + \varepsilon & \text{if } k \neq i, \end{cases} \qquad k = 1, \cdots, n, \tag{4.2}$$

such that for every $S \in \mathscr{D}(x)$,

$$e(S, y) < e(S, x) \tag{4.3}$$

and

$$\mathscr{D}(y) \subset \mathscr{D}(x). \tag{4.4}$$

This is a contradiction to our assumption that $x \in X_1(\Gamma)$.

LEMMA 4.2. *Let* $\Gamma = (N; v)$ *be a monotonic game and* $\Gamma_C = (C; \mathscr{W}_C)$ *a simple committee-game of* Γ. *If* $x \in X_1(\Gamma)$, *then there exists* $S \in \mathscr{D}(x)$ *such that* $S \cap C \in \mathscr{W}_C$.

Proof. The proof is immediate if $x(C) = 0$. If $i \in C$ and $x_i > 0$, let $S \in \mathscr{D}(x)$ such that $i \in S$ (see Lemma 4.1). If $S \cap C \notin \mathscr{W}_C$, then

$$e(S \setminus \{i\}, x) > e(S, x) \tag{4.5}$$

and, thus, $S \notin \mathscr{D}(x)$—a contradiction!

[1] We consider $\mathscr{D}(X)$ also as a set of coalitions; no confusion may be caused since there is a one-to-one correspondence between coalitions and their excesses.

DEFINITION 4.3. Given a monotonic game $\Gamma = (N; v)$, a simple committee-game $\Gamma_C = (C; \mathscr{W}_C)$ and an imputation x such that $x(C) \neq 0$, we denote

$$w^{(x)}(B) = \begin{cases} \frac{1}{x(C)} \max \{v(C \cup T) - x(T) : T \subset N \setminus C\} & \text{if } B \in \mathscr{W}_C, \\ \frac{1}{x(C)} \max \{v(T) - x(T) : T \subset N \setminus C\} & \text{if } B \notin \mathscr{W}_C. \end{cases} \tag{4.6}$$

We call $\Gamma_C^{(x)} = (C; w^{(x)})$ the modification of Γ_C with respect to x. We shall show that this modification, which does not depend on x_i for $i \in C$ (but depends on $x(C)$), does not affect the nucleolus of the committee-game if $x \in X_1(\Gamma)$. We use the notation $e_C^{(x)}(S, y)$, $\theta_C^{(x)}(y)$, etc. when these expressions refer to the game $\Gamma_C^{(x)}$.

Let us denote

$$\sigma(\Gamma) = \min \{\theta_1(x) : x \in X(\Gamma)\}, \tag{4.7}$$

$$w_1^{(x)} \equiv w^{(x)}(B) \tag{4.8}$$

for any $B \in \mathscr{W}_C$ and

$$w_0^{(x)} \equiv w^{(x)}(B) \tag{4.9}$$

for any $B \notin \mathscr{W}_C$.

LEMMA 4.4. *Let $\Gamma = (N; v)$ be a monotonic game and $\Gamma_C = (C; \mathscr{W}_C)$ a simple committee-game of Γ. If $x \in X_1(\Gamma)$ such that $x(C) \neq 0$, then*

$$\sigma(\Gamma_C^{(x)}) \geqq w_0^{(x)} \tag{4.10}$$

(see Definition 4.3 and (4.7)–(4.9)).

Proof. First, it follows from Definition 4.3 that

$$w_1^{(x)} \geqq 1 \tag{4.11}$$

(if $T = N \setminus C$, then $v(T \cup C) - x(T) = x(C)$). Let $y \in X(\Gamma_C)$ such that

$$\min \{y(B) : B \in \mathscr{W}_C\} = 1 - \sigma(\Gamma_C)$$

and let $\bar{x} \in X(\Gamma)$ such that

$$\bar{x}_k = \begin{cases} x(C) \cdot y_k & \text{if } k \in C, \\ x_k & \text{if } k \notin C. \end{cases} \tag{4.12}$$

It follows from Lemma 4.2 that

$$\begin{aligned} \theta_1(\bar{x}) &= \max \{e(S, \bar{x}) : S \subset N, S \cap C \in \mathscr{W}_C\} \\ &= \max \{v(C \cup T) - x(T) : T \subset N \setminus C\} \\ &\qquad - x(C) \cdot \min \{y(B) : B \in \mathscr{W}_C\} \\ &= x(C)[w_1^{(x)} - (1 - \sigma(\Gamma_C))]. \end{aligned} \tag{4.13}$$

Since $x(C) \cdot w_0^{(x)}$ is also an excess with respect to $\bar{x}$ (see Definition 4.3) it follows that

$$w_1^{(x)} - (1 - \sigma(\Gamma_C)) \geqq w_0^{(x)}. \tag{4.14}$$

Since for every $z \in X(\Gamma_C)$,

$$\min \{z(B) : B \in \mathscr{W}_C\} \leqq 1 - \sigma(\Gamma_C), \tag{4.15}$$

it follows from (4.14)–(4.15) that

$$\theta_1^{(x)}(z) = \max\{e_C^{(x)}(S, z): S \in \mathscr{W}_C\} \tag{4.16}$$

so that

$$\sigma(\Gamma_C^{(x)}) = w_1^{(x)} - (1 - \sigma(\Gamma_C)) \geqq w_0^{(x)}. \tag{4.17}$$

Remark 4.5. Since $\sigma(\Gamma_C) < 1$ it follows from (4.14) that

$$w_1^{(x)} > w_0^{(x)}. \tag{4.18}$$

Thus, the winning coalitions in Γ_C have (in the modified committee-game) a value which is still greater than that of the losing coalitions. Moreover, it also follows that

$$\sigma(\Gamma_C^{(x)}) \geqq \sigma(\Gamma_C). \tag{4.19}$$

LEMMA 4.6. *Let $\Gamma = (N; v)$ be a monotonic game and let $\Gamma_C = (C; \mathscr{W}_C)$ be a simple committee-game with no veto players. Under these conditions, for every $i \in C$ there is $S \in \mathscr{W}_C$ such that $i \notin S$ and $S \in \mathscr{D}_C^{(x)}(y)$ for every $y \in X_1(\Gamma_C^{(x)})$.*

Proof. (a) Suppose, per absurdum, that $i \in C$ and $y \in X_1(\Gamma_C^{(x)})$ such that for every $S \in \mathscr{W}_C$, $S \in \mathscr{D}_C^{(x)}(y)$ implies $i \in S$. Since i is not a veto player, there exists $S_0 \in \mathscr{W}_C$ such that $i \notin S_0$. Thus, $S_0 \notin \mathscr{D}_C^{(x)}(y)$ and, therefore, for every

$$S \in \mathscr{W}_C \cap \mathscr{D}_C^{(x)}(y),$$

$$e_C^{(x)}(S, y) > e_C^{(x)}(S_0, y). \tag{4.20}$$

It thus follows that for every $S \in \mathscr{W}_C \cap \mathscr{D}_C^{(x)}(y)$ there is $j \notin S$ ($j \neq i$) such that $y_j > 0$. Let ε be such that $0 < \varepsilon < \min\{y_j : j \neq i, y_j \neq 0\}$ and define y^* by

$$y_j^* = \begin{cases} y_j - \varepsilon & \text{if } j \neq i \text{ and } y_j \neq 0, \\ 0 & \text{if } y_j = 0, \\ 1 - y^*(N \setminus \{i\}) & \text{if } j = i. \end{cases} \tag{4.21}$$

It follows that for every $S \in \mathscr{D}_C^{(x)}(y)$,

$$e_C^{(x)}(S, y^*) < e_C^{(x)}(S, y). \tag{4.22}$$

If ε is small enough, then $\mathscr{W}_C \cap \mathscr{D}_C^{(x)}(y^*) \subset \mathscr{W}_C \cap \mathscr{D}_C^{(x)}(y)$ in contradiction to the assumption that $y \in X_1(\Gamma_C^{(x)})$. We have proved that for every $i \in C$ and $y \in X_1(\Gamma_C^{(x)})$ there is $S \in \mathscr{W}_C \cap \mathscr{D}_C^{(x)}(y)$ such that $i \in S$.

(b) The existence of S was proved for every $y \in X_1(\Gamma_C^{(x)})$. Particularly, that is true for every y in the set

$$X_1^*(\Gamma_C^{(x)}) \equiv \{y \in X_1(\Gamma_C^{(x)}) : (\forall z)(z \in X_1(\Gamma_C^{(x)}) \Rightarrow |\mathscr{D}_C^{(x)}(y)| \leqq |\mathscr{D}_C^{(x)}(z)|)\}. \tag{4.23}$$

It can be inferred from Lemma 3.1 that $y \in X_1^*$ and $z \in X_1$ imply $\mathscr{D}_C^{(x)}(y) \subset \mathscr{D}_C^{(x)}(z)$ and, therefore, $S \in \mathscr{D}_C^{(x)}(z)$ for every $z \in X_1(\Gamma_C^{(x)})$.

LEMMA 4.7. *Let $\Gamma = (N; v)$ be a monotonic game and let $\Gamma_C = (C; \mathscr{W}_C)$ be a simple committee-game. If $x \in X_1(\Gamma)$ is such that $x(C) > 0$ then the nucleolus of Γ_C coincides with the nucleolus of $\Gamma_C^{(x)}$ with respect to $X(\Gamma_C)$.*

Proof. (a) If there are veto players in Γ_C, then $\sigma(\Gamma_C) = 0$ and (4.14) implies $w_1^{(x)} - 1 \geqq w_0^{(x)}$. Thus, for every $y \in X(\Gamma_C)$ and $S \in \mathscr{W}_C$, $e_C^{(x)}(S, y) \geqq w_0^{(x)}$ and the respective lexicographical problems differ by a constant.

(b) Assume that there are no veto players in Γ_C. According to Definition 4.3 and (3.29), $\mathscr{R}(\Gamma_C^{(x)}) = \mathscr{R}(\Gamma_C) = \binom{C}{1} \cup \mathscr{W}_C^*$. It can be seen that $\binom{C}{1} \cup \mathscr{W}_C^*$ suffices to the nucleolus of $\Gamma_C^{(x)}$ (see Lemma 3.5). Suppose $\{i\} \notin \mathscr{W}_C$ and $S \in \mathscr{W}_C$ is such that $i \notin S$ and $y \in X_1(\Gamma_C^{(x)})$ implies $S \in \mathscr{D}_C^{(x)}(y)$ (see Lemma 4.6). Under these conditions, for every $y \in X_1(\Gamma_C^{(x)})$ (according to Lemmas 4.4 and 4.6)

$$\begin{aligned} e_C^{(x)}(S, y) &\geqq e_C^{(x)}(S \cup \{i\}, y) = \sigma(\Gamma_C^{(x)}) - y_i \\ &\geqq w_0^{(x)} - y_i = e_C^{(x)}(\{i\}, y). \end{aligned} \tag{4.24}$$

Since $\chi^{\{i\}} = \chi^{S \cup \{i\}} - \chi^S$, it follows, in view of (4.24), that the constraint $e_C^{(x)}(\{i\}, y) \leqq t$ may be omitted from the problem of the nucleolus of $\Gamma_C^{(x)}$ (see Remark 3.3). The collection $\mathscr{W}_C^*$ is thus nucleo-sufficient in $\Gamma_C^{(x)}$. That implies that the nuclei problems in the games $\Gamma_C^{(x)}$ and Γ_C have the same solution.

THEOREM 4.8. *Let $\Gamma = (N; v)$ be a monotonic game and let $\Gamma_C = (C; \mathscr{W}_C)$ be a simple committee-game of Γ. If v is the nucleolus of Γ and $v(C) \neq 0$, then $B_C v$ is the nucleolus of Γ_C.*

Proof. The nucleolus belongs to the kernel. In a simple game with veto players the nucleolus is the unique point in the kernel [3, Thm. 4.1]. Thus, if there are veto players in Γ_C, then our theorem follows from the compounding theorem with respect to the kernel (see [5, Thm. 6.1]).

Let us assume that Γ_C is free of veto players. Let v^C denote the nucleolus of Γ_C; it is the nucleolus of $\Gamma_C^{(v)}$ too (Lemma 4.7). Let $x^C = B_C v$ and suppose, per absurdum, that $x^C \neq v^C$. Let

$$\theta_C^{(v)}(v^C) = (e_C^{(v)}(S_1, v^C), \cdots, e_C^{(v)}(S_{2^{|C|}}, v^C)), \tag{4.25}$$

$$\theta_C^{(v)}(x^C) = (e_C^{(v)}(T_1, x^C), \cdots, e_C^{(v)}(T_{2^{|C|}}, x^C)). \tag{4.26}$$

Let i_0 be defined by

$$e_C^{(v)}(S_{i_0}, v^C) < e_C^{(v)}(T_{i_0}, x^C), \tag{4.27}$$

$$i < i_0 \Rightarrow e_C^{(v)}(S_i, v^C) = e_C^{(v)}(T_i, x^C). \tag{4.28}$$

Without loss of generality we assume that

$$i < i_0 \Rightarrow S_i = T_i \tag{4.29}$$

(see Lemma 3.1). Define an imputation y by

$$y_k = \begin{cases} v(C) \cdot x_k^C & \text{if } k \in C, \\ v_k & \text{if } k \notin C. \end{cases} \tag{4.30}$$

For every $B \subset C$,

$$e_C^{(v)}(B, v^C) = \frac{1}{v(C)} \max \{e(B \cup T, v) : T \subset N \setminus C\}, \tag{4.31}$$

$$e_C^{(v)}(B, x^C) = \frac{1}{v(C)} \max \{e(B \cup T, y) : T \subset N \setminus C\}. \tag{4.32}$$

Let $R_1, R_2 \subset N \setminus C$ be coalitions such that

$$e(R_1, v) = \max\{e(T, v): T \subset N \setminus C\}, \tag{4.33}$$

$$e(R_2 \cup C, v) = \max\{e(T \cup C, v): T \subset N \setminus C\}. \tag{4.34}$$

If $S_{i_0} \notin \mathscr{W}_C$ we denote $S_{i_0} \cup R_1$ by S^*, and if $S_{i_0} \in \mathscr{W}_C$, $S_{i_0} \cup R_2$ is denoted by S^*. For every $R \subset N \setminus C$ and i, if $S_i \in \mathscr{W}_C$, then

$$e(S_i \cup R, y) = v(C) \cdot e_C^{(v)}(S_i, v^C) + v(C \cup R) - v(R) - v(C) \cdot w_1^{(v)}, \tag{4.35}$$

and if $S_i \notin \mathscr{W}_C$, then

$$e(S_i \cup R, y) = v(C) \cdot e_C^{(v)}(S_i, v^C) + v(R) - v(R) - v(C) \cdot w_0^{(v)}. \tag{4.36}$$

Particularly (see Definition 4.3 and (4.33)–(4.34)),

$$e(S^*, y) = v(C) \cdot e_C^{(v)}(S_{i_0}, v^C). \tag{4.37}$$

Therefore, for every $i \geqq i_0$ and $R \subset N \setminus C$,

$$e(S^*, y) \geqq e(S_i \cup R, y). \tag{4.38}$$

Analogously, if $T_{i_0} \notin \mathscr{W}_C$ denote $T^* = T_{i_0} \cup R_1$ and if $T_{i_0} \in \mathscr{W}_C$ denote $T^* = T_{i_0} \cup R_2$. For every $i \geqq i_0$ and $R \subset N \setminus C$,

$$e(T^*, v) = v(C) \cdot e_C^{(v)}(T_{i_0}, x^C) \geqq e(T_i \cup R, x^C). \tag{4.39}$$

It follows from (4.27), (4.37) and (4.39) that

$$e(S^*, y) < e(T^*, v). \tag{4.40}$$

Also, (4.28)–(4.29) imply for every $i < i_0$ and $R \subset N \setminus C$,

$$e(S_i \cup R, y) = e(T_i \cup R, v). \tag{4.41}$$

It can be verified that (4.38)–(4.41) mean that $\theta(y)$ precedes $\theta(v)$ in the lexicographical order on R^{2^n} in contradiction to our assumption that v is the nucleolus of Γ.

Example 4.9. *The nucleolus of the sum of two simple games.* Let $\Gamma_i = (N_i; \mathscr{W}^i)$, $i = 1, 2$, be two monotonic simple games such that $N_1 \cap N_2 = \varnothing$. The *sum* $\Gamma_1 \oplus \Gamma_2$ is defined to be the game $\Gamma = (N; \mathscr{W})$ where $N = N_1 \cup N_2$ and $\mathscr{W} = \{S \subset N: S \cap N_1 \in \mathscr{W}^1 \text{ or } S \cap N_2 \in \mathscr{W}^2\}$. The *product* $\Gamma_1 \oplus \Gamma_2 = (N; \mathscr{W})$ where $\mathscr{W} = \{S \subset N: S \cap N_1 \in \mathscr{W}^1 \text{ and } S \cap N_2 \in \mathscr{W}^2\}$ was investigated and its nucleolus was characterized in [4]. Let $\Gamma = (N; \mathscr{W})$ be the sum $\Gamma = \Gamma_1 \oplus \Gamma_2$. We shall prove that for every $x \in X_1(\Gamma)$,

$$\max\{e(S, x): S \in \mathscr{W}^1\} = \max\{e(S, x): S \in \mathscr{W}^2\}. \tag{4.42}$$

Suppose, per absurdum, that for example,

$$\max\{e(S, x): S \in \mathscr{W}^1\} > \max\{e(S, x): S \in \mathscr{W}^2\}. \tag{4.43}$$

Thus,

$$\begin{aligned} \theta_1(x) &= \max\{e(S, x): S \in W\} \\ &= \max[\max\{e(S, x): S \cap N_1 \in \mathscr{W}^1\}, \max\{e(S, x): S \cap N_2 \in \mathscr{W}^2\}] \end{aligned} \tag{4.44}$$

(*cont.*)

$$= \max[\max\{e(S,x):S \in \mathscr{W}^1\}, \max\{e(S,x):S \in \mathscr{W}^2\}]$$
$$= \max\{e(S,x):S \in \mathscr{W}^1\}.$$

If $x(N_1) \neq 0$ let $x^1 = B_{N_1}x$ and if $x(N_1) = 0$ let x^1 be any point in $X_1(\Gamma_1)$. Also, let $x^2 = B_{N_2}x$ (it follows from (4.43) that $x(N_2) \neq 0$). Regard $x^i, i = 1, 2$, as an imputation in Γ by means of adding zeros for the players who do not belong to N_i. Thus,

$$x = x(N_1)x^1 + x(N_2)x^2. \tag{4.45}$$

There is $\varepsilon > 0$ such that the imputation

$$x_\varepsilon = (x(N_1) + \varepsilon)x^1 + (x(N_2) - \varepsilon)x^2 \tag{4.46}$$

satisfies (4.43) and, consequently, (4.44). Since for every $y \in X_1(\Gamma_1)$,

$$\min\{y(S):S \in \mathscr{W}^1\} > 0 \tag{4.47}$$

it follows that

$$\theta_1(x_\varepsilon) < \theta_1(x) \tag{4.48}$$

in contradiction to our assumption that $x \in X_1(\Gamma)$. Thus, (4.42) is proved. Let $v^i, i = 1, 2$, denote the nucleolus of Γ_i. According to Theorem 4.8 the nucleolus of Γ has the form

$$v = (1 - \alpha)v^1 + \alpha v^2 \tag{4.49}$$

where $0 \leqq \alpha \leqq 1$. It follows from (4.42) that

$$\begin{aligned} \alpha &= \frac{\min\{v^1(S):S \in \mathscr{W}^1\}}{\min\{v^1(S):S \in \mathscr{W}^1\} + \min\{v^2(S):S \in \mathscr{W}^2\}} \\ &= \frac{1 - \theta_1^1(v^1)}{2 - \theta_1^1(v^1) - \theta_1^2(v^2)}. \end{aligned} \tag{4.50}$$

This means that the computation of the nucleolus is equivalent to the computation of the nucleoluses of the summands. The coefficient α is a by-product of this computation.

5. The nucleolus of a compound game. According to the preceding section the nucleolus of a simple committee-game is the barycentric projection of the nucleolus of the grand game on that committee. In a compound game with simple components the set of players, N, is a union of disjoint committees. Thus, the nucleolus of the compound game is a convex combination of the nucleoluses of the component games (the nucleolus of a component game is changed to an imputation in the grand game by adding zeros). This combination depends only on the two highest levels of excesses in the components' nucleoluses. It is a solution of a lexicographical problem.

Let Γ be a monotonic compound game, $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$, where $\Gamma_i = (N_i; \mathscr{W}^i), i = 1, \cdots, m$, are simple games over pairwise disjoint sets and $\Gamma_0 = (M; u), M = \{1, \cdots, m\}$. Let v^i denote the nucleolus of $\Gamma_i, i = 1, \cdots, m$, and let v denote the nucleolus of Γ. According to Theorem 4.8 there are $\alpha_i \geqq 0$,

$i = 1, \cdots, m$, such that $\sum \alpha_i = u(M) = v(N)$ and

$$v = \alpha_1 v^{1*} + \cdots + \alpha_m v^{m*}. \tag{5.1}$$

By the $*$-notation an imputation in a committee-game is changed to an imputation in the grand game (by adding zeros). The problem is to compute the coefficients $\alpha_i, i = 1, \cdots, m$. For every component game $\Gamma_i, i = 1, \cdots, m$, let us denote the payoffs to the winning coalitions, in the nucleolus v^i, by

$$\omega_0^{(i)} < \omega_1^{(i)} < \cdots < \omega_{k_i}^{(i)} = 1. \tag{5.2}$$

The payoffs to the losing coalitions will be denoted by

$$0 = \lambda_0^{(i)} < \lambda_1^{(i)} < \cdots < \lambda_{l_i}^{(i)}. \tag{5.3}$$

For every $T \subset M$ denote

$$J(T) = \{j = (j_1, \cdots, j_m) : (\forall i \in T)(0 \leqq j_i \leqq k_i) \\ (\forall i \notin T)(0 \leqq j_i \leqq l_i)\}. \tag{5.4}$$

For every $T \subset M$ and $j \in J(T)$ let $a^{(T;j)}$ be an affine functional such that for every $\beta \in X(\Gamma_0)$,

$$a^{(T;j)}(\beta) = u(T) - \sum_{i \in T} \omega_{j_i}^{(i)} \beta_i - \sum_{i \notin T} \lambda_{j_i}^{(i)} \beta_i. \tag{5.5}$$

ASSERTION 5.1. *The combination* $\alpha = (\alpha_1, \cdots, \alpha_m)$*(see* (5.1)) *is the solution of the lexicographical problem with the functionals* $a^{(T;j)}$ $(T \subset M, j \in J(T))$ *and the polytope* $X(\Gamma_0)$.

The proof follows from the fact that for every $\beta \in X(\Gamma_0)$ the values $a^{(T;j)}(\beta)$ are the excesses with respect to the imputation $\beta_1 v^{1*} + \cdots + \beta_m v^{m*}$.

The number of functionals in the above problem can be decreased considerably. First, let us denote for every $T \subset M$,

$$J_1(T) = \{j \in J(T) : (\exists k)(\forall i)(i \neq k \Rightarrow j_i = 0) \wedge (j_k \in \{0, 1\})\}. \tag{5.6}$$

ASSERTION 5.2. *The set* $\{a^{(T;j)} : T \subset M, j \in J_1(T)\}$ *is sufficient (in the sense of Definition* 3.4) *to the lexicographical problem presented in Assertion* 5.1.

Proof. For every $T \subset M$ and $j \in J(T)$ denote

$$b^{(T;j)}(\beta) = a^{(T;j)}(\beta) - u(T). \tag{5.7}$$

Let $e_k, k = 1, \cdots, m$, denote the m-tuple in which $(e_k)_i = \delta_{ki}$ (Kronecker delta). Since $\{0, e_1, e_2, \cdots, e_m\}$ is an affine basis of R^m it follows that the linear functional $b^{(T;j)}$ is an affine combination of $b^{(T;0)}$ and $b^{(T;e_k)}, k = 1, \cdots, m$. For every k such that $j_k = 0$ the coefficient of $b^{(T;e_k)}$ in this combination must be equal to zero. Thus, for every $T \subset M$ and $j \in J(T)$ the functional $b^{(T;j)}$ is a linear combination of $b^{(T;0)}$ and $b^{(T;e_k)}, k = 1, \cdots, m, j_k \neq 0$. Moreover, for every $\beta \in X(\Gamma_0)$,

$$a^{(T;j)}(\beta) \leqq \min\{a^{(T;e_k)}(\beta) : 1 \leqq k \leqq m, j_k \neq 0\} \leqq a^{(T;0)}(\beta). \tag{5.8}$$

That implies that the set of functionals $a^{(T;j)}$ such that $j \in J_1(T)$ suffices.

Remark 5.3. If there are veto players in Γ_i, then there is only one level of payoffs to winning coalitions in the nucleolus of Γ_i. Thus, $k_i = 0$, $\omega_0^{(i)} = 1$ and for every

$j \in J_1(T)$ (where $i \in T$), $j_i = 0$. We shall show that it can also be assumed (i.e., $J_1(T)$ may be reduced furthermore) that for every i, if there are no veto players in Γ_i and if $i \notin T$ then for every $j \in J_1(T), j_i = 0$.

We denote

$$J^*(T) = \{j \in J_1(T): \text{For every } k \in T \text{ such that there are veto players in } \Gamma_k, j_k = 0, \text{ and for every } k \notin T \text{ such that there are no veto players in } \Gamma_k, j_k = 0\}. \tag{5.9}$$

ASSERTION 5.4. *The set* $\{a^{(T;j)}: T \subset M, j \in J^*(T)\}$ *is sufficient to the solution of the lexicographical problem presented in Assertion* 5.1.

Proof. It can be easily verified that at the first stage of the solution (see Remark 3.3) it is sufficient to use only the functionals of the form $a^{(T;0)}$. To see that, note that for every $j \in J(T)$ and $\beta \in X(\Gamma_0)$,

$$a^{(T;0)}(\beta) \geqq a^{(T;j)}(\beta). \tag{5.10}$$

Suppose $k \notin T \subset M$ and Γ_k is a game with no veto players. Let $\beta \in X_1$. According to Lemma 4.1 there exists $T^* \subset M$ such that $k \in T^*$ and $T^* \in \mathscr{D}(\beta)$ (see (4.1)). Let $l \in N_k$ be a player such that $v_1^k = \lambda_1^{(k)}$. Since l is not a veto player, there is $B \in \mathscr{W}^k$ such that $l \notin B$ and $B \in \mathscr{D}(v^k)$ (see the proof of Lemma 4.6). Clearly, $v^k(B) = \omega_0^{(k)}$ and also $v^k(B \cup \{l\}) = \omega_1^{(k)}$. The functional $a^{(T;e_k)}$ may, therefore, be omitted at the following stages. The last claim follows from the fact that for every $\beta \in X_1$,

$$a^{(T;e_k)}(\beta) \leqq \min\,[a^{(T^*;0)}(\beta), a^{(T^*;e_k)}(\beta), a^{(T;0)}(\beta)] \tag{5.11}$$

and $b^{(T;e_k)}$ (see (5.7)) is a linear combination of $b^{(T^*;e_k)}$, $b^{(T^*;0)}$ and $b^{(T;0)}$ (see Remark (3.3).

To conclude, the combination α which determines the nucleolus is the solution of the following problem.

Problem 5.5. Let $\Gamma_0 = (M; u)$ be an m-player game. Let $d = (d_1, \cdots, d_m)$ be an m-tuple of nonnegative integers and let ω^0 and ω^1 be two m-tuples such that $0 < \omega_i^0 < \omega_i^1 \leqq 1, i = 1, \cdots, m$. For every $T \subset M$ and $\beta \in X(\Gamma_0)$ let

$$f^{(T)}(\beta) = u(T) - \sum_{i \in T} \omega_i^0 \beta_i. \tag{5.12}$$

For every $k \in T$ such that $d_k = 0$ let

$$g^{(T;k)}(\beta) = u(T) - \sum_{i \in T \setminus \{k\}} \omega_i^0 \beta_i - \omega_k^1 \beta_k \tag{5.13}$$

and for every $k \notin T$ such that $d_k \neq 0$ let

$$g^{(T;k)}(\beta) = u(T) - \sum_{i \in T} \omega_i^0 \beta_i - \frac{\beta_k}{d_k}. \tag{5.14}$$

By $L[\Gamma_0, \omega^0, \omega^1, d]$ we denote the lexicographical problem with functionals $f^{(T)}$ and $g^{(T;k)}$ (where $T \subset M$ and either $k \in T$ and $d_k = 0$, or $k \notin T$ and $d_k \neq 0$).

The results are summed in the following theorem.

THEOREM 5.6. *Let* $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$ *be a monotonic compound game with simple components* (*see* (2.1)). *Let* v^i *be the nucleolus of* Γ_i *and let* d_i *be the number of* *veto players in* $\Gamma_i, i = 1, \cdots, m$. *Let*

$$\omega_i^0 = \min\{v^i(B): B \in \mathscr{W}^i\}, \qquad i = 1, \cdots, m, \tag{5.15}$$

$$\omega_i^1 = \min\{v^i(B): B \in \mathscr{W}^i, v^i(B) > \omega_i^0\}, \qquad i = 1, \cdots, m, \quad d_i = 0, \tag{5.16}$$

(for i such that $d_i \neq 0$ set $\omega_i^1 = 1$)

$$\omega^0 = (\omega_1^0, \cdots, \omega_m^0), \qquad \omega^1 = (\omega_1^1, \cdots, \omega_m^1). \tag{5.17}$$

Under these conditions the nucleolus of Γ *is*

$$v = \alpha_1 v^{1*} + \cdots + \alpha_m v^{m*}, \tag{5.18}$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$ *is the solution of the problem* $L[\Gamma_0, \omega^0, \omega^1, d]$ (Problem 5.5).

Example 5.7. Let $\Gamma = M_3[M_3 \otimes M_3, M_3 \oplus M_3, M_3[M_3, M_3, M_3]]$ (M_3 is the 3-person majority game) and let us compute the nucleolus of this 21-player game. According to our theorem we find by symmetry considerations that the nucleoluses of $M_3 \otimes M_3$, $M_3 \oplus M_3$ and $M_3[M_3, M_3, M_3]$ are, respectively, $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$. The minimal payoffs to winning coalitions (in the nucleoluses in the games are, respectively, $\frac{2}{3}, \frac{1}{3}, \frac{4}{9}$. The lexicographical problem is solved in one stage and the solution is $(\frac{2}{9}, \frac{4}{9}, \frac{1}{3})$. Thus, the nucleolus of Γ is

$$(\tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{2}{27}, \tfrac{2}{27}, \tfrac{2}{27}, \tfrac{2}{27}, \tfrac{2}{27}, \tfrac{2}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}).$$

Example 5.8. There are two generalizations of (2.1). L. S. Shapley (see [13, p. 40]) suggested

$$v(S) = \sum_{R \subset M} \sum_{T \subset R} (-1)^{|R \setminus T|} u(T) \cdot \min\{w_i(S \cap N_i): i \in R\} \tag{5.19}$$

and G. Owen (see [9]) defined

$$v(S) = \sum_{T \subset M} \prod_{i \in T} w_i(S \cap N_i) \prod_{i \notin T} (1 - w_i(S \cap N_i)) \cdot u(T). \tag{5.20}$$

Our results do not hold for these generalizations. Let $\Gamma = (N; v)$ be a 3-player game such that $v(12) = v(13) = \frac{1}{2}$, $v(123) = 1$ and $v(S) = 0$ for any other $S \subset \{1, 2, 3\}$. The nucleolus of Γ consists of the point $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. The nucleolus of the generalized product $\Gamma \otimes \Gamma$ (using either (5.19) or (5.20)) consists of the point $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Obviously, this point cannot be presented as

$$(1 - \alpha)(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0) + \alpha(0, 0, 0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}).$$

REFERENCES

[1] A. Kopelowitz, *Computation of the kernels of simple games and the nucleolus of n-person games*, Res. Memo. 31, Research Program in Game Theory and Mathematical Economics, Dept. of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 1967.

[2] M. Maschler and B. Peleg, *A characterization, existence proof and dimension bounds for the kernel of a game*, Pacific J. Math., 18 (1966), pp. 289–328.

[3] ———, *The structure of the kernel of a cooperative game*, this Journal, 15 (1967), pp. 569–604.

[4] N. Megiddo, *The kernel and the nucleolus of a product of simple games*, Israel J. Math., 9 (1971), pp. 210–221.

[5] ———, *Kernels of compound games with simple components*, Pacific J. Math., to appear.
[6] ———, *Tensor decomposition of cooperative games*, Res. Memo. 71, Research Program in Game Theory and Mathematical Economics, The Hebrew University of Jerusalem, Jerusalem, 1972.
[7] ———, *Nucleoluses of compound simple games. I: The nucleolus of the sun*, Res. Memo. 76, Research Program in Game Theory and Mathematical Economics, The Hebrew University of Jerusalem, Jerusalem, 1972.
[8] ———, *Nucleoluses of compound simple games. II: General compounds with simple components*, Res. Memo. 77, Research Program in Game Theory and Mathematical Economics, The Hebrew University of Jerusalem, Jerusalem, 1972.
[9] G. OWEN, *The tensor composition of non-negative games*, Advances in Game Theory, M. Dresher, L. S. Shapley and A. W. Tucker, eds., Annals of Mathematics Studies, No. 52, Princeton University Press, Princeton, 1964, pp. 307–327.
[10] D. SCHMEIDLER, *The nucleolus of a characteristic function game*, this Journal, 17 (1969), pp. 1163–1170.
[11] L. S. SHAPLEY, *Simple games: An outline of the descriptive theory*, Behavioral Sci., 7 (1962), pp. 59–66.
[12] ———, *Compound simple games. I: Solutions of sums and products*, RM-3192, The RAND Corporation, Santa Monica, Calif., 1962.
[13] ———, *Compound simple games. II: Some general composition theorems*, RM-3643, The RAND Corporation, Santa Monica, Calif., 1963.
[14] ———, *Solutions of compound simple games*, Advances in Game Theory, M. Dresher, L. S. Shapley and A. W. Tucker, eds., Annals of Mathematics Studies, No. 52, Princeton University Press, Princeton, 1964, pp. 267–305.
[15] ———, *Compound simple games. III: On committees*, RM-5438-PR, The RAND Corporation, 1967 (also in New Methods of Thought and Procedure, F. Zwycky and A. Wilson, eds., Springer, New York, 1968).