Null controllability of the structurally damped wave equation with moving point control

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Outline

1. Control of the structurally damped wave equation with moving point control
2. Control of the BBM equation with moving point control
3. Unique continuation property for BBM

Joint works with

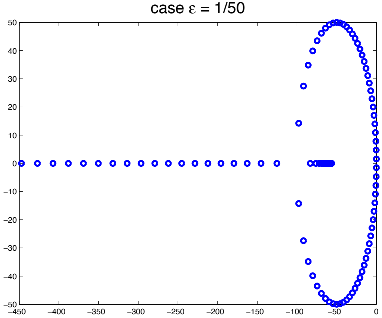
- **Philippe Martin** (Ecole des Mines, Paris)
- **Pierre Rouchon** (Ecole des Mines, Paris)
- **Bing-Yu Zhang** (University of Cincinnati)

Wave equation with structural damping

$$y_{tt} - y_{xx} - \varepsilon y_{txx} = 0$$
$$y(0,t) = y(1,t) = 0$$

$\varepsilon > 0$ strength of the structural (or internal) damping

- Spectrum: $\lambda_k^{\pm} = k^2\pi^2\varepsilon(-1 \pm \sqrt{1 - 4k^{-2}\pi^{-2}\varepsilon^{-2}})/2, \;\; k \in \mathbb{Z}$
- $\lambda_k^+ \sim -\varepsilon^{-1}, \qquad \lambda_k^- \sim -k^2\pi^2\varepsilon$



- Accumulation point in the spectrum: no spectral controllability, but approximate controllability (**LR-P. Rouchon ’07**)
- Phenomenon already noticed by
  - **D. Russell ’85** (Beam with structural damping)
  - **G. Leugering ’86** (viscoelasticity)
  - **S. Micu ’01** (linearized BBM)

Moving control

- Control whose support (a point, an interval) is **moving**; Introduced by **J.-L. Lions '92**
- Wave eq.: **Lions '92, Khapalov '95, Castro (preprint)**
- Heat eq.: **Khapalov '01, Castro-Zuazua '05**
- Here, we are concerned with

$$y_{tt} - y_{xx} - y_{txx} = b(x+ct)h(x,t)$$
$$x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \sim [0, 2\pi)$$

where b denotes δ_0, or $d\delta_0/dx$, or $b(\cdot) \in L^\infty(\mathbb{T})$, and $c \in \mathbb{R}$ is the (constant) velocity.

Control problem in a moving frame

Pick $c = -1$ for simplicity, and set $v(x,t) = y(x+t,t)$. Then

$$y_{tt} - y_{xx} - y_{txx} = b(x-t)h(x,t)$$
$$y(x,0) = y_0(x),\ y_t(x,0) = \xi_0(x)$$

is transformed into

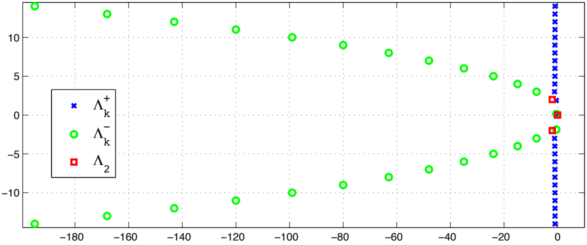
$$v_{tt} - 2v_{xt} - v_{txx} + v_{xxx} = b(x)h(x+t,t)$$
$$v(x,0) = y_0(x),\ v_t(x,0) = y_0'(x) + \xi_0(x)$$

The BBM term y_{txx} has generated the KdV term v_{xxx}!!

“New” spectrum

$$\lambda_k^{\pm} = (-(k^2 - 2ik) \pm \sqrt{k^4 - 4k^2})/2$$

- $\lambda_k^+ = -1 + ik + O(k^{-2})$ **hyperbolic part**
- $\lambda_k^- = -k^2 + 1 + ik + O(k^{-2})$ **parabolic part**
- $\lambda_0^{\pm} = 0,\ \lambda_2^{\pm} = -2 + 2i,\ \lambda_{-2}^{\pm} = -2 - 2i$ **dble eigenv.**



We expect at most a **null controllability in large time**

Main results (1)

Thm (P. Martin - LR - P. Rouchon)

Let $\omega \subset \mathbb{T}$ be any nonempty open set, and $T > 2\pi$.
Then any $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 15/2$, there exists a control $h \in L^2(\mathbb{T} \times (0, T))$ s.t. the solution y of

$$\begin{aligned} & y_{tt} - y_{xx} - y_{txx} = h(x,t)\mathbf{1}_\omega(x-t) \\ & (y, y_t)_{|t=0} = (y_0, \xi_0) \end{aligned}$$

satisfies $(y, y_t)_{|t=T} = (0, 0)$.

Main results (2)

Thm (P. Martin - LR - P. Rouchon)

Let $T > 2\pi$. Then for any $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 9/2$, there exists a control $h \in L^2(0, T)$ s.t. the solution y of

$$
\begin{aligned}
y_{tt} - y_{xx} - y_{txx} &= h(t)\delta_{x=t} \\
(y, y_t)_{|t=0} &= (y_0, \xi_0)
\end{aligned}
$$

satisfies $y(T) - [y(T)] = 0,\ y_t(T) = 0.$ $[y] = (2\pi)^{-1} \int_{\mathbb{T}} y(x)dx$

Sketch of the proof

- We reduce the problem to a moment problem (as in **Fattorini-Russell '71**).
- For the first result ($h = h(x,t)$), we first control to 0 the means of y and y_t in small time, and next use a control $h(x,t) = b(x)\tilde{h}(t)$, where the "controller" b takes the form

$$b(x) = \mathbf{1}_{(a,a+\sigma\pi)}(x) - \mathbf{1}_{(a+\sigma\pi,a+2\sigma\pi)}(x)$$

with σ a **quadratic irrational** number, so that

$$\hat{b}_0 = 0, \qquad |\hat{b}_k| > C/|k|^3 \text{ for } k \neq 0$$

Sketch of the proof (cont.)

- Two families $\{f_k\}$ and $\{g_k\}$ are said to be **biorthogonal** in $L^2(0,T)$ if we have

$$\int_0^T f_k(t)\overline{g_l(t)}dt = \delta_k^l \qquad \forall k, l$$

- We need to construct a biorthogonal family to the family of functions

$$\left(e^{\lambda_k^+ t}\right)_{k\in\mathbb{Z}} \cup \left(e^{\lambda_k^- t}\right)_{k\in\mathbb{Z}\setminus\{0,\pm 2\}} \cup \left\{te^{\lambda_2 t},\ te^{\lambda_{-2} t}\right\}$$

with $\lambda_k^{\pm} = (-(k^2 - 2ik) \pm \sqrt{k^4 - 4k^2})/2$ $\quad \lambda_k^+ = \lambda_k^-$ for $k = 0, \pm 2$
and to estimate carefully the L^2-norm of each function of the biorthogonal family

Step 1. Estimation of a canonical product

- We need to estimate the canonical product

$$P(z) = z\left(1-\frac{z}{i\lambda_2}\right)\left(1-\frac{z}{i\lambda_{-2}}\right)\prod_{k\in\mathbb{Z}\setminus\{0,\pm 2\}}\left(1-\frac{z}{i\lambda_k^+}\right)\prod_{k\in\mathbb{Z}\setminus\{0,\pm 2\}}\left(1-\frac{z}{i\lambda_k^-}\right)$$

- We show that P is an entire function of exponential type at most π, with

$$\begin{aligned} |P(x)| &\lesssim (1+|x|)^{-3}e^{\sqrt{2}\pi\sqrt{|x|}}, \quad x\in\mathbb{R} \\ |P'(i\lambda_k^+)| &\gtrsim |k|^{-3}e^{\sqrt{2}\pi\sqrt{|k|}}, \quad x\in\mathbb{Z}\setminus\{0,\pm 2\} \\ |P'(i\lambda_k^-)| &\gtrsim |k|^{-7}e^{\pi|k|^2}, \quad x\in\mathbb{Z}\setminus\{0,\pm 2\} \end{aligned}$$

- To do that, we use the theory of functions of **type sine (Levin)**

Functions of type sine

- An entire function $f(z)$ of exponential type π is of **type sine** if
  - its zeros are separated: $|\mu_k - \mu_l| > const$
  - $C^{-1}e^{\pi|y|} \leq |f(x+iy)| \leq Ce^{\pi|y|} \qquad |y| > H,\ x$
- From of a result of **Levin**:
  If $\mu_k = k + d_k$ with $d_0 = 0$, $d_k = d + O(k^{-1})$ $(d \in \mathbb{C})$ and the μ_k are pairwise $\neq$, then

  $$f(z) = z \prod_{k\in\mathbb{Z}^*} (1 - \frac{z}{\mu_k})$$

  is a function of type sine

Step 2: construction of a "good" multiplier

- If a canonical product P with roots $i\lambda_k$ is (say) bounded on the real axis, a biorthogonal family to the $e^{\lambda_k t}$'s is obtained by taking the inverse Fourier transform of the functions
$$\frac{P(z)}{P'(i\lambda_k)(z - i\lambda_k)}$$
- Here, we have to multiply P by an entire function $m(z)$ of exponential type to "balance" P on the real axis; namely, s.t.
$$|m(x)| \leq C(1 + |x|)e^{-\sqrt{2}\pi\sqrt{|x|}}$$
and with "almost" the same behavior on lines $\operatorname{Im} z = const$
- Following **Glass '10**, we use **Beurling-Malliavin** multiplier obtained by atomization of the measure $d\mu(t)$, where $\mu(t) = \mathbf{1}_{(B,\infty)}(at - b\sqrt{t})$), $B = (b/a)^2$, $a = T/(2\pi) - 1$, $b = \sqrt{2}$:
$$m(z) = \exp \int_0^\infty \log\left(1 - \frac{(z-i)^2}{t^2}\right) d[\mu(t)]$$

Step 3: Conclusion

- We construct the biorthogonal family by taking the inverse Fourier transform of some functions involving P and m. Invoque Paley-Wiener and Plancherel to get the required properties.
- The moment problem is solved explicitly, the control being expressed as a series of functions in the biorthogonal family.

II. The regularized long wave or BBM equation

$$u_t - u_{txx} + u_x + uu_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}$$

- Introduced by **Benjamin, Bona, Mahony** in 1972 as an alternative to Korteweg-de Vries (KdV) equation

  $$u_t + u_{xxx} + u_x + uu_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}$$

  for unidirectional propagation of water waves in channels
- **Nonlocal form**: $u_t = -A(u + u^2/2)$ where $A = (1 - \partial_x^2)^{-1}\partial_x$ (bounded in each H^s)
-
  - GWP in $H^1(\mathbb{R})$ (**Benjamin-Bona-Mahony '72**)
  - GWP in $L^2(\mathbb{R})$ and ill-posed in $H^s(\mathbb{R})$ for $s < 0$ (**Bona-Tzvetkov '09**)
  - GWP in $L^2(\mathbb{T})$ (**Roumégoux '10**)

BBM compared to KdV

Properties	KdV	BBM
Invariants	infinity	3
Integrability	Yes	No
Smoothing effect	in space	in time
GWP in $H^s(\mathbb{R})$	$s > -3/4$	$s \geq 0$
Numerics	Hard	Easy
Controllability of linearized eq. in $L^2(\mathbb{T})$	Exact	Approximate No spectral controllability

Control of BBM: some references

Linearized BBM:

$$u_t - u_{txx} + u_x = 0$$

- **S. Micu '01**: Cost of the control in the approximate controllability
- **X. Zhang, E. Zuazua '03**: weak stabilization
- **N. Adames, H. Leiva, J. Sanchez '08**: approximate controllability for $u_t - u_{txx} + au_{xx} = 0$
- **N. A. Larkin, M. P. Vishnevskii '08**: weak stabilization for $u_t - u_{txx} + uu_x = 0$
- Spectrum (for $x \in \mathbb{T}$): $\lambda_k = -ik/(k^2 + 1) \to 0$ as $k \to \infty$

Control problem in a moving frame

Pick $c = -1$ for simplicity, and set $v(x,t) = u(x+t,t)$. Then

$$u_t - u_{txx} + u_x + uu_x = b(x-t)h(x,t)$$
$$u(x,0) = u_0(x)$$

is transformed into the following **KdV-BBM** eq.

$$v_t - v_{txx} + v_{xxx} + vv_x = b(x)h(x+t,t)$$
$$v(x,0) = u_0(x).$$

The BBM term u_{txx} has generated the KdV term v_{xxx}!!
Spectrum: $\lambda_k = ik^3/(k^2+1)$; **spectral gap**!!

Moving control for BBM: exact controllability

Thm. (LR - B.-Y. Zhang)
Let $b \in C^\infty(\mathbb{T})$, $b \neq 0$, and $T > 2\pi$. Then there exists $\delta > 0$ such that for all $u_0, u_T \in H^1(\mathbb{T})$ with $||u_0||_{H^1} + ||u_T||_{H^1} < \delta$, there exists a control $h \in L^2(0, T; H^{-1}(\mathbb{T}))$ driving the sol. u of

$$u_t - u_{txx} + u_x + uu_x = b(x - t)h(x, t)$$

from u_0 at $t = 0$ to u_T at $t = T$.

Moving control for BBM: exp. stabilization

Thm. (LR - B.-Y. Zhang)
Let $b \in C^\infty(\mathbb{T})$, $b \neq 0$. Then there exist some positive numbers δ, C, λ such that for all $u_0 \in H^1(\mathbb{T})$ with $||u_0||_{H^1} < \delta$, the sol. u of

$$u_t - u_{txx} + u_x + uu_x = -b(x-t)(1-\partial_x^2)[b(x-t)u(x,t)]$$
$$u(x,0) = u_0(x)$$

satisfies

$$||u(t)||_{H^1} \leq Ce^{-\lambda t}||u_0||_{H^1}$$

Unique Continuation property

Hard! the linearized eq: $u_t - u_{xxt} + u_x = 0$ has for **principal symbol** $p(\xi, \tau) = \xi^2\tau$. Characteristic lines: $t = const$ **and** $x = const$

- **M. Davila, G. Perla-Menzala '98** Carleman estimate and UCP for BBM (but results not exact as stated)
- **S. Micu '01** UCP for $u_t - u_{txx} + u_x = 0$, assuming $u(0, t) = u(1, t) = 0 = \mathbf{u_x(1, t)}$
- **X. Zhang, E. Zuazua '03** UCP for $u_t - u_{txx} + p(x)u_x + q(x)u = 0$, assuming $u = 0$ on $\omega \times (0, T)$ and some hypotheses about p, q
- **M. Yamamoto '03** UCP for $u_t - u_{txx} + p(x, t)u_x + q(x, t)u = 0$, assuming $u_{|t=0} = 0$ and $u(1, t) = u_x(1, t) = 0$
- **Y. Mammeri '09**, UCP for KP-BBM-II (based on **Constantin**'s work on Camassa-Holm '05)

Bourgain approach

- If u solves $u_t - u_{txx} + u_x + f(u)_x = 0$ on $\mathbb{R} \times (0,T)$ and is supported in $(-L,L) \times (0,T)$, then $\hat{u} = \int_{\mathbb{R}} u e^{-i\xi x} dx$ is an entire function s.t.

$$\hat{u}_t = -i\xi(1+\xi^2)^{-1}(\hat{u} + \widehat{f(u)})$$

- The analysis at high frequencies ($\xi \to \infty$) works well for KdV, Schrödinger, not for BBM. Here, the problems occur at $\xi = \pm i$.
- However, we can use that method to prove the UCP for
  - $u_t - u_{txx} + uu_x = 0$ ("equal width eq.")
  - $u_t - u_{txx} + u_x + (u * u)_x = 0$

UCP for BBM

Consider

$$
\begin{aligned}
&u_t - u_{xxt} + u_x + uu_x = 0, \quad x \in \mathbb{T} \\
&u(x,0) = u_0(x)
\end{aligned}
$$

Thm. (LR-B.Y. Zhang)
Assume that $u_0 \in H^1(\mathbb{T})$ is s.t.

$$\int_{\mathbb{T}} u_0(x)dx \geq 0, \qquad ||u_0||_{L^\infty(\mathbb{T})} < 3.$$

If $u \equiv 0$ on $\omega \times (0,T)$, then $u_0 = 0$

Rmq. UCP false for any $u_0 \in L^\infty(\mathbb{T})$: Take
$u(x,t) = u_0(x) = \begin{cases} 0 & \text{if } x \in \omega \\ -2 & \text{otherwise} \end{cases}$

Conclusion and future directions of research

- The null controllability in large time of the wave eq. with structural damping has been derived in dim. 1.
  1. Is it true in less regular spaces?
  2. What's about the dimension 2?
- The local exact controllability and exponential stabilization of BBM with moving control have been derived.
  1. Can we obtain **global results**?
  2. **Global** UCP for KdV-BBM?
- Some UCP has been proved for BBM.
  1. Can we drop the two assumptions $\int_{\mathbb{T}} u_0 \geq 0$, $||u_0||_{L^\infty} < 3$?
  2. What sort of stability do we have when applying a (fixed) internal damping?