COLLOQUIUM MATHEMATICUM

VOL. XXI 1970 FASC. 2

NUMBER OF ALGEBRAIC OPERATIONS IN IDEMPOTENT GROUPOIDS

BY

J. DUDEK (WROCŁAW)

Let $\mathfrak{A}$ be an algebra. We say that an n-ary operation is *essentially n-ary* if it depends on every of its variables. We denote by $\omega_n = \omega_n(\mathfrak{A})$ the number of all essentially n-ary algebraic operations of $\mathfrak{A}$ (cf. Marczewski [2]). The aim of this paper is to find the minimal numbers ω_n for idempotent groupoids. Let us recall that for 2-dimensional proper diagonal algebras (cf. Płonka [4]) we have $\omega_2 = 2$ and $0 = \omega_3 = \omega_4 = \ldots$, and that for semilattices there is $\omega_n = 1$ for $n = 2, 3, \ldots$

The following shows that these groupoids are exceptional:

THEOREM. *For every idempotent groupoid G (with the exception of semilattices and the 2-dimensional proper diagonal algebras) we have $\omega_n(G) \geqslant n$ for $n \geqslant 3$.*

If $G = (G, \cdot)$ is an idempotent groupoid (and the operation $\cdot$ is essentially binary) we define the sequences of *simple* iterations of the operations $x\cdot y$ and $y\cdot x$ (cf. Marczewski [3]):

$$f_2(x_1, x_2) = x_1\cdot x_2, \quad f_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = f_2\big(f_n(x_1, x_2, \ldots, x_n), x_{n+1}\big),$$
$$g_2(x_1, x_2) = x_2\cdot x_1, \quad g_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = g_2\big(g_n(x_1, x_2, \ldots, x_n), x_{n+1}\big).$$

In the sequel f_n and g_n always denote these iterations.

LEMMA 1. *In every idempotent groupoid, if f_n is essentially n-ary, then f_{n+1} depends on the variables $x_3, x_4, \ldots, x_n$ and on one at least of the variables x_1, x_2 (an analogous statement is valid for g_{n+1}).*

Proof. The operation $x\cdot y$ is idempotent, so

$$f_{n+1}(x_2, x_2, x_3, \ldots, x_n, x_{n+1}) = f_n(x_2, x_3, \ldots, x_n, x_{n+1}).$$

This and the fact that f_n is essentially n-ary gives Lemma 1.

LEMMA 2. *If the operations f_3 and g_3 of an idempotent groupoid G are not essentially ternary, then G is a 2-dimensional proper diagonal algebra.*

Proof. Let us first remark that if f_3 (or g_3) is not essentially ternary, then the operation $x\cdot y$ is not commutative. Indeed, if it were commutative, Lemma 1 and the equality $f(x_1, x_2, x_3) = f(x_2, x_1, x_3)$ would result in contradiction with the assumption. Hence the operation $\cdot$ is not commutative and, according to Lemma 1, we have either

$$f_3(x_1, x_2, x_3) = x_2 \cdot x_3 \tag{1}$$

or

$$f_3(x_1, x_2, x_3) = x_1 \cdot x_3. \tag{2}$$

By applying (1) and Lemma 1 to the operation g_3 we get

$$x_1 \cdot x_2 = f_3(x_1, x_2, x_1 \cdot x_2) = g_3(x_2, x_1, x_2);$$

a contradiction, because there is either $x_1 \cdot x_2 = x_2 \cdot x_1$ or $x_1 \cdot x_2 = x_2$.

Analogous proof shows that equality (2) contradicts the equality $g_3(x_1, x_2, x_3) = x_3 \cdot x_2$, and so we come to the conclusion that G satisfies the equalities $x \cdot x = x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot z$ of a 2-dimensional proper diagonal algebra (cf. Płonka [3]); thus Lemma 2 is proved.

By (φ) and (ψ) we denote propositions saying that all of the operations f_n (or g_n respectively) are essentially n-ary.

LEMMA 3. *Each idempotent non-diagonal groupoid G satisfies either (φ) or (ψ).*

Proof. Suppose that non-(φ) and non-(ψ). Then there exist the least integers n and m such that the operations f_n and g_m do not depend on all of their variables. According to Lemma 1 the following equalities are fulfilled in the groupoid G:

$$f_n(x_1, x_2, \ldots, x_n) = f_{n-1}(x_1, x_3, x_4, \ldots, x_{n-1}, x_n) \tag{3}$$

or

$$f_n(x_1, x_2, \ldots, x_n) = f_{n-1}(x_2, x_3, x_4, \ldots, x_{n-1}, x_n), \tag{4}$$

and

$$g_m(x_1, x_2, \ldots, x_m) = g_{m-1}(x_2, x_3, x_4, \ldots, x_{m-1}, x_m) \tag{5}$$

or

$$g_m(x_1, x_2, \ldots, x_m) = g_{m-1}(x_1, x_3, x_4, \ldots, x_{m-1}, x_m). \tag{6}$$

Without loss of generality (because of Lemma 2 and of the assumption of this lemma) we can take $n \geqslant 4$.

Putting $x_n = f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ in (3) and (4) we get

$$\begin{aligned} &f_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ &\quad = g_3\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big), \end{aligned} \tag{7}$$

$$\begin{aligned} &f_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ &\quad = g_3\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big). \end{aligned} \tag{8}$$

If we now multiply (7) on the left $m-3$ times by $f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ and after every multiplication use (3), we get

$$(9) \qquad f_{n-1}(x_1, x_2, \ldots, x_{n-1}) = g_m\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big).$$

If (6) is satisfied, then the right-hand side of the last equality is independent of x_2, and so (3) and (6) contradict each other; and if (5) and (9) are satisfied, then

$$\begin{aligned} f_{n-1}(x_1, x_2, \ldots, x_{n-1}) &= g_m\big(x_{n-1}, f_{n-2}(x_1, \ldots, x_{n-2}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \\ &\qquad f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big) \\ &= g_m\big(x_{n-2}, f_{n-3}(x_1, x_2, \ldots, x_{n-3}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \\ &\qquad f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big) \\ &= \ldots = g_m\big(x_2, x_1, f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \\ &\qquad f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big) \\ &= g_{m-1}\big(x_1, f_{n-2}(x_1, x_3, \ldots, x_{n-1}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \\ &\qquad \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big). \end{aligned}$$

This proves that $f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ does not depend on x_2, and so (3) and (5) cannot be satisfied simultaneously. If we assume (4), we get from (8) in an analogous way the formula

$$(10) \qquad f_{n-1}(x_1, x_2, \ldots, x_{n-1}) = g_m\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, \ldots, x_{n-1})\big).$$

If (4) and (6) are satisfied, then we get a contradiction in view of (10), because $f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ depends on x_1 and the right-hand side of (10) does not. If (5) and (10) are satisfied, then the right-hand side of (10) takes the form

$$\begin{aligned} &g_{m-1}\big(f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \\ &\qquad \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big) \\ &= g_m\big(x_{n-2}, f_{n-3}(x_1, x_2, \ldots, x_{n-3}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \\ &\qquad f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big) \\ &= \ldots = g_{m-1}\big(x_1 \cdot x_2, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \\ &\qquad \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big) \end{aligned}$$

$$= g_m\big(x_2, x_1, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$
$$= g_{m-1}\big(x_1, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$
$$= f_{n-1}(x_1, x_2, \ldots, x_{n-1}) = g_m\big(x_1, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$
$$= f_{n-2}(x_2, x_3, \ldots, x_{n-1}).$$

So (5) and (4) cannot be satisfied simultaneously, because otherwise f_{n-1} would not depend on its first variable. Thus Lemma 3 has been proved.

LEMMA 4. *If an idempotent non-diagonal groupoid G satisfies non-(φ) (or non-(ψ), respectively), then $\omega_m(G) \geqslant m$ for $m = 2, 3, \ldots$*

Proof. Suppose that non-(φ) (the proof is analogous if we suppose that non-(ψ)). Then either (3) or (4) is satisfied and we conclude from Lemma 3 that $g_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1})$, where $m = 3, 4, \ldots$ and $k = 2, 3, \ldots, m$, are essentially m-ary. Suppose that for different integers k and l we have

$$g_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1}) = g_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}). \tag{11}$$

Using (3) and (11) we get

$$f_{n-1}\big(g_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_n\big(x_{k-1}, g_{m-1}(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-2}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_{n-1}(x_{k-1}, y_2, y_3, \ldots, y_{n-1})$$
$$= f_{n-1}\big(g_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_n\big(x_{l-1}, g_{m-1}(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-2}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_{n-1}(x_{l-1}, y_2, y_3, \ldots, y_{n-1}).$$

It means that f_{n-1} does not depend on its first variable. Using analogously (11) and (4) we come to a contradiction with the assumption on the operation f_{n-1}. So we have $\omega_m \geqslant m$ for $m = 2, 3, \ldots$

LEMMA 5. *If $f(x_1, x_2, \ldots, x_n)$ is an arbitrary essentially n-ary algebraic operation of the idempotent groupoid G which satisfies (φ) or (ψ), and if $f(x \cdot y, x_2, x_3, \ldots, x_n)$ does not depend on all of its variables, then $\omega_m(G) \geqslant m$ for $m = 2, 3, \ldots$*

Proof. In virtue of the assumption on the operations f and $f(x \cdot y, x_2, x_3, \ldots, x_n)$ one of the following equalities holds in G:

$$f(x \cdot y, x_2, x_3, \ldots, x_n) = f(y, x_2, x_3, \ldots, x_n), \tag{12}$$

$$f(x \cdot y, x_2, x_3, \ldots, x_n) = f(x, x_2, x_3, \ldots, x_n). \tag{13}$$

Suppose that (φ) is satisfied (the proof is analogous if we suppose that (ψ) is satisfied), and suppose that for different k and l and for some m we have

$$(14) \qquad f_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1}) = f_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}).$$

Then

$$\begin{aligned} f\big(f_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1}), y_2, y_3, \ldots, y_n\big) \\ = f\big(f_{m-1}(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-2})\cdot x_{k-1}, y_2, y_3, \ldots, y_n\big) \\ = f\big(f_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}), y_2, y_3, \ldots, y_n\big) \\ = f\big(f_{m-1}(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-2})\cdot x_{l-1}, y_2, y_3, \ldots, y_n\big). \end{aligned}$$

If (12) or (13) holds, we conclude from the last equality that f does not depend on its first variable, what obviously contradicts the assumption. Hence (14) does not hold for $k \neq l$, and so $\omega_m(G) \geqslant m$.

LEMMA 6. *If f is an arbitrary essentially n-ary algebraic operation of the idempotent groupoid G which satisfies (φ) and for a k_0 we have $\omega_{k_0}(G) < k_0$, then the operation*

$$f\big(f_m(x_1, x_2, \ldots, x_m), y_2, y_3, \ldots, y_n\big)$$

depends on all of its variables; if G satisfies (ψ), then the operation

$$f\big(g_m(x_1, x_2, \ldots, x_m), y_2, y_3, \ldots, y_n\big)$$

depends on all of its variables.

We give a proof by induction on m in the case in which G satisfies (φ). If $m = 2$, the thesis follows from Lemma 5. Suppose that the thesis of the lemma holds for a fixed $m \geqslant 2$. Then, according to the inductive assumption and Lemma 5, the operation

$$f\big(g_{m+1}(x_1, x_2, \ldots, x_{m+1}), y_2, y_3, \ldots, y_n\big) = f\big(g_m(x_2\cdot x_1, x_3, \ldots, x_{m+1}), y_2, y_3, \ldots, y_n\big)$$

depends on all of its variables. Thus the lemma is proved.

Let us now introduce a definition: the operation $f(x_1, x_2, \ldots, x_n)$ *admits the permutation* $\sigma \in S_n$ if

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

The permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n-k+1 & n-k+2 & \ldots & n \\ k & k+1 & \ldots & n & 1 & & k-1 \end{pmatrix}$$

will be denoted by σ_n^k $(k = 2, 3, \ldots, n)$.

LEMMA 7. *If the idempotent groupoid is non-abelian, the operation f_p (and g_p, analogously) does not admit any permutation σ_p^k for any prime p.*

Proof. Let $p \geqslant 5$ and suppose that for a k_0 the operation f_p admits $\sigma_p^{k_0}$. Then f_p admits every permutation σ_p^k, because p is prime.

First we prove that the binary operation

$$f_{p+1}(\underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}, y, y, \ldots, y)$$

is commutative. Indeed,

$$\begin{aligned} f_{p+1}(x, x, \ldots, x, y, y, \ldots, y) &= f_p(x, x, \ldots, x, y, y, \ldots, y)\cdot y \\ &= f_{p-1}(y, y, \ldots, y, x, x, \ldots, x)\cdot y = f_p(y, y, \ldots, y, x, x, \ldots, x, y) \\ &= f_p(y, y, \ldots, y, x, x, \ldots, x) = f_{p+1}(y, y, \ldots, y, \underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}), \end{aligned}$$

whence

$$\begin{aligned} x\cdot y &= f_p(x, x, \ldots, x, y) = f_p(y, x, x, \ldots, x) \\ &= f_{(p-1)/2}\big(f_{p+1}(\underbrace{y, y, \ldots, y}_{(p+1)/2 \text{ times}}, x, x, \ldots, x), \underbrace{x, x, \ldots, x}_{(p-3)/2 \text{ times}}\big) \\ &= f_{(p-1)/2}\big(f_{p+1}(\underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}, y, y, \ldots, y), x, x, \ldots, x\big) \\ &= f_{(p-1)/2}\big(f_{(p+3)/2}(x, x, \underbrace{y, y, \ldots, y}_{(p+1)/2 \text{ times}}), x, x, \ldots, x\big) \\ &= f_p(x, y, y, \ldots, y, x, x, \ldots, x) = f_p(x, x, \ldots, x, y, y, \ldots, y) \\ &= f_{p+1}(\underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}, y, y, \ldots, y). \end{aligned}$$

It means that the operation $x\cdot y$ is commutative, which contradicts the assumption.

For $p = 2$ the lemma is obvious, and if $p = 3$ and f_3 admits the permutation σ_3^k, then the following equalities hold:

$$f_3(x_1, x_2, x_3) = f_3(x_2, x_3, x_1) = f_3(x_3, x_1, x_2).$$

Putting $x_3 = x_1\cdot x_2$ in the last equality, we have

$$\begin{aligned} x_1\cdot x_2 &= f_3(x_1\cdot x_2, x_1, x_2) = f_3(x_1, x_2, x_1)\cdot x_2 = f_3(x_1, x_1, x_2)\cdot x_2 \\ &= f_3(x_1, x_2, x_3) = f_3(x_2, x_2, x_1) = x_2\cdot x_1, \end{aligned}$$

a contradiction.

LEMMA 8. *If*

$$f = f(x_1, x_2, \ldots, x_n) \quad \textit{and} \quad g = g(y_1, y_2, \ldots, y_m)$$

are idempotent operations, if f does not admit any permutation σ_n^i and g does not admit any permutation σ_m^j, then the operation

$$\hat{f}(g)(x_1, x_2, \ldots, x_{n\cdot m})$$
$$= f\big(g(x_1, x_2, \ldots, x_m), g(x_{m+1}, x_{m+2}, \ldots, x_{2m}), \ldots, g(x_{(n-1)m+1}, \ldots, x_{n\cdot m})\big)$$

does not admit any permutation $\sigma_{n\cdot m}^k$.

Proof. Suppose that $\hat{f}(g)$ admits the permutation $\sigma_{n\cdot m}^k$ and let $k = (s-1)m+1$, where $s = 2, 3, \ldots, n$. Then

$$f(x_1, x_2, \ldots, x_n)$$
$$= \hat{f}(g)(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_n, x_n, \ldots, x_n)$$
$$= \hat{f}(g)(x_s, x_s, \ldots, x_s, x_{s+1}, x_{s+1}, \ldots, x_{s+1}, \ldots, x_n, x_n,$$
$$\ldots, x_n, x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_{s-1}, x_{s-1}, \ldots, x_{s-1})$$
$$= f(x_s, x_{s+1}, \ldots, x_n, x_1, x_2, \ldots, x_{s-1}),$$

which contradicts the assumption of the lemma. And if $k \neq (s-1)m+1$ for $s = 2, 3, \ldots, n$, then there exists an r such that $2 \leqslant r \leqslant m$ and

$$g(x_1, x_2, \ldots, x_m)$$
$$= \hat{f}(g)(x_1, x_2, \ldots, x_m, x_1, x_2, \ldots, x_m, \ldots, x_1, x_2, \ldots, x_m)$$
$$= \hat{f}(g)(x_r, x_{r+1}, \ldots, x_m, x_1, x_2, \ldots, x_{r-1}, x_r, x_{r+1},$$
$$\ldots, x_m, x_1, x_2, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots, x_m, x_1, x_2, \ldots, x_{r-1})$$
$$= g(x_r, x_{r+1}, \ldots, x_m, x_1, x_2, \ldots, x_{r-1})$$

which contradicts the assumption on the operation g. Thus Lemma 8 is proved.

Now for each prime p we define inductively some sequences of algebraic operations of the groupoid G; namely,

$$F_1^{(p)}(x_1, x_2, \ldots, x_p) = f_p(x_1, x_2, \ldots, x_p),$$
$$F_{n+1}^{(p)}(x_1, x_2, \ldots, x_{p^{n+1}}) = \hat{F}_n^{(p)}(f_p)(x_1, x_2, \ldots, x_{p^{n+1}}).$$

Analogously, we define the sequence $G_n^{(p)}$ starting with g_p.

LEMMA 9. *If an idempotent groupoid G satisfies (φ) and $\omega_{k_0}(G) < k_0$ for a k_0, then for every p and n the operations $F_n^{(p)}$ depend on all their variables (analogous proposition holds for $G_n^{(p)}$ if (ψ)).*

Proof is by induction on n and uses Lemma 6.

LEMMA 10. *For every idempotent non-abelian groupoid the operations $F_n^{(p)}$ and $G_n^{(p)}$ do not admit any permutation σ_{pn}^k.*

Proof immediately follows from Lemmas 7 and 8.

Now we can prove the theorem in the case of non-abelian (idempotent and non-diagonal) groupoid. Suppose that we have $\omega_{k_0}(G) < k_0$ for a $k_0 = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ ($p_1, p_2, \ldots, p_k$ are primes). According to Lemmas 3 and 4 we can assume that the groupoid G satisfies (φ) and (ψ), and from Lemmas 9 and 10 we conclude that $F_{n_j}^{(p_j)}$ $(j = 1, 2, \ldots, k)$ depend on all their variables and do not admit any permutation of the type σ_n^i. Then, in view of Lemmas 6 and 8, the operations

$$H_1 = F_{n_1}^{(p_1)}, \quad H_2 = \hat{H}_1(F_{n_2}^{(p_2)}), \ldots, H_k = \hat{H}_{k-1}(F_{n_k}^{(p_k)})$$

depend on all their variables and also do not admit any permutation of the type σ_n^i; however, the operation H_k is essentially k_0-ary and does not admit any permutation $\sigma_{k_0}^i$, whence $\omega_{k_0}(G) \geqslant k_0$, which ends the proof of the theorem in this case.

Now we take the case of an abelian groupoid.

LEMMA 11. *If G is an idempotent abelian groupoid, then, for every n, $f_n = g_n$ and f_n is essentially n-ary.*

Proof. For $n = 2$ the lemma is obvious. Idempotency and commutativity of the operation $\cdot$ and Lemma 1 imply that if f_n is essentially n-ary, then f_{n+1} is essentially $(n+1)$-ary.

LEMMA 12. *If G is an idempotent abelian non-associative groupoid, then for $n \geqslant 3$ the operation f_n does not admit any permutation $\varrho_k = (k, k+1)$ of the set $\{1, 2, \ldots, n\}$, where $2 \leqslant k \leqslant n-1$.*

Proof. The operation $x \cdot y$ is not associative and hence we have the thesis in case $n = 3$. Suppose that f_n does not admit any permutation ϱ_k $(2 \leqslant k \leqslant n-1)$ and let

$$f_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = f_n(x_1, x_2, \ldots, x_n) \cdot x_{n+1}$$

admit the permutation ϱ_k for $3 \leqslant k \leqslant n$. Then, putting $x_1 = x_2$ in the last equality, we get the contradiction with the inductive assumption. And if

$$f_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = f_{n+1}(x_1, x_3, x_2, x_4, \ldots, x_n, x_{n+1}),$$

then for $x_{n+1} = f_n(x_1, x_2, \ldots, x_{n-1}, x_n)$ we have

$$\begin{aligned} f_n(x_1, x_2, \ldots, x_n) &= f_n(x_1, x_3, x_2, x_4, \ldots, x_n) \cdot f_n(x_1, x_2, x_3, x_4, \ldots, x_n) \\ &= f_n(x_1, x_2, x_3, x_4, \ldots, x_n) \cdot f_n(x_1, x_3, x_2, x_4, \ldots, x_n) \\ &= f_n(x_1, x_3, x_2, x_4, \ldots, x_n) \cdot f_n(x_1, x_3, x_2, x_4, \ldots, x_n) \\ &= f_n(x_1, x_3, x_2, x_4, \ldots, x_n). \end{aligned}$$

This proves that f_n admits the permutation ϱ_2, which contradicts the inductive assumption.

LEMMA 13. *For every idempotent abelian non-associative groupoid G the operations f_n do not admit any of the permutations σ_n^k.*

Proof. In fact, if for $2 \leqslant k \leqslant n-1$ the operation f_n admitted the permutation σ_n^k, then, the operation $x \cdot y$ being commutative, the operation f_n would admit the permutation ϱ_k, which contradicts Lemma 12. And if

$$f_n(x_1, x_2, \ldots, x_n) = f_n(x_n, x_1, x_2, \ldots, x_{n-1}),$$

f_n would admit ϱ_2, which also contradicts Lemma 12.

If the abelian idempotent groupoid is not a semilattice, the theorem follows from Lemmas 11 and 13.

Let us remark that the equality in the thesis of the theorem cannot be improved as there exist groupoids for which $\omega_n = n$ (Płonka [5]).

I would like to thank Professor E. Marczewski and Dr. S. Fajtlowicz for their valuable remarks which helped me in the preparation of this paper. The results of this paper were announced in [1].

REFERENCES

[1] J. Dudek, *The number of algebraic operations in idempotent groupoids*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys., 27 (1969), p. 477-479.

[2] E. Marczewski, *Independence in abstract algebras, results and problems*, Colloquium Mathematicum 14 (1966), p. 169-188.

[3] — *Remarks on symmetrical and quasi-symmetrical operations*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys., 12 (1964), p. 723-727.

[4] J. Płonka, *Diagonal algebras*, Fundamenta Mathematicae 58 (1966), p. 309-321.

[5] — *Algebras in which $\omega_n = n$* (to appear).

Reçu par la Rédaction le 23. 4. 1969