NUMBER VARIANCE OF RANDOM ZEROS ON COMPLEX MANIFOLDS

BERNARD SHIFFMAN AND STEVE ZELDITCH

Abstract. We show that the variance of the number of simultaneous zeros of m i.i.d. Gaussian random polynomials of degree N in an open set $U \subset \mathbb{C}^m$ with smooth boundary is asymptotic to $N^{m-1/2}\,\nu_{mm}\,\mathrm{Vol}(\partial U)$, where ν_{mm} is a universal constant depending only on the dimension m. We also give formulas for the variance of the volume of the set of simultaneous zeros in U of $k < m$ random degree-N polynomials on $\mathbb{C}^m$. Our results hold more generally for the simultaneous zeros of random holomorphic sections of the N-th power of any positive line bundle over any m-dimensional compact Kähler manifold.

Contents

1. Introduction

This article is concerned with the asymptotic statistics of the number $\mathcal{N}_N^U(p_1^N,\dots,p_m^N)$ of zeros in an open set $U \subset \mathbb{C}^m$ of a full system $\{p_j^N\}$ of m Gaussian random polynomials (or more generally, of sections of a holomorphic line bundle over a Kähler manifold M_m) as the degree $N \to \infty$. In earlier work [SZ1], we proved that the zeros become uniformly distributed in U with respect to the natural volume form. The main result of this article (Theorem 1.1) gives an asymptotic formula for the variance of $\mathcal{N}_N^U(p_1^N,\dots,p_m^N)$ for open sets with piecewise smooth boundary. We also give analogous results for the volume of the simultaneous zero set of $k < m$ polynomials or sections (Theorem 1.4). Our results show that the zeros of a random system are close to the expected distribution, i.e. number statistics are 'self-averaging', and moreover the degree of self-averaging increases with the dimension.

Research of the first author partially supported by NSF grants DMS-0100474 and DMS-0600982; research of the second author partially supported by NSF grants DMS-0302518 and DMS-0603850.

To introduce our results, let us start with the case of polynomials in m variables. By homogenizing, we may identify the space of polynomials of degree N in m variables with the space $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ of holomorphic sections of the N-th power of the hyperplane section bundle over $\mathbb{CP}^m$. This space carries a natural $\mathrm{SU}(m+1)$-invariant inner product and associated Gaussian measure γ_N. To each m-tuple of degree N polynomials $(p_1^N, \dots, p_m^N)$, we associate its zero set $\{p_1^N(z) = \cdots = p_m^N(z) = 0\}$, which is almost always discrete, and thus obtain a random point process on $\mathbb{CP}^m$. We denote by

$$Z_{p_1^N,\dots,p_m^N} := \{z \in M : p_1^N(z) = \cdots = p_m^N(z) = 0\}$$

the set of zeros and by

$$\langle \left[Z_{p_1^N,\dots,p_m^N}\right], \psi\rangle := \sum_{z\in Z_{p_1^N,\dots,p_m^N}} \psi(z), \quad \psi \in C(M) \tag{1}$$

the sum of point masses at the joint zeros. It easily follows from the $\mathrm{SU}(m+1)$-invariance of γ_N that the expected value of this measure is a multiple of the Fubini-Study volume form, i.e.

$$\mathbf{E}\left[Z_{p_1^N,\dots,p_m^N}\right] = N^m \left(\frac{1}{\pi}\,\omega_{\mathrm{FS}}\right)^m , \tag{2}$$

where $\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar\partial \log |z|^2$ is the Fubini-Study metric on $\mathbb{CP}^m$. Here, $\mathbf{E}(X)$ denotes the expected value of a random variable X.

Given a measurable set U, the random variable counting the number of zeros of the polynomial system in U is defined by

$$\mathcal{N}_N^U(p_1^N, \dots, p_m^N) := \#\{z \in U : p_1^N(z) = \cdots = p_m^N(z) = 0\}.$$

Clearly, $\mathcal{N}_N^U$ is discontinuous along the set of polynomials having a zero on the boundary ∂U. Integrating (2) over U gives

$$\mathbf{E}(\mathcal{N}_N^U) = N^m \int_U \left(\frac{1}{\pi}\,\omega_{\mathrm{FS}}\right)^m . \tag{3}$$

Formula (3) has a counterpart for Gaussian random holomorphic sections of powers of any Hermitian holomorphic line bundle (L, h) with positive curvature Θ_h over any m-dimensional compact Kähler manifold M with Kähler form $\omega = \frac{i}{2}\Theta_h$. The Hermitian inner product

$$\langle \sigma_1, \bar\sigma_2\rangle = \int_M h^N(\sigma_1, \bar\sigma_2)\, \frac{1}{m!}\omega^m , \qquad \sigma_1, \sigma_2 \in H^0(M, L^N) , \tag{4}$$

induces the complex Gaussian probability measure

$$d\gamma_N(s^N) = \frac{1}{\pi^m} e^{-|c|^2} dc , \qquad s^N = \sum_{j=1}^{d_N} c_j S_j^N , \tag{5}$$

on the space $H^0(M, L^N)$ of holomorphic sections of L^N, where $\{S_1^N, \dots, S_{d_N}^N\}$ is an orthonormal basis for $H^0(M, L^N)$, and dc denotes $2d_N$-dimensional Lebesgue measure. The Gaussian measure γ_N given by (4)–(5) is called the *Hermitian Gaussian measure induced by h*. The Gaussian ensembles $(H^0(M, L^N), \gamma_N)$ were used in [SZ1, SZ2, BSZ1, BSZ2]; for the case of polynomials in one variable, they are equivalent to the SU(2) ensembles studied in [BBL, Ha, NV] and elsewhere. In [SZ1], we showed that the expected value of the corresponding random variable $\mathcal{N}_N^U$ (where $U \subset M$) on the ensemble $(H^0(M, L^N)^m, \gamma_N^m)$ has the asymptotics

$$\frac{1}{N^m}\,\mathbf{E}(\mathcal{N}_N^U) = \int_U \left(\frac{i}{2\pi}\,\Theta_h\right)^m + \,O\left(\frac{1}{N}\right)\,. \tag{6}$$

Thus, zeros of Gaussian random systems of sections of L^N become uniformly distributed with respect to the curvature volume form $\frac{1}{m!}\omega^m$, as $N \to \infty$.

Our main result gives an asymptotic formula for the *number variance*

$$\mathrm{Var}(\mathcal{N}_N^U) := \mathbf{E}\big(\mathcal{N}_N^U - \mathbf{E}(\mathcal{N}_N^U)\big)^2$$

in this general setting:

THEOREM **1.1.** *Let (L, h) be a positive Hermitian holomorphic line bundle over a compact m-dimensional Kähler manifold M. We give $H^0(M, L^N)$ the Hermitian Gaussian measure induced by h and the Kähler form $\omega = \frac{i}{2}\Theta_h$. Let U be a domain in M with piecewise $\mathcal{C}^2$ boundary and no cusps. Then for m independent random sections $s_j^N \in H^0(M, L^N)$, $1 \le j \le m$, the variance of the random variable*

$$\mathcal{N}_N^U(s_1^N, \dots, s_m^N) := \#\{z \in U : s_1^N(z) = \cdots = s_m^N(z) = 0\}$$

is given by

$$\mathrm{Var}\big(\mathcal{N}_N^U\big) = N^{m-1/2}\,\Big[\nu_{mm}\,\mathrm{Vol}_{2m-1}(\partial U) + O(N^{-\frac{1}{2}+\varepsilon})\Big]\ ,$$

where ν_{mm} is a universal positive constant. In particular, $\nu_{11} = \frac{\zeta(3/2)}{8\pi^{3/2}}$.

Here, we say that U has piecewise $\mathcal{C}^k$ boundary without cusps if for each boundary point $z_0 \in \partial U$, there exists a (not necessarily convex) closed polyhedral cone $K \subset \mathbb{R}^{2m}$ and a $\mathcal{C}^k$ diffeomorphism $\rho : V \to \rho(V) \subset \mathbb{R}^{2m}$, where V is a neighborhood of z_0, such that $\rho(V \cap \overline{U}) = \rho(V) \cap K$. By $O(N^{-\frac{1}{2}+\varepsilon})$, we mean a term whose magnitude is less than $C_p N^p$ for some constant $C_p \in \mathbb{R}^+$ (depending on M, L, h, U as well as p), for all $p > -\frac{1}{2}$.

As mentioned above, a special case of Theorem 1.4 gives statistics for the number of zeros of systems of polynomials of degree N. Identifying polynomials on $\mathbb{C}^m$ of degree N with $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ endowed with the Fubini-Study metric, we obtain an orthonormal basis of monomials (see [BSZ2, SZ1]):

$$\left\{ \tbinom{N}{J}^{1/2} z_1^{j_1}\cdots z_m^{j_m} \right\}_{|J|\le N} \quad \left(J = (j_1, \dots, j_m),\ |J| = j_1 + \cdots + j_m,\ \tbinom{N}{J} = \tfrac{N!}{(N-|J|)!j_1!\cdots j_m!}\right)$$

The polynomial case of Theorem 1.1 then takes the form:

COROLLARY **1.2.** *Consider the Gaussian random polynomials*

$$p_l^N(z_1, \dots, z_m) = \sum_{\{J\in\mathbb{N}^m : |J|\le N\}} c_{J,l}\,\tbinom{N}{J}^{1/2} z_1^{j_1}\cdots z_m^{j_m} \qquad (1 \le l \le m)\ ,$$

where the $c_{J,l}$ are independent complex Gaussian random variables with mean 0 and variance 1. Let U be a domain in $\mathbb{C}^m$ with piecewise $\mathcal{C}^2$ boundary and no cusps. Then the variance of the number $\mathcal{N}_N^U$ of zeros in U of the degree-N polynomial system $(p_1^N, \dots, p_m^N)$ is given by

$$\mathrm{Var}\big(\mathcal{N}_N^U\big) = N^{m-1/2}\,\Big[\nu_{mm}\,\mathrm{Vol}_{2m-1}^{\mathbb{CP}^m}(\partial U) + O(N^{-\frac{1}{2}+\varepsilon})\Big]\,,$$

where $\mathrm{Vol}_{2m-1}^{\mathbb{CP}^m}$ denotes the hypersurface volume with respect to the Fubini-Study metric on $\mathbb{CP}^m$.

The variance $\mathrm{Var}(\mathcal{N}_N^U)$ measures the fluctuations in the number of zeros in U of random systems of polynomials or sections. Theorem 1.1 implies that the number of zeros in U is 'self-averaging' in the sense that its fluctuations are of smaller order than its typical values. Recalling (6), we have:

Corollary **1.3.** *Under the hypotheses of Theorem 1.1 or Corollary 1.2,*

$$\frac{\left[\mathrm{Var}\left(\mathcal{N}_N^U\right)\right]^{1/2}}{\mathbf{E}\left(\mathcal{N}_N^U\right)} \sim N^{-\frac{m}{2}-\frac{1}{4}} \to 0 \qquad as \;\; N \to \infty \, .$$

By Corollary 1.3 and the Borel-Cantelli Lemma applied to the sets $\{N^{-2m}(\mathcal{N}_N^U - \mathbf{E}\mathcal{N}_N^U)^2 > \varepsilon\}$, the zeros of a *random sequence* $\{(s_1^N, \dots, s_m^N) : N = 1, 2, 3, \dots\}$ of systems almost surely become uniformly distributed:

$$\frac{1}{N^m}\, \mathcal{N}_N^U(s_1^N, \dots, s_m^N) \to m! \,\mathrm{Vol}(U) \qquad a.s.$$

Our proof of Theorem 1.1 also yields asymptotic formulas for the variance of volumes of simultaneous zero sets $Z_{s_1^N,\dots,s_k^N} \cap U$, where

$$Z_{s_1^N,\dots,s_k^N} := \{z \in M : s_1^N(z) = \cdots = s_k^N(z) = 0\} \, .$$

of $k < m$ sections $s_1^N, \dots, s_k^N$. For N sufficiently large (so that L^N is base point free), $Z_{s_1^N,\dots,s_k^N}$ is almost always a complex codimension k submanifold of M (by Bertini's Theorem) and

$$\mathrm{Vol}_{2m-2k}(Z_{s_1^N,\dots,s_k^N} \cap U) = \int_{Z_{s_1^N,\dots,s_k^N} \cap U} \frac{1}{(m-k)!}\, \omega^{m-k} \, . \tag{7}$$

We have the following asymptotic formula for the variance of the volume in (7):

Theorem **1.4.** *Let $1 \le k \le m$. With the same notation and hypotheses as in Theorem 1.1, for k independent random sections $s_j^N \in H^0(M, L^N)$, $1 \le j \le k$, we have*

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N} \cap U]\big) = N^{2k-m-1/2} \left[\nu_{mk}\, \mathrm{Vol}_{2m-1}(\partial U) + O(N^{-\frac{1}{2}+\varepsilon})\right] \, ,$$

where ν_{mk} is a universal constant; in particular,

$$\nu_{m1} = \frac{\pi^{m-5/2}}{8}\, \zeta(m + \tfrac{1}{2}) \, .$$

Theorem 1.1 is the case $k = m$ of Theorem 1.4, where the 0-dimensional volume is the counting measure:

$$\mathrm{Vol}_0(Z_{s_1^N,\dots,s_m^N} \cap U) = \mathcal{N}_N^U(s_1^N, \dots, s_m^N) \, .$$

In §4, we simultaneously prove Theorems 1.1 and 1.4.

The constants ν_{mk} in Theorem 1.4 must be nonnegative, since variances (of nonconstant random variables) are positive. We prove in §4.1 that ν_{mm} is positive in the point case of Theorem 1.1, and we know that ν_{m1} is positive by the computation of its value. We conjecture, but do not prove here, that $\nu_{mk} > 0$ for all k. Although we do not prove that our asymptotics results in these intermediate cases are sharp we include them here because they follow with no additional effort from the proofs in the point case and because our analysis of the point case makes use of induction on the codimension.

In the remainder of the introduction, we discuss related results on the variance problems studied in this article and indicate some key ideas in the proofs. In particular, we indicate which aspects of the results and methods are essentially complex analytical and which aspects are mainly probabilistic.

The first results on number variance in domains appear to be due to Forrester and Honner [FH] for certain one-dimensional Gaussian ensembles of random complex polynomials. They gave an intuitive derivation of the leading term of the asymptotic formula in dimension one, which is proved in Theorem 1.1. Peres-Virag have precise results on numbers of zeros of Gaussian random analytic functions on the unit disc for a certain ensemble with a determinantal zero point process [PV]. To our knowledge, there are no prior results on number or volume variance in higher dimensions.

Variance asymptotics have also been studied for smooth analogues of numbers statistics, namely for the random variables (1) with $\psi \in C^\infty(M)$. A rather simple and non-sharp estimate on the variance for the smooth linear statistics was given in our article [SZ1] to show that the codimension-one zeros of a random sequence $\{\mathbf{s}^N\}$ almost surely become uniformly distributed. Smooth linear statistics were then studied in depth for certain model one-dimensional Gaussian analytic functions by Sodin-Tsirelson [ST]) as a key ingredient in their proof of asymptotic normality for linear statistics. For their model ensembles, they gave a sharp estimate for the variance of (Z_{s^N}, φ) and determined the leading term. (The constant $\frac{\zeta(3)}{16\pi}$ was given for model ensembles in a private communication from M. Sodin.)

We now discuss some key ideas in the proofs, and also their relation to Sodin-Tsirelson [ST] and to our prior work [BSZ1, SZ3]. Apart from a rather routine computation (Lemma 3.3) for the expected value $\mathbf{E}\big(\log|Y_1|\,\log|Y_2|\big)$ where Y_j are complex normal random variables, the principal ingredients in our work are purely complex analytical. The key ones are:

- A bipotential formula for the variance current of one section (Theorem 3.1) or several sections (Theorem 3.13), and the closely related formulas for the pair correlation current $\mathbf{K}_{21}^N$ (cf. (79)) in the codimension one case (Proposition 3.10);
- Analysis of the singularities along the diagonal of the pair correlation and variance currents, particularly in the point case (maximal codimension) where these currents contain a delta-function along the diagonal in $M \times M$ (Theorem 3.15). This analysis is necessary to verify the formula $\mathbf{K}_{2k}^N = \big[\mathbf{K}_{21}^N\big]^{\wedge k}$ (see (82)), where $\mathbf{K}_{2k}^N$ is the pair correlation current for the simultaneous zeros of k random sections or polynomials of degree N, and to define the product $\big[\mathbf{K}_{21}^N\big]^{\wedge k}$.
- Application of the rapid (in fact, exponentially fast) off-diagonal asymptotics of the Szegő kernel of [SZ2] as the degree $N \to \infty$ to obtain asymptotics of the variance current and number variance.

Let us discuss these items in more detail. The 'bipotential' for the pair correlation current was introduced in [BSZ1]; in the notation used here, the bipotential is a function $Q_N(z,w)$ such that

$$\Delta_z \Delta_w Q_N(z,w) = K_{21}^N(z,w)\,, \tag{8}$$

where K_{21}^N is the 'pair correlation function' (see (79)–(81)) for the zeros of degree N polynomials on $\mathbb{CP}^m$ or of holomorphic sections of $L^N \to M$. In this article, we show that $Q_N(z,w)$ is actually a *pluri-bipotential* for the *variance current* $\mathbf{Var}(Z_{s^N})$ of Z_{s^N} (see Theorem 3.1), so we can use $Q_N(z,w)$ to give an explicit formula (Theorems 3.11) for the variance of the zeros of $k \le m$ independent sections of $H^0(M, L^N)$, for any codimension k.

The existence of the bipotential makes essential use of the complex analyticity of the polynomials and sections, in particular the Poincaré-Lelong formula. It is not clear if there exists a useful generalization of (8) to the non-holomorphic setting. Moreover, the analysis of the singularities of the variance current requires a detailed study of intersections of complex analytic varieties and currents of fixed degree. Part of the length of this paper is due to the lack of a prior reference for the relevant facts on smoothing and intersection of currents. We hope that the analysis in §3 will be useful in other applications.

The bipotential for the variance was also used implicitly by Sodin-Tsirelson [ST], where it is defined as a power series in the Szegő kernel for $\mathcal{O}(N) \to \mathbb{CP}^1$. They used it to obtain the first sharp formula for the variance of certain model one-dimensional random analytic functions, and further used it to prove asymptotic normality of smooth linear statistics.

From the bipotential formulas, and the analysis of the singularities of the variance current, the asymptotics of the variance are reduced to the off-diagonal asymptotics for the Szegő kernel $\Pi_N(z,w)$ (two point function). Our main results are proved by applying the off-diagonal asymptotics of $\Pi_N(z,w)$ in [SZ2] to obtain asymptotics of $Q_N(z,w)$ and then of the variance. One consequence of these asymptotics, which is of independent interest, is that the normalized pair correlation function $\widetilde{K}^N_{2m}$ rapidly approaches 1 off the diagonal as N increases:

$$\widetilde{K}^N_{2m}(z,w) = 1 + O\left(N^{-\infty}\right) , \quad \text{for } \operatorname{dist}(z,w) \geq N^{-1/2+\varepsilon} . \tag{9}$$

(See Corollary 3.17 for a precise statement and the definition of $\widetilde{K}^N_{2m}$.) These asymptotics make essential use of the holomorphic setting, and moreover of the positive curvature of the line bundles involved. The two-point function $\Pi_N(z,w)$ and consequently the correlation functions may behave quite differently in non-holomorphic cases (the two point kernel can decay at only a power law rate in real cases) or even for random polynomials of degree N on domains in $\mathbb{C}^m$ with the flat metric, i.e. with an inner product independent of N. Subsequent to [SZ2], sharper off-diagonal estimates for $\Pi_N(z,w)$ with exponentially small remainder estimates away from the diagonal (i.e., when $\operatorname{dist}(z,w) \gg \frac{1}{\sqrt{N}}$) were also given in [DLM, MM], and they would improve the decay of correlations in (9); we state the result as above, since the estimates of [SZ2] already suffice for our applications.

The paper is organized as follows: In §2, we review the formulas for the expected zero currents and describe the asymptotics of the Szegő kernel for powers of a line bundle. In §3, we define the variance current (in codimension one) and introduce the pluri-bipotential $Q_N(z,w)$ for the variance current and study its off-diagonal asymptotics. Next, we provide our explicit formula for the variance (Theorem 3.11). In §4, this formula and the asymptotics of the pluri-bipotential are applied to prove Theorems 1.1 and 1.4 on number and volume variance. Finally, in the Appendix (§5), we review and to some degree sharpen the derivation of the off-diagonal asymptotics in §2.2.

We end the introduction with a word on the relation of this article to its predecessors posted on arXiv.org. The first predecessor of this article is our preprint [SZ3], in which we proved the codimension $k=1$ case of Theorem 1.4. This prior article did not contain results on the point case in higher dimensions since, as we wrote there, "new technical ideas seem to be necessary to obtain limit formula for the intersections of the random zero currents Z_{s_j}." The present article furnishes the necessary new methods (cf. §3). The preprint [SZ3] also extended the Sodin-Tsirelson asymptotic normality result for smooth statistics [ST] to general one-dimensional ensembles and to codimension one zero sets in higher dimensions. It remains an interesting open problem to generalize asymptotic normality to the point case in higher dimension. The original arXiv.org posting of the present article (arxiv.org/abs/math/0608743v1) also contains results on smooth linear statistics and on random entire functions on $\mathbb{C}^m$ and on certain noncompact complete Kähler manifolds; for the sake of brevity, these results will be presented elsewhere.

Acknowledgment: We thank M. Sodin for discussions of his work with B. Tsirelson.

2. Background

In this section we review the basic facts about the distribution of zeros and the asymptotic properties of Szegő kernels.

2.1. **Expected distribution of zeros.**

In this section, we review the formula for the expected simultaneous zero current of $k \le m$ independent Gaussian random sections of the tensor powers $L^N = L^{\otimes N}$ of a holomorphic line bundle L over an m-dimensional complex manifold M (Corollary 2.3). In order to give a simple proof of our formula by induction on the codimension k of the zero set, we state our result in a more general form (Proposition 2.2). In the point case $k = m$, this formula was given by Edelman-Kostlan [EK, Theorem 8.1] (for the essentially equivalent case of trivial line bundles) using integral-geometric methods.

Throughout this paper, we let (L, h) be a Hermitian holomorphic line bundle over a compact complex manifold M. We let $\mathcal{S}$ be a subspace of $H^0(M, L)$, endowed with an (arbitrary) Hermitian inner product. The inner product induces the complex Gaussian probability measure

$$d\gamma(s) = \frac{1}{\pi^m} e^{-|c|^2} dc\,, \qquad s = \sum_{j=1}^{n} c_j S_j\,, \tag{10}$$

on $\mathcal{S}$, where $\{S_j\}$ is an orthonormal basis for $\mathcal{S}$ and dc is $2n$-dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2n$ real variables $\operatorname{Re} c_j, \operatorname{Im} c_j$ $(j = 1, \dots, n)$ are independent Gaussian random variables with mean 0 and variance $\frac{1}{2}$; i.e.,

$$\mathbf{E} c_j = 0, \quad \mathbf{E} c_j c_k = 0, \quad \mathbf{E} c_j \bar{c}_k = \delta_{jk}\,.$$

We let

$$\Pi_{\mathcal{S}}(z, z) = \mathbf{E}_\gamma \left(\|s(z)\|_h^2 \right) = \sum_{j=1}^{n} \|S_j(z)\|_h^2\,, \qquad z \in M\,, \tag{11}$$

denote the 'Szegő kernel' for $\mathcal{S}$ on the diagonal. We now consider a local holomorphic frame e_L over a trivializing chart U, and we write $S_j = f_j e_L$ over U. Any section $s \in \mathcal{S}$ may then be written as

$$s = \langle c, F \rangle e_L\,, \quad \text{where} \quad F = (f_1, \dots, f_k)\,, \quad \langle c, F \rangle = \sum_{j=1}^{n} c_j f_j\,. \tag{12}$$

If $s = f e_L$, its Hermitian norm is given by $\|s(z)\|_h = a(z)^{-\frac{1}{2}} |f(z)|$ where

$$a(z) = \|e_L(z)\|_h^{-2}\,. \tag{13}$$

Recall that the curvature form of (L, h) is given locally by

$$\Theta_h = \partial\bar\partial \log a\,,$$

and the *Chern form* $c_1(L, h)$ is given by

$$c_1(L, h) = \frac{\sqrt{-1}}{2\pi} \Theta_h = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log a\,. \tag{14}$$

Using standard notation, we let $\mathcal{E}^{p,q}(X)$ and $\mathcal{D}^{p,q}(X)$ denote the spaces of $\mathcal{C}^\infty$ (p, q)-forms and compactly supported $\mathcal{C}^\infty$ (p, q)-forms, respectively, on a complex manifold X, and we let

$\mathcal{D}'^{p,q}(X) = \mathcal{D}^{m-p,m-q}(X)'$ denote the space of (p,q)-currents on X; $(T,\varphi) = T(\varphi)$ denotes the pairing of $T \in \mathcal{D}'^{p,q}(X)$ and $\varphi \in \mathcal{D}^{m-p,m-q}(X)$. If Y is a complex submanifold of X of codimension p, we let $[Y] \in \mathcal{D}'^{p,p}(X)$ denote the current of integration over Y given by $([Y],\varphi) = \int_Y \varphi$. (For a further description of currents on complex manifolds, see [GH, Ch. 3].)

We now suppose that $\mathcal{S}$ is *base point free*; i.e., the set $\{z \in M : s(z) = 0,\ \forall s \in \mathcal{S}\}$ is empty. By Bertini's theorem (or by an application of Sard's theorem), for almost all k-tuples $(s_1,\dots,s_k) \in \mathcal{S}^k$, the simultaneous zero set $\{z \in M : s_1(z) = \cdots = s_k(z) = 0\}$ is a complex submanifold of M of codimension k, and we let $Z_{s_1,\dots,s_k} := \big[\{s_1(z) = \cdots = s_k(z) = 0\}\big]$ denote the current of integration over the zero set:

$$(Z_{s_1,\dots,s_k},\varphi) = \int_{\{s_1(z)=\cdots=s_k(z)=0\}} \varphi(z), \qquad \varphi \in \mathcal{D}^{m-k,m-k}(M)\ .$$

The current $Z_{s_1,\dots,s_k}$ is well-defined for almost all $s_1,\dots,s_k$, and we regard $Z_{s_1,\dots,s_k}$ as a current-valued random variable. For the point case $k = m$, $Z_{s_1,\dots,s_m}$ is a measure-valued random variable:

$$(Z_{s_1,\dots,s_m},\varphi) = \sum_{s_1(z)=\cdots=s_m(z)=0} \varphi(z), \qquad \varphi \in \mathcal{D}(M)\ .$$

The current of integration Z_s over the zeros of one section $s \in \mathcal{S}$, written locally as $s = fe_L$, is then given by the *Poincaré-Lelong formula* (see [GH, p. 388]):

$$Z_s = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log|f| = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log\left\|s^N\right\|_h + c_1(L,h)\ , \tag{15}$$

where the second equality is a consequence of (13)–(14).

We begin with a general form of the *probabilistic Poincaré-Lelong formula* from [SZ1] (see also [BSZ1, BSZ2]) for the expected value of a random zero divisor:

PROPOSITION **2.1.** *Let (L,h) be a Hermitian line bundle on a compact Kähler manifold M. Let $\mathcal{S}$ be a base-point-free subspace of $H^0(M,L)$ endowed with a Hermitian inner product and we let γ be the induced Gaussian probability measure on $\mathcal{S}$. Then the expected zero current of a random section $s \in \mathcal{S}$ is given by*

$$\mathbf{E}_\gamma(Z_s) \;=\; \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \Pi_{\mathcal{S}}(z,z) + c_1(L,h)\ .$$

The formula of the proposition was essentially given in [SZ1, Prop. 3.1]. For completeness, we give a short proof: It suffices to verify the identity over a trivializing neighborhood U. As above, we let $\{S_j\}$ be an orthonormal basis for $\mathcal{S}$, we write $S_j = f_j e_L$ (over U), and we let $F = (f_1,\dots,f_n)$. As in [SZ1], we then write $F(z) = |F(z)|u(z)$ so that $|u| \equiv 1$ and

$$\log|\langle c,F\rangle| = \log|F| + \log|\langle c,u\rangle|\ . \tag{16}$$

Thus by (15), we have

$$\begin{aligned}\big(\mathbf{E}_\gamma(Z_s),\varphi\big) &= \frac{\sqrt{-1}}{\pi}\int_{\mathcal{S}} \big(\log|\langle c,F\rangle|, \partial\bar\partial\varphi\big)\, d\gamma \\ &= \frac{\sqrt{-1}}{\pi}\big(\log|F|, \partial\bar\partial\varphi\big) + \frac{\sqrt{-1}}{\pi}\int_{\mathcal{S}} \big(\log|\langle c,u\rangle|, \partial\bar\partial\varphi\big)\, d\gamma\ ,\end{aligned}$$

for all test forms $\varphi \in \mathcal{D}^{m-1,m-1}(U)$.

A key point is that $\langle c, u(z)\rangle$ is a standard (mean 0, variance 1) complex Gaussian random variable for all $z \in U$ (since $u(z)$ is unit vector) and hence $\mathbf{E}\big(\log|\langle c, u(z)\rangle|\big)$ is a universal constant C independent of z.

Thus

$$\begin{aligned}\int_{\mathcal{S}} \left(\log|\langle c,u\rangle|, \partial\bar\partial\varphi\right) d\gamma &= \int_{\mathcal{S}} d\gamma(c) \int_M \log|\langle c,u\rangle|\, \partial\bar\partial\varphi \\ &= \int_M \left[\int_{\mathcal{S}} \log|\langle c,u\rangle| d\gamma(c)\right] \partial\bar\partial\varphi \\ &= C\int_M \partial\bar\partial\varphi \;=\; 0\,.\end{aligned}$$

Fubini's Theorem can be applied above since

$$\int_{M\times\mathcal{S}} \big|\log|\langle c,u\rangle|\,\partial\bar\partial\varphi\big|\; d\gamma_N(c) = \left(\int_{\mathbb{C}} \big|\log|\zeta|\big|\,\frac{1}{\pi}\,e^{-|\zeta|^2}d\zeta\right)\left(\int_M |\partial\bar\partial\varphi|\right) < +\infty\,.$$

Therefore

$$\mathbf{E}_\gamma(Z_s) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|F|^2 = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\left(\log\sum\|S_j\|_h^2 + N\log a\right)\,. \tag{17}$$

The formula of the proposition then follows from (11), (14) and (17). □

Remark: Proposition 2.1 also holds if M is noncompact and without the assumption that $\mathcal{S}$ is base point free or even finite dimensional.

Next, we give a general result on the expected value of the simultaneous zero current of k independent random holomorphic sections:

PROPOSITION **2.2.** *Let $(L,h)\to M$, $\mathcal{S}\subset H^0(M,L)$, and γ be given as in Proposition 2.1, and let $1\le k\le m$. Then the expected value of the simultaneous zero current of k independent random sections $s_1,\dots,s_k$ in $\mathcal{S}$ is given by*

$$\mathbf{E}_{\gamma^k}\left(Z_{s_1,\dots,s_k}\right) = \left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\Pi_{\mathcal{S}}(z,z) + c_1(L,h)\right)^k\,.$$

The proposition is a formal consequence of Proposition 2.1 and the independence of the sections s_j, but needs a proof since the wedge products of currents is not always defined. We give here a simple induction proof without using the theory of wedge products of singular currents.

Proof. Let ω be the Kähler form on M. We first note that

$$\left(Z_{s_1,\dots,s_k}, \omega^{m-k}\right) = \int_M c_1(L,h)^k\wedge\omega^{m-k}\,, \tag{18}$$

whenever the Z_{s_j} are smooth and intersect transversely. The identity (18) is a consequence of the fact that the current $Z_{s_1,\dots,s_k}$ and the smooth form $c_1(L,h)^k$ are in the same de Rham cohomology class. Equation (18) can also be verified by induction: the case $k=1$ follows immediately from the Poincaré-Lelong formula (15) and the fact that ω is closed; assuming the result for $k-1$ sections on Z_{s_1}, we have

$$\left(Z_{s_1}\cap Z_{s_2,\dots,s_k}, \omega^{m-k}\right) = \int_{Z_{s_1}} c_1(L,h)^{k-1}\wedge\omega^{m-k} = \int_M c_1(L,h)\wedge c_1(L,h)^{k-1}\wedge\omega^{m-k}\,,$$

which gives (18) for k sections.

Now let $\varphi \in \mathcal{D}^{m-k,m-k}(M)$ be a test form. By (18) and the formula for the volume of complex submanifolds (7), we then have

$$\begin{aligned}\left|\big(Z_{s_1,\dots,s_k},\varphi\big)\right| &= \left|\int_{Z_{s_1,\dots,s_k}}\varphi\right| \le \sup\|\varphi\|\,\mathrm{Vol}(Z_{s_1,\dots,s_k})\\ &= \frac{\sup\|\varphi\|}{(m-k)!}\big(Z_{s_1,\dots,s_k},\omega^{m-k}\big) = \frac{\sup\|\varphi\|}{(m-k)!}\int_M c_1(L,h)^k\wedge\omega^{m-k},\end{aligned}$$

for almost all $s_1,\dots,s_k$. Thus the random variable $\big(Z_{s_1,\dots,s_k},\varphi\big)$ is L^∞, so its expected value is well defined.

We must show that

$$\mathbf{E}_{\gamma^k}\big(Z_{s_1,\dots,s_k},\varphi\big) = \int_M \alpha^k\wedge\varphi\ , \tag{19}$$

where

$$\alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\Pi_{\mathcal{S}}(z,z) + c_1(L,h)\ .$$

We verify (19) by induction on k: For $k=1$, Proposition 2.1 yieldds (19). Let $k\ge 2$ and suppose that (19) has been verified for $k-1$ sections. Choose $s_1\in\mathcal{S}$ such that Z_{s_1} is a submanifold, and let $M' = Z_{s_1}$, $s'_j = s_j|_{M'}$, $\mathcal{S}'=\mathcal{S}|_{M'}$. We give $\mathcal{S}'$ the Gaussian measure $\gamma' := \rho_*\gamma$, where $\rho:\mathcal{S}\to\mathcal{S}'$ is the restriction map, and we note that

$$\Pi_{\mathcal{S}'}(z',z') = \mathbf{E}_{\gamma'}\left(\|s'(z)\|_h^2\right) = \mathbf{E}_{\gamma}\left(\|s(z)\|_h^2\right) = \Pi_{\mathcal{S}}(z',z')\ ,\quad \text{for } z'\in M'\ .$$

By the inductive assumption applied to $M',\mathcal{S}'$, and noting that $Z_{s_1,\dots,s_k} = Z_{s'_2,\dots,s'_k}$, we have

$$\int_{\mathcal{S}^{k-1}}\big(Z_{s_1,\dots,s_k},\varphi\big)\,d\gamma(s_2)\cdots d\gamma(s_k) = \mathbf{E}_{\gamma'^{k-1}}\big(Z_{s'_2,\dots,s'_k},\varphi\big) = \int_{Z_{s_1}}\alpha^{k-1}\wedge\varphi\ . \tag{20}$$

We average (20) over s_1 and apply Proposition 2.1 to conclude that

$$\int_{\mathcal{S}^k}\big(Z_{s_1,\dots,s_k},\varphi\big)\,d\gamma(s_1)\cdots d\gamma(s_k) = \int_{\mathcal{S}}\big(Z_{s_1},\alpha^{k-1}\wedge\varphi\big)\,d\gamma(s_1) = \big(\alpha\,,\alpha^{k-1}\wedge\varphi\big) = \int_M\alpha^k\wedge\varphi,$$

which gives (19). □

2.1.1. *Powers of a positive line bundle.* We now specialize Proposition 2.2 to our case of interest. We suppose that the line bundle (L,h) is *positive*, i.e. the $(1,1)$-form $c_1(L,h)$ is everywhere positive definite, and we give M the Kähler form $\omega = \frac{i}{2}\Theta_h = \pi c_1(L,h)$. Recall that it is a consequence of the Kodaira embedding theorem that for sufficiently large integers N, the spaces of global sections of the tensor powers $L^N = L^{\otimes N}$ of the line bundle are base point free. (In fact, the global sections give a projective embedding [GH, §1.4]. The Kodaira embedding theorem is also a consequence of the Tian-Yau-Zelditch theorem [Ca, Ti, Ze2]; see (29).) We recall that the Hermitian metric h on L induces Hermitian metrics h^N on L^N, and we have

$$c_1(L^N,h^N) = N\,c_1(L,h) = \frac{N}{\pi}\,\omega\,.$$

We give $H^0(M,L^N)$ the Hermitian inner product induced by the metrics h,ω, as defined by (4); this inner product induces the Hermitian Gaussian measure γ_N given by (5). Considering the spaces $\mathcal{S}_N = H^0(M, L^N)$, we have the Szegő kernels (on the diagonal)

$$\Pi_N(z,z) := \Pi_{\mathcal{S}_N}(z,z) = \sum_{j=1}^{d_N} \|S_j^N(z)\|_{h^N}^2 \,, \tag{21}$$

where $\{S_1^N, \dots, S_{d_N}^N\}$ is an orthonormal basis for $H^0(M, L^N)$ with respect to the Hermitian inner product (4). These Szegő kernels were analyzed in [Ze2, BSZ1, SZ1, SZ2] by viewing them as orthogonal projectors on $\mathcal{L}^2(X)$, where $X \to M$ is the circle bundle of unit vectors of L^{-1}. We give this description of Π_N in §2.2 below.

Applying Proposition 2.2 to the line bundles L^N and the spaces $H^0(M, L^N)$ of holomorphic sections, we obtain:

Corollary **2.3.** *Let $(L,h) \to (M,\omega)$ be as in Theorem 1.1, and let γ_N be the Hermitian Gaussian measure on $H^0(M, L^N)$. Then for $1 \le k \le m$ and N sufficiently large, we have*

$$\mathbf{E}_{\gamma_N^k}\left(Z_{s_1^N,\dots,s_k^N}\right) = \left(\mathbf{E}_{\gamma_N} Z_{s^N}\right)^k = \left(\frac{i}{\pi}\partial\bar\partial \log \Pi_N(z,z) + \frac{N}{\pi}\omega\right)^k .$$

2.2. **Off-diagonal asymptotics for the Szegő kernel.** As in [Ze2, SZ1, BSZ1] and elsewhere, we analyze the Szegő kernel for $H^0(M, L^N)$ by lifting it to the circle bundle $X \xrightarrow{\pi} M$ of unit vectors in the dual bundle $L^{-1} \to M$ with respect to h. In the standard way (loc. cit.), sections of L^N lift to equivariant functions on X. Then $s \in H^0(M, L^N)$ lifts to a CR holomorphic functions on X satisfying $\hat s(e^{i\theta}x) = e^{iN\theta}\hat s(x)$. We denote the space of such functions by $\mathcal{H}_N^2(X)$. The *Szegő projector* is the orthogonal projector $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}_N^2(X)$, which is given by the *Szegő kernel*

$$\Pi_N(x,y) = \sum_{j=1}^{d_N} \widehat{S}_j^N(x)\overline{\widehat{S}_j^N(y)} \qquad (x, y \in X) \,.$$

(Here, the functions $\widehat{S}_j^N$ are the lifts to $\mathcal{H}_N^2(X)$ of the orthonormal sections S_j^N; they provide an orthonormal basis for $\mathcal{H}_N^2(X)$.)

Further, the covariant derivative ∇s of a section s lifts to the horizontal derivative $\nabla_h \hat s$ of its equivariant lift $\hat s$ to X; the horizontal derivative is of the form

$$\nabla_h \hat s = \sum_{j=1}^m \left(\frac{\partial \hat s}{\partial z_j} - A_j \frac{\partial \hat s}{\partial \theta}\right) dz_j. \tag{22}$$

For further discussion and details on lifting sections, we refer to [SZ1].

Our pluri-bipotential for the variance described in §3 is based on the *normalized Szegő kernels*

$$P_N(z,w) := \frac{|\Pi_N(z,w)|}{\Pi_N(z,z)^{\frac12}\Pi_N(w,w)^{\frac12}} \,, \tag{23}$$

where we write

$$|\Pi_N(z,w)| := |\Pi_N(x,y)| \,, \quad z = \pi(x),\ w = \pi(y) \in M \,.$$

In particular, on the diagonal we have $\Pi_N(z,z) = \Pi_N(x,x) > 0$. Note that $\Pi_N(z,z) = \Pi_{\mathcal{S}}(z,z)$ as defined in (21) with $\mathcal{S} = H^0(M, L^N)$.

In this section, we use the off-diagonal asymptotics for $\Pi_N(x,y)$ from [SZ2] to provide the off-diagonal estimates for the normalized Szegő kernel $P_N(z,w)$ that we need for our variance formulas. Our estimates are of two types: (1) 'near-diagonal' asymptotics (Propositions 2.7–2.8) for $P_N(z,w)$ where the distance $\mathrm{dist}(z,w)$ between z and w satisfies an upper bound

$\mathrm{dist}(z,w) \leq b\left(\frac{\log N}{N}\right)^{1/2}$ $(b \in \mathbb{R}^+)$; (2) 'far-off-diagonal' asymptotics (Proposition 2.6) where $\mathrm{dist}(z,w) \geq b\left(\frac{\log N}{N}\right)^{1/2}$.

To describe the scaling asymptotics for the Szegő kernel at a point $z_0 \in M$, we choose a neighborhood U of z_0, a local normal coordinate chart $\rho : U, z_0 \to \mathbb{C}^m, 0$ centered at z_0, and a *preferred* local frame at z_0, which we defined in [SZ2] to be a local frame e_L such that

$$\|e_L(z)\|_h = 1 - \tfrac{1}{2}\|\rho(z)\|^2 + \cdots \ . \tag{24}$$

For $u = (u_1, \dots, u_m) \in \rho(U),\ \theta \in (-\pi, \pi)$, we let

$$\tilde\rho(u_1, \dots, u_m, \theta) = \frac{e^{i\theta}}{|e_L^*(\rho^{-1}(u))|_h} e_L^*(\rho^{-1}(u)) \in X\,, \tag{25}$$

so that $(u_1, \dots, u_m, \theta) \in \mathbb{C}^m \times \mathbb{R}$ give local coordinates on X. As in [SZ2], we write

$$\Pi_N^{z_0}(u,\theta;v,\varphi) = \Pi_N(\tilde\rho(u,\theta), \tilde\rho(v,\varphi))\ .$$

Note that $\Pi_N^{z_0}$ depends on the choice of coordinates and frame; we shall assume that we are given normal coordinates and local frames for each point $z_0 \in M$ and that these normal coordinates and local frames are smooth functions of z_0. The scaling asymptotics of $\Pi_N^{z_0}(u,\theta;v,\varphi)$ lead to the model Heisenberg Szegő kernel

$$\Pi_N^{\mathbf{H}}(z,\theta;w,\varphi) = e^{iN(\theta-\varphi)} \sum_{k\in\mathbb{N}^m} S_k(z)\overline{S_k(w)} = \frac{N^m}{\pi^m} e^{iN(\theta-\varphi) + Nz\cdot\bar w - \frac{N}{2}(|z|^2+|w|^2)} \tag{26}$$

of level N for the Bargmann-Fock space of functions on $\mathbb{C}^m$ (see [BSZ2]).

We shall apply the following (near and far) off-diagonal asymptotics from [SZ2]:

THEOREM **2.4.** *Let $(L,h) \to (M,\omega)$ be as in Theorem 1.1, and let $z_0 \in M$. Then using the above notation,*

i) $N^{-m}\Pi_N^{z_0}(\frac{u}{\sqrt N}, \frac{\theta}{N}; \frac{v}{\sqrt N}, \frac{\varphi}{N})$

$$= \Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)\left[1 + \textstyle\sum_{r=1}^k N^{-r/2} p_r(u,v) + N^{-(k+1)/2} R_{Nk}(u,v)\right]\ ,$$

where the p_r are polynomials in (u,v) of degree $\leq 5r$ (of the same parity as r), and

$$|\nabla^j R_{Nk}(u,v)| \leq C_{jk\varepsilon b} N^\varepsilon \quad \text{for} \quad |u| + |v| < b\sqrt{\log N}\,,$$

for $\varepsilon, b \in \mathbb{R}^+$, $j,k \geq 0$. Furthermore, the constant $C_{jk\varepsilon b}$ can be chosen independently of z_0.

ii) *For $b > \sqrt{j + 2k + 2m}$, $j,k \geq 0$, we have*

$$\left|\nabla_h^j \Pi_N(z,w)\right| = O(N^{-k}) \qquad \text{uniformly for } \ \mathrm{dist}(z,w) \geq b\sqrt{\frac{\log N}{N}}\ .$$

Here $\nabla^j R = \left\{\frac{\partial^j R}{\partial u^{K'} \partial v^{K''}} : |K'| + |K''| = j\right\}$, and $\nabla_h^j = (\nabla_h)^j$ denotes the j-th iterated horizontal covariant derivative; see (22). Theorem 2.4 is equivalent to equations (95)–(96) in [SZ2], where the result was shown to hold for almost-complex symplectic manifolds. (The remainder in (i) was given for $v = 0$, but the proof holds without any change for $v \neq 0$. Also the statement of the result was divided into the two cases where the scaled distance is less or more, respectively, than $N^{1/6}$ instead of $\sqrt{\log N}$ in the above formulation, which is more useful for our purposes.) A description of the polynomials p_r in part (i) is given in [SZ2], but we only need the $k = 0$ case in this paper. For the benefit of the reader, we give a proof of Theorem 2.4 in §5.

Remark: The Szegő kernel actually satisfies the sharper 'Agmon decay estimate' away from the diagonal:

$$\nabla^j \Pi_N(z,\theta;w,\varphi) = O\left(e^{-A_j\sqrt{N}\,\mathrm{dist}(z,w)}\right)\,, \qquad j \geq 0\,. \tag{27}$$

In particular,

$$|\Pi_N(z,w)| = O\left(e^{-A\sqrt{N}\,\mathrm{dist}(z,w)}\right)\,. \tag{28}$$

A short proof of (28) is given in [Be, Th. 2.5]; similar estimates were established by M. Christ [Ch], H. Delin [De], and N. Lindholm[Li]. (See also [DLM, MM] for off-diagonal exponential estimates in a more general setting.) We do not need Agmon estimates for this paper; instead Theorem 2.4 suffices.

It follows from Theorem 2.4(i)) with $k = 1$ that on the diagonal, the Szegő kernel is of the form

$$\Pi_N(z,z) = \frac{1}{\pi^m} N^m (1 + O(N^{-1}))\,, \tag{29}$$

which comprises the leading terms of the Tian-Yau-Zelditch asymptotic expansion of the Szegő kernel [Ca, Ti, Ze2]. Applying (29), we obtain the asymptotic formula from [SZ1] for the expected simultaneous zero currents:

Proposition **2.5.** [SZ1, Prop. 4.4] *Let $(L,h) \to (M,\omega)$ be as in Theorem 1.1, and let $1 \leq k \leq m$. Then for independent random sections $s_1^N,\dots,s_k^N$ in $H^0(M,L^N)$, we have*

$$\mathbf{E}\big(Z_{s_1^N,\dots,s_k^N}\big) = \frac{N^k}{\pi^k}\omega^k + O\left(N^{k-1}\right).$$

Proof. By (29), $\partial\bar\partial \log \Pi_N(z,z) = O(N^{-1})$. The asymptotics for $\mathbf{E}\big(Z_{s_1^N,\dots,s_k^N}\big)$ then follow from the formula of Corollary 2.3. □

We now state our far-off-diagonal decay estimate for $P_N(z,w)$, which follows immediately from Theorem 2.4(ii) and (29).

Proposition **2.6.** *Let $(L,h) \to (M,\omega)$ be as in Theorem 1.1, and let $P_N(z,w)$ be the normalized Szegő kernel for $H^0(M,L^N)$ given by (23). For $b > \sqrt{j+2k}$, $j,k \geq 0$, we have*

$$\nabla^j P_N(z,w) = O(N^{-k}) \qquad \textit{uniformly for } \ \mathrm{dist}(z,w) \geq b\sqrt{\frac{\log N}{N}}\,.$$

The normalized Szegő kernel P_N also satisfies Gaussian decay estimates valid very close to the diagonal. To give the estimate, we write by abuse of notation,

$$P_N(z_0+u, z_0+v) := P_N(\rho^{-1}(u), \rho^{-1}(v)) = \frac{|\Pi_N^{z_0}(u,0;v,0)|}{\Pi_N^{z_0}(u,0;u,0)^{1/2}\,\Pi_N^{z_0}(v,0;v,0)^{1/2}}\,. \tag{30}$$

As an immediate consequence of Theorem 2.4(i), we have:

Proposition **2.7.** *Let $P_N(z,w)$ be as in Proposition 2.6, and let $z_0 \in M$. For $b, \varepsilon > 0$, $j \geq 0$, there is a constant $C_j = C_j(\varepsilon, b)$, independent of the point z_0, such that*

$$\begin{aligned} P_N\left(z_0 + \tfrac{u}{\sqrt{N}}, z_0 + \tfrac{v}{\sqrt{N}}\right) &= e^{-\frac12|u-v|^2}[1 + R_N(u,v)] \\ &\qquad |\nabla^j R_N(u,v)| \leq C_j\, N^{-1/2+\varepsilon} \quad \textit{for } \ |u|+|v| < b\sqrt{\log N}\,. \end{aligned}$$

As a corollary we have:

Proposition 2.8. *The remainder R_N in Proposition 2.7 satisfies*

$$|R_N(u,v)| \le \frac{C_2}{2}\,|u-v|^2 N^{-1/2+\varepsilon}, \quad |\nabla R_N(u)| \le C_2\,|u-v|\,N^{-1/2+\varepsilon}, \quad \textit{for}\ \ |u|+|v| < b\sqrt{\log N}.$$

Proof. Since $P_N(z_0+u, z_0+v) \le 1 = P_N(z_0+u, z_0+u)$, we conclude that $R_N(u,u) = 0$, $dR_N|_{(u,u)} = 0$, and thus by Proposition 2.7,

$$|\nabla R_N(u,v)| \le \sup_{0\le t\le 1} |\nabla^2 R_N(u,(1-t)u+tv)|\,|u-v| \le C_2\,|u-v|\,N^{-1/2+\varepsilon}\ .$$

Similarly,

$$|R_N(u,v)| \le \tfrac{1}{2} \sup_{0\le t\le 1} |\nabla^2 R_N(u,(1-t)u+tv)|\,|u-v|^2 \le \frac{C_2}{2}\,|u-v|^2\,N^{-1/2+\varepsilon}\ .$$

□

3. A pluri-bipotential for the variance

Our proof of Theorems 1.1 is based on a pluri-bipotential given implicitly in [SZ1] for the variance current for random zeros in codimension one. More generally, for random codimension k zeros, we define the *variance current* of $Z_{s_1^N,\dots,s_k^N}$ to be the current

$$\mathbf{Var}\big(Z_{s_1^N,\dots,s_k^N}\big) := \mathbf{E}\big(Z_{s_1^N,\dots,s_k^N} \boxtimes Z_{s_1^N,\dots,s_k^N}\big) - \mathbf{E}\big(Z_{s_1^N,\dots,s_k^N}\big) \boxtimes \mathbf{E}\big(Z_{s_1^N,\dots,s_k^N}\big) \in \mathcal{D}'^{2k,2k}(M\times M). \tag{31}$$

Here we write

$$S \boxtimes T = \pi_1^* S \wedge \pi_2^* T \in \mathcal{D}'^{p+q}(M\times M)\ , \qquad \text{for}\ \ S \in \mathcal{D}'^p(M),\ T\in \mathcal{D}'^q(M)\ ,$$

where $\pi_1, \pi_2 : M\times M \to M$ are the projections to the first and second factors, respectively. The variance for the 'smooth zero statistics' is given by:

$$\mathrm{Var}\big(Z_{s_1^N,\dots,s_k^N}, \varphi\big) = \Big(\mathbf{Var}\big(Z_{s_1^N,\dots,s_k^N}\big),\ \varphi\boxtimes\varphi\Big)\ . \tag{32}$$

Conversely, (32) can be taken as an equivalent definition of the variance current in terms of $\mathrm{Var}\big(Z_{s_1^N,\dots,s_k^N}, \varphi\big)$.

Theorem 3.1 below gives a *pluri-bipotential* for the variance current in codimension one, i.e. a function $Q_N \in L^1(M\times M)$ such that

$$\mathbf{Var}\big(Z_{s^N}\big) = (i\partial\bar\partial)_z\,(i\partial\bar\partial)_w\,Q_N(z,w)\ . \tag{33}$$

To describe our pluri-bipotential $Q_N(z,w)$, we define the function

$$\widetilde G(t) := -\frac{1}{4\pi^2}\int_0^{t^2} \frac{\log(1-s)}{s}\,ds \ = \ \frac{1}{4\pi^2}\sum_{n=1}^\infty \frac{t^{2n}}{n^2}\ , \qquad 0\le t\le 1. \tag{34}$$

Alternately,

$$\widetilde G(e^{-\lambda}) = -\frac{1}{2\pi^2}\int_\lambda^\infty \log(1-e^{-2s})\,ds\ , \qquad \lambda \ge 0\ . \tag{35}$$

The function $\widetilde G$ is a modification of the function G defined in [BSZ1]; see (57).

Theorem 3.1. *Let $(L, h) \to (M, \omega)$ be as in Theorems 1.1. Let $Q_N : M \times M \to [0, +\infty)$ be the function given by*

$$Q_N(z, w) = \widetilde{G}(P_N(z, w)) = -\frac{1}{4\pi^2} \int_0^{P_N(z,w)^2} \frac{\log(1-s)}{s} \, ds \ , \tag{36}$$

where $P_N(z, w)$ is the normalized Szegő kernel given by (23)*. Then*

$$\mathbf{Var}\big(Z_{s^N}\big) = (i\partial\bar\partial)_z \, (i\partial\bar\partial)_w \, Q_N(z, w) \ .$$

Theorem 3.1 says that

$$\mathrm{Var}(Z_{s^N}, \varphi) = \big(-\partial_z\bar\partial_z\partial_w\bar\partial_w Q_N, \ \varphi \boxtimes \varphi \big) = \int_M \int_M Q_N(z, w) \, (i\partial\bar\partial\varphi(z)) \, (i\partial\bar\partial\varphi(w)) \ , \tag{37}$$

for test forms $\varphi \in \mathcal{D}^{m-1,m-1}(M)$. We note that Q_N is $\mathcal{C}^\infty$ off the diagonal, but is only $\mathcal{C}^1$ and not $\mathcal{C}^2$ at all points on the diagonal in $M \times M$, as the computations in §3.1 show. Additionally, its derivatives of order ≤ 4 are in $L^{m-\varepsilon}(M \times M)$ (see Lemma 3.7).

To begin the proof of the theorem, we write

$$\Psi_N = (S_1^N, \ldots, S_{d_N}^N) \in H^0(M, L^N)^{d_N} \ , \tag{38}$$

where $\{S_j^N\}$ is an orthonormal basis of $H^0(M, L^N)$. As in the proof of Proposition 2.1, we write

$$\Psi_N(z) = |\Psi_N(z)| \, u_N(z) \ , \tag{39}$$

where $|\Psi_N| := (\sum_j \|S_j^N\|^2_{h^N})^{1/2}$, so that $|u_N| \equiv 1$. For $c = (c_1, \ldots, c_{d_N})$, we write

$$\begin{aligned} \langle c, u_N(z)\rangle &= \left\langle c, \frac{1}{|\Psi_N(z)|} \Psi_N(z) \right\rangle = \frac{1}{|\Psi_N(z)|} \sum_{j=1}^{d_N} c_j \, S_j^N(z) \in L_z^N \ , \\ |\langle c, u_N(z)\rangle| &= \|\langle c, u_N(z)\rangle\|_{h^N} \ . \end{aligned}$$

Lemma 3.2.

$$\mathbf{Var}(Z_{s^N}) = -\frac{1}{\pi^2}\partial_z\bar\partial_z\partial_w\bar\partial_w \int_{\mathbb{C}^{d_N}} \log|\langle c, u_N(z)\rangle| \, \log|\langle c, u_N(w)\rangle| \, d\gamma_N(c) \ .$$

Proof. We write sections $s^N \in H^0(M, L^N)$ as

$$s^N = \sum_{j=1}^{d_N} c_j S_j^N = \langle c, \Psi_N\rangle \ , \qquad c = (c_1, \ldots, c_{d_N}) \ . \tag{40}$$

Writing $\Psi_N = F e_L^{\otimes N}$, where e_L is a local nonvanishing section of L, and recalling that

$$\omega = \frac{i}{2}\Theta_h = -i\partial\bar\partial \log \|e_L\|_h \ ,$$

we have by (15),

$$\begin{aligned} Z_{s^N} &= \frac{i}{\pi}\partial\bar\partial \log|\langle c, F\rangle| = \frac{i}{\pi}\partial\bar\partial \log|\langle c, \Psi_N\rangle| - \frac{i}{\pi}\partial\bar\partial \log\|e_L^{\otimes N}\|_h \\ &= \frac{i}{\pi}\partial\bar\partial \log|\langle c, \Psi_N\rangle| + \frac{N}{\pi}\,\omega \ . \end{aligned} \tag{41}$$

Consider the random current

$$\widehat{Z}_N := \frac{i}{\pi}\partial\bar\partial \log|\langle c, \Psi_N\rangle| = Z_{s^N} - \frac{N}{\pi}\omega \, . \tag{42}$$

It follows immediately from the definition (31) of variance currents that

$$\mathbf{Var}(\widehat{Z}_N) = \mathbf{E}(\widehat{Z}_N \boxtimes \widehat{Z}_N) - \mathbf{E}(\widehat{Z}_N) \boxtimes \mathbf{E}(\widehat{Z}_N) = \mathbf{Var}(Z_{s^N}) \ .$$

By (17), we have

$$\mathbf{E}(\widehat{Z}_N) = \frac{i}{\pi}\partial\bar\partial \log|\Psi_N| \ , \tag{43}$$

whereas by (42), we have

$$\begin{aligned}\mathbf{E}(\widehat{Z}_N \boxtimes \widehat{Z}_N) &= -\frac{1}{\pi^2}\int_{\mathbb{C}^{d_N}} \partial_z\bar\partial_z\partial_w\bar\partial_w \log|\langle c, \Psi_N(z)\rangle| \log|\langle c, \Psi_N(w)\rangle|\, d\gamma_N(c) \\ &= -\frac{1}{\pi^2}\,\partial_z\bar\partial_z\partial_w\bar\partial_w \int_{\mathbb{C}^{d_N}} \log|\langle c, \Psi_N(z)\rangle| \log|\langle c, \Psi_N(w)\rangle|\, d\gamma_N(c) \ .\end{aligned} \tag{44}$$

Recalling (39), we have

$$\begin{aligned}\log|\langle \Psi_N(z), c\rangle| \log|\langle \Psi_N(w), c\rangle| &= \log|\Psi_N(z)| \log|\Psi_N(w)| + \log|\Psi_N(z)| \log|\langle c, u_N(w)\rangle| \\ &\quad + \log|\Psi_N(w)| \log|\langle c, u_N(z)\rangle| \\ &\quad + \log|\langle c, u_N(w)\rangle| \log|\langle c, u_N(z)\rangle| \ ,\end{aligned} \tag{45}$$

which decomposes (44) into four terms. By (43), the first term contributes

$$-\frac{1}{\pi^2}\,\partial\bar\partial \log|\Psi_N(z)| \wedge \partial\bar\partial\log|\Psi_N(w)| = \mathbf{E}(\widehat{Z}_N) \boxtimes \mathbf{E}(\widehat{Z}_N)\,.$$

The c-integral in the second term is independent of w and hence the second term vanishes when applying $\partial_w\bar\partial_w$. The third term likewise vanishes when applying $\partial_z\bar\partial_z$. Therefore, the fourth term gives the variance current $\mathbf{Var}(Z_{s^N})$. □

To complete the proof of Theorem 3.1, we use the following probability lemma, which gives the c-integral of Lemma 3.2:

Lemma **3.3.** *Let (Y_1, Y_2) be joint complex Gaussian random variables with mean 0 and* $\mathbf{E}(|Y_1|^2) = \mathbf{E}(|Y_2|^2) = 1$. *Then*

$$\mathbf{E}\big(\log|Y_1| \log|Y_2|\big) = G\big(\left|\mathbf{E}(Y_1\overline{Y}_2)\right|\big) \ ,$$

where

$$G(t) := \frac{\gamma^2}{4} - \frac{1}{4}\int_0^{t^2} \frac{\log(1-s)}{s}\, ds \ , \quad 0 \le t \le 1 \qquad (\gamma = \textit{Euler's constant})\,.$$

Proof. By replacing Y_1 with $e^{i\alpha}\, Y_1$, we can assume without loss of generality that $\mathbf{E}(Y_1\overline{Y}_2) \ge 0$. We can write

$$\begin{aligned} Y_1 &= \Xi_1 \ , \\ Y_2 &= (\cos\theta)\,\Xi_1 + (\sin\theta)\,\Xi_2 \ ,\end{aligned}$$

where Ξ_1, Ξ_2 are independent joint complex Gaussian random variables with mean 0 and variance 1, and $\cos\theta = \mathbf{E}(Y_1\overline{Y}_2)$. Then

$$\mathbf{E}(\log|Y_1| \log|Y_2|) = G(\cos\theta) \ , \tag{46}$$

where

$$G(\cos\theta) = \frac{1}{\pi^2}\int_{\mathbb{C}^2} \log|\Xi_1| \log|\Xi_1\cos\theta + \Xi_2\sin\theta|\ e^{-(|\Xi_1|^2+|\Xi_2|^2)}\, d\Xi_1\, d\Xi_2 \ . \tag{47}$$

The computation of $G(\cos\theta)$ was essentially given in [BSZ1, §4.1]. We repeat this computation here for the readers' convenience: Write $\Xi_1 = r_1 e^{i\alpha}$, $\Xi_2 = r_2 e^{i(\alpha+\varphi)}$, so that (47) becomes

$$G(\cos\theta) = \frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^{2\pi} r_1 r_2 e^{-(r_1^2+r_2^2)}\log r_1 \log|r_1\cos\theta + r_2 e^{i\varphi}\sin\theta|\,d\varphi\,dr_1\,dr_2\ .$$

Evaluating the inner integral by Jensen's formula, we obtain

$$\int_0^{2\pi}\log|r_1\cos\theta + r_2\sin\theta e^{i\varphi}|\,d\varphi = \begin{cases} 2\pi\log(r_1\cos\theta) & \text{for } \ r_2\sin\theta \le r_1\cos\theta \\ \\ 2\pi\log(r_2\sin\theta) & \text{for } \ r_2\sin\theta \ge r_1\cos\theta \end{cases}$$

Hence

$$G(\cos\theta) = 4\int_0^\infty\int_0^\infty r_1 r_2 e^{-(r_1^2+r_2^2)}\log r_1 \log\max(r_1\cos\theta, r_2\sin\theta)\,dr_1\,dr_2.$$

We make the change of variables $r_1 = \rho\cos\varphi$, $r_2 = \rho\sin\varphi$ to get

$$G(\cos\theta) = 4\int_0^\infty\int_0^{\pi/2}\rho^3 e^{-\rho^2}\log(\rho\cos\varphi)\log\max(\rho\cos\varphi\cos\theta,\rho\sin\varphi\sin\theta)\cos\varphi\sin\varphi\,d\varphi\,d\rho\ .$$

Since

$$\log\max(\rho\cos\varphi\cos\theta,\rho\sin\varphi\sin\theta) = \log(\rho\cos\varphi\cos\theta) + \log^+(\tan\varphi\tan\theta)\ ,$$

we can write $G = G_1 + G_2$, where

$$G_1(\cos\theta) = 4\int_0^\infty\int_0^{\pi/2}\rho^3 e^{-\rho^2}\log(\rho\cos\varphi)\log(\rho\cos\varphi\cos\theta)\cos\varphi\sin\varphi\,d\varphi\,d\rho\ , \tag{48}$$

$$G_2(\cos\theta) = 4\int_0^\infty\int_{\pi/2-\theta}^{\pi/2}\rho^3 e^{-\rho^2}\log(\rho\cos\varphi)\log(\tan\varphi\tan\theta)\cos\varphi\sin\varphi\,d\varphi\,d\rho\ . \tag{49}$$

From (48), $G_1(\cos\theta) = C_1 + C_2\log\cos\theta$. Substituting

$$\cos\theta = e^{-\lambda}\ ,$$

we obtain

$$G_1(e^{-\lambda}) = C_1 - C_2\lambda\ . \tag{50}$$

We now evaluate G_2. Since the integrand in (49) vanishes when $\varphi = \pi/2 - \theta$, we have

$$\frac{d}{d\lambda}G_2(e^{-\lambda}) = 4\left(\frac{d}{d\lambda}\log\tan\theta\right)\int_0^\infty\int_{\pi/2-\theta}^{\pi/2}\rho^3 e^{-\rho^2}\log(\rho\cos\varphi)\cos\varphi\sin\varphi\,d\varphi\,d\rho\ .$$

Since

$$\frac{d}{d\lambda}\log\tan\theta = \frac{1}{2}\frac{d}{d\lambda}\log(e^{2\lambda}-1) = \frac{1}{1-e^{-2\lambda}}\ ,$$

we have

$$\frac{d}{d\lambda}G_2(e^{-\lambda}) = \frac{4}{1-e^{-2\lambda}}\ (I_1 + I_2)\ ,$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty \int_{\pi/2-\theta}^{\pi/2} \rho^3 e^{-\rho^2} (\log\rho)\cos\varphi\sin\varphi\,d\varphi\,d\rho \;=\; C_3\sin^2\theta \;=\; C_3(1-e^{-2\lambda})\ ,\\
I_2 &= \int_0^\infty \int_{\pi/2-\theta}^{\pi/2} \rho^3 e^{-\rho^2} (\log\cos\varphi)\cos\varphi\sin\varphi\,d\varphi\,d\rho\\
&= \frac12\int_{\pi/2-\theta}^{\pi/2}(\log\cos\varphi)\cos\varphi\sin\varphi\,d\varphi \;=\; \frac12\int_0^{\sin\theta} t\log t\,dt\\
&= \frac18(\sin^2\theta\log\sin^2\theta-\sin^2\theta) \;=\; \frac18(1-e^{-2\lambda})\left[\log(1-e^{-2\lambda})-1\right]\ .
\end{aligned}$$

Thus

$$\frac{d}{d\lambda}G_2(e^{-\lambda}) = \frac12\log(1-e^{-2\lambda}) + 4\,C_3 - \frac12\ . \tag{51}$$

Combining (50)–(51), we have

$$G(e^{-\lambda}) = C_4 + C_5\,\lambda + \frac12\int_0^\lambda \log(1-e^{-2s})\,ds\ . \tag{52}$$

By (47),

$$G(0) = \left[\mathbf{E}(\log|\Xi_1|)\right]^2 = \left[2\int_0^\infty (\log r)\,e^{-r^2}\,r\,dr\right]^2 = \frac{\gamma^2}{4}\ .$$

Substituting $\lambda=\infty$ in (52), we conclude that $C_5=0$ and

$$G(e^{-\lambda}) = \frac{\gamma^2}{4} - \frac12\int_\lambda^\infty \log(1-e^{-2s})\,ds\ , \tag{53}$$

or equivalently,

$$G(t) = \frac{\gamma^2}{4} - \frac14\int_0^{t^2}\frac{\log(1-s)}{s}\,ds \qquad (0\le t\le 1)\ . \tag{54}$$

□

Proof of Theorem 3.1: Fix points $z,w\in M$, and let $x,y\in X$ with $\pi(x)=z,\ \pi(y)=w$. We apply Lemma 3.3 with $Y_1=\langle c,\hat u_N(x)\rangle$, $Y_2=\langle c,\hat u_N(y)\rangle$. Since $|\langle c,\hat u_N\rangle| = |\langle c,u_N\rangle|\circ\pi$, we have

$$\log|Y_1| = \log|\langle c,u_N(z)\rangle|\,,\quad \log|Y_2| = \log|\langle c,u_N(w)\rangle|\,.$$

To determine $\mathbf{E}(Y_1\overline{Y}_2)$, we note that for a random $\hat s^N=\sum c_j\widehat S_j^N\in\mathcal{H}^2_N(X)$,

$$\mathbf{E}\left(\hat s(x)\,\overline{\hat s(y)}\right) = \sum_{j,k=1}^{d_N}\mathbf{E}(c_j\bar c_k)\,\widehat S_j^N(x)\,\overline{\widehat S_k^N(y)} = \sum_{j=1}^{d_N}\widehat S_j^N(x)\,\overline{\widehat S_j^N(y)} = \Pi_N(x,y)\ . \tag{55}$$

Since

$$\langle c,\hat u_N(x)\rangle = \frac{\langle c,\widehat\Psi_N(x)\rangle}{\left|\widehat\Psi_N(x)\right|} = \frac{\hat s^N(x)}{\Pi_N(x,x)^{1/2}}\ ,$$

we have by (55),

$$\mathbf{E}(Y_1\overline{Y}_2) = \frac{\Pi_N(x,y)}{\Pi_N(x,x)^{1/2}\Pi_N(y,y)^{1/2}}\ ,$$

and recalling (23),

$$\left|\mathbf{E}(Y_1\overline{Y}_2)\right| = P_N(z,w)\,. \tag{56}$$

Therefore, by Lemma 3.3 and (56),

$$\int_{\mathbb{C}^{d_N}} \log|\langle u_N(z),c\rangle|\,\log|\langle u_N(w),c\rangle|\,d\gamma_N(c) = \mathbf{E}\big(\log|Y_1|\,\log|Y_2|\big) = G\big(P_N(z,w)\big)\,.$$

By (34) and (54),

$$\widetilde{G}(t) = \frac{1}{\pi^2}\left[G(t) - \frac{\gamma^2}{4}\right]\,, \tag{57}$$

and hence, recalling that $\widetilde{G}\circ P_N = Q_N$,

$$\frac{1}{\pi^2}\int_{\mathbb{C}^{d_N}} \log|\langle u_N(z),c\rangle|\,\log|\langle u_N(w),c\rangle|\,d\gamma_N(c) = Q_N(z,w) + C\,. \tag{58}$$

Theorem 3.1 follows by combining Lemma 3.2 and (58). □

3.1. **Asymptotics of the pluri-bipotential.** We now use the Szegő kernel off-diagonal asymptotics to describe the N-asymptotics for the variance current $\mathbf{Var}(Z_{s^N})$ (Lemma 3.9). We also need to know the behavior of the variance current near the diagonal. We showed in [BSZ2, (107)] that the codimension-one scaling limit pair correlation K_{21}^{∞} grows like $|z-w|^{-2}$ near the diagonal (for dimension $m \geq 2$). Our computation of the variance current asymptotics also gives this growth rate for the variance current (Lemma 3.7) as well as for its scaling limit (Lemma 3.9).

We begin by noting that the pluri-bipotential decays rapidly away from the diagonal:

LEMMA **3.4.** *For $b > \sqrt{j+q+1}$, $j \geq 0$, we have*

$$|\nabla^j Q_N(z,w)| = O\left(\frac{1}{N^q}\right)\,,\quad \textit{for}\ \ \mathrm{dist}(z,w) \geq \frac{b\sqrt{\log N}}{\sqrt{N}}\,.$$

Proof. We recall from (36) that $Q_N = \widetilde{G}\circ P_N$, where $\widetilde{G}$ is analytic at 0 (with radius of convergence 1) and $\widetilde{G}(t) = O(t^2)$. The estimate then follows from Proposition 2.6 with $k = \left\lfloor \frac{q+1}{2}\right\rfloor$. □

Applying Lemma 3.4 to the pluri-bipotential formula for the variance of Theorem 3.1, we conclude that the variance current decays rapidly away from the diagonal.

We next show the near-diagonal estimate:

LEMMA **3.5.** *For $b \in \mathbb{R}^+$, we have*

$$Q_N\Big(z_0, z_0 + \frac{v}{\sqrt{N}}\Big) = \widetilde{G}(e^{-\frac12|v|^2}) + O(N^{-1/2+\varepsilon})\,,\qquad \textit{for}\ \ |v| \leq b\sqrt{\log N}.$$

Proof. Since $P_N(z_0,z_0) = 1$ and $\widetilde{G}'(t) \to \infty$ as $t \to 1$, we need a short argument: let

$$\Lambda_N = -\log P_N\,. \tag{59}$$

Recalling (35), we write,

$$F(\lambda) := \widetilde{G}(e^{-\lambda}) = -\frac{1}{2\pi^2}\int_\lambda^\infty \log(1-e^{-2s})\,ds \qquad (\lambda \geq 0)\,, \tag{60}$$

so that

$$Q_N = F\circ\Lambda_N\,. \tag{61}$$

By Proposition 2.8,

$$\Lambda_N\left(z_0, z_0 + \frac{v}{\sqrt{N}}\right) = \tfrac12|v|^2 + \widetilde{R}_N(v)\,, \tag{62}$$

where

$$\widetilde{R}_N = -\log(1+R_N) = O(|v|^2 N^{-1/2+\varepsilon}) \quad \text{for} \quad |v| < b\sqrt{\log N}\ . \tag{63}$$

By (60),

$$0 < -F'(\lambda) = -\frac{1}{2\pi^2}\log(1-e^{-2\lambda}) \le \frac{1}{2\pi^2}\left(1+\log^+\frac{1}{\lambda}\right)\ . \tag{64}$$

Since $\frac12|v|^2 + \widetilde{R}_N(v) = |v|^2\left(\frac12 + o(N)\right)$, it follows from (62)–(64) that

$$\begin{aligned} Q_N\Big(z_0, z_0 + \frac{v}{\sqrt{N}}\Big) &= F\left(\tfrac12|v|^2 + \widetilde{R}_N(v)\right) \\ &= F\left(\tfrac12|v|^2\right) + O\left(\left[1+\log^+\frac{1}{|v|}\right]\widetilde{R}_N(v)\right) \\ &= \widetilde{G}(e^{-\frac12|v|^2}) + O(N^{-1/2+\varepsilon})\ , \qquad \text{for} \quad |v| \le b\sqrt{\log N}\ . \end{aligned}$$

□

We shall use the following notation: for a current T on $M\times M$, we write

$$\partial T = \partial_1 T + \partial_2 T\ , \qquad \partial_1 = \sum dz_j\,\frac{\partial}{\partial z_j}\ , \quad \partial_2 = \sum dw_j\,\frac{\partial}{\partial w_j}\ ,$$

where $z_1,\dots,z_m$ are local coordinates on the first factor, and $w_1,\dots,w_m$ are local coordinates on the second factor of $M\times M$. We similarly write

$$\bar\partial T = \bar\partial_1 T + \bar\partial_2 T\ .$$

In particular, we shall write $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$ in place of $\partial_z\bar\partial_z\partial_w\bar\partial_w Q_N(z,w)$ to avoid confusion when we change variables.

Next we compute the leading term of the N-asymptotics of $\bar\partial_1\bar\partial_2 Q_N$ and $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$. We choose normal coordinates at a point $z_0\in M$, and we recall that in terms of these coordinates, we have

$$Q_N = F\circ\Lambda_N, \quad \Lambda_N(z,w) = \frac{N}{2}|w-z|^2 + \widetilde{R}_N(\sqrt{N}\,z, \sqrt{N}\,w)\ , \tag{65}$$

where $\widetilde{R}_N$ is given by (63). We now write $A_N(z,w)\approx B_N(z,w)$ when

$$A_N(z,w) - B_N(z,w) = O\left(N^{-\frac12+\varepsilon}\,|B_N(z,w)|\right) \quad \text{for} \quad |z|+|w| < b\sqrt{\frac{\log N}{N}}\ .$$

By (65), we have:

$$\bar\partial_2 Q_N(z,w) \ \approx\ \tfrac{N}{2}\,F'(\Lambda_N(z,w))\left[(w-z)\cdot d\bar w\right]\ , \tag{66}$$

$$\bar\partial_1\bar\partial_2 Q_N(z,w) \ \approx\ -\tfrac{N^2}{4}\,F''(\Lambda_N(z,w))\left[(w-z)\cdot d\bar z\right]\left[(w-z)\cdot d\bar w\right]\ , \tag{67}$$

$$\begin{aligned} \bar\partial_1\partial_2\bar\partial_2 Q_N(z,w) \ \approx\ & -\tfrac{N^3}{8}\,F^{(3)}(\Lambda_N(z,w))\left[(w-z)\cdot d\bar z\right]\left[(\bar w-\bar z)\cdot dw\right]\left[(w-z)\cdot d\bar w\right] \\ & -\tfrac{N^2}{4}\,F''(\Lambda_N(z,w))\left\{\left[d\bar z\cdot dw\right]\left[(w-z)\cdot d\bar w\right] + \left[(w-z)\cdot d\bar z\right]\left[dw\cdot d\bar w\right]\right\}\ , \end{aligned} \tag{68}$$

and hence

$$\begin{aligned} \partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(0,w) \ \approx\ & \tfrac{N^4}{16}F^{(4)}(\Lambda_N(0,w))\ (\bar w\cdot dz)(w\cdot d\bar z)(\bar w\cdot dw)(w\cdot d\bar w) \\ & + \tfrac{N^3}{8}F^{(3)}(\Lambda_N(0,w))\left[(dz\cdot d\bar z)(\bar w\cdot dw)(w\cdot d\bar w) + (w\cdot d\bar z)(\bar w\cdot dw)(dz\cdot d\bar w)\right. \\ & \qquad \left. + (\bar w\cdot dz)(d\bar z\cdot dw)(w\cdot d\bar w) + (\bar w\cdot dz)(w\cdot d\bar z)(dw\cdot d\bar w)\right] \\ & + \tfrac{N^2}{4}\,F''(\Lambda_N(0,w))\left[(d\bar z\cdot dw)(dz\cdot d\bar w) + (dz\cdot d\bar z)(dw\cdot d\bar w)\right]\ . \end{aligned} \tag{69}$$

Differentiating (64), we have

$$F''(\lambda) = \frac{1}{\pi^2}\,\frac{1}{e^{2\lambda}-1}\;, \tag{70}$$

and hence

$$F^{(j)}(\lambda) = O(e^{-2\lambda}) \qquad (\lambda > 1), \tag{71}$$

for $j \geq 0$. Furthermore, by (64),

$$F'(\lambda) = \frac{1}{2\pi^2}\log\lambda + \eta(\lambda), \qquad \eta \in \mathcal{C}^\infty([0,+\infty))\;,$$

and therefore

$$F^{(j+1)}(\lambda) = (-1)^{j+1}\frac{(j-1)!}{2\pi^2}\,\lambda^{-j} + O(1) \qquad (\lambda > 0), \tag{72}$$

for $j \geq 1$.

We now use the above computation to describe the singularity of the variance current near the diagonal. We first recall an elementary fact:

Lemma **3.6.** *Let $u \in \mathcal{C}^1((\mathbb{R}^p\smallsetminus\{0\})\times\mathbb{R}^q)$, and let $1 \leq j \leq p+q$. Suppose that $\partial u/\partial x_j \in L^1(\mathbb{R}^{p+q})$ and $u(x) = o\left(|\pi_1(x)|^{-p+1}\right)$, where $\pi_1 : \mathbb{R}^{p+q} \to \mathbb{R}^p$ is the projection. Then the distribution derivative $\partial u/\partial x_j \in \mathcal{D}'(\mathbb{R}^{p+q})$ is given by the pointwise derivative, i.e.*

$$\int_{(\mathbb{R}^p\smallsetminus\{0\})\times\mathbb{R}^q} u\,\frac{\partial\varphi}{\partial x_j} = -\int_{(\mathbb{R}^p\smallsetminus\{0\})\times\mathbb{R}^q}\frac{\partial u}{\partial x_j}\,\varphi \qquad \forall\ \varphi\in\mathcal{D}(\mathbb{R}^{p+q})\;.$$

Proof. Let $U_\varepsilon = \{x \in \mathbb{R}^{p+q} : |\pi_1(x)| > \varepsilon\}$. The identity follows by integrating $u\,\frac{\partial\varphi}{\partial x_j}$ by parts over U_ε, and noting that the boundary term

$$\int_{\partial U_\varepsilon} u\varphi\, dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_{p+q}$$

goes to zero as $\varepsilon \to 0$. □

Lemma **3.7.** *There exists a constant $C_m \in \mathbb{R}^+$ (depending only on the dimension m) and $N_0 = N_0(M) \in \mathbb{Z}^+$ such that for $N \geq N_0$, we have:*

i) *The coefficients of the current $\bar\partial_1\bar\partial_2 Q_N$ are locally bounded functions (given by pointwise differentiation of Q_N), and we have the pointwise estimate*

$$|\bar\partial_1\bar\partial_2 Q_N(z,w)| \leq C_m N \quad \text{for}\ \ 0 < |w-z| < b\sqrt{\frac{\log N}{N}}.$$

ii) *If $m \geq 2$, the coefficients of the current $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$ are locally L^{m-1} functions, and we have the estimate*

$$|\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z,w)| \leq \frac{C_m N}{|w-z|^2} \quad \text{for}\ \ 0 < |w-z| < b\sqrt{\frac{\log N}{N}}.$$

Proof. We take $z = z_0 = 0$. By Propositions 2.7–2.8, we can choose N_0 such that

$$\Lambda_N(0,w) \geq \frac{N}{3}|w|^2 \quad \text{for}\ \ |w| < b\sqrt{\frac{\log N}{N}}\;, \quad N \geq N_0\;. \tag{73}$$

By applying the chain rule as in (66)–(69), we conclude that for each $N \geq N_0$,

$$\nabla Q_N(z,w) = O(|w-z|\log|w-z|), \qquad \nabla^2 Q_N(z,w) = O(\log|w-z|),$$
$$\nabla^j Q_N(z,w) = O(|w-z|^{-j+2}) \quad \text{for}\ \ j \geq 3. \tag{74}$$

Hence, the partial derivatives of Q_N of order ≤ 3 are in L^1_{loc}, and the same holds for the fourth order derivatives if $m \ge 2$. By repeatedly applying Lemma 3.6 with $\pi_1(x) = w - z$, we conclude that the currents $\bar\partial_1\bar\partial_2 Q_N$ and $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$ have locally L^1 coefficients. The upper bound in (i) follows from (67), (70) and (73), and hence the coefficients of $\bar\partial_1\bar\partial_2 Q_N$ are actually in L^∞_{loc}. The upper bound in (ii) similarly follows from (69), (72) and (73), and hence the coefficients of $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$ are in L^{m-1}_{loc}. □

These computations show that Q_N is $\mathcal{C}^1$ and has vanishing first derivatives on the diagonal in $M \times M$, but Q_N is not $\mathcal{C}^2$ along the diagonal. Lemma 3.7(i) says that $\bar\partial_1\bar\partial_2 Q_N$ is bounded; however, a similar computation shows that $\partial_2\bar\partial_2 Q_N(z,w) \ge c \log|z-w|$, for a positive constant c. When $m > 1$, $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N \sim |w-z|^{-2}$; but when $m = 1$, $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$ is a measure with a singular component along the diagonal, and off the diagonal there is cancellation in (69) and $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N \sim |w-z|^2$ (see [BSZ1, Th. 4.2]).

Making the change of variables

$$w = \frac{v}{\sqrt{N}}$$

in (67) and again applying (65) and (72), we obtain the following asymptotic formulas:

Lemma **3.8.** *For N sufficiently large,*

$$\bar\partial_1\bar\partial_2 Q_N(0, \tfrac{v}{\sqrt{N}}) = -\tfrac{\sqrt{N}}{4} F''(\tfrac12|v|^2)(v\cdot d\bar z)(v\cdot d\bar v) + O(N^\varepsilon) \in T^{*1,1}_{z_0}(M)\otimes \mathcal{D}'^{1,1}(\mathbb{C}^m)\ ,$$

for $0 < |v| < b\sqrt{\log N}$.

Proof. By (62)–(63) and (72) with $j = 2$, we have

$$\begin{aligned} F''\left(\Lambda_N(0,\tfrac{v}{\sqrt{N}})\right) &= F''(\tfrac12|v|^2) + F^{(3)}\left(\left[\tfrac12 + O(N^{-1/2+\varepsilon})\right]|v|^2\right)\cdot O\left(|v|^2 N^{-1/2+\varepsilon}\right) \\ &= F''(\tfrac12|v|^2) + O\left(|v|^{-2}N^{-1/2+\varepsilon}\right). \end{aligned} \tag{75}$$

The formula follows from (67) and (75). □

Lemma **3.9.** *For N sufficiently large,*

$$-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z_0, z_0 + \tfrac{v}{\sqrt{N}}) = N\,\mathbf{Var}^{z_0}_\infty(v) + O\left(|v|^{-2}\,N^{1/2+\varepsilon}\right) \quad \text{for}\ \ 0<|v|<b\sqrt{\log N}\ , \tag{76}$$

where $\mathbf{Var}^{z_0}_\infty \in T^{*1,1}_{z_0}(M)\otimes\mathcal{D}'^{1,1}(\mathbb{C}^m)$ *is given by*

$$\begin{aligned} \mathbf{Var}^{z_0}_\infty(v) \ :=\ & -\tfrac{1}{16}F^{(4)}(\tfrac12|v|^2)\ (\bar v\cdot dz)(v\cdot d\bar z)(\bar v\cdot dv)(v\cdot d\bar v) \\ & -\tfrac18 F^{(3)}(\tfrac12|v|^2)\left[(dz\cdot d\bar z)(\bar v\cdot dv)(v\cdot d\bar v) + (v\cdot d\bar z)(\bar v\cdot dv)(dz\cdot d\bar v)\right. \\ & \qquad \left. +(\bar v\cdot dz)(d\bar z\cdot dv)(v\cdot d\bar v) + (\bar v\cdot dz)(v\cdot d\bar z)(dv\cdot d\bar v)\right] \\ & -\tfrac14 F''(\tfrac12|v|^2)\left[(d\bar z\cdot dv)(dz\cdot d\bar v) + (dz\cdot d\bar z)(dv\cdot d\bar v)\right]. \end{aligned} \tag{77}$$

Furthermore,

$$\mathbf{Var}^{z_0}_\infty(v) = \begin{cases} O(|v|^{-2}) & \text{for}\ \ |v|>0 \\ O(|v|^4\,e^{-|v|^2}) & \text{for}\ \ |v|>1 \end{cases}\ . \tag{78}$$

Proof. Formula (77) follows by the same argument as in the proof of Lemma 3.8, applying (69) in place of (67). The estimate (78) follows by applying (71)–(72) to (77). □

The current $\mathbf{Var}^{z_0}_\infty$ has L^1_{loc} coefficients if $m \ge 2$, but contains the singular term $\pi\delta_{z_0}(w)$ if $m = 1$ (see [SZ1, Theorem 4.1]).

3.2. **The pair correlation current.** The pair correlation current gives the correlation for the zero densities at two points of M. It is defined to be

$$\mathbf{K}^N_{2k} := \mathbf{E}\big(Z_{s^N_1,\dots,s^N_k} \boxtimes Z_{s^N_1,\dots,s^N_k}\big) \in \mathcal{D}'^{2k,2k}(M\times M)\ . \tag{79}$$

Thus by Corollary 2.3,

$$\mathbf{Var}\big(Z_{s^N_1,\dots,s^N_k}\big) = \mathbf{K}^N_{2k} - \mathbf{E}\big(Z_{s^N_1,\dots,s^N_k}\big)\boxtimes \mathbf{E}\big(Z_{s^N_1,\dots,s^N_k}\big) = \mathbf{K}^N_{2k} - \big[\mathbf{E}\big(Z_{s^N}\big)\boxtimes \mathbf{E}\big(Z_{s^N}\big)\big]^k\ . \tag{80}$$

As a consequence of Theorem 3.1, we have the following formula for the case $k=1$:

PROPOSITION **3.10.** *The pair correlation current in codimension one is given by*

$$\mathbf{K}^N_{21} = -\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N + \mathbf{E}Z_{s^N}\boxtimes \mathbf{E}Z_{s^N}\ .$$

For $m\geq 2$, the coefficients of the current $\mathbf{K}^N_{21}$ are in L^{m-1}_{loc}.

Proof. The formula for $\mathbf{K}^N_{21}$ is an immediate consequence of Theorem 3.1 and (80). By Lemma 3.7(ii), the coefficients of the current $\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N$ are in L^{m-1}_{loc}, and hence the same holds for the coefficients of $\mathbf{K}^N_{21}$. □

We note that the pair correlation function $K^N_{2k}(z,w)$, which gives the probability density of zeros occurring at both z and w (see [BSZ1, BSZ2]), can be obtained from the pair correlation current:

$$K^N_{2k}\left(\tfrac{1}{m!}\,\omega^m\boxtimes \tfrac{1}{m!}\,\omega^m\right) = \mathbf{K}^N_{2k}\wedge\left(\tfrac{1}{(m-k)!}\,\omega^{m-k}\boxtimes \tfrac{1}{(m-k)!}\,\omega^{m-k}\right)\ . \tag{81}$$

The advantage of the pair correlation current is that, because of the independence of the s^N_j, the codimension-k correlation current is the k-th exterior power of the corresponding codimension-one current, i.e.

$$\mathbf{K}^N_{2k} = \big[\mathbf{K}^N_{21}\big]^{\wedge k}\ . \tag{82}$$

This formula is analogous to the corresponding identity $\mathbf{E}(Z_{s^N_1,\dots,s^N_k}) = \mathbf{E}(Z_{s^N})^k$ for the expected value in Corollary 2.3; both formulas hold since the s^N_j are independent random sections. However, in the case of correlation currents, the right side of (82) is not well defined along the diagonal in $M\times M$, since $\mathbf{K}^N_{21}$ is singular on the diagonal. We shall show that for $k<m$, the current $\mathbf{K}^N_{2k}$ has L^1 coefficients and (82) holds, with $\big[\mathbf{K}^N_{21}\big]^{\wedge k}$ given by pointwise multiplication. However, for the point case $k=m$ of Theorem 1.1, the pair correlation current $\mathbf{K}^N_{2m}$ contains a singular measure supported on the diagonal (see Theorem 3.15), and the right side of (82) must be interpreted as a limit of smooth currents. The singularities of $\mathbf{K}^N_{21}$ necessitate a more complicated proof, using a smoothing method. (One can also define in an analogous way the n-point correlation currents $\mathbf{K}^N_{nk}$, which satisfy the identity $\mathbf{K}^N_{nk} = \big[\mathbf{K}^N_{n1}\big]^{\wedge k}$.)

3.3. **Explicit formula for the variance.** We shall write

$$\begin{aligned}\Phi_k &= \tfrac{1}{(m-k)!}\omega^{m-k} \quad \text{for}\ \ 1\leq k\leq m-1,\\ \Phi_m &= 1\ ,\end{aligned} \tag{83}$$

so that

$$\mathrm{Vol}_{2m-2k}\left(Z_{s^N_1,\dots,s^N_k}\cap U\right) = \left(Z_{s^N_1,\dots,s^N_k}\,,\chi_U\,\Phi_k\right)\ ,\quad \text{for}\ \ 1\leq k\leq m\ .$$

By Corollary 2.3, the expected volume of the codimension-k zero current is given by

$$\mathbf{E}\big(\mathrm{Vol}_{2m-2k}[Z_{s^N_1,\dots,s^N_k}\cap U]\big) = \int_U \big(\mathbf{E}Z_{s^N}\big)^k\wedge\Phi_k\ , \tag{84}$$

where

$$\mathbf{E}Z_{s^N} = \frac{i}{\pi}\partial\bar\partial \log \Pi_N(z,z) + \frac{N}{\pi}\omega\ . \tag{85}$$

In this section, we prove the following integral formula for the volume and number variance:

THEOREM **3.11.** *The variance in Theorem 1.4 is given by:*

$$\begin{aligned}&\mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N}\cap U]\big)\\ &\quad= \sum_{j=1}^k (-1)^j\binom{k}{j}\int_{\partial U\times\partial U} \bar\partial_1\bar\partial_2 Q_N\wedge\big(\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\big)^{j-1}\wedge\big(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N}\big)^{k-j}\wedge(\Phi_k\boxtimes\Phi_k),\end{aligned}$$

for N sufficiently large, where Q_N is given by (36), $\mathbf{E}Z_{s^N}$ *is given by* (85) *and the integrands are in $L^1(\partial U\times\partial U)$.*

In particular, for the one-dimensional case $k=m=1$, we have

$$\mathrm{Var}\big(\mathcal{N}_N^U\big) = -\int_{\partial U\times\partial U}\bar\partial_z\bar\partial_w Q_N(z,w)\ .$$

Theorem 3.11 follows formally from Theorem 3.1 and equations (80) and (82). To verify the formula rigorously, we must show that the currents $\bar\partial_1\bar\partial_2 Q_N\wedge\big[\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\big]^{j-1}$ $(1\le j\le m)$, which are smooth forms off the diagonal in $M\times M$, are well defined and have L^1 coefficients; i.e., they impart no mass to the diagonal. To do this, we use the asymptotics of $Q_N(z,w)$ as $|z-w|\to 0$ given in Lemma 3.7. In §4, we shall use the N asymptotics of Lemmas 3.8–3.9 together with Theorem 3.11 to prove Theorem 1.4.

DEFINITION **3.12.** *We say that a current $u\in\mathcal{D}'^{p,q}(X)$ on a Kähler manifold X is an L^1 current on X if its local coefficients are L^1 functions and $\int_X |u|\,d\mathrm{Vol}_X<+\infty$. (The second condition is redundant if X is compact.) If $\{u_n\}$ is a sequence of L^1 currents on X, we say that $u_n\to u$ in L^1 if $\int_X|u_n-u|\,d\mathrm{Vol}_X\to 0$.*

We shall prove Theorem 3.11 by approximating χ_U by smooth cut-off functions, and then applying the following explicit formula for the variance current:

THEOREM **3.13.** *Let $1\le k\le m$. Then for N sufficiently large,*

$$\mathbf{Var}\big(Z_{s_1^N,\dots,s_k^N}\big) \;=\; \partial_1\partial_2\left[\sum_{j=1}^k(-1)^j\binom{k}{j}\bar\partial_1\bar\partial_2 Q_N\wedge\big(\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\big)^{j-1}\wedge\big(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N}\big)^{k-j}\right].$$

where the current inside the brackets is an L^1 current on $M\times M$ given by pointwise multiplication, Q_N is given by (36), *and* $\mathbf{E}Z_{s^N}$ *is given by* (85). *Furthermore,* $\mathbf{Var}\big(Z_{s_1^N,\dots,s_k^N}\big)$ *is an L^1 current on $M\times M$ if $k\le m-1$.*

We need the following smoothing result for our proof of Theorem 3.13:

PROPOSITION **3.14.** *Let $(L,h)\to M$ be as in Theorem 1.1, with $\dim M=m\ge 2$. Then there is a positive integer N_0 such that the following holds for all $N\ge N_0$:*

- *for $1\le j\le m-1$, $\mathbf{K}_{2j}^N$ is an L^1 current on $M\times M$, and is given by the pointwise formula $\mathbf{K}_{2j}^N=(\mathbf{K}_{21}^N)^j$.*

- *for all points $P_0 \in M\times M$, there exist a neighborhood $\Omega\subset M\times M$ of P_0 and smooth forms*
$$S_\varepsilon K^N_{21}\in\mathcal{E}^{2,2}(\Omega)\qquad (0<\varepsilon<1)$$
such that
 i) $\partial_1(S_\varepsilon K^N_{21})=\partial_2(S_\varepsilon K^N_{21})=0$;
 ii) *for* $1\le j\le m-1$, $(S_\varepsilon\mathbf{K}^N_{21})^j\to\mathbf{K}^N_{2j}|_\Omega$ *in* L^1, *as* $\varepsilon\to 0$;
 iii) *for* $2\le k\le m$, $\mathbf{K}^N_{21}\wedge\big(S_\varepsilon\mathbf{K}^N_{21}\big)^{k-1}\to\mathbf{K}^N_{2k}|_\Omega$ *weakly, as* $\varepsilon\to 0$.

We postpone the proof of Proposition 3.14 to the next section, and we now prove Theorem 3.13, assuming Proposition 3.14: The case $m=1$ of Theorem 3.13 is Theorem 3.1. So we let $m\ge 2$. It suffices to consider a test form $\varphi\in\mathcal{D}^{2m-2k,2m-2k}(\Omega)$, where $\Omega\subset M\times M$ is as in Proposition 3.14. By Propositions 3.10 and 3.14 (recalling that $\bar\partial_1\bar\partial_2 Q_N$ has L^∞ coefficients by Lemma 3.7), we have

$$\begin{aligned}(\mathbf{K}^N_{2k},\varphi) &= \lim_{\varepsilon\to 0}\big(\mathbf{K}^N_{21}\wedge[S_\varepsilon\mathbf{K}^N_{21}]^{k-1},\,\varphi\big)\\ &= \lim_{\varepsilon\to 0}\int_\Omega\Big\{\bar\partial_1\bar\partial_2 Q_N\wedge[S_\varepsilon\mathbf{K}^N_{21}]^{k-1}\wedge\partial_1\partial_2\varphi\ +\ [\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N}]\wedge[S_\varepsilon\mathbf{K}^N_{21}]^{k-1}\wedge\varphi\Big\}\\ &= \int_\Omega\bar\partial_1\bar\partial_2 Q_N\wedge(-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N+\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^{k-1}\wedge\partial_1\partial_2\varphi\\ &\quad+\int_\Omega(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})\wedge(-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N+\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^{k-1}\wedge\varphi\,. \end{aligned}\tag{86}$$

Expanding the integrand and recalling (80), we then have

$$\begin{aligned}&\Big(\mathbf{Var}\big(Z_{s_1^N,\dots,s_k^N}\big),\,\varphi\Big)\\ &\quad= \big(\mathbf{K}^N_{2k},\,\varphi\big)-\int(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^k\wedge\varphi\\ &\quad= \sum_{j=1}^{k}\binom{k-1}{j-1}\int\bar\partial_1\bar\partial_2 Q_N\wedge(-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N)^{j-1}\wedge(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^{k-j}\wedge\partial_1\partial_2\varphi\\ &\qquad+\sum_{j=1}^{k-1}\binom{k-1}{j}\int\bar\partial_1\bar\partial_2 Q_N\wedge(-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N)^{j-1}\wedge(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^{k-j}\wedge\partial_1\partial_2\varphi\\ &\quad= \sum_{j=1}^{k}\binom{k}{j}\int\bar\partial_1\bar\partial_2 Q_N\wedge(-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N)^{j-1}\wedge(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^{k-j}\wedge\partial_1\partial_2\varphi\,.\end{aligned}$$

This is the formula of Theorem 3.13; to complete the proof of the theorem, it remains only to prove Proposition 3.14.

3.3.1. *Off-diagonal decay of the correlation current.* We now use 3.13–3.14 to give a more explicit formula for the pair correlation current in higher codimension and describe its off-diagonal asymptotics. (The results of this section are presented here for their general interest and are not needed for the proof of the variance formulas.)

We note that the correlation currents $\mathbf{K}^N_{2k}$ are smooth forms away from the diagonal in $M\times M$. We now show that $\mathbf{K}^N_{2k}$ has no mass on the diagonal for $k<m$, while $\mathbf{K}^N_{2m}$ contains a 'delta-function' along the diagonal:

THEOREM **3.15.** *Let $(L,h) \to M$ be as in Theorem 1.1. Then for N sufficiently large, we have:*

i) *for $1 \le k \le m-1$, the correlation current for $\mathbf{K}^N_{2k}$ is an L^1 current on $M \times M$ given by*

$$\mathbf{K}^N_{2k} = \left[-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N + \mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right]^k \ ,$$

where Q_N is given by (36), and $\mathbf{E}Z_{s^N}$ is given by (85);

ii) $\mathbf{K}^N_{2m} = (-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N + \mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N})^m\big|_{M\times M\smallsetminus\Delta} + \mathrm{diag}_*(\mathbf{E}Z_{s^N})^m$ *,*

where $\Delta = \{(z,z) : z \in M\}$ is the diagonal in $M \times M$, and $\mathrm{diag} : M \to M \times M$ *is the diagonal map* $\mathrm{diag}(z) = (z,z)$.

Proof. Part (i) is an immediate consequence of Proposition 3.10 and the first conclusion of Proposition 3.14. To verify (ii), we regard the current $\mathbf{K}^N_{2m} \in \mathcal{D}'^{4m}(M \times M)$ (which is of order 0 by its definition) as a measure on $M \times M$. Since Q_N is $\mathcal{C}^\infty$ in $M \times M \smallsetminus \Delta$, it follows from Theorem 3.13 (or from (86)) that $\mathbf{K}^N_{2m} = (-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N + \mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N})^m$ on $M \times M \smallsetminus \Delta$. Hence it suffices to show that (ii) holds on Δ, i.e., for any Borel set $A \subset M$,

$$\left(\mathbf{K}^N_{2m}\,,\,\chi_{\mathrm{diag}(A)}\right) = \left((\mathbf{E}Z_{s^N})^m\,,\,\chi_A\right) . \tag{87}$$

To verify (87), we note that

$$\left(Z_{s_1^N,\dots,s_m^N} \boxtimes Z_{s_1^N,\dots,s_m^N}\,,\,\chi_{\mathrm{diag}(A)}\right) = \#\left\{z \in A : s_1^N(z) = \cdots = s_m^N(z) = 0\right\} = \left(Z_{s_1^N,\dots,s_m^N}\,,\,\chi_A\right).$$

Taking expectations and recalling Corollary 2.3, we then have

$$\begin{aligned}\left(\mathbf{K}^N_{2m}\,,\,\chi_{\mathrm{diag}(A)}\right) &= \mathbf{E}\left(Z_{s_1^N,\dots,s_m^N} \boxtimes Z_{s_1^N,\dots,s_m^N}\,,\,\chi_{\mathrm{diag}(A)}\right)\\ &= \left(\mathbf{E}Z_{s_1^N,\dots,s_m^N}\,,\,\chi_A\right) = \left((\mathbf{E}Z_{s^N})^m\,,\,\chi_A\right) ,\end{aligned}$$

and therefore (ii) holds on Δ and hence on all of $M \times M$. □

THEOREM **3.16.** *Let $(L,h) \to M$ be as in Theorem 1.1 and let $1 \le k \le m$. For $b > \sqrt{q+2k+3} \ge 3$, the pair correlation currents satisfy the off-diagonal asymptotics*

$$\mathbf{K}^N_{2k} = \left(\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right)^k + O\left(N^{-q}\right) \ , \quad \textit{for}\ \ \mathrm{dist}(z,w) \ge \frac{b\sqrt{\log N}}{\sqrt{N}} \ .$$

In particular,

$$\mathbf{K}^N_{2k} = \left(\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right)^k + O\left(N^{-\infty}\right) \ , \quad \textit{for}\ \ \mathrm{dist}(z,w) \ge N^{-1/2+\varepsilon} \ .$$

Proof. By Lemma 3.4, for $b > \sqrt{q+2k+3}$, we have

$$\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N = O\left(\frac{1}{N^{q+2k-2}}\right) \ , \quad \text{for}\ \ \mathrm{dist}(z,w) \ge \frac{b\sqrt{\log N}}{\sqrt{N}} \ . \tag{88}$$

Since the statement only pertains to the off-diagonal, by Theorem 3.15 and (88),

$$\mathbf{K}^N_{2k} = \left[-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N + \mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right]^k = \left[\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N} + O\left(\frac{1}{N^{q+2k-2}}\right)\right]^k , \tag{89}$$

for $\mathrm{dist}(z,w) \ge \frac{b\sqrt{\log N}}{\sqrt{N}}$. By Proposition 2.5, $\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N} = O(N^2)$, and the desired asymptotics follow immediately from (89). □

In particular, the point case of Theorem 3.16 yields the asymptotics

Corollary **3.17.** *Let $(L,h)\to M$ be as in Theorem 1.1, and define the normalized pair correlation function by*

$$\widetilde{K}^N_{2m}=\frac{\mathbf{K}^N_{2m}}{(\mathbf{E}Z_{s^N}\boxtimes\mathbf{E}Z_{s^N})^m}\ .$$

Then for $b>\sqrt{q+5}$, $q\ge 1$, we have

$$\widetilde{K}^N_{2m}(z,w)=1+O\left(N^{-q}\right),\quad \textit{for}\ \ \mathrm{dist}(z,w)\ge\frac{b\sqrt{\log N}}{\sqrt{N}}\ .$$

In particular,

$$\widetilde{K}^N_{2m}(z,w)=1+O\left(N^{-\infty}\right)\ ,\quad \textit{for}\ \ \mathrm{dist}(z,w)\ge N^{-1/2+\varepsilon}\ .$$

Corollary 3.17 can also be obtained from Theorem 2.4 (or [SZ2, (95)–(96)]), using the argument in Section 4.1 of [BSZ2].

3.4. **Smoothing $\mathbf{K}^N_{21}$: Proof of Proposition 3.14.** We shall use the following fact about averaging currents of integration over a smooth family $\{Y_t\}$ of submanifolds:

Lemma **3.18.** *Let X and Ω be complex manifolds of dimension m and n respectively, and let X' be a complex submanifold of X. Let $\mathcal{Y}$ be a complex submanifold of $M\times\Omega$ such that the projections $\pi_1:\mathcal{Y}\to M$ and $\pi_2:\mathcal{Y}\to\Omega$ are submersions, and let $Y_t=\pi_1\left(\pi_2^{-1}\{t\}\right)\subset X$, for $t\in\Omega$. Then for all $\alpha\in\mathcal{D}^{n,n}(\Omega)$, we have*

- $\displaystyle\int_{t\in\Omega}[Y_t]\,\alpha(t)\in\mathcal{E}^{p,p}(X)$*, where* $p=\mathrm{codim}\,\mathcal{Y}=\mathrm{codim}_X Y_t$ *;*
- *for almost all $t\in\Omega$, $X'\cap Y_t$ is a complex submanifold of X of codimension $p':=p+\mathrm{codim}\,X'$, and*

$$\int_{t\in\Omega}[X'\cap Y_t]\,\alpha(t)=[X']\wedge\int_{t\in\Omega}[Y_t^j]\,\alpha(t)\in\mathcal{D}'^{p',p'}(X)\ .$$

Proof. Let $\mathcal{Y}'=\pi_1^{-1}(X')=\mathcal{Y}\cap(X'\times\Omega)$, and consider the commutative diagram

$$\begin{array}{ccccc} & \mathcal{Y}' & \overset{\hat\iota}{\hookrightarrow} & \mathcal{Y} & \\ \pi_1'\swarrow & & \pi_1\swarrow & & \searrow\pi_2 \\ X' & \overset{\iota}{\hookrightarrow} & X & & \Omega \end{array}\ , \tag{90}$$

where $\pi_1'=\pi|_{\mathcal{Y}'}$, and $\iota,\hat\iota$ are the inclusions. Since π_2 is a submersion, Y_t is a smooth submanifold of X for all $t\in\Omega$. For a test form $\varphi\in\mathcal{D}^{m-p,m-p}(X)$, we have

$$\left(\int_{t\in\Omega}[Y_t]\,\alpha(t)\,,\,\varphi\right)\overset{\mathrm{def}}{=}\int_\Omega\alpha(t)\int_{Y_t}\varphi=\int_{\mathcal{Y}}\pi_2^*\alpha\wedge\pi_1^*\varphi=(\pi_{1*}\pi_2^*\alpha,\varphi)\ ,$$

and thus

$$\int_{t\in\Omega}[Y_t]\,\alpha(t)=\pi_{1*}\pi_2^*\alpha\ , \tag{91}$$

which is a smooth form, since the push-forward of a smooth current via a submersion is smooth.

Since π_1 is a submersion, $\mathcal{Y}'$ is a submanifold, and hence by Sard's theorem, the set of critical values of the map $\pi_2':=\pi_2|_{\mathcal{Y}'}$ has measure zero. Therefore $\pi_2'^{-1}(t)=(X'\cap Y_t)\times\{t\}$ is a submanifold for almost all $t\in\Omega$.

Now suppose that $\varphi \in \mathcal{D}^{m-p',m-p'}(X)$. Let S denote the set of critical points of the map $\pi'_2 : \mathcal{Y}' \to \Omega$, and let $E = \pi'_2(S \cap \operatorname{Supp} \pi'^*_1 \varphi)$, which is a closed subset of measure zero in Ω. Let $\rho_n \in \mathcal{C}^\infty(\Omega)$ such that $\rho_n \geq 0$, $\rho_n \nearrow \chi_{\Omega \smallsetminus E}$, and $\operatorname{Supp} \rho_n \cap E = \emptyset$. As before, we have

$$\int_\Omega \rho_n(t)\alpha(t) \int_{X' \cap Y_t} \varphi = \int_{\mathcal{Y}' \smallsetminus S} \pi'^*_2(\rho_n \alpha) \wedge \pi'^*_1 \varphi = \int_{\mathcal{Y}'} \pi'^*_2(\rho_n \alpha) \wedge \pi'^*_1 \varphi \,.$$

We claim that

$$\int_\Omega \rho_n(t)\alpha(t) \int_{X' \cap Y_t} \varphi \;\to\; \int_\Omega \alpha(t) \int_{X' \cap Y_t} \varphi \,, \tag{92}$$

$$\int_{\mathcal{Y}'} \pi'^*_2(\rho_n \alpha) \wedge \pi'^*_1 \varphi \;\to\; \int_{\mathcal{Y}'} \pi'^*_2 \alpha \wedge \pi'^*_1 \varphi \,, \tag{93}$$

as $n \to \infty$, and hence

$$\int_\Omega \alpha(t) \int_{X' \cap Y_t} \varphi = \int_{\mathcal{Y}'} \pi'^*_2 \alpha \wedge \pi'^*_1 \varphi \,. \tag{94}$$

To verify (92)–(94), we first consider the case $\alpha \geq 0$ and $\varphi = \beta^{m-p'}$, where β is a (compactly supported) semi-positive (1,1)-form on X, and apply monotone convergence to obtain (92)–(93) for this case. Since the right side of (94) is given by the integral of a compactly supported smooth form over $\mathcal{Y}'$, both sides of (94) are finite, and then (92)–(93) hold in the general case by dominated convergence.

Thus,

$$\left(\int_{t \in \Omega} [X' \cap Y_t]\, \alpha(t) \,,\, \varphi \right) = \int_\Omega \alpha(t) \int_{X' \cap Y_t} \varphi = \int_{\mathcal{Y}'} \pi'^*_2 \alpha \wedge \pi'^*_1 \varphi = \big([\mathcal{Y}'] \wedge \pi^*_2 \alpha ,\, \pi^*_1 \varphi \big),$$

and therefore

$$\int_{t \in \Omega} [X' \cap Y_t]\, \alpha(t) = \pi_{1*}\big([\mathcal{Y}'] \wedge \pi^*_2 \alpha \big) \,. \tag{95}$$

Furthermore, since π_1 is a submersion, π^*_1 is well-defined on currents and

$$\pi_{1*}\big([\mathcal{Y}'] \wedge \pi^*_2 \alpha \big) = \pi_{1*}\big(\pi^*_1 [X'] \wedge \pi^*_2 \alpha \big) = [X'] \wedge \pi_{1*} \pi^*_2 \alpha \,. \tag{96}$$

The identity of the lemma follows from (95)–(96) and (91). □

We now proceed to the proof of Proposition 3.14. By abuse of notation, we let Z_{s^N} denote the zero set of a section s^N as well as the current of integration over the zero set. We choose N_0 such that if $N \geq N_0$, the zero sets $Z_{s_1^N}, \dots, Z_{s_m^N}$ are almost always smooth and intersect transversely. (This holds if the Kodaira map for $H^0(M, L^N)$ is an embedding.)

We begin by smoothing currents (locally) on M. Let $a \in M$ be arbitrary and consider a coordinate chart $\tau_a : V_a \xrightarrow{\approx} B_r := \{z \in \mathbb{C}^m : \|z\| < r\}$, with $a \in V_a \subset M$, $\tau(a) = 0$. We let $U_a = \tau_a^{-1}(B_{r/2})$. To simplify our argument below, we choose the biholomorphism τ as follows: Embed $M \subset \mathbb{CP}^q$, and choose projective coordinates $(\zeta_0 : \cdots : \zeta_q)$ in $\mathbb{CP}^q$ such that

- $a = (1 : 0 : \cdots : 0)$,
- $\{x \in M : \zeta_j(x) = 0 \text{ for } 0 \leq j \leq m\} = \emptyset$,
- the projection $\pi_a : M \to \mathbb{CP}^m$, $x \mapsto (\zeta_0(x) : \cdots : \zeta_m(x))$ has nonsingular Jacobian at a.

We choose a neighborhood V'_a of a such that π_a is injective on V'_a and $\pi_a(V'_a) \subset \mathbb{C}^m = \mathbb{CP}^m \smallsetminus \{\zeta_0 = 0\}$. We then choose $r > 0$ such that $B_r \subset \pi_a(V'_a)$, and we let $V_a = \tau_a^{-1}(B_r)$ and $\tau_a = \pi_a|_{V_a}$.

The advantage of this construction is that degree bounds in M push forward under τ_a to degree bounds in $\mathbb{CP}^m$. In particular, if X is an algebraic hypersurface in M, then

$$\tau_a(X \cap V_a) \subset \pi_a(X), \quad \deg_{\mathbb{CP}^m} \pi_a(X) = \deg_{\mathbb{CP}^q} X\ . \tag{97}$$

(The well-known formula (97) is easily verified by recalling that the degree of a subvariety X in projective space is the number of points in the intersection of X with a generic linear subspace of complementary dimension.)

Let $\psi_\varepsilon(z) = \varepsilon^{-2m}\psi(z/\varepsilon)$ be an approximate identity on $\mathbb{C}^m$, with $\psi \in \mathcal{C}^\infty(\mathbb{C}^m)$ and $\operatorname{Supp}\psi \subset B_{r/2}$. We consider the local smoothing operator $S^a_\varepsilon : \mathcal{D}'^{1,1}(M) \to \mathcal{E}^{1,1}(U_a)$ given by convolution in the τ coordinates:

$$S^a_\varepsilon u = u * \psi_\varepsilon := \tau_a^*\big[\tau_{a*}(u|_{V_a}) * \psi_\varepsilon\big] \in \mathcal{E}^{1,1}(U_a)\ , \quad \text{for} \ \ u \in \mathcal{D}'^{1,1}(M)\,. \tag{98}$$

(Note that $\tau_{a*}(u|_{V_a}) \in \mathcal{D}'^{1,1}(B_r)$, and hence its convolution with ψ_ε is well-defined on $B_{r/2}$ for $0 < \varepsilon < 1$.)

Now suppose that $P_0 = (a, b) \in M \times M$, and let $\tau_a : V_a \xrightarrow{\approx} B_r,\ \tau_b : V_b \xrightarrow{\approx} B_r$ be as above, and let $\Omega = U_a \times U_b$. We consider the approximate identity $\widetilde{\psi}_\varepsilon(z, w) = \psi_\varepsilon(z)\psi_\varepsilon(w)$ on $\mathbb{C}^{2m}$ and we similarly define $S_\varepsilon : \mathcal{D}'^{2,2}(M) \to \mathcal{E}^{2,2}(\Omega)$ by

$$(S_\varepsilon \widetilde{u}) = \widetilde{u} * \widetilde{\psi}_\varepsilon := \tau^*\big[\tau_*(\widetilde{u}|_{V_a\times V_b}) * \widetilde{\psi}_\varepsilon\big] \in \mathcal{E}^{2,2}(\Omega)\,, \quad \text{for} \ \ \widetilde{u} \in \mathcal{D}'^{2,2}(M \times M)\,, \tag{99}$$

where $\tau = \tau_a \times \tau_b : V_a \times V_b \xrightarrow{\approx} B_r \times B_r$.

Lemma **3.19.** *For $N \geq N_0$,*

$$S_\varepsilon K^N_{21} = \mathbf{E}\big(S_\varepsilon(Z_{s^N} \boxtimes Z_{s^N})\big) = \mathbf{E}\big(S^a_\varepsilon Z_{s^N} \boxtimes S^b_\varepsilon Z_{s^N}\big) \in \mathcal{E}^{2,2}(\Omega)\,.$$

Proof. We have

$$S_\varepsilon K^N_{21} = \big[\mathbf{E}(Z_{s^N} \boxtimes Z_{s^N})\big] * \widetilde{\psi}_\varepsilon \ = \ \mathbf{E}\big[(Z_{s^N} \boxtimes Z_{s^N}) * \widetilde{\psi}_\varepsilon\big] = \mathbf{E}\big(S_\varepsilon(Z_{s^N} \boxtimes Z_{s^N})\big).$$

Furthermore,

$$\mathbf{E}\big[(Z_{s^N} \boxtimes Z_{s^N}) * \widetilde{\psi}_\varepsilon\big] = \mathbf{E}\big[(Z_{s^N} * \psi_\varepsilon) \boxtimes (Z_{s^N} * \psi_\varepsilon)\big] \ = \ \mathbf{E}\big(S^a_\varepsilon Z_{s^N} \boxtimes S^b_\varepsilon Z_{s^N}\big)\,.$$

□

Lemma **3.20.** *Let $2 \leq k \leq m$, $N \geq N_0$. For almost all $(s^N_1, \dots, s^N_k) \in H^0(M, L^N)^k$, we have*

$$[Z_{s^N_1} \boxtimes Z_{s^N_1}] \wedge S_\varepsilon[Z_{s^N_2} \boxtimes Z_{s^N_2}] \wedge \cdots \wedge S_\varepsilon[Z_{s^N_k} \boxtimes Z_{s^N_k}] \to Z_{s^N_1,\dots,s^N_k} \boxtimes Z_{s^N_1,\dots,s^N_k}\ ,$$

weakly in $\mathcal{D}'^{2k,2k}(\Omega)$, as $\varepsilon \to 0$.

Proof. It suffices to consider the case where the $Z_{s^N_j}$ are smooth and intersect transversely. We let $Y_j = \big(Z_{s^N_j} \times Z_{s^N_j}\big) \cap (V_a \times V_b)$, and we identify $V_a \times V_b$ with $B_r \times B_r$ via the biholomorphism τ. Under this identification, $\Omega = B_{r/2} \times B_{r/2} \subset \mathbb{C}^{2m}$. For $t \in \mathbb{C}^{2m}$, let $T_t : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ denote the translation $T_t(w) = w + t$, so that

$$S_\varepsilon Y_j = \int_{t\in\Omega} [T_{-t}\, Y_j]\, \widetilde{\psi}_\varepsilon(t)\, \nu(t) \in \mathcal{E}^{2,2}(\Omega)\,,$$

where ν is the Euclidean volume form on $\mathbb{C}^{2m}$.

Suppose that X' is a complex submanifold of Ω. We first show by induction that $X' \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k$ is a complex submanifold of Ω for almost all $t_2, \ldots, t_k$, and

$$[X'] \wedge S_\varepsilon Y_2 \wedge \cdots \wedge S_\varepsilon Y_k = \int_{\Omega^{k-1}} [X' \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k]\, \widetilde{\psi}_\varepsilon(t_2) \cdots \widetilde{\psi}_\varepsilon(t_k)\, \nu(t_2) \wedge \cdots \wedge \nu(t_k). \quad (100)$$

To verify (100) for $k = 2$, we let $\mathcal{Y} = \{(z,t) \in \Omega \times \Omega : z - t \in Y_2\}$. Then $\mathcal{Y}$ is smooth and the two projections $\pi_1 : \mathcal{Y} \to \Omega$, $\pi_2 : \mathcal{Y} \to \Omega$ are submersions. Furthermore, $\pi_1(\pi_2^{-1}\{t\}) = T_t Y_2$. Hence by Lemma 3.18 with $X = \Omega$ and $\alpha = \widetilde{\psi}_\varepsilon \nu$, the intersection $X' \cap T_{t_2} Y_2$ is a complex submanifold for almost all $t_2 \in \Omega$, and (100) holds for $k = 2$. For the inductive step, let $k > 2$ and suppose that (100) has been verified for $k - 1$. Let $t_1, \ldots, t_{k-1}$ be parameters in Ω such that $X' \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_{k-1}} Y_{k-1}$ is a complex submanifold of Ω. By Lemma 3.18 with X' replaced by $X' \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_{k-1}} Y_{k-1}$ and $\mathcal{Y} = \{(z,t) \in \Omega \times \Omega : z - t \in Y_k\}$, we conclude that $X' \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k$ is a complex submanifold of Ω for almost all t_k, and

$$\int_\Omega \left[X' \cap Y^1_{t_1} \cap \cdots \cap Y^{k-1}_{t_{k-1}} \cap Y^k_{t_k}\right] \psi_\varepsilon(t_k)\, \nu(t_k) = \left[X' \cap Y^1_{t_1} \cap \cdots \cap Y^{k-1}_{t_{k-1}}\right] \wedge S_\varepsilon Y_k\ .$$

Integrating over $t_1, \ldots, t_{k-1}$ and applying the inductive assumption, we obtain (100).

Setting $X' = Y_1$ in (100), we have

$$\big[Y_1\big] \wedge S_\varepsilon Y_2 \wedge \cdots \wedge S_\varepsilon Y_k = \int_{\Omega^{k-1}} [Y_1 \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k]\, \widetilde{\psi}_\varepsilon(t_2) \cdots \widetilde{\psi}_\varepsilon(t_k)\, \nu(t_2) \wedge \cdots \wedge \nu(t_k). \quad (101)$$

Now choose $\varepsilon_0 > 0$ such that $Y_1, T_{t_2} Y_2, \ldots, T_{t_k} Y_k$ intersect transversely whenever $|t_j| < \varepsilon_0$ for $2 \leq j \leq k$. Let $\varphi \in \mathcal{D}^{2m-2k,2m-2k}(\Omega)$ be a test form. Since the submanifolds $Y_1 \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k$ vary smoothly as the parameters $t_2, \ldots, t_k$ vary in the ε_0-ball, it follows from an argument using the implicit function theorem that the map

$$(t_2, \ldots, t_k) \mapsto \int_{Y_1 \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k} \varphi = ([Y_1 \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k]\,, \varphi)$$

is continuous (and in fact is $\mathcal{C}^\infty$) for $|t_j| < \varepsilon_0$. Therefore by (101),

$$\big(\big[Y_1\big] \wedge S_\varepsilon Y_2 \wedge \cdots \wedge S_\varepsilon Y_k, \varphi\big) \to ([Y_1 \cap Y_2 \cap \cdots \cap Y_k]\,, \varphi) \qquad \text{as } \ \varepsilon \to 0\ ;$$

i.e.,

$$\big[Y_1\big] \wedge S_\varepsilon Y_2 \wedge \cdots \wedge S_\varepsilon Y_k \to [Y_1 \cap Y_2 \cap \cdots \cap Y_k] \qquad \text{weakly,} \quad \text{as } \ \varepsilon \to 0\ . \quad (102)$$

□

LEMMA **3.21.** *There exists a positive constant $C < +\infty$ such that for all $N \geq N_0$, $\varphi \in \mathcal{D}^{2m-2k,2m-2k}(\Omega)$, and $0 < \varepsilon < 1$, we have*

$$\Big|[Z_{s_1^N} \boxtimes Z_{s_1^N}] \wedge S_\varepsilon [Z_{s_2^N} \boxtimes Z_{s_2^N}] \wedge \cdots \wedge S_\varepsilon [Z_{s_k^N} \boxtimes Z_{s_k^N}], \varphi)\Big| \leq C N^{2k} \|\varphi\|_\infty$$

for almost all $(s_1^N, \ldots s_k^N) \in H^0(M, L^N)^k$.

Proof. Fix $s_1, \ldots, s_k$ so that $Z_{s_1^N}, \ldots, Z_{s_k^N}$ intersect transversely, and let $Y_j = \big(Z_{s_j^N} \times Z_{s_j^N}\big) \cap (V_a \times V_b)$, as in the proof of Lemma 3.20. By (101) it suffices to show that

$$([Y_1 \cap T_{t_2} Y_2 \cap \cdots \cap T_{t_k} Y_k]\,, \varphi) \leq C N^{2k} \|\varphi\|_\infty\ , \quad (103)$$

for almost all $(t_2, \dots, t_k) \in \Omega^{k-1}$. To verify (103), it in turn suffices to show that

$$\mathrm{Vol}_{M\times M}\left(Y_1 \cap T_{t_2}Y_2 \cap \cdots \cap T_{t_k}Y_k\right) \le CN^{2k} \,. \tag{104}$$

We write $t_j = (t'_j, t''_j)$, $Z^a_j = Z_{s^N_j} \cap V_a$, $Z^b_j = Z_{s^N_j} \cap V_b$, so that $T_{t_j}Y_j = T_{t'_j}Z^a_j \times T_{t''_j}Z^b_j$. Then

$$\begin{aligned}\mathrm{Vol}_{M\times M}&\left(Y_1 \cap T_{t_2}Y_2 \cap \cdots \cap T_{t_k}Y_k\right)\\ &= \mathrm{Vol}_M(Z^a_1 \cap T_{t'_2}Z^a_2 \cap \cdots \cap T_{t'_k}Z^a_k)\, \mathrm{Vol}_M(Z^b_1 \cap T_{t''_2}Z^b_2 \cap \cdots \cap T_{t''_k}Z^b_k)\,. \end{aligned}\tag{105}$$

Since $\tau_a : Z^a_j \hookrightarrow \pi_a(Z_{s^N_j}) \subset \mathbb{CP}^m$ is injective and the translations $T_{t'_j}$ extend to automorphisms of $\mathbb{CP}^m$, we have

$$\begin{aligned}\mathrm{Vol}_M\left(Z^a_1 \cap T_{t'_2}Z^a_2 \cap \cdots \cap T_{t'_k}Z^a_k\right) &\le C_1\, \mathrm{Vol}_{\mathbb{CP}^m}\Big(\pi_a(Z_{s^N_1}) \cap T_{t'_2}\pi_a(Z_{s^N_2}) \cap \cdots \cap T_{t'_k}\pi_a(Z_{s^N_k})\Big)\\ &= \frac{\pi^k C_1}{k!}\prod_{j=1}^k \deg_{\mathbb{CP}^m}\pi_a(Z_{s^N_j})\,.\end{aligned}$$

However, by (97),

$$\deg_{\mathbb{CP}^m}\pi_a(Z_{s^N_j}) = \deg_{\mathbb{CP}^q} Z_{s^N_j} = \frac{1}{\pi^{m-1}}\int_{Z_{s^N_j}} \omega^{m-1}_{\mathbb{CP}^q} = \frac{1}{\pi^{m-1}}\int_M N\, c_1(L,h)\wedge \omega^{m-1}_{\mathbb{CP}^q} = C_2 N,$$

and hence

$$\mathrm{Vol}_M\left(Z^a_1 \cap T_{t'_2}Z^a_2 \cap \cdots \cap T_{t'_k}Z^a_k\right) \le C_3 N^k \,. \tag{106}$$

The bound (104) follows from (105)–(106). □

LEMMA **3.22.** *Let $f_j \in L^n_{loc}(\mathbb{R}^k)$ for $1 \le j \le n$. Let $f^\varepsilon_j = f_j * \psi_\varepsilon$, where ψ_ε is a compactly supported smooth approximate identity. Then*

$$\prod_{j=0}^n f^\varepsilon_j \xrightarrow{L^1_{loc}} \prod_{j=0}^n f_j \qquad as\ \ \varepsilon \to 0\,.$$

Proof. We use the generalized Hölder inequality:

$$\sum_{j=1}^n \frac{1}{p_j} = 1 \implies \|f_1\cdots f_n\|_1 \le \|f_1\|_{p_1}\cdots\|f_n\|_{p_n}\,. \tag{107}$$

We can assume without loss of generality that the f_j have compact support and hence $f_j \in L^n(\mathbb{R}^k)$, for $1 \le j \le n$. By (107) with $p_j = n$, we then have

$$\begin{aligned}\|f^\varepsilon_1&\cdots f^\varepsilon_n - f_1\cdots f_n\|_1\\ &\le \|(f^\varepsilon_1 - f_1)f^\varepsilon_2\cdots f^\varepsilon_n\|_1 + \|f_1(f^\varepsilon_2 - f_2)f^\varepsilon_3\cdots f^\varepsilon_n\|_1 + \cdots + \|f_1\cdots f_{n-1}(f^\varepsilon_n - f_n)\|_1\\ &\le \|(f^\varepsilon_1 - f_1)\|_n\,\|f^\varepsilon_2\|_n\cdots\|f^\varepsilon_n\|_n + \cdots + \|f_1\|_n\cdots\|f_{n-1}\|_n\,\|(f^\varepsilon_n - f_n)\|_n \to 0\,.\end{aligned}$$

□

Part (i) of Proposition 3.14 is an immediate consequence of Lemma 3.19. Next we show part (iii): By Lemma 3.19 and the independence of the s^N_j, we have

$$\begin{aligned}\mathbf{K}^N_{21}\wedge\left(S_\varepsilon\mathbf{K}^N_{21}\right)^{k-1} &= \mathbf{E}\big(Z_{s^N_1}\boxtimes Z_{s^N_1}\big)\wedge\mathbf{E}\big(S_\varepsilon(Z_{s^N_2}\boxtimes Z_{s^N_2})\big)\wedge\cdots\wedge\mathbf{E}\big(S_\varepsilon(Z_{s^N_k}\boxtimes Z_{s^N_k})\big)\\ &= \mathbf{E}\big([Z_{s^N_1}\boxtimes Z_{s^N_1}]\wedge S_\varepsilon(Z_{s^N_2}\boxtimes Z_{s^N_2})\wedge\cdots\wedge S_\varepsilon(Z_{s^N_k}\boxtimes Z_{s^N_k})\big)\,.\end{aligned}$$

Therefore, for a test form $\varphi \in \mathcal{D}^{m-k,m-k}(\Omega)$, we have

$$\Big(\mathbf{K}_{21}^N \wedge \big(S_\varepsilon \mathbf{K}_{21}^N\big)^{k-1}, \varphi\Big)$$
$$= \int_{H^0(M,L^N)^k} \Big([Z_{s_1^N} \boxtimes Z_{s_1^N}] \wedge S_\varepsilon(Z_{s_2^N} \boxtimes Z_{s_2^N}) \wedge \cdots \wedge S_\varepsilon(Z_{s_k^N} \boxtimes Z_{s_k^N}), \varphi\Big) \left[\prod_{j=1}^k d\gamma_N(s_j^N)\right]. \quad (108)$$

By Lemma 3.21, the integrand in (108) is uniformly bounded, and hence by (79), Lemma 3.20 and Lebesgue dominated convergence, we have

$$\Big(\mathbf{K}_{21}^N \wedge \big(S_\varepsilon \mathbf{K}_{21}^N\big)^{k-1}, \varphi\Big) \to \int_{H^0(M,L^N)^k} \Big(Z_{s_1^N,\dots,s_k^N} \boxtimes Z_{s_1^N,\dots,s_k^N}, \varphi\Big) \left[\prod_{j=1}^k d\gamma_N(s_j^N)\right] = \big(\mathbf{K}_{2k}^N, \varphi\big),$$

as $\varepsilon \to 0$, verifying part (iii).

To complete the proof of the proposition, we recall from Proposition 3.10 that the current $\mathbf{K}_{21}^N$ has L^{m-1}_{loc} coefficients (if $m \geq 2$) and hence by Lemma 3.22,

$$(S_\varepsilon \mathbf{K}_{21}^N)^j \to (\mathbf{K}_{21}^N)^j|_\Omega \quad \text{in} \quad L^1(\Omega), \quad \text{for} \quad 1 \leq j \leq m-1\,. \quad (109)$$

Since L^1 convergence implies weak convergence, it follows from (iii) and (109) that $\mathbf{K}_{2j}^N|_\Omega = (\mathbf{K}_{21}^N)^j|_\Omega$ and hence (ii) holds. This completes the proof of Proposition 3.14. □

3.5. **Completion of the proof of Theorem 3.11.** We recall that

$$\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N} \cap U] = (Z_{s_1^N,\dots,s_k^N}, \chi_U \Phi_k) = (Z_{s_1^N,\dots,s_k^N}, \chi_{\overline{U}}\, \Phi_k) \qquad a.s. \quad (110)$$

(To verify the second equality in (110), we note that $(Z_{s_1^N,\dots,s_k^N}, \chi_{\partial U}\Phi_k) = 0$ almost surely, since $\mathbf{E}\big(Z_{s_1^N,\dots,s_k^N}, \chi_{\partial U}\Phi_k\big) = 0$ by Corollary 2.3.) We now approximate $\chi_{\overline{U}}$ by a sequence of $\mathcal{C}^\infty$ functions $\chi_n : M \to \mathbb{R}$, $n = 1, 2, 3, \dots$, satisfying:

- $0 \leq \chi_n \leq 1$,
- $\sup |d\chi_n| = O(n)$,
- $\chi_n|_{\overline{U}} \equiv 1$,
- $\chi_n(w) = 0$ for $\mathrm{dist}(U, w) > 1/n$.

To construct χ_n, we choose $\rho \in \mathcal{C}^\infty(\mathbb{R})$ such that $\rho(t) = 1$ for $t \leq \frac{1}{3}$, $\rho(t) = 0$ for $t \geq \frac{2}{3}$, and $0 \leq \rho \leq 1$. Let $\chi_n^0(w) = \rho(n\,\mathrm{dist}(U, w))$. If ∂U is smooth, then χ_n^0 is smooth, for n sufficiently large, and we can take $\chi_n = \chi_n^0$. Otherwise, the Lipschitz constant of χ_n^0 is $O(n)$, and we can smooth χ_n^0 to obtain our desired $\mathcal{C}^\infty$ function χ_n.

Then $\chi_n \to \chi_{\overline{U}}$ pointwise, and hence for all $(s_1^N, \dots, s_k^N)$, we have by Lebesgue dominated convergence,

$$(Z_{s_1^N,\dots,s_k^N}, \chi_n \Phi_k) \to (Z_{s_1^N,\dots,s_k^N}, \chi_{\overline{U}}\, \Phi_k) = \mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N} \cap \overline{U}] \qquad \text{as} \quad n \to \infty\,.$$

Therefore (again by dominated convergence),

$$\mathrm{Var}\big(Z_{s_1^N,\dots,s_k^N}, \chi_n \Phi_k\big) \to \mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N} \cap \overline{U}]\big) = \mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N} \cap U]\big) \quad (111)$$

as $n \to \infty$. To complete the proof of Theorem 3.11, it suffices by (111) and Theorem 3.13 with $\varphi = \chi_n \Phi_k$ to show that

$$\int_{M\times M} \bar\partial_1\bar\partial_2 Q_N \wedge \left(\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\right)^{j-1} \wedge \left(\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right)^{k-j} \wedge \left(\partial[\chi_n\Phi_k] \boxtimes \partial[\chi_n\Phi_k]\right)$$
$$= \int_{M\times M} \bar\partial_1\bar\partial_2 Q_N \wedge \left(\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\right)^{j-1} \wedge \left(\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right)^{k-j} \wedge (\Phi_k \boxtimes \Phi_k) \wedge (d\chi_n \boxtimes d\chi_n)$$
$$\to \int_{\partial U\times\partial U} \bar\partial_1\bar\partial_2 Q_N \wedge \left(\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\right)^{j-1} \wedge \left(\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right)^{k-j} \wedge (\Phi_k \boxtimes \Phi_k). \qquad (112)$$

To verify (112), let

$$f = \bar\partial_1\bar\partial_2 Q_N \wedge \left(\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N\right)^{j-1} \wedge \left(\mathbf{E}Z_{s^N} \boxtimes \mathbf{E}Z_{s^N}\right)^{k-j} \wedge (\Phi_k \boxtimes \Phi_k)\ .$$

We must show that $f|_{\partial U\times\partial U}$ is L^1 and

$$\int_{M\times M} f \wedge (d\chi_n \boxtimes d\chi_n) \to \int_{\partial U\times\partial U} f\ . \qquad (113)$$

By Lemma 3.7, we have

$$|f(z,w)| = O\left(N^j \operatorname{dist}(z,w)^{-2j+2}\right) \le O\left(N^j \operatorname{dist}(z,w)^{-2m+2}\right)\ . \qquad (114)$$

Since ∂U is a finite union of $\mathcal{C}^2$ submanifolds of M of real dimension $2m-1$, it follows from (114) that f is L^1 on $\partial U \times \partial U$.

Let $\delta > 0$ and consider the cut-off function $\lambda_\delta(z,w) = \rho(\delta^{-1}\operatorname{dist}(z,w))$, where ρ is as above. Then $\lambda_\delta \in \mathcal{C}^\infty(M\times M)$ for δ sufficiently small, $\lambda_\delta(z,w) = 0$ if $\operatorname{dist}(z,w) > \delta$, and $\lambda_\delta(z,w) = 1$ if $\operatorname{dist}(z,w) < \delta/3$. We decompose the integral in (113):

$$\int_{M\times M} f \wedge (d\chi_n \boxtimes d\chi_n) = \int_{M\times M} \lambda_\delta\, f \wedge (d\chi_n \boxtimes d\chi_n) + \int_{M\times M} (1-\lambda_\delta)\, f \wedge (d\chi_n \boxtimes d\chi_n)\,. \quad (115)$$

Since $(1-\lambda_\delta)\, f$ is smooth and $\chi_n \to \chi_U$, it follows immediately by applying Stokes' theorem twice that

$$\int_{M\times M} (1-\lambda_\delta)\, f \wedge (d\chi_n \boxtimes d\chi_n) \to (d\chi_U \boxtimes d\chi_u, (1-\lambda_\delta)\, f) = \int_{\partial U\times\partial U} (1-\lambda_\delta)\, f\ . \qquad (116)$$

To complete the proof, we must show that the $\lambda_\delta\, f$ integrals are uniformly small. For $z_0 \in M,\ n \in \mathbb{Z}^+,\ \delta > 0$, we write

$$\begin{aligned} V(z_0,n,\delta) &:= \{w \in M : \operatorname{dist}(z,w) < \delta,\ w \in \operatorname{Supp}(d\chi_n)\} \\ &\subset \{w : \operatorname{dist}(z,w) < \delta,\ \operatorname{dist}(U,w) < 1/n\}\,. \end{aligned}$$

Since ∂U is piecewise smooth, we can choose $\delta_0 > 0,\ n_0 \in \mathbb{Z}^+$ such that for all $z_0 \in M$:

- the exponential map $\exp_{z_0} : T_{z_0}(M) \to M$ is injective on the δ_0-ball $B_{\delta_0}(z_0) = \{v \in T_{z_0}(M) : |v| < \delta_0\}$;
- there exists real hyperplanes $P_1, \dots, P_q$, such that

$$V(z_0,n,\delta_0) \subset \bigcup_{j=1}^{q} \exp_{z_0}\left(\{v + tu_j \in B_{\delta_0}(z_0) : v \in P_j,\ |t_j| < 2/n\}\right), \qquad (117)$$

  for all $n > n_0$, where u_j is a unit normal to P_j.

Here, q is the maximal number of facets of the polyhedral cones locally diffeomorphic to open sets of ∂U, as described after the statement of Theorem 1.1. (If ∂U is smooth, then $q = 1$.)

Since $j \le m$ and $|d\chi_n| = O(1/n)$, we then have by (114) and (117),

$$\begin{aligned}\left|\int_M \lambda_\delta(z_0,w) f(z_0,w)\wedge d\chi_n(w)\right| &\le \left|\int_{\{z_0\}\times V(z_0,n,\delta)} f(z_0,w)\wedge d\chi_n(w)\right|\\ &\le Cn\int_{V(z_0,n,\delta)} \mathrm{dist}(z_0,w)^{-2m+2}\, d\mathrm{Vol}_M\\ &\le C'n\int_{\{x\in\mathbb{R}^{2m}:|x|<\delta,\ |x_1|<2/n\}} |x|^{-2m+2}\,dx\\ &\le 4C'\int_{\{y\in\mathbb{R}^{2m-1}:|y|<\delta\}} |y|^{-2m+2}\,dy \;=\; C''\delta\,,\end{aligned}$$

where C, C', C'' are constants independent of z_0 (but depending on m, U, N). Here, $f(z_0, w)$ is regarded as a $(2m-1)$-form (in the w variable) with values in $T^{*2m-1}_{z_0}(M)$. Therefore,

$$\begin{aligned}\left|\int_{M\times M}\lambda_\delta\, f\wedge(d\chi_n\boxtimes d\chi_n)\right| &= \left|\int_{\{z\in M:\mathrm{dist}(U,z)<1/m\}} d\chi_n(z)\int_{\{z\}\times M}\lambda_\delta(z,w)\, f(z,w)\wedge d\chi_n(w)\right|\\ &\le C''\delta\int_{\{z\in M:\mathrm{dist}(U,z)<1/m\}} |d\chi_n(z)|\, d\mathrm{Vol}_{\partial U}(z)\\ &\le C''\delta\,\sup|d\chi_n|\,\mathrm{Vol}(\{z\in M:\mathrm{dist}(U,z)<1/n\})\,.\end{aligned}$$

Since $\sup|d\chi_n| = O(n)$ and the volume of the shell $\{z \in M : \mathrm{dist}(U, z) < 1/n\}$ is $O(1/n)$, it follows that

$$\left|\int_{M\times M}\lambda_\delta\, f\wedge(d\chi_n\boxtimes d\chi_n)\right| \le C'''\delta \qquad \forall\ n>n_0\,. \tag{118}$$

Then (113) follows from (115), (116) and (118), which completes the proof of Theorem 3.11. □

4. Variance of zeros in a domain: Proof of Theorems 1.1 and 1.4

We now use Theorem 3.11 together with the asymptotics of the pluri-bipotential Q_N to prove Theorem 1.4.

By Theorem 3.11 and Proposition 2.5, we have

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N}\cap U]\big) = \sum_{j=1}^k \binom{k}{j} V_j^N(U)\,, \tag{119}$$

where

$$\begin{aligned}V_j^N(U) &= \left(\frac{N}{\pi}\right)^{2k-2j}\int_{\partial U\times\partial U}\bar\partial_1\bar\partial_2 Q_N(z,w)\wedge\big[-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z,w)\big]^{j-1}\\ &\qquad\wedge\left[\omega(z)^{k-j}+O\left(\frac{1}{N}\right)\right]\wedge\left[\omega(w)^{k-j}+O\left(\frac{1}{N}\right)\right]\wedge\Phi_k(z)\wedge\Phi_k(w)\\ &= \frac{1}{(m-k)!^2}\left(\frac{N}{\pi}\right)^{2k-2j}\int_{\partial U}\Upsilon_j^N\wedge\left[\omega^{m-j}+O\left(\frac{1}{N}\right)\right]\,,\end{aligned} \tag{120}$$

where Φ_k is given by (83), and

$$\begin{aligned}\Upsilon_j^N(z) &:= \int_{\{z\}\times\partial U} \bar\partial_1\bar\partial_2 Q_N(z,w)\wedge\big[-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z,w)\big]^{j-1}\wedge\left[\omega(w)^{m-j}+O\left(\frac{1}{N}\right)\right] \\ &\qquad \in T_z^{*j-1,j}(M)\,. \end{aligned} \tag{121}$$

By Lemma 3.4,

$$\bar\partial_1\bar\partial_2 Q_N(z,w)\wedge\big[\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z,w)\big]^{j-1} = O(N^{-m})\,,\quad \text{for}\ \ \mathrm{dist}(z,w) > b\sqrt{\tfrac{\log N}{N}}\,, \tag{122}$$

where we choose $b=\sqrt{2m+3}$. Thus we can approximate $\Upsilon_j^N(z)$ by restricting the integration in (121) to the set of $w\in\partial U$ with $\mathrm{dist}(z,w)<b\sqrt{\frac{\log N}{N}}$.

To evaluate $\Upsilon_j^N(z_0)$ at a fixed point $z_0\in\partial U$, we choose normal holomorphic coordinates $\{w_1,\dots,w_m\}$ centered at z_0 and defined in a neighborhood V of z_0, and we make the change of variables $w_j=\frac{v_j}{\sqrt N}$ as in §2.2. Since $\omega=\frac{i}{2}\partial\bar\partial\log a=\frac{i}{2}\partial\bar\partial\,[|w|^2+O(|w|^3)]$, we note that

$$\omega\left(z_0+\frac{v}{\sqrt N}\right)=\frac{i}{2}\sum\left[\delta_{jk}+O\left(\frac{|v|}{\sqrt N}\right)\right]\frac{1}{N}dv_j\wedge d\bar v_k=\frac{i}{2N}\partial\bar\partial|v|^2+O\left(\frac{|v|}{N^{3/2}}\right)\,, \tag{123}$$

for $|v|\le b\sqrt{\log N}$. We then have

$$\begin{aligned}\Upsilon_j^N(z_0) &= N^{j-m}\int_{\left\{|v|\le b\sqrt{\log N}:\, z_0+\frac{v}{\sqrt N}\in\partial U\right\}} -\bar\partial_1\bar\partial_2 Q_N(z_0,z_0+\tfrac{v}{\sqrt N}) \\ &\qquad \wedge\big[-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z_0,z_0+\tfrac{v}{\sqrt N})\big]^{j-1}\wedge\big[(\tfrac{i}{2}\partial\bar\partial|v|^2)^{m-j}+O\left(\tfrac{1}{N}\right)\big]\,. \end{aligned} \tag{124}$$

Applying the asymptotics of Lemmas 3.8–3.9 to (124), we obtain the formula

$$\begin{aligned}\Upsilon_j^N(z_0) &= N^{2j-m-1/2}\left[\int_{\left\{|v|\le b\sqrt{\log N}:\, z_0+\frac{v}{\sqrt N}\in\partial U\right\}} \tfrac14\, F''(\tfrac12|v|^2)\,(v\cdot d\bar z)\wedge(v\cdot d\bar v) \right.\\ &\qquad \left.\wedge\,(\mathbf{Var}_\infty^{z_0})^{j-1}\wedge\left(\frac{i}{2}\partial\bar\partial|v|^2\right)^{m-j} + O(N^{-1/2+\varepsilon})\right]\,. \end{aligned} \tag{125}$$

We first consider the case where ∂U is $\mathcal{C}^2$ smooth (without corners). We can choose our holomorphic normal coordinates $\{w_j\}$ so that the real hyperplane $\{\mathrm{Im}\,w_1=0\}$ is tangent to ∂U at z_0. We can then write (after shrinking the neighborhood V if necessary),

$$U\cap V=\{w\in V:\mathrm{Im}\,w_1+\varphi(w)>0\}\,,$$

where $\varphi:V\to\mathbb{R}$ is a $\mathcal{C}^2$ function of $(\mathrm{Re}\,w_1,\,w_2,\dots,w_m)$ such that $\varphi(0)=0,\ d\varphi(0)=0$.

We consider the *nonholomorphic* variables

$$\widetilde w=\tau(w):=(w_1+i\varphi(w),w_2,\dots,w_m)\,, \tag{126}$$

so that $\partial U=\{\mathrm{Im}\,\widetilde w_1=0\}$. We next make the change of variables

$$\widetilde v=\tau_N(v):=\sqrt N\,\tau\left(\frac{v}{\sqrt N}\right)=\sqrt N\,\widetilde w=v\left[1+O\left(\frac{v}{\sqrt N}\right)\right]$$

in the integral (125) to obtain

$$\Upsilon_j^N(z_0) = N^{2j-m-1/2}\Big[\int_{\{\widetilde v\in B_N^{2m-1}\}} \tfrac14 F''(\tfrac12|\widetilde v|^2)\,(\widetilde v\cdot d\bar z)\wedge(\widetilde v\cdot d\overline{\widetilde v}) \wedge (\mathbf{Var}_\infty^{z_0}(\widetilde v))^{j-1}\wedge\left(\tfrac{i}{2}\partial\bar\partial|\widetilde v|^2\right)^{m-j} + O(N^{-1/2+\varepsilon})\Big], \tag{127}$$

where

$$\left\{v\in\mathbb{R}\times\mathbb{C}^{m-1} : |v|<(b-1)\sqrt{\log N}\right\}\subset B_N^{2m-1}\subset\left\{v\in\mathbb{R}\times\mathbb{C}^{m-1} : |v|<(b+1)\sqrt{\log N}\right\}.$$

By (70) and (78), we have $F''(\frac12|v|^2)|v|^2\{\mathbf{Var}_\infty^{z_0}(v)\}^{j-1}=O(e^{-|v|^2})$ for $|v|>1$, and hence

$$\int_{|\widetilde v|>(b-1)\sqrt{\log N}}\left|F''(\tfrac12|\widetilde v|^2)\,(\widetilde v\cdot d\bar z)\wedge(\widetilde v\cdot d\overline{\widetilde v})\wedge(\mathbf{Var}_\infty^{z_0}(\widetilde v))^{j-1}\wedge\left(\tfrac{i}{2}\partial\bar\partial|\widetilde v|^2\right)^{m-j}\right| = O\left(N^{-(b-1)^2+\varepsilon}\right)=O\left(N^{-1}\right).$$

Thus we can replace the B_N^{2m-1} integral in (127) with the affine integral over $\mathbb{R}\times\mathbb{C}^{m-1}$, so that

$$\Upsilon_j^N(z_0)=N^{2j-m-1/2}\left[\Upsilon_j^\infty(z_0)+O(N^{-1/2+\varepsilon})\right], \tag{128}$$

where

$$\begin{aligned}\Upsilon_j^\infty(z_0) &:= \int_{\mathbb{R}\times\mathbb{C}^{m-1}} \tfrac14 F''(\tfrac12|v|^2)\,(v\cdot d\bar z)\wedge(v\cdot d\bar v)\wedge(\mathbf{Var}_\infty^{z_0})^{j-1}\wedge\left(\tfrac{i}{2}\partial\bar\partial|v|^2\right)^{m-j}\\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}\times\mathbb{C}^{m-1}}\frac{1}{e^{|v|^2}-1}\,(v\cdot d\bar z)\wedge(v\cdot d\bar v)\wedge(\mathbf{Var}_\infty^{z_0})^{j-1}\wedge\left(\tfrac{i}{2}\partial\bar\partial|v|^2\right)^{m-j}. \end{aligned}\tag{129}$$

The integral in (129) is independent of the point z_0 and hence

$$(\Upsilon_j^\infty\wedge\omega^{m-j})(z_0)=c_{mj}\,dx_1\wedge(\tfrac{i}{2}dz_2\wedge d\bar z_2)\wedge\cdots\wedge(\tfrac{i}{2}dz_m\wedge d\bar z_m)=c_{mj}\,d\mathrm{Vol}_{\partial U,z_0}, \tag{130}$$

where c_{mj} is a universal constant.

Substituting (128) and (130) in (120), we have

$$V_j^N(U)=\frac{1}{(m-k)!^2}\left(\frac{1}{\pi^{2k-2j}}\right)N^{2k-m-1/2}\left[\int_{\partial U}c_{mj}\,d\mathrm{Vol}_{\partial U,z_0}+O\left(N^{-1/2+\varepsilon}\right)\right]. \tag{131}$$

Combining (119) and (131), we obtain the formula of Theorem 1.4 with

$$\nu_{mk}=\frac{1}{(m-k)!^2}\sum_{j=1}^k\binom{k}{j}\frac{c_{mj}}{\pi^{2k-2j}}, \tag{132}$$

for the case where ∂U is smooth.

We now compute the coefficient ν_{m1} in the codimension-one case: By (129),

$$\begin{aligned}\Upsilon_1^\infty(z_0) &= \frac{1}{4\pi^2}\sum_{j,k}\left[\int_{\mathbb{R}\times\mathbb{C}^{m-1}}\frac{v_jv_kd\bar v_k}{e^{|v|^2}-1}\wedge\left(\tfrac{i}{2}\partial\bar\partial|v|^2\right)^{m-1}\right]d\bar z_j\\ &= \frac{(m-1)!}{4\pi^2}\sum_{j=1}^m\left[\int_{\mathbb{R}\times\mathbb{C}^{m-1}}\frac{v_jv_1}{e^{|v|^2}-1}\,d\mathrm{Vol}_{\mathbb{R}\times\mathbb{C}^{m-1}}(v)\right]d\bar z_j.\end{aligned}\tag{133}$$

By (119)–(120) with $k=1$ and (128), we have

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N}\cap U]\big)=\frac{N^{3/2-m}}{(m-1)!^2}\left[\int_{\partial U}\Upsilon_1^\infty(z)\wedge\omega(z)^{m-1}+O\left(N^{-1/2+\varepsilon}\right)\right]. \tag{134}$$

Since $d\mathrm{Vol}_{\partial U}(z_0) = dx_1 \wedge \frac{1}{(m-1)!}\omega(z_0)^{m-1}$, only the $j=1$ term in (133) contributes to the integral in (134), and we then have

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N} \cap U]\big) = N^{3/2-m}\left[\int_{\partial U} \nu_{m1} d\mathrm{Vol}_{\partial U} + O\left(N^{-1/2+\varepsilon}\right)\right], \tag{135}$$

where

$$\begin{aligned}
\nu_{m1} &= \frac{1}{4\pi^2}\int_{\mathbb{R}^{2m-1}} \frac{v_1^2}{e^{|v|^2}-1}\,dv = \frac{1}{4\pi^2(2m-1)}\int_{\mathbb{R}^{2m-1}} \frac{|v|^2}{e^{|v|^2}-1}\,dv \\
&= \frac{1}{4\pi^2(2m-1)}\,\frac{2\pi^{m-1/2}}{\Gamma(m-1/2)}\int_0^\infty \frac{r^{2m}}{e^{r^2}-1}\,dr \\
&= \frac{\pi^{m-5/2}}{4\,\Gamma(m+1/2)}\sum_{k=1}^\infty \int_0^\infty e^{-kr^2}\,r^{2m}\,dr \\
&= \frac{\pi^{m-5/2}}{4\,\Gamma(m+1/2)}\sum_{k=1}^\infty \frac{\Gamma(m+1/2)}{2\,k^{m+1/2}} = \frac{\pi^{m-5/2}}{8}\,\zeta\Big(m+\tfrac{1}{2}\Big)\,,
\end{aligned}$$

as stated in the theorem.

It remains to verify the general case where ∂U is piecewise smooth (without cusps). Let S denote the set of singular points ('corners') of ∂U, and let S_N be the small neighborhood of S given by

$$S_N = \left\{ z \in \partial U : \mathrm{dist}(z,S) < \frac{b'\sqrt{\log N}}{\sqrt{N}} \right\}\,,$$

where $b' > 0$ is to be chosen below. We shall show that:

i) (128) holds uniformly for $z_0 \in \partial U \smallsetminus S_N$;
ii) $\sup_{z\in\partial U} |\Upsilon_j^N(z)| = O\left(N^{2j-m-1/2+\varepsilon}\right)$, for $1 \le j \le k$.

Let us assume (i)–(ii) for now. Since $\mathrm{Vol}_{2m-1} S_N = O\left(\frac{\sqrt{\log N}}{\sqrt{N}}\right)$, the estimate (ii) implies that

$$\int_{S_N} \Upsilon_j^N \wedge \omega^{m-j} = O(N^{2j-m-1+\varepsilon})\,,$$

and hence by (120),

$$V_j^N(U) = \frac{1}{(m-k)!^2}\left(\frac{N}{\pi}\right)^{2k-2j}\int_{\partial U\smallsetminus S_N} \Upsilon_j^N \wedge \left[\omega^{m-j} + O\left(\frac{1}{N}\right)\right] + O\left(N^{2k-m-1+\varepsilon}\right)\,.$$

It then follows from (i) and (130) that

$$\begin{aligned}
V_j^N(U) &= \frac{N^{2k-m-1/2}}{(m-k)!^2\,\pi^{2k-2j}}\left[\int_{\partial U\smallsetminus S_N} \Upsilon_j^\infty \wedge \omega^{m-j} + O(N^{-1/2+\varepsilon})\right] \\
&= \frac{c_{mj}\,N^{2k-m-1/2}}{(m-k)!^2\,\pi^{2k-2j}}\left[\mathrm{Vol}(\partial U \smallsetminus S_N) + O(N^{-1/2+\varepsilon})\right].
\end{aligned}$$

Then by (119)

$$\begin{aligned}
\mathrm{Var}\big(\mathrm{Vol}_{2m-2k}[Z_{s_1^N,\dots,s_k^N} \cap U]\big) &= N^{2k-m-1/2}\left[\nu_{mk}\,\mathrm{Vol}_{2m-1}(\partial U \smallsetminus S_N) + O(N^{-\frac{1}{2}+\varepsilon})\right] \\
&= N^{2k-m-1/2}\left[\nu_{mk}\,\mathrm{Vol}_{2m-1}(\partial U) + O(N^{-\frac{1}{2}+2\varepsilon})\right]\,,
\end{aligned}$$

which is our desired formula.

It remains to prove (i)–(ii). To verify (i), for each point $z_0 \in \partial U \smallsetminus S$, we choose holomorphic coordinates $\{w_j\}$ and non-holomorphic coordinates $\{\widetilde{w}_j\}$ as above. We can choose these coordinates on a geodesic ball V_{z_0} about z_0 of a fixed radius $R > 0$ independent of the point z_0, but if z_0 is near a corner, ∂U will coincide with $\{\operatorname{Im} \widetilde{w}_1 = 0\}$ only in a small neighborhood of z_0. To be precise, we let D_{z_0} denote the connected component of $V_{z_0} \cap \partial U \smallsetminus S$ containing z_0. Then we choose $\varphi \in \mathcal{C}^2(V_{z_0})$ with $\varphi(0) = 0,\ d\varphi(0) = 0$, such that

$$\{w \in V_{z_0} : \operatorname{Im} w_1 + \varphi(w) = 0\} = \{\operatorname{Im} \widetilde{w}_1 = 0\} \supset D_{z_0}\ . \tag{136}$$

We let

$$C = \sup_{z \in \partial U \smallsetminus S} \frac{\operatorname{dist}(z, S)}{\operatorname{dist}(z, \partial U \smallsetminus D_z)} \geq 1\ .$$

Choose $N_0 > 0$ such that $b\sqrt{\frac{\log N_0}{N_0}} < R$; then

$$\left\{ w \in \partial U : \operatorname{dist}(z_0, w) < b\sqrt{\frac{\log N}{N}} \right\} \subset V_{z_0}\ , \quad \text{for} \quad N \geq N_0\ .$$

We recall that our assumption that ∂U is piecewise $\mathcal{C}^2$ *without cusps* means that $\overline{U}$ is locally $\mathcal{C}^2$ diffeomorphic to a polyhedral cone, which implies that $C < +\infty$. We now let $b' = Cb$, where $b = \sqrt{2m+3}$ as before.

Consider any point $z_0 \in \partial U \smallsetminus S_N,\ N \geq N_0$. Then

$$\operatorname{dist}(z_0, \partial U \smallsetminus D_{z_0}) \geq \frac{\operatorname{dist}(z_0, S)}{C} \geq \frac{b'\sqrt{\log N}}{C\sqrt{N}} = \frac{b\sqrt{\log N}}{\sqrt{N}}\ .$$

Thus by our far-off-diagonal decay estimate (122), the points in $\partial U \smallsetminus D_{z_0}$ contribute negligibly to the integral in (125), so that integral can be taken over the set

$$\left\{ |v| \leq b\sqrt{\log N} :\ z_0 + \frac{v}{\sqrt{N}} \in D_{z_0} \right\}\ ,$$

which is mapped by τ_N into $\mathbb{R} \times \mathbb{C}^{m-1}$. Then (127) holds, and (128) follows as before.

To verify (ii), we must show that the integral in the right side of (125),

$$\widetilde{\Upsilon}_j^N(z_0) := \int_{\left\{ |v| \leq b\sqrt{\log N} :\, z_0 + \frac{v}{\sqrt{N}} \in \partial U \right\}} \tfrac{1}{4} F''(\tfrac{1}{2}|v|^2)\, (v \cdot d\bar{z}) \wedge (v \cdot d\bar{v}) \wedge (\mathbf{Var}_\infty^{z_0})^{j-1} \wedge \left(\tfrac{i}{2}\partial\bar{\partial}|v|^2\right)^{m-j}\ ,$$

is $O(N^\varepsilon)$ uniformly for $z_0 \in \partial U$. By Lemma 3.9, $\mathbf{Var}_\infty^{z_0}(v) = O(|v|^{-2})$. Furthermore,

$$\left| \frac{1}{e^{|v|^2} - 1}\, (v \cdot d\bar{z}) \wedge (v \cdot d\bar{v}) \right| \leq \frac{\sqrt{m}\, |v|^2}{e^{|v|^2} - 1} \leq \sqrt{m}$$

(using Euclidean norms in the z and v variables), and hence

$$|\widetilde{\Upsilon}_j^N(z_0)| \leq A_{jm} \int_{\left\{ |v| \leq b\sqrt{\log N} :\, z_0 + \frac{v}{\sqrt{N}} \in \partial U \right\}} |v|^{-2j+2}\, d\mathrm{Vol}^E_{2m-1}(v)\ , \tag{137}$$

for universal constants A_{jm},where Vol^E denotes Euclidean volume. Rewriting (137) in terms of the original variables $w = z_0 + \frac{v}{\sqrt{N}}$, we have

$$|\widetilde{\Upsilon}_j^N(z_0)| \leq A_{jm}\, N^{-j+m+1/2} \int_{\left\{ w \in \partial U :\, |w - z_0| \leq b\sqrt{\frac{\log N}{N}} \right\}} |w - z_0|^{-2j+2}\, d\mathrm{Vol}^E_{2m-1}(w)\ .$$

For each point $P \in \partial U$, we choose a closed neighborhood $V_P \in M$ of P and a $\mathcal{C}^2$ diffeomorphism $\rho_P : V_P \to \mathbb{R}^{2m}$ mapping $V_P \cap \partial U$ to the boundary of a polyhedral cone $K_P \subset \mathbb{R}^{2m}$. Then for N sufficiently large, for all $z_0 \in \partial U$, the set $\left\{ w \in \partial U : |w - z_0| \le b\sqrt{\frac{\log N}{N}} \right\}$ is contained in one of the V_P. We then make the (nonholomorphic) coordinate change $\widetilde{w} = \rho_P(w)$. Since the diffeomorphisms ρ_P have bounded distortion, we then have

$$|\widetilde{\Upsilon}_j^N(z_0)| \le A'_{jm}\, N^{-j+m+1/2} \int_{\left\{\widetilde{w}\in\partial K_P:\, |\widetilde{w}-\widetilde{z}_0|\le b\sqrt{\frac{\log N}{N}}\right\}} |\widetilde{w} - \widetilde{z}_0|^{-2j+2}\, d\mathrm{Vol}^E_{2m-1}(\widetilde{w}) \,. \tag{138}$$

Let $q \in \mathbb{Z}^+$ be the maximum number of facets in ∂K_P. We easily see that

$$\begin{aligned} \int_{\left\{\widetilde{w}\in\partial K_P:\, |\widetilde{w}-\widetilde{z}_0|\le b\sqrt{\frac{\log N}{N}}\right\}} |\widetilde{w} - \widetilde{z}_0|^{-2j+2}\, d\mathrm{Vol}^E_{2m-1}(\widetilde{w}) \ &\le \ q \int_{\left\{x\in\mathbb{R}^{2m-1}:\, |x|\le b\sqrt{\frac{\log N}{N}}\right\}} |x|^{-2j+2}\, dx \\ &= \ \text{const.} \times \left(\frac{\log N}{N}\right)^{m-j-1/2} \,. \end{aligned} \tag{139}$$

Combining (138)–(139), we conclude that $\widetilde{\Upsilon}_j^N(z_0) = O(N^{\varepsilon})$, which verifies (ii) and completes the proof of Theorem 1.4 for the general case where ∂U has corners. □

4.1. **Positivity of the constant in Theorem 1.1.** The constants ν_{mk} could be obtained from (129)–(130) and (132). Since the computation is rather difficult, we instead outline a proof that $\nu_{mm} > 0$.

Since ν_{mm} is universal, we shall verify that it is positive using the following example: Let $M = \mathbb{C}^m/\mathbb{Z}^{2m}$ with principal polarization $L \to M$, i.e. L has a metric h with $\omega = \frac{i}{2}\Theta_h = \frac{\pi i}{2}\partial\bar{\partial}|z|^2$. Let U be the image of the strip $\{z \in \mathbb{C}^m : 0 < \operatorname{Im} z_1 < \frac{1}{p}\}$ under the covering map $\pi : \mathbb{C}^m \to \mathbb{C}^m/\mathbb{Z}^{2m}$, where p is a prime greater than m. The Szegő kernel for this example is essentially given by the Heisenberg Szegő kernel near the diagonal: Using the notation of Theorem 2.4, we have

$$N^{-m}\Pi_N^{z_0}\left(\frac{u}{\sqrt{N}}, \frac{\theta}{N}; \frac{v}{\sqrt{N}}, \frac{\varphi}{N}\right) = \Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)\left[1 + O(N^{-\infty})\right] \quad \text{for} \ \ |u| + |v| < b\sqrt{\log N}\,. \tag{140}$$

Equation (140) can be verified in two ways. One method is to note that the coefficients of the polynomials p_r of Theorem 2.4 are curvature invariants (of order ≥ 1), and thus must vanish since $(\mathbb{C}^m/\mathbb{Z}^m, \omega)$ is flat. Alternately, we can use the explicit construction of the Szegő kernel on the unit circle bundle of $L \to \mathbb{C}^m/\mathbb{Z}^m$ as the projection of $\Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)$. To describe this construction, we recall that the unit circle bundle of the Hermitian line bundle (L, h) may be identified with the quotient of the reduced Heisenberg group $\mathbf{H}^m_{red}$ by a co-compact sub-lattice $\bar{\Gamma}$. Here, $\mathbf{H}^m_{red}$ is the quotient of the simply connected Heisenberg group $\mathbf{H}^m$ by the integer subgroup of its center. The lattice $\bar{\Gamma}$ is the embedding of $\mathbb{Z}^{2m}$ into the reduced Heisenberg group by the splitting homomorphism

$$s : \mathbb{Z}^{2m} \to \mathbf{H}_m^{red} \qquad s(j,k) = (j, k, e^{\frac{i}{2}\langle j,k\rangle}).$$

The degree-N Szegő kernel of the quotient is the projection of the degree-N Szegő kernel of the universal cover (as a consequence of the existence of a spectral gap), as discussed in [Ze1, Theorem 6.1]; i.e.,

$$\Pi_N(x,y) = \sum_{\gamma\in\bar{\Gamma}} \Pi_N^{\mathbf{H}}(u,\theta;\gamma\cdot(v,\varphi)) \tag{141}$$

It is immediate from the explicit formula (26) for $\Pi_N^{\mathbf{H}}$ that, for all u,θ,v,φ,

$$\sum_{\gamma\neq 0}|\Pi_N^{\mathbf{H}}(u,\theta;\gamma\cdot(v,\varphi))|\leq \sum_{J\in\mathbb{Z}^m, J\neq 0} e^{-N|J|^2}=O(e^{-CN}),\quad C>0.$$

Thus (141) yields (140).

In particular, (140) implies that

$$\Pi_N(z,z) = \frac{1}{\pi^m}N^m\left[1+O(N^{-\infty})\right], \tag{142}$$

$$P_N\left(z_0+\tfrac{u}{\sqrt{N}}, z_0+\tfrac{v}{\sqrt{N}}\right) = e^{-\frac12|u-v|^2}[1+O(N^{-\infty})] \quad\text{for}\quad |u|+|v|<b\sqrt{\log N}\,. \tag{143}$$

By Corollary 2.3 and (142), we have

$$\begin{aligned}\mathbf{E}\big(\mathcal{N}_N^U\big) &= \int_U\left(\frac{i}{\pi}\partial\bar\partial\log\Pi_N(z,z)+\frac{N}{\pi}\omega\right)^m\\ &= N^m\int_U\frac{1}{\pi^m}\,\omega^m+O(N^{-\infty})\\ &= \frac{m!}{p}N^m+O(N^{-\infty})\,.\end{aligned} \tag{144}$$

Since the random variable $\mathcal{N}_N^U$ is integer-valued, it follows from (144) that

$$\mathrm{Var}\big(\mathcal{N}_N^U\big)\geq\frac{1}{p^2}+O(N^{-\infty})\quad\text{for}\quad N\not\equiv 0\ (p)\,. \tag{145}$$

On the other hand, by repeating the arguments in §3.1, using instead the $O(N^{-\infty})$ error from (143), we have

$$\bar\partial_1\bar\partial_2 Q_N(0,\tfrac{v}{\sqrt{N}}) = -\frac{\sqrt{N}}{4}F''(\tfrac12|v|^2)(v\cdot d\bar z)(v\cdot d\bar v)+O(N^{-\infty})\,, \tag{146}$$

$$-\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z_0,z_0+\tfrac{v}{\sqrt{N}}) = N\,\mathbf{Var}_\infty^{z_0}(v)+O\left(|v|^{-2}\,N^{-\infty}\right)\,, \tag{147}$$

for $0<|v|<b\sqrt{\log N}$. We then repeat the argument in §4, using (146)–(147) for the near-diagonal estimate. As for the far estimate, for any $K>0$, we choose b_K sufficiently large so that by Lemma 3.4, we have

$$\bar\partial_1\bar\partial_2 Q_N(z,w)\wedge\left[\partial_1\bar\partial_1\partial_2\bar\partial_2 Q_N(z,w)\right]^{j-1}=O(N^{-K})\,,\quad\text{for}\ \ \mathrm{dist}(z,w)>b_K\sqrt{\tfrac{\log N}{N}}\,.$$

Since ∂U is flat, in place of (126), we simply set $\widetilde w=w$, and we conclude from the argument in §4 that

$$\mathrm{Var}\big(\mathcal{N}_N^U\big)=N^{m-1/2}\left[\nu_{mm}\,\mathrm{Vol}_{2m-1}(\partial U)+O(N^{-\infty})\right]\,. \tag{148}$$

Positivity of ν_{mm} follows immediately from the inequalities (145) and (148). This completes the proof of Theorem 1.1 □

5. Appendix: Proof of Theorem 2.4

In this appendix, we sketch the proof of the off-diagonal Szegő asymptotics theorem. The argument is essentially contained in [SZ2], but we add some details relevant to the estimates in Theorem 2.4.

The Szegő kernels $\Pi_N(x,y)$ are the Fourier coefficients of the total Szegő projector $\Pi(x,y):\mathcal{L}^2(X)\to\mathcal{H}^2(X)$; i.e. $\Pi_N(x,y)=\frac{1}{2\pi}\int e^{-iN\theta}\Pi(e^{i\theta}x,y)\,d\theta$. The estimates for $\Pi_N(z,w)$ are then based on the Boutet de Monvel-Sjöstrand construction of an oscillatory integral parametrix for the Szegő kernel:

$$\begin{aligned} \Pi(x,y) &= S(x,y) + E(x,y)\,, \\ \text{with }\ S(x,y) &= \textstyle\int_0^\infty e^{it\psi(x,y)} s(x,y,t)dt\,, \qquad E(x,y) \in \mathcal{C}^\infty(X\times X)\,. \end{aligned} \tag{149}$$

The amplitude has the form $s \sim \sum_{k=0}^\infty t^{m-k} s_k(x,y) \in S^m(X\times X\times\mathbb{R}^+)$. The phase function ψ is of positive type, and as described in [BSZ2], is given by:

$$\psi(z,\theta,w,\varphi) = i\left[1 - \frac{a(z,\bar w)}{\sqrt{a(z)}\sqrt{a(w)}}\, e^{i(\theta-\varphi)}\right]\,, \tag{150}$$

where $a\in\mathcal{C}^\infty(M\times M)$ is an almost holomorphic extension of the function $a(z,\bar z) := a(z)$ on the anti-diagonal $A = \{(z,\bar z) : z\in M\}$, i.e., $\bar\partial a$ vanishes to infinite order along A. We recall from (13) that $a(z)$ describes the Hermitian metric on L in our preferred holomorphic frame at z_0, so by (24), we have $a(u) = 1 + |u|^2 + O(|u|^3)$, and hence

$$a(u,\bar v) = 1 + u\cdot\bar v + O(|u|^3 + |v|^3)\,. \tag{151}$$

For further background and notation on complex Fourier integral operators we refer to [BSZ2] and to the original paper of Boutet de Monvel and Sjöstrand [BS].

As above, denote the N-th Fourier coefficient of these operators relative to the S^1 action by $\Pi_N = S_N + E_N$. Since E is smooth, we have $E_N(x,y) = O(N^{-\infty})$, where $O(N^{-\infty})$ denotes a quantity which is uniformly $O(N^{-k})$ on $X\times X$ for all positive k. Then, $E_N(z,w)$ trivially satisfies the remainder estimates in Theorem 2.4.

Hence it is only necessary to verify that the oscillatory integral

$$S_N(x,y) = \int_0^{2\pi} e^{-iN\theta} S(e^{i\theta}x,y)d\theta = \int_0^\infty\int_0^{2\pi} e^{-iN\theta + it\psi(e^{i\theta}x,y)} s(e^{i\theta}x,y,t)d\theta dt \tag{152}$$

satisfies Theorem 2.4. This follows from an analysis of the stationary phase method and remainder estimate for the rescaled parametrix

$$S_N^{z_0}\left(\frac{u}{\sqrt N},0;\frac{v}{\sqrt N},0\right) = N\int_0^\infty\int_0^{2\pi} e^{iN\left(-\theta + t\psi\left(\frac{u}{\sqrt N},\theta;\frac{v}{\sqrt N},0\right)\right)} s\left(\frac{u}{\sqrt N},\theta;\frac{v}{\sqrt N},0,Nt\right)d\theta dt\,, \tag{153}$$

where we changed variables $t\mapsto Nt$. For background on the stationary phase method when the phase is complex we refer to [Hö]. We are particularly interested in the dependence of the stationary phase expansion and remainder estimate on the parameters (u,v) satisfying the constraints in (i)-(ii) of Theorem 2.4.

To clarify the constraints, we recall from [SZ2, (95)] that the Szegő kernel satisfies the following far from diagonal estimates:

$$\left|\nabla_h^j \Pi_N(z,w)\right| = O(N^{-K}) \qquad \text{for all }\ j,K \text{ when }\ \mathrm{dist}(z,w) \ge \frac{N^{1/6}}{\sqrt N}\,. \tag{154}$$

Hence we may assume from now on that $z = z_0 + \frac{u}{\sqrt N}$, $w = z_0 + \frac{v}{\sqrt N}$ with

$$|u| + |v| \le \delta N^{1/6} \tag{155}$$

for a sufficiently small constant $\delta > 0$.

By (150)–(151), the rescaled phase in (153) has the form:

$$\widetilde{\Psi} := t\psi\left(\frac{u}{\sqrt{N}},\theta;\frac{v}{\sqrt{N}},0\right)-\theta = it\left[1-\frac{a\left(\frac{u}{\sqrt{N}},\frac{\bar v}{\sqrt{N}}\right)}{a\left(\frac{u}{\sqrt{N}},\frac{\bar u}{\sqrt{N}}\right)^{\frac12} a\left(\frac{v}{\sqrt{N}},\frac{\bar v}{\sqrt{N}}\right)^{\frac12}}\,e^{i\theta}\right]-\theta \tag{156}$$

and the N-expansion

$$\widetilde{\Psi} = it[1-e^{i\theta}]-\theta-\frac{it}{N}\psi_2(u,v)e^{i\theta}+tR_3^{\psi}(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}})e^{i\theta}\,, \tag{157}$$

where

$$\psi_2(u,v) = u\cdot\bar v-\tfrac12(|u|^2+|v|^2) = -\tfrac12|u-v|^2+i\,\mathrm{Im}\,(u\cdot\bar v)\ .$$

After multiplying by iN, we move the last two terms of (157) into the amplitude. Indeed, we absorb all of $\exp\{(\psi_2+iNR_3^{\psi})te^{i\theta}\}$ into the amplitude so that (153) is an oscillatory integral

$$N\int_0^\infty\int_0^{2\pi} e^{iN\Psi(t,\theta)}A(t,\theta;z_0,u,v)d\theta dt+O(N^{-\infty}) \tag{158}$$

with phase

$$\Psi(t,\theta) := it(1-e^{i\theta})-\theta \tag{159}$$

and with amplitude

$$A(t,\theta;z_0,u,v) := e^{te^{i\theta}\psi_2(u,v)+ite^{i\theta}NR_3^{\psi}(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}})}\,s\big(\frac{u}{\sqrt{N}},\theta;\frac{v}{\sqrt{N}},0,Nt\big). \tag{160}$$

The phase Ψ is independent of the parameters (u,v), satisfies $\mathrm{Re}\,(i\Psi) = -t(1-\cos\theta)\le 0$ and has a unique critical point at $\{t=1,\theta=0\}$ where it vanishes.

The factor $e^{te^{i\theta}\psi_2(u,v)}$ is of exponential growth in some regions. However, since it is a rescaling of a complex phase of positive type, the complex phase $iN\Psi+te^{i\theta}\psi_2(u,v)$ is of positive type,

$$\mathrm{Re}\,(iN\Psi+te^{i\theta}\psi_2(u,v))<0 \tag{161}$$

once the cubic remainder $Nte^{i\theta}R_3^{\psi}(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}})$ is smaller than $iN\Psi+te^{i\theta}\psi_2(u,v)$, which occurs for all (t,θ,u,v) when (u,v) satisfy (155) with δ sufficiently small.

To estimate the joint rate of decay in (N,u,v), we follow the stationary phase expansion and remainder estimate in Theorem 7.7.5 of [Hö], with extra attention to the unbounded parameter u.

The first step is to use a smooth partition of unity $\{\rho_1(t,\theta),\rho_2(t,\theta)\}$ to decompose the integral (153) into a region $(1-\varepsilon,1+\varepsilon)_t\times(-\varepsilon,\varepsilon)_\theta$ containing the critical point and one over the complementary set containing no critical point. We claim that the ρ_2 integral is of order $N^{-\infty}$ and can be neglected. This follows by repeated partial integration as in the standard proof together with the fact that the exponential factors in (161) decay, so that the estimates are integrable and uniform in u.

We then apply [Hö] Theorem 7.7.5 to the ρ_1 integral. The first term of the stationary phase expansion equals $N^m e^{te^{i\theta}\psi_2(u,v)}$ and the remainder satisfies

$$|\widehat{R}_J(P_0,u,v,N)|\le CN^{-m+J}\sum_{|\alpha|\le 2J+2}\sup_{t,\theta}|D^\alpha_{t,\theta}\rho_1 A(t,\theta;P_0,u,v)|. \tag{162}$$

From the formula in (160) and the fact that s is a symbol, A has a polyhomogeneous expansion of the form

$$\begin{aligned} A(t,\theta;P_0,u,v) &= \rho_1(t,\theta)e^{te^{i\theta}\psi_2(u,v)}N^m\left[\sum_{n=0}^{K} N^{-n/2}f_n(u,v;t,\theta,P_0)+R_K(u,v,t,\theta)\right], \\ & \qquad |\nabla^j R_{Nk}(u,v)| \le C_{jk\varepsilon b}e^{\varepsilon(|u|^2+|v|^2)}N^{-\frac{K+1}{2}}). \end{aligned} \tag{163}$$

The exponential remainder factor $e^{\varepsilon(|u|^2+|v|^2)}$ comes from the fact $\mathrm{Re}\, e^{i\theta}\psi_2 = \cos\theta \mathrm{Re}\,\psi - \sin\theta \mathrm{Im}\,\psi$ with $\mathrm{Re}\,\psi \le 0$ and $|\sin\theta| < \varepsilon$ on the support of ρ_1. Hence, the supremum of the amplitude in a neighborhood of the stationary phase set (in the support of ρ_1) is bounded by $e^{\varepsilon|\mathrm{Im}\,\psi_2|}$. The remainder term is smaller than the main term asymptotically as $N\to\infty$ as long as (u,v) satisfies (155). Part(i) of Theorem 2.4 is an immediate consequence of (163) since $e^{\varepsilon(|u|^2+|v|^2)} \le N^\varepsilon$ for $|u|+|v| \le \sqrt{\log N}$.

To prove part (ii), we may assume from (154)–(155) that $\sqrt{\log N} \le |u|+|v| \le \delta\, N^{1/6}$. In this range the asymptotics (163) are valid. We first rewrite the horizontal z-derivatives $\frac{\partial^h}{\partial z_j}$ as u_j derivatives, which for L^N have the form $\sqrt{N}\frac{\partial}{\partial u_j} - NA_j(\frac{u}{\sqrt{N}})$ and thus ∇_h contributes a factor of $\sqrt{N}$. We thus obtain an asymptotic expansion and remainder for $\nabla_h^j \Pi_N(z,w)$ by applying ∇_h^j to the expansion (i) with $k=0$:

$$\Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)\left[1+N^{-1/2}R_{N0}(u,v)\right].$$

The operator ∇_h^j contributes a factor of $N^{j/2}$ to each term, and thus

$$\begin{aligned} \left|\nabla_h^j \Pi_N(z,w)\right| &= O\left(N^{m+j/2}\, e^{-(1-\varepsilon)\frac{|u|^2+|v|^2}{2}}\right) \\ &= O(N^{-k}) \qquad \text{uniformly for} \quad |u|^2+|v|^2 \ge (j+2k+2m+\varepsilon')\log N\,, \end{aligned}$$

where $\varepsilon' = (j+2k+2m+1)\varepsilon$. □

References

[Be] B. Berndtsson, Bergman kernels related to Hermitian line bundles over compact complex manifolds, *Explorations in Complex and Riemannian eometry: A Volume Dedicated to Robert E. Greene*, Contemporary Math., vol. 332, Amer. Math. Soc., Providence, RI, 2003.

[BSZ1] P. Bleher, B. Shiffman and S. Zelditch, Poincaré-Lelong approach to universality and scaling of correlations between zeros, *Comm. Math. Phys.* 208 (2000), 771–785.

[BSZ2] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.* 142 (2000), 351–395.

[BBL] E. Bogomolny, O. Bohigas, and P. Leboeuf, Quantum chaotic dynamics and random polynomials, *J. Statist. Phys.* 85 (1996), 639–679.

[BS] L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, *Asterisque* 34–35 (1976), 123–164.

[Ca] D. Catlin, The Bergman kernel and a theorem of Tian, in: *Analysis and Geometry in Several Complex Variables*, G. Komatsu and M. Kuranishi, eds., Birkhäuser, Boston, 1999.

[Ch] M. Christ, On the $\bar\partial$ equation in weighted L^2 norms in $\mathbb{C}^1$, *Jour. Geom. Anal.* 3 (1991), 193–230.

[DLM] X. Dai, K. Liu and X. Ma, On the asymptotic expansion of Bergman kernel, *J. Differential Geom.* 72 (2006), 1–41.

[De] H. Delin, Pointwise estimates for the weighted Bergman projection kernel in $\mathbf{C}^n$, using a weighted L^2 estimate for the $\overline{\partial}$ equation, *Ann. Inst. Fourier (Grenoble)* 48 (1998), 967–997.

[EK] A. Edelman and E. Kostlan, How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc.* 32 (1995), 1–37.

[FH] P. J. Forrester and G. Honner, Exact statistical properties of the zeros of complex random polynomials, *J. Phys. A* 32 (1999), 2961–2981.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York (1978).

[Ha] J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state, *J. Phys. A* 29 (1996), L101–L105.

[Hö] L. Hörmander, *The Analysis of Linear Partial Differential Operators, I*, Springer Verlag, N.Y., 1983.

[Kac] M. Kac, On the average number of real roots of a random algebraic equation, II, *Proc. London Math. Soc.* 50 (1949), 390–408.

[Li] N. Lindholm, Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel, *J. Funct. Anal.* 182 (2001), 390–426.

[MM] X. Ma and G. Marinescu, Generalized Bergman kernels on symplectic manifolds, preprint (math.DG/0411559).

[NV] S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space, *J. Statist. Phys.* 92 (1998), 431–518.

[PV] Y. Peres and B. Virag, Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process, *Acta Math.* 194 (2005), 1–35.

[SZ1] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, *Comm. Math. Phys.* 200 (1999), 661–683.

[SZ2] B. Shiffman and S. Zelditch, Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds, *J. Reine Angew. Math.* 544 (2002), 181–222.

[SZ3] B. Shiffman and S. Zelditch, Number variance of random zeros, preprint (math.CV/0512652).

[SZ4] B. Shiffman and S. Zelditch, Complex zeros of random fewnomials, in preparation.

[So] M. Sodin, Zeros of Gaussian analytic functions, *Math. Res. Lett.* 7 (2000), 371–381.

[ST] M. Sodin and B. Tsirelson, Random complex zeros, I. Asymptotic normality, *Israel J. Math.* 144 (2004), 125–149.

[Ti] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, *J. Diff. Geometry* 32 (1990), 99–130.

[Ze1] S. Zelditch, Index and dynamics of quantized contact transformations, *Ann. Inst. Fourier (Grenoble)* 47 (1997), 305–363.

[Ze2] S. Zelditch, Szegő kernels and a theorem of Tian, *Internat. Math. Res. Notices* 1998 (1998), 317–331.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: shiffman@math.jhu.edu, zelditch@math.jhu.edu