

Number Operators for Representations of the Canonical Commutation Relations

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Abstract. A *number operator* for a representation of the canonical commutation relations is defined as a self-adjoint operator satisfying an exponentiated form of the equation $Na^* = a^*(N + I)$, where a^* is an arbitrary creation operator. When N exists it may be chosen to have spectrum $\{0, 1, 2, \dots\}$ (in a direct sum of Fock representations) or $\{0, \pm 1, \pm 2, \dots\}$ (otherwise). Examples are given of representations having number operators, and a necessary and sufficient condition is given for a direct-product representation to have a number operator.

Introduction

The Fock representation of the canonical commutation relations has a total occupation number operator N . One way of completely describing N is to say

(i) it is self-adjoint

(ii) its spectrum is $\{0, 1, 2, \dots\}$

and

(iii) it satisfies the commutation relation,

$$Na^*(\varphi) = a^*(\varphi)(N + I) \quad (0.1)$$

in a suitably rigorous form. Here $a^*(\varphi)$ is the creation operator for a wavefunction φ , and (0.1) is to hold for all φ .

In fact, the only representations of the canonical commutation relations which have a number operator N satisfying (i)–(iii) are direct sums of Fock representations [2, 4, 5].

If we relax the requirements on N by eliminating the assumption (ii) about the spectrum, then there exist other representations of the canonical commutation relations possessing such number operators. We call them *particle representations*.

In Section 1 we discuss general properties of particle representations. For a *strange* particle representation (other than a direct sum of Fock

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representations) the number operator is always unbounded below (Theorem 1.3). Furthermore, given a strange particle representation, one can always select a number operator which has every integer (negative as well as positive) as an eigenvalue (Theorem 1.3).

In Sections 3 and 4 we consider direct-product representations of the canonical commutation relations as described by KLAUDER, McKENNA, and WOODS [11] and by STREIT [20]. We determine precisely which ones are particle representations; they are the ones having in the representation space a vector $\varphi_1 \otimes \varphi_2 \otimes \cdots$ where each φ_j is a multiple of a Hermite function (Theorem 3.3). We discuss the problem of extending these representations so that they are defined over a Hilbert space, showing that this can be done in a smooth way if and only if the indices on the Hermite functions are bounded (Theorem 4.7). We understand that M. REED [14] has considered similar and related questions about direct-product representations, but his work was not yet available at the time of this writing.

In Section 5 we discuss a class of particle representations which includes the extreme universally invariant representations as described by SHALE and SEGAL [18] and the representations corresponding to a non-relativistic infinite free Bose gas as described by ARAKI and WOODS [1]. They have generating functionals of the form

$$\mu(z) = \exp[-1/4 \|Tz\|^2],$$

where $T \geq I$. The corresponding representation is a direct sum of Fock representations if and only if $T^2 - I$ is trace class (Theorem 5.1).

1. Number Operators

We consider representations of the canonical commutation relations over a space \mathfrak{H} of test functions. \mathfrak{H} is assumed to be a complex inner product space, with the imaginary part of the inner product serving as the commutator bracket. This means that the commutation relations, in the Weyl form, are

$$W(z) W(z') = \exp\left[\frac{1}{2}i \operatorname{Im}(z, z')\right] W(z + z'), \quad (1.1)$$

where z and z' are arbitrary elements of \mathfrak{H} , and (z, z') is their inner product (linear on the left).

By a *representation of the Weyl relations* (or a *Weyl system*) over \mathfrak{H} we mean a map W from \mathfrak{H} into the unitary operators on some complex Hilbert space \mathfrak{R} such that the *Weyl relation* (1.1) is satisfied, and, in addition, for each fixed $z \in \mathfrak{H}$, the function $W(tz)$ of the real variable t is weakly continuous at 0. For a description of the motivation for this definition and its connection with other formulations of the commutation relations, see [2] or [19].

If W is a representation of the Weyl relations over \mathfrak{H} and $z \in \mathfrak{H}$, one can define the associated creation operator $\alpha^*(z)$ to be the closure of $2^{-1/2} [R(z) - iR(iz)]$, where $R(z)$ is the self-adjoint generator of the group $t \rightarrow W(tz)$. In case W_F is the Fock-Cook representation [9, 3] and N is the total occupation number operator, a suitable exponentiated form of the commutation relation

$$N\alpha_F^*(z) = \alpha_F^*(z)(N + I) \quad (1.2)$$

is satisfied. To be precise, for each $z \in \mathfrak{H}$ the relation

$$e^{itN} W_F(z) e^{-itN} = W_F(e^{it}z) \quad (1.3)$$

is satisfied for all $t \in \mathbb{R}$ [3, 2].

As has been suggested by SEGAL [16, 19 p. 64], if one has a Weyl system W and an operator N satisfying the indicated commutation relations, then that N should have a physical interpretation as an occupation number operator. Accordingly, we take the commutation relation as a definition of a number operator, and then we investigate the properties of such operators.

1.1 Definition. Let W be a Weyl system over \mathfrak{H} on \mathfrak{R} . A self-adjoint operator N on \mathfrak{R} is a *number operator for W* if

$$e^{itN} W(z) e^{-itN} = W(e^{it}z), \quad (1.4)$$

for all $z \in \mathfrak{H}$, $t \in \mathbb{R}$.

This definition differs in two respects from the definition

$$N = \sum_{k=1}^{\infty} \alpha^*(e_k) \alpha(e_k),$$

where $\{e_k\}$ is an orthonormal basis of \mathfrak{H} . First, the infinite sum can converge in certain strange representations where Eq. (1.4) fails to hold [2]. These representations agree with the Fock-Cook representation on a dense subspace of \mathfrak{H} . Second, we shall see that number operators (in the sense of Definition 1.1) exist in many physically interesting representations where the infinite sum fails to converge. In fact the sum $\sum \alpha^*(e_k) \alpha(e_k)$ exists only in representations which are direct sums of those indicated above (i.e. those which agree with the Fock-Cook representation on a fixed dense subspace) [2, 4]. The distinction between the two definitions arises from the fact that when the sum exists, it is a non-negative operator, whereas number operators are not generally bounded below.

One difficulty with Definition 1.1 is that a number operator need not have integer eigenvalues. To see this, suppose W_F is the Fock-Cook representation of the Weyl relations on \mathfrak{R}_F , and N is the usual number operator. Let \mathfrak{R} be an infinite dimensional Hilbert space and let A be a self-adjoint operator on \mathfrak{R} having continuous spectrum. Then the self-

adjoint generator of the group $t \rightarrow e^{itA} \otimes e^{itN}$ is a number operator for the Weyl system $I \otimes W_F$ acting on $\mathcal{R} \otimes \mathcal{R}_F$. (I is the identity operator.) This number operator for $I \otimes W_F$ is easily seen to have no eigenvectors.

In this example one sees a feature which occurs in general; namely the representation $I \otimes W_F$ also has another number operator $I \otimes N$ which does have integer eigenvalues.

1.2 Lemma. *If a Weyl system W has a number operator N , then it has another number operator N' whose spectrum is a subset of the integers.*

Proof. Since, for every $z \in \mathfrak{H}$, $e^{2\pi i N} W(z) e^{-2\pi i N} = W(e^{2\pi i} z) = W(z)$, the unitary $U = e^{2\pi i N}$ commutes with all the $W(z)$'s. Now U has a spectral resolution $U = \int_0^1 e^{2\pi i \theta} dF(\theta)$ where the spectral projections $F(\theta)$ commute with every bounded operator which commutes with U (see [22], p. 307). Thus each $F(\theta)$ commutes with all the $W(z)$'s and also with e^{itN} , $t \in \mathbb{R}$. Let $A = -\int_0^1 \theta dF(\theta)$, and

$$V(t) = e^{itN} e^{itA}.$$

Then V is a continuous one-parameter unitary group, and

$$\begin{aligned} V(t) W(z) V(-t) &= e^{itN} [e^{itA} W(z) e^{-itA}] e^{-itN} \\ &= e^{itN} W(z) e^{-itN}, \end{aligned}$$

so the self adjoint generator N' of V is a number operator for W . The spectrum of N' is a subset of the integers since

$$e^{2\pi i N'} = e^{2\pi i N} e^{2\pi i A} = e^{2\pi i N} U^{-1} = I. \quad \blacksquare$$

Actually the spectrum of N' looks like $\{n_0, n_0 + 1, n_0 + 2, \dots\}$ or $\{\dots - 2, -1, 0, 1, 2, \dots\}$. It appears easy enough to prove this: Take an eigenvector φ of N' , with eigenvalue n . Then $a^*(z)\varphi$ should be, according to (1.2), an eigenvector of N' with eigenvalue $n + 1$. However it is important to realize that (1.2) is symbolic, not rigorous, so to make this argument correct we would have to check that φ is in the domain of $a^*(z)$ and that $a^*(z)\varphi$ is in the domain of N . Instead of this, we shall prove the desired result, and more, using only bounded operators.

1.3 Theorem. *Suppose W is a Weyl system with a number operator N . If the spectrum of N is bounded below, then W is a direct sum of Fock-Cook representations. Otherwise W has a number operator N' whose spectrum is the set of all integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$.*

Proof. Define A and N' as in the proof of Lemma 1.2. Since $N = N' + A$ in the sense of strong sum of commuting operators, and the spectrum of A is a subset of $[-1, 0]$, the spectrum of N' is bounded below if and only if that of N is. If the spectrum of N' is bounded below by the integer n , then $N' + nI$ is a number operator for W whose spectrum

consists of non-negative integers. By Theorem 1 of [2], p. 64, W is a direct sum of Fock-Cook representations.

The possibility that N' is not bounded below is handled by the next lemma.

1.4 Lemma. *If N' is a number operator whose spectrum is unbounded below and consists of integers, then its spectrum is $\{0, \pm 1, \pm 2, \dots\}$.*

Proof. Fix a unit vector $z_0 \in \mathfrak{H}$. Then by the Stone-von Neumann Theorem [12] (or see Ref. [2], p. 27), the representation of the Weyl relations over \mathbb{C} given by $\alpha \rightarrow W(\alpha z_0)$ is unitarily equivalent to a direct sum of copies of the Schrödinger representation W_s . So we may assume that $\mathfrak{R} = \mathfrak{R}_1 \otimes \mathfrak{R}_s$, where \mathfrak{R}_s is the representation space $L^2(\mathbb{R})$ for the Schrödinger representation, and

$$W(\alpha z_0) = I \otimes W_s(\alpha).$$

Let N_s be the usual number operator $\frac{1}{2}(P^2 + Q^2 - 1)$ for the Schrödinger representation. Then, writing $U(t) = (I \otimes e^{itN_s}) e^{-itN'}$, we have $U(t) W(\alpha z_0) U(-t) = W(\alpha z_0)$ for all $\alpha \in \mathbb{C}$, so $U(t)$ commutes with all the operators $I \otimes W_s(\alpha)$, $\alpha \in \mathbb{C}$. Thus $U(t)$ must lie in the commutator of the algebra $\{I \otimes W_s(\alpha) : \alpha \in \mathbb{C}\}''$. Since the Schrödinger representation is irreducible this commutator consists of all operators of the form $A_1 \otimes I$, so we have $U(t) = U_1(t) \otimes I$ or $e^{itN'} = U_1(t) \otimes e^{itN_s}$. Thus $U_1(t)$ is a continuous one-parameter unitary group; call its self-adjoint generator A . The spectrum of A is a subset of the integers since $e^{2\pi i A} \otimes I = I$. Furthermore A cannot be bounded below, because N_s is non-negative, and we are assuming N' is not bounded below.

Now we can prove that any integer m is in the spectrum of N' . Select an integer $m_0 \leq m$ belonging to the spectrum of A . Since the spectrum of N_s is $\{0, 1, 2, \dots\}$, $(m - m_0)$ is in the spectrum of N_s . But the spectrum of N' is the sum of that of A and that of N_s , so $m = m_0 + (m - m_0)$ is in the spectrum of N' . ■

The existence of a number operator imparts a particle interpretation to the vectors in the representation space. To see this consider a Weyl system W acting on \mathfrak{R} , and suppose no subrepresentation of W is unitarily equivalent to the Fock-Cook representation. Then if W has a number operator N , we may suppose, according to Theorem 1.3, that N has spectrum $\{0, \pm 1, \pm 2, \dots\}$. We may think of any eigenvector v of N with eigenvalue 0 as a "ground" state, even though it has an infinite number of "bare" particles with probability one [2]. Then an eigenvector of N with eigenvalue $n > 0$ has n more particles than v has, and an eigenvector with eigenvalue $-n < 0$ has n fewer particles than v . In

fact, using the spectral representation $N = \sum_{n=-\infty}^{\infty} n P_n$, we may asso-

ciate with any unit vector $x \in \mathfrak{R}$ the probability $\langle P_n x, x \rangle$ that the number of particles in x differs from the number in v by n . For this reason we adopt the following terminology.

1.5 Definition. A representation W of the Weyl relations which has a number operator is called a *particle representation*. A number operator for W is called *normalized* if its spectrum is either $\{0, \pm 1, \pm 2, \dots\}$ or $\{0, 1, 2, \dots\}$.

2. Generating Functionals for Particle Representations

We review briefly the definition of *generating functional*. Terms not defined here are explained in Ref. [2], p. 44–45.

Let W be a Weyl system over \mathfrak{H} . Define for each finite-dimensional subspace \mathcal{M} of H the weakly-closed algebra $\mathfrak{A}_{\mathcal{M}}(W) = \{W(z) : z \in \mathcal{M}\}''$. Then the *Weyl algebra* $\mathfrak{A}(W)$ is the C^* -algebra generated by all the $\mathfrak{A}_{\mathcal{M}}(W)$'s as \mathcal{M} varies over the finite-dimensional subspaces of \mathfrak{H} . As a C^* -algebra, \mathfrak{A} is independent of W [16].

Given any state E of \mathfrak{A} , the Gelfand-Segal construction [10, 15], [6] yields a cyclic representation π_E of \mathfrak{A} on a Hilbert space \mathfrak{R}_E with a normalized cyclic vector v_E such that

$$E(A) = \langle \pi_E(A) v_E, v_E \rangle$$

for all $A \in \mathfrak{A}$.

Assuming E is *regular*, which means that E is strongly continuous on the unit ball of each $\mathfrak{A}_{\mathcal{M}}(W)$, \mathcal{M} finite-dimensional, then the operators $W_E(z) = \pi_E(W(z))$ form a representation of the Weyl relations whose Weyl algebra is $\pi_E(\mathfrak{A})$. Furthermore the complex-valued function μ on \mathfrak{H} defined by

$$\mu(z) = \langle W_E(z) v_E, v_E \rangle = E(W(z))$$

completely determines E and is called the *generating functional* of E [17].

Suppose W is a *particle* representation of the Weyl relations (Defn. 1.5). If N is a normalized number operator, and v is any eigenvector of N , then the generating functional

$$\mu(z) = \langle W(z) v, v \rangle$$

is *invariant under changes of phase*:

$$\mu(e^{it}z) = \mu(z) \quad \text{for all } z \in \mathfrak{H}, \quad t \in \mathbb{R}. \quad (2.1)$$

In fact,

$$\begin{aligned} \mu(e^{it}z) &= \langle e^{itN} W(z) e^{-itN} v, v \rangle \\ &= \langle W(z) e^{-itN} v, e^{-itN} v \rangle \\ &= \mu(z). \end{aligned}$$

It is easy to see that every particle representation is a direct sum of cyclic representations whose cyclic vectors are eigenvectors of N , and hence whose generating functionals satisfy (2.1). Conversely, as we prove below, any generating functional μ which has the property (2.1) corresponds to a particle representation. Since it is quite easy to exhibit generating functionals which are invariant under changes of phase, and it is also easy to determine whether a given functional has that property, this observation is quite helpful in studying particle representations.

We now proceed to a proof of the statement made above about generating functionals which satisfy (2.1).

2.1 Proposition. *Let μ be a generating functional which is invariant under changes of phase (2.1). Then the Weyl system W_E determined by μ via the Gelfand-Segal construction has a number operator which annihilates the cyclic vector v_E .*

Proof. A theorem of SEGAL [16] shows that the map $W(z) \rightarrow W(e^{it}z)$ induces an automorphism γ_t of the Weyl algebra \mathfrak{A} . The condition (2.1) implies that the regular state E determined by μ is invariant under γ_t :

$$E(\gamma_t(A)) = E(A) \quad \text{for all } A \in \mathfrak{A}.$$

Now it is an easily checked property of the Gelfand-Segal construction that the invariance of E under γ_t implies the existence of a unitary $U(t)$ on the representation space \mathfrak{R}_E which leaves the cyclic vector v_E invariant and which implements the automorphism:

$$U(t) A U(t)^{-1} = \gamma_t(A).$$

In fact $U(t)$ is defined by

$$U(t) A v_E = \gamma_t(A) v_E, \quad \text{for all } A \in \mathfrak{A}.$$

Clearly $U(t+t') = U(t)U(t')$, so U is a one-parameter group of unitary operators. To prove the existence of a number operator N , we just have to show that $U(t)$ is a strongly continuous function of t at $t = 0$. Then its self-adjoint generator N will be a number operator since

$$e^{itN} W(z) e^{-itN} = \gamma_t(W(z)) = W(e^{it}z).$$

To do this it suffices to prove that for all $z \in \mathfrak{H}$

$$\lim_{t \rightarrow 0} \|[U(t) - I] W(z) v_E\| = 0.$$

But

$$\begin{aligned} \|(U(t) - I) W(z) v_E\|^2 &= 2 - 2 \operatorname{Re} \langle U(t) W(z) v_E, W(z) v_E \rangle \\ &= 2 - 2 \operatorname{Re} \langle W(-z) W(e^{it}z) v_E, v_E \rangle \\ &= 2 - 2 \operatorname{Re} \mu((e^{it} - 1)z) \exp \left[\frac{1}{2} i \sin t \|z\|^2 \right]. \end{aligned}$$

This $\rightarrow 0$ as $t \rightarrow 0$ because μ is continuous on finite-dimensional subspaces and its value at 0 is 1. ■■

If a generating functional is not invariant under change of phase, then the corresponding representation may or may not have a number operator. In future work I hope to present a criterion by which one can tell directly from μ whether or not a number operator exists.

3. Number Operators for Direct-Product Representations

Suppose $\mathfrak{B} = \{e_1, e_2, \dots\}$ is an orthonormal basis of \mathfrak{H} , and \mathcal{V} is the set of all finite linear combinations of vectors in \mathfrak{B} . We consider direct-product representations of the Weyl relations over \mathcal{V} following KLAUDER, MCKENNA, and WOODS [11]. Our goal will be to determine which of them are particle representations.

For each $n = 1, 2, \dots$ let \mathfrak{R}_n be the representation space $L^2(\mathbb{R})$ for the Schrödinger Weyl system W_s over \mathbb{C} . Denote by \mathfrak{R} the complete infinite tensor product space $\mathfrak{R}_1 \otimes \mathfrak{R}_2 \otimes \dots$ [13].

If $z \in \mathcal{V}$, say $z = \sum_{i=1}^n z_i e_i$, then defining

$$W(z) = W_s(z_1) \otimes \dots \otimes W_s(z_n) \otimes I \otimes I \otimes \dots,$$

we get a representation of the Weyl relations over \mathcal{V} acting on \mathfrak{R} . This representation leaves invariant each incomplete infinite tensor product space. In fact, if $\psi = \psi_1 \otimes \psi_2 \otimes \dots$ is a decomposable vector in \mathfrak{R} such that $\|\psi_n\| = 1$ for all n , then the incomplete infinite tensor product space \mathfrak{R}_ψ whose distinguished vector is ψ is defined as the closed subspace spanned by vectors $\varphi = \varphi_1 \otimes \varphi_2 \otimes \dots$ such that $\sum_n |1 - \langle \varphi_n, \psi_n \rangle|$ converges. Since, for $z \in \mathcal{V}$, $W(z)$ changes only a finite number of factors in φ , such a $W(z)$ maps \mathfrak{R}_ψ into \mathfrak{R}_ψ .

So for each ψ , by restricting W to \mathfrak{R}_ψ we get a Weyl system W_ψ on \mathfrak{R}_ψ , which we shall call a *direct-product* representation. It is known [11] that each W_ψ is irreducible, and that W_ψ is unitarily equivalent to W_φ if and only if ψ is weakly equivalent to φ (i.e. $\sum |1 - |\langle \varphi_n, \psi_n \rangle||$ converges).

One might guess that the self-adjoint generator N of the unitary group

$$U(t) = e^{itN_s} \otimes e^{itN_s} \otimes e^{itN_s} \otimes \dots \quad (3.1)$$

is a number operator for the whole representation W because $U(t) W(z) U(-t) = W(e^{it} z)$, for $z \in \mathcal{V}$. (Here N_s is the usual number operator for the Schrödinger representation.) However it is easy to see that $t \rightarrow U(t)$ is not weakly continuous at zero, so it has no self-adjoint generator. In fact if $\varphi \in L^2(\mathbb{R})$ is a normalized eigenfunction of N_s with eigenvalue 1, then $\langle U(t) [\varphi \otimes \varphi \otimes \dots], \varphi \otimes \varphi \otimes \dots \rangle$ is one when t is an integer multiple of 2π , zero otherwise.

To find the representations W_ψ for which a number operator does exist, we first look for those ψ such that the generating functional

$$\mu_\psi(z) = \langle W_\psi(z) \psi, \psi \rangle \quad (3.2)$$

is invariant under change of phase.

3.1 Proposition. *Suppose $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$, where each $\|\psi_k\| = 1$. The generating functional μ_ψ given by (3.2) is invariant under change of phase if and only if each ψ_k is an eigenfunction of N_s (i.e. each ψ_k is a multiple of some Hermite function).*

Proof. If each ψ_k is an eigenfunction of N_s then the generating functional

$$\mu_k(z) = \langle W_s(z) \psi_k, \psi_k \rangle, \quad z \in \mathbb{C} \quad (3.3)$$

is invariant under change of phase for each k . Then if $z = \sum_{k=1}^n z_k e_k$, we have

$$\begin{aligned} \mu_\psi(e^{it}z) &= \prod_{k=1}^n \mu_k(e^{it}z_k) \\ &= \prod_{k=1}^n \mu_k(z_k) \\ &= \mu_\psi(z). \end{aligned}$$

On the other hand, if μ_ψ is invariant under change of phase, then each μ_k , as given in (3.3), will have the same property. By Prop. 2.1, this implies there exists a number operator for the Schrödinger representation which annihilates ψ_k . Because W_s is irreducible, any number operator for it differs from N_s by an additive constant. So ψ_k is an eigenfunction of N_s . ■■

3.2 Corollary. *If $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$ and each ψ_k is an eigenfunction of N_s , then the direct-product representation W_ψ has a number operator.*

Proof. This follows immediately from the proposition, using Prop. 2.1 and the fact that W_ψ is irreducible (so that ψ is a cyclic vector). ■■

It is easy to exhibit explicitly the number operator N for W_ψ which annihilates ψ . In fact, if n_k is the eigenvalue of N_s corresponding to ψ_k ($N_s \psi_k = n_k \psi_k$), then

$$e^{itN} = \exp it(N_s - n_1 I) \otimes \exp it(N_s - n_2 I) \otimes \cdots$$

[This is proved by observing that the operator on the right leaves \mathcal{R}_ψ invariant, and (1.4) is satisfied.] So we see that N is obtained from $\sum a^*(e_k) a(e_k)$ by subtracting a constant multiple of the identity. The constant is infinite, except in the case where all but a finite number of the n_k 's are zero, which is the case that W_ψ is unitarily equivalent to the Fock-Cook representation (Theorem 1.3; or this can be proved directly by calculating the generating functional).

The representations W_ψ singled out by Prop. 3.1 were among the first strange representations to be discussed; they are unitarily equivalent to the discrete representations of WIGHTMAN and SCHWEBER [21]. It may

appear that Prop. 3.1 identifies these representations as the only direct-product representations having a number operator. However, there remains the possibility that some direct-product representation is a particle representation, but that no number operator for it annihilates any vector of the form $\psi_1 \otimes \psi_2 \otimes \cdots$. This is excluded by the next result.

3.3 Theorem. *The only direct-product representations of the Weyl relations which have number operators are the discrete representations, i.e. those with a vector $\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots$ in the representation space such that each φ_k is an eigenfunction of N_s (i.e. is a multiple of a Hermite function).*

Proof. Suppose $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$, each $\|\psi_k\| = 1$, and W_ψ has a number operator N , assumed normalized.

Step 1. For every $t \in \mathbb{R}$, $U(t)\psi$ is weakly equivalent to ψ , where $U(t)$ is defined in (3.1).

Proof of Step 1: For each $t \in \mathbb{R}$, define $V(t) = U(t)e^{-itN}$. Considered as a map from \mathfrak{R}_ψ to $\mathfrak{R}_{U(t)\psi}$, $V(t)$ is unitary. Moreover, for each $z \in \mathcal{V}$

$$\begin{aligned} V(t)^{-1} W_{U(t)\psi}(z) V(t) &= e^{itN} [U(-t) W_{U(t)\psi}(z) U(t)] e^{itN} \\ &= e^{itN} W_\psi(e^{-it} z) e^{itN} \\ &= W_\psi(z). \end{aligned}$$

This shows that for each t $V(t)$ establishes a unitary equivalence between W_ψ and $W_{U(t)\psi}$. It follows [11] that for each t $U(t)\psi$ is weakly equivalent to ψ .

Step 2. There exist real constants a_1, a_2, \dots such that

$$e^{itN} = \exp it(N_s - a_1 I) \otimes \exp it(N_s - a_2 I) \otimes \cdots \quad (3.4)$$

Proof of Step 2: By Step 1 and the definition of weak equivalence, we know that $\sum_{k=1}^{\infty} |1 - |\mu_k(t)||$ converges for each t , where

$$\mu_k(t) = \langle e^{itN_s} \psi_k, \psi_k \rangle.$$

Now each $\mu_k(t)$ is the characteristic function (Fourier transform) of a probability measure, as one sees by using the spectral resolution of N_s . It follows from a theorem in probability theory (e.g. Doob [8] Th. 2.7) that there exist real constants a_1, a_2, \dots such that

$$\sum_{k=1}^{\infty} |1 - e^{-ia_k t} \mu_k(t)| < \infty \quad (3.5)$$

for all t .

For each real s , let $Y(s) = e^{-ia_1 s} \otimes e^{-ia_2 s} \otimes \cdots$, a unitary operator on \mathfrak{R} [13] which commutes with all the $U(t)$'s. Now (3.5) says $U(t)Y(t)\psi \in \mathfrak{R}_\psi$, so we may restrict $U(t)Y(t)$ to \mathfrak{R}_ψ getting a unitary operator $Z(t)$.

Z is easily seen to be a one-parameter group, and $Z(t)$ is weakly measurable in t since

$$Z(t) = \text{st-lim}_{n \rightarrow \infty} \exp it(N_s - a_1 I) \otimes \cdots \otimes \exp it(N_s - a_n I) \otimes I \otimes I \otimes \cdots.$$

Since \mathfrak{R}_ψ is separable this implies that $t \rightarrow Z(t)$ has a self-adjoint generator N' . Clearly N' is a number operator for W_ψ because

$$\begin{aligned} Z(t) W_\psi(z) Z(-t) &= U(t) W_\psi(z) U(-t) \\ &= W_\psi(e^{itz}). \end{aligned}$$

Then, since $e^{itN'} e^{-itN}$ commutes with all the $W_\psi(z)$'s, the irreducibility of the representation implies that N' differs from N by a constant multiple of the identity. Hence by changing the real number a_1 selected above, we may suppose $N' = N$. This gives (3.4).

Step 3. The constants a_1, a_2, \dots in (3.4) may be selected to be integers n_1, n_2, \dots .

Proof of Step 3: Since each $\mu_k(2\pi) = 1$, we have from (3.5)

$$\sum |1 - e^{-2\pi i a_k}| < +\infty. \quad (3.6)$$

If we write $a_k = n_k + b_k$, where n_k is an integer and $-1/2 < b_k \leq 1/2$, then (3.6) says $\sum |1 - e^{-2\pi i b_k}| < +\infty$, which implies [13] that $\sum |b_k|$ converges. So we have

$$e^{itN} = \exp\left(-it \sum_k b_k\right) \exp it(N_s - n_1 I) \otimes \exp it(N_s - n_2 I) \otimes \cdots.$$

Taking $t = 2\pi$, we see that $\sum b_k$ is an integer, which we may incorporate into n_1 . We then have

$$e^{itN} = \exp it(N_s - n_1 I) \otimes \exp it(N_s - n_2 I) \otimes \cdots. \quad (3.7)$$

Step 4. If h_k is the k th Hermite function, then ψ is weakly equivalent to

$$h = h_{n_1} \otimes h_{n_2} \otimes h_{n_3} \otimes \cdots$$

where n_1, n_2, \dots are the integers in (3.7).

Proof of Step 4: Using (3.7) and the fact that $e^{itN} \psi \in \mathfrak{R}_\psi$, we have

$$\langle e^{itN} \psi, \psi \rangle = \prod_{k=1}^{\infty} \langle \exp it(N_s - n_k I) \psi_k, \psi_k \rangle. \quad (3.8)$$

Using the spectral theorem we see that the function $t \rightarrow \langle e^{itN} \psi, \psi \rangle$ is the characteristic function of a probability measure, and likewise $t \rightarrow \langle \exp it(N_s - n_k I) \psi_k, \psi_k \rangle$ is the characteristic function of a probability measure m_k . In fact, if P_n is the projection of $L^2(\mathbb{R})$ onto the one-dimensional subspace spanned by the Hermite function h_n , then

$N_s = \sum_{n=0}^{\infty} n P_n$. Hence m_k assigns measure $\langle P_n \psi_k, \psi_k \rangle$ to the integer $n - n_k$, $n = 0, 1, 2, \dots$

Now we use the Kolmogorov three series theorem (see [8], p. 111, or [20]) which tells us that since the infinite product of the characteristic functions of the dm_k 's converges to a characteristic function, we have

$$\sum_k \int_{|x|>c} dm_k(x) < +\infty. \quad (3.9)$$

Here c is any positive number; for our purposes we take $c = 1/2$. Then

$$\begin{aligned} \int_{|x|<c} dm_k(x) &= m_k(\{0\}) \\ &= \langle P_{n_k} \psi_k, \psi_k \rangle \\ &= |\langle h_{n_k}, \psi_k \rangle|^2. \end{aligned}$$

So

$$\int_{|x|>c} dm_k(x) = 1 - |\langle h_{n_k}, \psi_k \rangle|^2$$

and (3.9) says

$$\sum_{k=1}^{\infty} (1 - |\langle h_{n_k}, \psi_k \rangle|^2) < \infty.$$

Since each $|\langle h_{n_k}, \psi_k \rangle| \leq 1$, this implies the convergence of

$$\sum (1 - |\langle h_{n_k}, \psi_k \rangle|),$$

which is the definition of weak equivalence of ψ with $h_{n_1} \otimes h_{n_2} \otimes \dots$.

Step 5. The theorem is now proved, since if ψ is weakly equivalent to h , then [13] there exist constants c_1, c_2, \dots such that

$$\sum |1 - \langle c_k h_{n_k}, \psi_k \rangle|$$

converges. Then $\varphi = c_1 h_{n_1} \otimes c_2 h_{n_2} \otimes \dots \in \mathbb{R}_\varphi$, and each $\varphi_k = c_k h_{n_k}$ is an eigenfunction of N_s . ■■

4. Continuity Properties of Discrete Representations

The direct-product representations, as described in Section 3, are defined only over the space \mathcal{V} , which is the algebraic span of a basis. But for physical applications such a space is too small; generally one needs a representation defined over a space of test functions or over a complete space. So it is of interest to inquire which of the discrete representations can be extended from \mathcal{V} to \mathfrak{H} . And for our purposes it is not sufficient to prove abstractly that such an extension exists, since we would want the extended Weyl system over \mathfrak{H} to have a number operator. Examples are known [2] of Weyl systems over \mathfrak{H} which have no number operator, yet whose restrictions to \mathcal{V} do have number operators.

The most natural idea is to extend the representation by continuity. For the case of a direct-product representation W , STREET [20] has determined the precise set of $z = \sum_{j=1}^{\infty} z_j e_j$ in \mathfrak{H} to which the representation

can be extended via the formula

$$W(z) = \text{st-lim}_{n \rightarrow \infty} W\left(\sum_{j=1}^n z_j e_j\right). \quad (4.1)$$

However, to use his criterion to determine whether or not a particular discrete representation can be extended to every $z \in \mathfrak{H}$ using (4.1) is much more difficult than proceeding directly. So our method is independent of STREIT's Theorem. The result is that some of them can be extended to all of \mathfrak{H} via (4.1) and some cannot be.

4.1 Definition. Let \mathfrak{H} be an inner product space and W a Weyl system over \mathfrak{H} . W is *continuous* (on all of \mathfrak{H}) if the map $z \rightarrow W(z)$ is continuous from the metric topology of \mathfrak{H} into the weak operator topology.

We recall that every Weyl system is continuous on finite-dimensional subspaces of \mathfrak{H} , but examples are known [2, 20] of Weyl systems which are not continuous on all of \mathfrak{H} . Our interest in continuous Weyl systems lies in the fact that they may be easily extended from dense subspaces to the whole space, and if the original representation had a number operator, so will the extended one. There is nothing difficult about these results, and the first is essentially proved elsewhere [1], but we give the proofs here for later reference.

4.2 Lemma. Let \mathfrak{H} be an inner product space and \mathcal{V} a dense subspace of \mathfrak{H} . A continuous Weyl system over \mathcal{V} has a unique extension to a continuous Weyl system over \mathfrak{H} .

Proof. We just have to prove the existence of a continuous extension, since such an extension is clearly unique and satisfies the Weyl relation (1.1). Since every representation is a direct sum of cyclic representations it suffices to consider a cyclic continuous representation W over \mathcal{V} acting on, say, \mathfrak{R} .

We must show that if $z_0 \in \mathfrak{H}$, and $\{z_n\}$ is any sequence in \mathcal{V} converging to z_0 , then the sequence $\{W(z_n)x\}$ is a Cauchy sequence in \mathfrak{R} for every $x \in \mathfrak{R}$. Since the $W(z_n)$'s are unitary, it actually suffices to prove this only for those x lying in a total subset S of \mathfrak{R} . For S we choose $\{W(z)v : z \in \mathcal{V}\}$, where v is a unit cyclic vector. The Weyl relation (1.1) then gives directly

$$\begin{aligned} & \| [W(z_m) - W(z_n)] W(z)v \|^2 \\ &= 2 - 2 \operatorname{Re} \left[\mu(z_n - z_m) \exp \frac{1}{2} i \operatorname{Im} \{ (z_n, z_m) + 2(z_n - z_m, z) \} \right], \end{aligned} \quad (4.2)$$

where $\mu(z_n - z_m) = \langle W(z_n - z_m)v, v \rangle$. This $\rightarrow 0$ as $m, n \rightarrow \infty$ since $z_n - z_m \in \mathcal{V}$ and $\|z_n - z_m\| \rightarrow 0$, so that by the continuity of W at 0 in \mathcal{V} , $W(z_n - z_m) \rightarrow I$.

Hence we know there is an operator $W(z_0)$ such that for any sequence $\{z_n\}$ in \mathcal{V} converging to z_0 , $\text{st-lim } W(z_n) = W(z_0)$. Since $W(z_0)$ is the limit of unitaries, it is isometric. But since $W(-z_0)$ also exists, and the Weyl relation shows it is the inverse of $W(z_0)$, we know $W(z_0)$ is unitary. ■■

4.3 Lemma. *Let W be a continuous representation of the Weyl relations over \mathfrak{H} , and \mathcal{V} a dense subspace of \mathfrak{H} . If the restriction of W to \mathcal{V} has a number operator N , then N is a number operator for W over \mathfrak{H} .*

Proof. If $z \in \mathfrak{H}$, and $\{z_n\}$ is a sequence in \mathcal{V} converging to z , then

$$\begin{aligned} e^{itN} W(z) e^{-itN} &= \text{st-lim}_{n \rightarrow \infty} e^{itN} W(z_n) e^{-itN} \\ &= \text{st-lim}_{n \rightarrow \infty} W(e^{it} z_n) \\ &= W(e^{it} z). \quad \blacksquare \end{aligned}$$

Now we need a practical criterion for deciding whether or not a representation is continuous, and this is given in the next result.

4.4 Proposition. *Let W be a cyclic Weyl system over \mathcal{V} on \mathfrak{R} , let v be a unit cyclic vector and μ the generating functional*

$$\mu(z) = \langle W(z) v, v \rangle.$$

W is continuous on all of \mathcal{V} if and only if μ is continuous at $0 \in \mathcal{V}$.

4.5 Corollary. *A Weyl system is continuous if and only if it is continuous at zero.*

Proof of Proposition (sufficiency). Suppose μ is continuous at 0. If $z_0 \in \mathcal{V}$, and $\{z_n\}$ is any sequence in \mathcal{V} converging to z_0 , we must prove that $\text{st-lim}_{n \rightarrow \infty} W(z_n) = W(z_0)$. This is done exactly as in the proof of Lemma 4.2, except that in (4.2) we replace z_m by z_0 . ■

Now we use these observations to analyse the discrete representations. In this case \mathcal{V} is the algebraic span of the orthonormal basis $\{e_1, e_2, \dots\}$. Each discrete representation is unitarily equivalent to a W_h , where h has the form $h = h_{n_1} \otimes h_{n_2} \otimes \dots$, and h_n is the n th Hermite function. The generating functional

$$\mu(z) = \langle W_h(z) h, h \rangle \quad (4.3)$$

is entirely determined by the functions μ_n on \mathbb{C} defined by

$$\mu_n(\alpha) = \langle W_s(\alpha) h_n, h_n \rangle, \alpha \in \mathbb{C}. \quad (4.4)$$

For if $z = \sum_{j=1}^p \alpha_j e_j \in \mathcal{V}$, then

$$\mu(z) = \prod_{j=1}^p \mu_{n_j}(\alpha_j). \quad (4.5)$$

The functions μ_n are easily calculated using the fact that

$$h_n = (n!)^{-1/2} C^n h_0,$$

where $h_0(x) = \pi^{-1/4} e^{-(1/2)x^2}$, and C is the creation operator for the Schrödinger representation. We omit the details, and give the result:

$$\mu_n(\alpha) = \sum_{k=0}^n \binom{n}{k} \frac{1}{k!} \left(-\frac{|\alpha|^2}{2} \right)^k e^{-(1/4)|\alpha|^2}. \quad (4.6)$$

Here $\binom{n}{k}$ is the binomial coefficient.

Knowing the explicit form of the generating functionals, we can determine their continuity properties. The result is this:

4.6 Proposition. *Let $h = h_{n_1} \otimes h_{n_2} \otimes \cdots$, where h_n is the n th Hermite function. The generating functional μ defined on \mathcal{V} by (4.3) is continuous at 0 if and only if the sequence n_1, n_2, \dots is bounded.*

As an immediate corollary of 4.2–4.4, 4.6 we have

4.7 Theorem. *Let $h = h_{n_1} \otimes h_{n_2} \otimes \cdots$, where h_n is the n th Hermite function, and let W_h be the direct-product representation of the Weyl relations over \mathcal{V} , which acts on the infinite tensor product space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes \cdots$ with distinguished vector h . Then W_h has a continuous extension to a particle representation (Defn. 1.5) on \mathfrak{S} (= completion of \mathcal{V}) if and only if the occupation numbers n_1, n_2, n_3, \dots are bounded.*

For the purpose of proving Proposition 4.6 we will need the following simple inequality.

4.8 Lemma. *Let μ_n be the generating functional (4.3). If $|\alpha| \leq 2^{-n}$, then*

$$1 \geq \mu_n(\alpha) \geq \exp(-2^n |\alpha|^2). \quad (4.7)$$

Proof. Since μ_n is a generating functional, $|\mu_n(\alpha)| \leq 1$ for all α , and since μ_n is real [cf. (4.6)], half of the inequality is proved. For the other half, write

$$\mu_n(\alpha) = (1 + b_n(\alpha)) e^{-(1/4)|\alpha|^2}, \quad (4.8)$$

where

$$b_n(\alpha) = \sum_{k=1}^n \binom{n}{k} \frac{1}{k!} \left(-\frac{|\alpha|^2}{2} \right)^k.$$

Then for $|\alpha| < 2^{-n}$

$$\begin{aligned} |b_n(\alpha)| &\leq \frac{|\alpha|^2}{2} \left(\sum_{k=1}^n \binom{n}{k} \right) = \frac{|\alpha|^2}{2} (2^n - 1) \\ &\leq |\alpha|^2 2^{n-1} < 1/2. \end{aligned}$$

Hence

$$|\log(1 + b_n(\alpha))| \leq 2|b_n(\alpha)| \leq (2^n - 1) |\alpha|^2,$$

so

$$1 + b_n(\alpha) \geq \exp[-(2^n - 1) |\alpha|^2].$$

In view of (4.8) this gives

$$\mu_n(\alpha) \geq \exp(-2^n |\alpha|^2),$$

and the lemma is proved. ■■

Proof of Proposition 4.6 (sufficiency). Suppose the sequence n_1, n_2, \dots is bounded above by the integer M . Then if $z = \sum_{j=1}^p \alpha_j e_j \in \mathcal{V}$ is such that $\|z\|^2 < 2^{-2M}$, then each $|\alpha_j| < 2^{-M}$ so it follows from Lemma 4.8 that

$$1 \geq \mu_{n_j}(\alpha_j) \geq \exp(-2^M |\alpha_j|^2),$$

for $j = 1, \dots, p$.

Hence, using (4.5), if $\|z\|^2 < 2^{-2M}$ we have

$$1 \geq \mu(z) \geq \prod_{j=1}^p \exp(-2^M |\alpha_j|^2) = \exp(-2^M \|z\|^2).$$

This shows μ is continuous at zero in \mathcal{V} .

(Necessity). Suppose the sequence n_1, n_2, \dots is unbounded. We will show that for any $\delta > 0$ there exists $z \in \mathcal{V}$ such that $\|z\| < \delta$ and yet

$$|1 - \mu(z)| > 1/4. \quad (4.9)$$

This proves μ is not continuous at 0.

So suppose $\delta > 0$ is given. Since the sequence n_1, n_2, \dots is unbounded we can find n_j such that $(2/n_j)^{1/2} < \delta$, and we suppose $n_j \geq 3$. Select $z = (2/n_j)^{1/2} e_j$. Then $\|z\| < \delta$ and we see that (4.9) is true as follows (we drop the subscript j , writing n for n_j):

$$\begin{aligned} \mu(z) &= \mu_n((2/n)^{1/2}) \\ &= \left[1 - \binom{n}{1} \frac{1}{n} + r \right] \exp\left(-\frac{1}{2n}\right) \\ &= r \exp\left(-\frac{1}{2n}\right), \end{aligned}$$

where

$$r = \sum_{k=2}^n \binom{n}{k} \frac{1}{k!} \left(-\frac{1}{n}\right)^k.$$

Thus we have $|\mu(z)| < |r|$. But

$$\begin{aligned} |r| &\leq \sum_{k=2}^n \frac{n!}{(n-k)! k!} \frac{1}{k!} \frac{1}{n^k} \\ &\leq \sum_{k=2}^n \frac{n!}{n^k (n-k)!} \frac{1}{(k!)^2} \\ &\leq \sum_{k=2}^n \frac{1}{(k!)^2} < e - 2 < \frac{3}{4}, \end{aligned}$$

so $|\mu(z)| < 3/4$ which implies $|1 - \mu(z)| > 1/4$. ■■

It is still thinkable that the discrete representations corresponding to unbounded occupation numbers n_1, n_2, \dots might be extendable to all of \mathfrak{H} via (4.1), even though the Weyl system is not continuous. But an easy modification of the proof just completed shows that this does not happen. In general, such a representation can be extended by (4.1) to a Weyl system on a subspace strictly larger than \mathcal{V} , but not to all of \mathfrak{H} .

5. Some Other Particle Representations

If T is a self-adjoint operator on \mathfrak{H} such that $T \geq I$, and \mathscr{D} is its domain, then a generating functional on \mathscr{D} is given by

$$\mu(z) = \exp \left[-\frac{1}{4} \|Tz\|^2 \right], \quad z \in \mathscr{D}. \quad (5.1)$$

Special examples of representations having generating functionals of this form are those given by the extreme universally invariant states found by SHALE, in which case T is a constant operator (see SEGAL [18]), and the states of an infinite free nonrelativistic Bose gas discussed by ARAKI and WOODS [1].

Since the μ given in (5.1) is invariant under change of phase, by Prop 2.1 there is a number operator for the cyclic representation of the Weyl relations determined by μ . Using a construction due essentially to ARAKI and WOODS, it is possible to exhibit explicitly the representation and its number operator. This construction also proves that (5.1) is actually a generating functional.

Let $A = \frac{1}{2}(T^2 - I)$, and let β be any conjugation of \mathfrak{H} which commutes with A . (β is anti-linear, and $\beta^2 = I$; such a conjugation always exists.) Now let \mathcal{M} be the closure of the range of $A^{1/2}$, a subspace of \mathfrak{H} , and denote by \mathfrak{H}_F (resp. \mathcal{M}_F) the Fock-Cook space constructed over \mathfrak{H} (resp. \mathcal{M}). For $z \in \mathscr{D}$, it is easy to see that $[I + A]^{1/2} z$ and $A^{1/2} \beta z$ are both defined, so we may define a unitary operator $W(z)$ on $\mathfrak{H}_F \otimes \mathcal{M}_F$ by

$$W(z) = W_F((I + A)^{1/2} z) \otimes W_F(A^{1/2} \beta z). \quad (5.2)$$

Here the first W_F is the Fock-Cook representation of the Weyl relations over \mathfrak{H} , and the second is the analogous representation over \mathcal{M} .

Direct calculations show that W is a representation of the Weyl relations over \mathscr{D} and that the function μ given in (5.1) satisfies

$$\mu(z) = \langle W(z) v_0 \otimes v_0, v_0 \otimes v_0 \rangle,$$

where v_0 is the zero-particle state in the Fock-Cook representation. Also simple modifications of the proofs in ARAKI-WOODS [1] show that $v_0 \otimes v_0$ is cyclic for $W(z)$, and W is a factor representation, reducible unless $A = 0$.

In case $A \neq 0$, an explicit normalized number operator for W is the closure of $N_F \otimes I - I \otimes N_F$, where the first N_F is the usual number operator for the Fock-Cook representation over \mathfrak{H} , and the second is the analogous operator for \mathcal{M} . In fact

$$\begin{aligned} & (e^{itN_F} \otimes e^{-itN_F}) W(z) (e^{-itN_F} \otimes e^{itN_F}) \\ &= W_F(e^{it} [I + A]^{1/2} z) \otimes W_F(e^{-it} A^{1/2} \beta z) \\ &= W_F([I + A]^{1/2} (e^{it} z)) \otimes W_F(A^{1/2} \beta (e^{it} z)) \\ &= W(e^{it} z), \end{aligned}$$

which shows that the self-adjoint generator of the group $t \rightarrow e^{itN_F} \otimes e^{-itN_F}$ is a number operator N for W , and this is the operator described above. Since e^{itN} leaves invariant the cyclic vector $v_0 \otimes v_0$ whose generating functional is μ , this is the operator one obtains using the construction described in the proof of Prop. 2.1.

The simplicity of constructing N disguises the fact that it is not really a natural number operator for W . First of all, N is not affiliated with the weakly-closed algebra \mathfrak{B} generated by the $W(z)$'s, i.e. $e^{itN} \notin \mathfrak{B}$. Hence it is difficult to think of N as an observable or as a renormalization of the formal operator $\sum_{k=1}^{\infty} a_k^* a_k$, since the finite sums $\sum_{k=1}^n a_k^* a_k$ are all affiliated with \mathfrak{B} . To see that $e^{itN} \notin \mathfrak{B}$, observe that when $A \neq 0$ it is always possible to find non-zero $z_0 \in \mathcal{D}$ such that $[I + A]^{1/2} z_0 \in \mathcal{M}$. Then the Weyl relations show that the operator

$$W_F(A^{1/2} \beta z_0) \otimes W_F([I + A]^{1/2} z_0)$$

commutes with all the $W(z)$'s; but it does not commute with e^{itN} except when t is an integer multiple of 2π .

A second peculiarity of N , related to the first, is that its spectrum always consists of all integers (positive and negative). However for certain choices of T the cyclic representation (5.2) determined by μ is actually unitarily equivalent to a direct sum of Fock-Cook representations. So for these representations it is possible to find a *non-negative* number operator, in which case the operator N selected above is a particularly unnatural choice.

We conclude with a determination of which T 's give rise to a direct sum of Fock-Cook representations. The proof uses the fact that such representations are the only finite-particle representations [2].

5.1 Theorem. *Suppose $T \geq I$ is a self-adjoint operator with domain $\mathcal{D} \subset \mathfrak{H}$. The cyclic particle representation W of the Weyl relations determined by the generating functional*

$$\mu(z) = \exp \left[-\frac{1}{4} \|Tz\|^2 \right], \quad z \in \mathcal{D}$$

is (unitarily equivalent to) a direct sum of Fock-Cook representations if and only if $T^2 - I$ has convergent trace.

Proof. First we show that if $A = \frac{1}{2}(T^2 - I)$ does not have pure point spectrum, then the representation (5.2) corresponding to μ is not a direct sum of Fock-Cook representations. Let \mathfrak{H}_c be the continuous subspace for A , i.e. the orthogonal complement of the subspace of \mathfrak{H} spanned by the eigenvectors of A . It suffices to show that the $W(z)$'s for $z \in \mathfrak{H}_c$ give a Weyl system which is not a direct sum of Fock-Cook representations over the same subspace \mathfrak{H}_c . So there is no loss of generality in assuming A has no point spectrum, i.e. $\mathfrak{H}_c = \mathfrak{H}$.

We shall use the following lemma which is proved, but not stated, by ARAKI and WOODS [1].

5.2 Lemma. *Suppose the weakly-closed algebra \mathfrak{B} of operators on \mathfrak{R} is a factor other than all bounded operators on \mathfrak{R} , and v is a cyclic vector for \mathfrak{B} . Suppose further that there exists a unitary operator U on \mathfrak{R} such that $U\mathfrak{B}U^{-1} = \mathfrak{B}$ and v is the unique eigenvector of U . Then \mathfrak{B} is not type I.*

We apply this lemma to the algebra $\mathfrak{B} = \{W(z) : z \in \mathscr{D}\}''$, which is a reducible factor, as mentioned earlier. For v we take the cyclic vector $v_0 \otimes v_0$. For U we take the operator $V \otimes V^{-1}$ where

$$V = \bigoplus_{n=0}^{\infty} (e^{iA})^{\otimes n}$$

a unitary operator on the Fock-Cook space \mathfrak{H}_F . Since A has only continuous spectrum, the same is true of $(e^{iA})^{\otimes n}$, so that V has only one eigenvector, v_0 . Hence U has only one eigenvector, namely $v = v_0 \otimes v_0$.

Furthermore, for $z \in \mathscr{D}$

$$\begin{aligned} UW(z)U^{-1} &= V W_F([I + A]^{1/2} z) V^{-1} \otimes V^{-1} W_F(A^{1/2} \beta z) V \\ &= W_F(e^{iA} [I + A]^{1/2} z) \otimes W_F(e^{-iA} A^{1/2} \beta z) \\ &= W(e^{iA} z). \end{aligned}$$

This shows $U\mathfrak{B}U^{-1} = \mathfrak{B}$, and then the Lemma 5.2 says \mathfrak{B} is not Type I. But the algebra generated by a direct sum of Fock-Cook representations is Type I, since the Fock-Cook representation is irreducible. So if A has continuous spectrum, the representation is not a direct sum of Fock-Cooks.

We are thus reduced to the case that A (or T) has pure point spectrum. So let $\{e_1, e_2, \dots\}$ be an orthonormal basis of \mathfrak{H} consisting of eigenvectors of T :

$$T e_j = t_j e_j.$$

For simplicity we shall first consider μ as defined only on the set \mathscr{V} of finite linear combinations of the basis vectors e_1, e_2, \dots .

Let E be the regular state of the Weyl algebra over \mathscr{V} whose generating functional is μ . According to Theorem 4, p. 77 [2], the cyclic representation determined by E is a direct sum of Fock representations if and only if the functions $\psi_{\mathcal{M}}(t) = E(e^{itN(\mathcal{M})})$ converge uniformly in t as $\mathcal{M} \rightarrow \mathscr{V}$ through the finite-dimensional subspace of \mathscr{V} . (Here $N(\mathcal{M})$ is the usual number operator over \mathcal{M} .) In the present case, since every finite-dimensional subspace of \mathscr{V} is contained in some $\mathcal{M}_k = \text{span}\{e_1, \dots, e_k\}$, it can be shown that this is equivalent to the convergence of the sequence $\psi_k(t) = E(\exp itN(\mathcal{M}_k))$ to a characteristic function.

Now if $z = \sum_{j=1}^n z_j e_j \in \mathcal{V}$, then the generating functional μ factors as

$$\mu(z) = \prod_{j=1}^n \mu_j(z_j) \quad (5.3)$$

where

$$\mu_j(\alpha) = \exp \left[-\frac{1}{4} t_j^2 |\alpha|^2 \right], \quad \alpha \in \mathbb{C}. \quad (5.4)$$

Let \mathfrak{A}_j be the weakly-closed algebra generated by the Weyl operators corresponding to z 's lying in the one-dimensional subspace $[e_j]$ spanned by e_j . And let E_j be the state of \mathfrak{A}_j whose generating functional is μ_j (5.4). Then by (5.3) and the regularity of E it follows that if $A_j \in \mathfrak{A}_j$, $j = 1, \dots, n$, we have

$$E(A_1 A_2 \dots A_n) = \prod_{j=1}^n E_j(A_j).$$

But the operator $\exp(itN(\mathcal{M}_k))$ is just such a product. In fact (e.g. [2] p. 35, 37)

$$\exp(itN(\mathcal{M}_k)) = \exp(itN([e_1])) \dots \exp(itN([e_k])),$$

so

$$\psi_k(t) = E(\exp(itN(\mathcal{M}_k))) \quad (5.5)$$

$$= \prod_{j=1}^k \varphi_j(t)$$

where

$$\varphi_j(t) = E_j(\exp(itN([e_j])). \quad (5.6)$$

Thus we are reduced to finding necessary and sufficient conditions for the infinite product $\prod \varphi_j$ to converge to a characteristic function.

These are given by the Kolmogorov Three-series Theorem if we know the measure whose characteristic function is φ_j . For this we need the explicit formula for E_j as given by SEGAL [18], Theorem 1. Namely

$$E_j(A) = (1 - c_j) \sum_{n=0}^{\infty} c_j^n \text{trace}(A P_n(j)).$$

where c_j is selected between 0 and 1 so that $t_j^2 = \frac{(1 + c_j)}{(1 - c_j)}$, and $P_n(j)$ is the projection onto the n -particle subspace of $N([e_j])$. Then, using (5.6), we see that the measure whose characteristic function is φ_j has mass $(1 - c_j) c_j^n$ at n , $n = 0, 1, 2, \dots$. The Three-series Theorem (e.g. [20]) then says that $\prod \varphi_j$ converges if and only if these three series converge:

$$\sum_j c_j, \quad \sum_j c_j/(1 - c_j), \quad \text{and} \quad \sum_j \left[\left(\frac{c_j}{1 - c_j} \right)^2 + \frac{c_j}{1 - c_j} \right].$$

The convergence of these three is equivalent to that of $\sum c_j/(1 - c_j)$ and since $t_j^2 - 1 = 2c_j/(1 - c_j)$, this is equivalent to the convergence of the trace of $T^2 - I$.

It follows that if $T^2 - I$ is *not* a trace class operator, then the representation W over \mathcal{V} is not a direct sum of Fock-Cook representations over \mathcal{V} . In this case the representation over $\mathcal{D} \supset \mathcal{V}$ cannot be a direct sum of Fock-Cook representations over \mathcal{D} .

Conversely, if $T^2 - I$ is trace class, then the representation W over \mathcal{V} is a direct sum of Fock-Cook representations over \mathcal{V} . Since T is bounded, $\mathcal{D} = \mathfrak{H}$ so the representation W is defined on all of \mathfrak{H} , and by Prop. 4.4 it is continuous on all of \mathfrak{H} . Since a direct sum of Fock-Cook representations is also continuous on all of \mathfrak{H} , the two agree everywhere. ■

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