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Numerical Analogs to the Schwarz Alternating Procedure

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ALTERNATING PROCEDURE

1. Introduction

The alternating procedure of H.A. Schwarz [1] was introduced to prove the solvability of the Dirichlet problem for Laplace's equation on a plane region which is the union of two overlapping plane regions, provided that the Dirichlet problem is solvable on each of these two regions, and that their boundaries intersect at non-zero angles.

Nevanlinna [2] removes the necessity that the boundaries intersect at a non-zero angle. He also gives a very good outline of the procedure as originally applied by Schwarz. Schwarz was able to prove convergence of his alternating procedure with the aid of what we shall in the example call "the $q<1$ property." As a result, the error at the nth step of the procedure is less than $O(q_1^n q_2^n)$ where q_1 and q_2 are numbers less than 1. Nevanlinna in eliminating the condition of a non-zero angle of intersection eliminates the $q<1$ property, and hence eliminates any estimate of the rate of convergence. However, it is exactly this property which will make the method of interest to us for numerical purposes. Nevanlinna also points out that, since the Dirichlet solution for each region is known in explicit integral form the alternating procedure need involve only the values on the boundaries of the two regions, that the boundary value problem of Schwarz is equivalent to an integral equation of one less dimension than the original problem, and that the alternating procedure is nothing other than a solution of this equation through successive approximations.

The success of Schwarz's method as an existence proof depends only upon certain general algebraic conditions which hold true for many other partial differential equations and which are independent of dimensionality. Thus, the variety of possible applications seems endless. Kantorovich [3] gives an analysis of some abstract conditions under which the method succeeds.

We will be interested only in the problem of evaluating solutions already known to exist, rather than in proving existence. Schwarz's method presents some intriguing possibilities for numerical methods. Firstly, quite simple explicit solutions by classical methods are often known for simple regions such as rectangles or circles. Also, better numerical solutions, from the standpoint of the algebraic work involved, are often known for certain types of regions than for others. By Schwarz's method we may be able to extend these classical results and these algebraic advantages to more complicated regions. Secondly, the convergence rate $O(q_1 q_2)^n$ is very good and moreover the alternating procedure may involve one less dimension than the original problem. Thus, the procedure should require very little work.

In section two we consider the abstract formulation of a Dirichlet problem on two overlapping regions, and for its numerical solution a corresponding abstract numerical problem, which can be solved by a numerical analog to the Schwarz alternating procedure. The success of this method will be seen to depend only upon four formal hypotheses; from these follow rates of convergence, stability of the numerical method, convergence to the solution of the Dirichlet problem, etc.

Sections three through five are specific examples of the method. In each case the abstract notation of section two can be employed, and the four formal hypotheses can be proven to hold true.

2. The Abstract Numerical Schwarz Alternating Procedure

The concept of a Dirichlet problem. We discuss the notion of an abstract Dirichlet, or first boundary value, problem. Let R be a set of points, finite or infinite in number, which we call a region. R^0 will denote a certain subset of R, called the interior of R. Let $\partial R = R - R^0$ be non-empty, and be called the boundary of R. In any specific case the concepts of interior and boundary will coincide with their usual definitions. The abstractness here is prompted by a desire to include, for example, such problems as finite difference equations, in which we have a finite grid of points, or network problems, in which the points are nodes of an electrical network (See [4]).

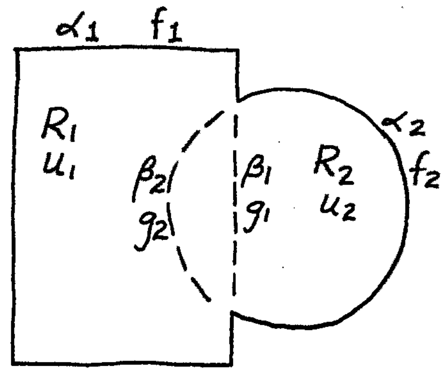
Consider functions u defined on R, and functions f defined on ∂R. Let E_R be a condition upon the functions u on R; if u satisfies E_R we say $u \in E_R$. The Dirichlet problem for the condition E_R and the boundary function f is to find $u \in E_R$ such that $u = f$ on ∂R. We assume that the condition E_R is sufficiently restrictive that such u for a given f is unique if it exists. Then we may introduce the notation $u = P(f)$; the Dirichlet operator P maps each boundary function f, for which a solution exists, onto its unique Dirichlet solution u.

The Dirichlet problem on the union of two regions. Let R_1, R_2 be two regions and let R be the region $R = R_1 \cup R_2$. Let $\alpha_1 = \partial R_1 \cap \partial R$, $\alpha_2 = \partial R_2 \cap \partial R$, $\beta_1 = \partial R_1 \cap R_2^0$, $\beta_2 = \partial R_2 \cap R_1^0$. Suppose that $\beta_1 \neq \emptyset$.

$\beta_2 \neq \emptyset$ and $\alpha_1 = \partial R_1 - \beta_1, \alpha_2 = \partial R_2 - \beta_2$.

Consider a fixed Dirichlet problem, for which the boundary functions and solution functions are either real-valued or complex-valued.

Consider a fixed boundary function f on ∂R. Let u be its Dirichlet solution on R. Let $f_1, f_2 = f$ on α_1, α_2 respectively, and let $f_1, f_2 = 0$ on β_1, β_2 respectively. Let $g_1, g_2 = u$ on β_1, β_2 respectively, and $g_1, g_2 = 0$ on α_1 and α_2 respectively. Then $(f_1 + g_1)$ and $(f_2 + g_2)$ are boundary functions on ∂R_1 and ∂R_2 respectively. Let u_1 be the restriction of u to R_1, and u_2 be the restriction of u to R_2. The notation is summarized in the schematic below.



Suppose that then u_1 and u_2 are the solutions of the Dirichlet problems on R_1 and R_2 for the boundary functions $(f_1 + g_1)$ and $(f_2 + g_2)$. That is

$$u = u_1 = P_1 (g_1 + f_1) \text{ on } R_1,$$

$$u = u_2 = P_2 (g_2 + f_2) \text{ on } R_2.$$

The limited problem of decomposition. It is, therefore, clear that our problem is equivalent to finding $u = g_1$ on β_1 and $u = g_2$ on β_2, for then the problem has been decomposed into the two separate problems of evaluating $u_1 = P_1(g_1 + f_1)$ on R_1 and $u_2 = P_2(g_2 + f_2)$ on R_2. We concentrate, henceforth, in section 2 upon the limited problem of evaluating g_1 and g_2 to within arbitrary accuracy. Notice that as a particular case of the above equations we have that g_1 and g_2 satisfy the equations

$$g_1 = P_2(g_2 + f_2) \text{ on } \beta_1,$$

$$g_2 = P_1(g_1 + f_1) \text{ on } \beta_2.$$

The numerical approximations. Let $\dot{\partial} R_1$, $\dot{\partial} R_2$ denote a finite grid of points on ∂R_1, ∂R_2 respectively. In general the notation $(\dot{})$ when applied to a set denotes the grid points of that set. We have then the finite sets $\dot{\beta}_1, \dot{\beta}_2, \dot{\alpha}_1, \dot{\alpha}_2$. Let n_1, n_2, m_1, m_2 be the respective number of points in these sets. The notation $(\dot{})$ when applied to a function denotes its restriction to the grid points where it is defined, that is, the finite-dimensional vector whose components are the values of the function at the grid points where it is defined. We have then the functions or vectors $\dot{g}_1$, $\dot{g}_2$, $\dot{f}_1$, $\dot{f}_2$. Vectors, other than those derived by $(\dot{})$ from functions, will be denoted by capital letters. As a general rule H's will denote general boundary vectors, and G's will denote vectors defined on the interior β arcs, and identically zero on the exterior α arcs.

Let P_1 on R_1 be approximated by a finite dimensional operator T_1. That is, for boundary data h_1 on ∂R_1, then $T_1(h_1)$ is an approximation to $P_1(h_1)$. Assume that T_1 operates upon every vector H_1 defined on ∂R_1. Now $T_1(H_1)$ may or may not be defined at every point in R_1°, but in particular it is defined at the points of $\dot{\beta}_2$ in R_1°, and there is denoted by $\dot{T}_1(H_1)$. $\dot{T}_1$ is then an operator which maps any vector H_1 defined on $\dot{\partial} R_1$ into a vector $G_2 = \dot{T}_1(H_1)$ defined on $\dot{\beta}_2$. Likewise, let $\dot{T}_2$ be an approximation on $\dot{\beta}_1$ to P_2.

Let $\| \;\|$ applied to a function denote its uniform norm. That is, for a real or complex valued function or vector h defined on a set α, $\| h \|_\alpha = \sup_\alpha |h|$. Often we will write just $\| h \|$.

<u>The formal hypotheses.</u> We now state four hypotheses in terms of our abstract notation.

I. f_1, $f_2 = f$ on α_1, α_2 respectively; f_1, $f_2 = 0$ on β_1, β_2 respectively; g_1, $g_2 = 0$ on α_1, α_2 respectively; and g_1, g_2 satisfy the equations

$$(2.1) \quad \begin{cases} g_1 = P_2\,(g_2 + f_2) \text{ on } \beta_1, \\ g_2 = P_1\,(g_1 + f_1) \text{ on } \beta_2. \end{cases}$$

II. $\dot{T}_1$, $\dot{T}_2$ are linear operators, mapping the spaces of $n_1 + m_1$, $n_2 + m_2$ dimensional vectors defined on $\dot{\partial} R_1$, $\dot{\partial} R_2$ respectively into the spaces of n_2, n_1 dimensional vectors defined on $\dot{\beta}_2$, $\dot{\beta}_1$ respectively.

III. $\dot{T}_1$, $\dot{T}_2$ are convergent approximations to P_1, P_2 for the boundary data at hand. That is,

$$\dot{P}_1(g_1+f_1) - \dot{T}_1(\dot{g}_1+\dot{f}_1) = E_2 \text{ on } \dot{\beta}_2,$$

$$\dot{P}_2(g_2+f_2 - \dot{T}_2(\dot{g}_2+\dot{f}_2) = E_1 \text{ on } \dot{\beta}_1,$$

where a-priori bounds are known for the two errors $\| E_1 \|_{\dot{\beta}_1}$ and $\| E_2 \|_{\dot{\beta}_2}$. Moreover these two errors can be made arbitrarily small by choosing the dimensionality of our approximation sufficiently large; that is, there exists a sequence of grids or partitions $\dot{\partial} R_1$, $\dot{\partial} R_2$ of ∂R_1, ∂R_2 such that these errors approach zero as the number of points in the grid is increased.

IV. A. The $Q<1$ property holds for the $\dot{T}_1$, $\dot{T}_2$ and the arcs $\dot{\beta}_1$, $\dot{\beta}_2$; that is, there exist numbers Q_1, Q_2 such that $Q_1 Q_2 < 1$, and such that

$$\| \dot{T}_1 (G_1) \|_{\dot{\beta}_2} \leq Q_1 \| G_1 \|_{\dot{\beta}_1},$$

$$\| \dot{T}_2 (G_2) \|_{\dot{\beta}_1} \leq Q_2 \| G_2 \|_{\dot{\beta}_2},$$

for all vectors G_1, G_2 defined on $\dot{\beta}_1$, $\dot{\beta}_2$ and zero on $\dot{\alpha}_1$, $\dot{\alpha}_2$ respectively.

IV. B. For each partition $\dot{\partial} R_1$, $\dot{\partial} R_2$ of our sequence of partitions in III there exists a particular pair Q_1, Q_2 satisfying IVA. Moreover there exist numbers q_1, q_2 independent of the particular partition such that $q_1 q_2 < 1$ and $Q_1 \leq q_1$ and $Q_2 \leq q_2$ for all of these Q_1, Q_2.

The numerical problem. We wish to find G_1, G_2 on $\dot{\beta}_1, \dot{\beta}_2$ such that G_1, $G_2 = 0$ on $\dot{\alpha}_1$, $\dot{\alpha}_2$ respectively, and satisfy

$$(2.2) \quad \begin{cases} G_1 = \dot{T}_2 (G_2 + \dot{f}_2) \text{ on } \dot{\beta}_1, \\ G_2 = \dot{T}_1 (G_1 + \dot{f}_1) \text{ on } \dot{\beta}_2. \end{cases}$$

We now derive an error equation.

Lemma 2.1 If I, II hold, and G_1, G_2 are a solution to problem (2.2), and $Z_1 = \dot{g}_1 - G_1$, $Z_2 = \dot{g}_2 - G_2$, then Z_1, $Z_2 = 0$ on $\dot{\alpha}_1$, $\dot{\alpha}_2$ respectively and satisfy

$$Z_1 = \dot{T}_2 (Z_2) + E_1 \text{ on } \dot{\beta}_1,$$

$$(2.3) \qquad Z_2 = \dot{T}_1 (Z_1) + E_2 \text{ on } \dot{\beta}_2.$$

Proof: By I, g_1 and g_2 satisfy equation (2.1) on β_1, β_2. Restricting these equations to $\dot{\beta}_1$ and $\dot{\beta}_2$ we have

$$\dot{g}_1 = \dot{P}_2 (g_2 + f_2) \text{ on } \dot{\beta}_1,$$
$$\dot{g}_2 = \dot{P}_1 (g_1 + f_1) \text{ on } \dot{\beta}_2.$$

Since G_1, G_2 satisfy equation (2.2) we have

$$\dot{g}_1 - Z_1 = \dot{T}_2 (\dot{g}_2 - Z_2 + \dot{f}_2) \text{ on } \dot{\beta}_1,$$
$$\dot{g}_2 - Z_2 = \dot{T}_1 (\dot{g}_1 - Z_1 + \dot{f}_1) \text{ on } \dot{\beta}_2.$$

By the linearity from II, and subtracting, we got

$$Z_1 = \dot{T}_2 (Z_2) + \dot{P}_2 (g_2 + f_2) - \dot{T}_2(\dot{g}_2 + \dot{f}_2) \text{ on } \dot{\beta}_1,$$
$$Z_2 = \dot{T}_1 (Z_1) + \dot{P}_1 (g_1 + f_1) - \dot{T}_1(\dot{g}_1 + \dot{f}_2) \text{ on } \dot{\beta}_2.$$

Using the notation E_1, E_2 of hypothesis III we have the above error equation. ∎

<u>Lemma 2.2</u> If II holds, then problem (2.2) is a special case of the following problem: Given arbitrary vectors F_1, F_2 on $\dot{\beta}_1$, $\dot{\beta}_2$, find G_1, G_2 on $\dot{\beta}_1$, $\dot{\beta}_2$, and $= 0$ on $\dot{\alpha}_1$, $\dot{\alpha}_2$ such that

$$(2.2') \begin{cases} G_1 = \dot{T}_2(G_2) + F_1 \text{ on } \dot{\beta}_1 , \\ G_2 = \dot{T}_1(G_1) + F_2 \text{ on } \dot{\beta}_2 . \end{cases}$$

<u>Proof</u>: Let $F_1 = \dot{T}_2(\dot{f}_2)$ and $F_2 = \dot{T}_1(\dot{f}_1)$ for problem (2.2). ||

We find an a-apriori bound for equation (2.2')

<u>Lemma 2.3</u> If II, IVA hold, and G_1, G_2 satisfy (2.2') then

$$\|G_1\|_{\dot{\beta}_1} \leq \frac{1}{(1 - Q_1 Q_2)} (\|F_1\|_{\dot{\beta}} + Q_2 \|F_2\|_{\dot{\beta}_2})$$

$$\|G_2\|_{\dot{\beta}_2} \leq \frac{1}{(1 - Q_1 Q_2)} (\|F_2\|_{\dot{\beta}_2} + Q_1 \|F_1\|_{\dot{\beta}_1}).$$

As an immediate corollary, the solution to (2.2') and hence to (2.2), is unique.

<u>Proof</u>: Applying IVA to equation (2.2') we get

$$\|G_1\| \leq Q_2 \|G_2\| + \|F_1\| ,$$
$$\|G_2\| \leq Q_1 \|G_1\| + \|F_2\| ;$$

or

$$\|G_1\| \leq Q_2 Q_1 \|G_1\| + Q_2 \|F_2\| + \|F_1\| .$$
$$\|G_2\| \leq Q_1 Q_2 \|G_2\| + Q_1 \|F_1\| + \|F_2\| .$$

Since by IVA, $Q_1 Q_2 < 1$, we immediately get the desired inequalities. Now if G_1', G_2' are another solution to (2.2'), then by the linearity $G_1' - G_1$ and $G_2' - G_2$ satisfy an equation of form (2.2') with $F_1 = 0$, $F_2 = 0$. By the a-apriori bound the only solution to this homogeneous equation is $G_1' - G_1 = 0$, $G_2' - G_2 = 0$. ||

<u>Lemma 2.4</u> If I, II, III, and IVA hold; g_1, g_2 satisfy to (2.1) of hypothesis I; and G_1, G_2 is the solution to problem (2.2), then

$$\| Z_1\|_{\dot{\beta}_1} = \| \dot{g}_1 - G_1\|_{\dot{\beta}_1} \leq \frac{1}{(1 - Q_1 Q_2)} \left(\| E_1\|_{\dot{\beta}_1} + Q_2 \| E_2\|_{\dot{\beta}_2} \right).$$

$$\| Z_2\|_{\dot{\beta}_2} = \| \dot{g}_2 - G_2\|_{\dot{\beta}_2} \leq \frac{1}{(1 - Q_1 Q_2)} \left(\| E_2\|_{\dot{\beta}_2} + Q_1 \| E_1\|_{\dot{\beta}_1} \right).$$

<u>Proof:</u> Consider the error equation from Lemma 2.1. Then by the a-priori bound of lemma 2.3 the above inequality follows. ||

<u>Corollary 2.4</u> If, moreover, IVB is satisfied; then G_1, G_2 converge to the Dirichlet solution $\dot{g}_1$, $\dot{g}_2$. That is, there exists a sequence of grids or partitions $\partial \dot{R}_1$, $\partial \dot{R}_2$ of ∂R_1, ∂R_2 such that $\| \dot{g}_1 - G_1\|$ and $\| \dot{g}_2 - G_2\|$ approach zero as the number of points in the partition is increased. ||

<u>Proof:</u> We can replace the Q_1, Q_2 in lemma 2.4 by the q_1, q_2 of IVB. The result follows immediately from III.

<u>The numerical Schwarz alternating procedure.</u> We consider now the algebraic solution of the numerical problem (2.2) or (2.2'). We define a sequence of approximate solutions G_1^n, G_2^n to (2.2'). Let G_1^n, G_2^n be = 0 on $\dot{\alpha}_1$, $\dot{\alpha}_2$ respectively, for all n considered, and let

$$\left\{ G_1^0 \text{ be arbitrary on } \dot{\beta}_1, \right.$$

$$(2.4) \quad \left\{ \begin{aligned} G_2^n &= \dot{T}_1 (G_1^{n-1}) + F_2 \text{ on } \dot{\beta}_2, \\ G_1^n &= \dot{T}_2 (G_2^n) + F_1 \text{ on } \dot{\beta}_1, \end{aligned} \right\} \quad n = 1, 2, \ldots .$$

This shall be called the numerical Schwarz alternating procedure.

Lemma 2.5 If II and IVA hold, then there exists a solution G_1, G_2 of problem (2.2').

Proof: We could show that G_1^n, G_2^n converge to a solution; however, as will be pointed out in Lemma 2.8, problem (2.2') can be written as a set of simultaneous linear equations. From Lemma 2.3 we have that the solution is unique for abitrary F_1, and F_2. We simply point out that uniqueness is equivalent to existence for a finite set of simultaneous equations. $\|$

We now show that G_1^n, G_2^n converge to the solution G_1, G_2 of (2.2') and find a bound for the algebraic error at the nth step, in terms of the initial error $\|G_1^o - G_1\|_{\dot\beta_1}$.

Lemma 2.6 If II and IVA hold, then

$$\|G_1^n - G_1\|_{\dot\beta_1} \leq (Q_1 Q_2)^n \|G_1^o - G_1\|_{\dot\beta_1} ,$$
$$\|G_2^n - G_2\|_{\dot\beta_2} \leq Q_1(Q_1 Q_2)^{n-1} \|G_1^o - G_1\|_{\dot\beta_1} .$$

Proof: Subtracting equations (2.2') for G_1, G_2 from the defining equations (2.4) for G_1^n, G_2^n we get by the linearity,

$$(G_2^n - G_2) = \dot T_1 (G_1^{n-1} - G_1) \text{ on } \dot\beta_2 ,$$
$$(G_1^n - G_1) = \dot T_2 (G_2^n - G_2) \text{ on } \dot\beta_1 , \quad n = 1, 2, \cdots .$$

By hypothesis IVA we get

$$\|G_2^n - G_2\| \leq Q_1 \|G_1^{n-1} - G_1\| ,$$
$$\|G_1^n - G_1\| \leq Q_2 \|G_2^n - G_2\| .$$

Combining these,

$$\|G_1^n - G_1\| \leq Q_2 Q_1 \|G_1^{n-1} - G_1\| \leq \cdots \leq (Q_2 Q_1)^n \|G_1^o - G_1\| ,$$
$$\|G_2^n - G_2\| \leq Q_1 \|G_1^{n-1} - G_1\| \leq Q_1 (Q_2 Q_1)^{n-1} \|G_1^o - G_1\| . \quad \|$$

Now $\dot{T}_1(G_1^n)$ is itself only evaluated approximately at each step of the numerical alternating procedure, either because of round-off error, or because the operator $\dot{T}_1$ is not known in explicit form, but instead must itself be solved by some approximate iterative method at each step. Instead of G_1^n, G_2^n of procedure (2.4) we actually calculate J_1^n, J_2^n defined by

$$(2.5)\quad \begin{cases} J_1^o = G_1^o \text{ on } \dot{\beta}_1, \\ J_2^n = \dot{T}_1(J_1^{n-1}) + F_2 + \epsilon_2^n \text{ on } \dot{\beta}_2 \\ J_1^n = \dot{T}_2(J_2^n) + F_1 + \epsilon_1^n \text{ on } \dot{\beta}_1 \end{cases} \quad n = 1, 2, \ldots.$$

We prove that the numerical alternating procedure is stable against such errors.

Lemma 2.7 If II and IVA hold, and G_1^n, G_2^n are defined by (2.4) while J_1^n, J_2^n are defined by (2.5), then

$$\left.\begin{array}{l} \| J_1^n - G_1^n \|_{\dot{\beta}_1} \\ \| J_2^n - G_2^n \|_{\dot{\beta}_2} \end{array}\right\} \leq \frac{1 + \max(Q_1, Q_2)}{1 - Q_1 Q_2} \|\epsilon\|$$

where $\|\epsilon\|$ is the maximum error $\|\epsilon_1^k\|_{\dot{\beta}_1}$ or $\|\epsilon_2^k\|_{\dot{\beta}_2}$ for all $k \leq n$.

Proof: Let $V_1 = J_1^n - G_1^n$ and $V_2 = J_2^n - G_2^n$. Then V satisfies

$$\begin{cases} V_1^o = 0, \\ V_2^n = \dot{T}_1(V_1^{n-1}) + \epsilon_2^n \\ V_1^n = \dot{T}_2(V_2^n) + \epsilon_1^n \end{cases} \quad n = 1, 2, \ldots.$$

Now we use superposition. Let $k = 1, 2, \ldots$ and define

W_{1k}^n, W_{2k}^n, Y_{1k}^n, Y_{2k}^n by

$$\begin{cases} W_{2k}^n = 0 \\ W_{1k}^n = 0 \end{cases}, \quad n < k;$$

$$\begin{cases} W_{2k}^k = \epsilon_2^k \\ W_{1k}^k = \dot{T}_2(W_{2k}^k) \end{cases} \quad (10)$$

$$\begin{cases} W_{2k}^n = \dot{T}_1(W_{1k}^{n-1}) \\ W_{1k}^n = \dot{T}_2(W_{2k}^n) \end{cases}, \quad n > k;$$

$$\begin{cases} Y_{2k}^n = 0 \\ Y_{1k}^n = 0 \end{cases}, \quad n < k;$$

$$\begin{cases} Y_{2k}^k = 0 \\ Y_{1k}^k = \epsilon_1^k \end{cases}$$

$$\begin{cases} Y_{2k}^n = \dot{T}_1(Y_{1k}^{n-1}) \\ Y_{1k}^n = \dot{T}_2(Y_{2k}^n) \end{cases}, \quad n > k.$$

Then by IVA we have that

$$\begin{cases} \| W_{2k}^k \| = \| \epsilon_2^k \| \\ \| W_{1k}^k \| = Q_2 \| \epsilon_2^k \| \end{cases}$$

$$\left.\begin{cases} \| W_{2k}^n \| \le Q_1 \| W_{1k}^{n-1} \| \\ \| W_{1k}^n \| \le Q_2 \| W_{2k}^n \| \end{cases}\right\} n > k.$$

$$\therefore \| W_{2k}^n \| \le (Q_1 Q_2)^1 \| W_{2k}^{n-1} \| \le \ldots \le (Q_1 Q_2)^{n-k} \| \epsilon_2^k \|$$

$$\therefore \| W_{1k}^n \| \le Q_2 \| W_{2k}^n \| \le \ldots \le Q_2 (Q_1 Q_2)^{n-k} \| \epsilon_2^k \|$$

Thus

$$\left\| \sum_{k=1}^{n} W_{2k}^{n} \right\| \le \sum_{k=1}^{n} \| W_{2k}^{n} \| \le \sum_{k=1}^{n} (Q_1 Q_2)^{n-k} \| \epsilon_2^{k} \|.$$
$$\le (1 + Q_1 Q_2 + \quad + (Q_1 Q_2)^{n-1})$$
$$\le \frac{1}{1 - Q_1 Q_2} \| \epsilon \| .$$

And likewise we get

$$\left\| \sum_{k=1}^{n} W_{1k}^{n} \right\| \le \frac{Q_2}{1 - Q_1 Q_2} \| \epsilon \| .$$

Again we have by IVA

$$\{ \| Y_{1k}^{k} \| = \| \epsilon_1^{k} \|$$
$$\left. \begin{cases} \| Y_{2k}^{n} \| \le Q_1 \, \| Y_{1k}^{n-1} \| \\ \| Y_{1k}^{n} \| \le Q_2 \, \| Y_{2k}^{n} \| \end{cases} \right\} n > k.$$

$$\therefore \| Y_{1k}^{n} \| \le Q_1 Q_2 \, \| Y_{1k}^{n-1} \| \le \dots \le (Q_1 Q_2)^{n-k} \| \epsilon_1^{k} \|.$$
$$\therefore \| Y_{2k}^{n} \| \le Q_1 \| Y_{1k}^{n-1} \| \le Q_1 (Q_1 Q_2)^{n-1-k} \| \epsilon_1^{k} \|, \quad k = 1, \dots n-1.$$

Thus we got

$$\left\| \sum_{k=1}^{n} Y_{1k}^{n} \right\| \le \sum_{k=1}^{n} (Q_1 Q_2)^{n-k} \| \epsilon_1^{k} \|$$
$$\le \frac{1}{1 - Q_1 Q_2} \| \epsilon \| \quad ;$$

and $$\left\| \sum_{k=1}^{n} Y_{2k}^{n} \right\| \le \sum_{k=1}^{n-1} \| Y_{2k}^{n} \| \le \sum_{k=1}^{n-1} Q_1 (Q_1 Q_2)^{n-k-1} \| \epsilon_1^{k} \|$$
$$\le \frac{Q_1}{1 - Q_1 Q_2} \| \epsilon \| .$$

But $$V_2^{n} = \sum_{k=1}^{n} W_{2k}^{n} + \sum_{k=1}^{n} Y_{2k}^{n} .$$
$$V_1^{n} = \sum_{k=1}^{n} W_{1k}^{n} + \sum_{k=1}^{n} Y_{1k}^{n} .$$

Thus $$\| V_2^{n} \| \le \left(\frac{1 + Q_1}{1 - Q_1 Q_2} \right) \| \epsilon \| ,$$
$$\| V_1^{n} \| \le \left(\frac{Q_2 + 1}{1 - Q_1 Q_2} \right) \| \epsilon \| .$$

||

Next we point out that our problem (2.2) or (2.2') can be written as a set of simultaneous linear equations. Let $1, \ldots, n_1$; $n_1+1, \ldots, n_1+m_1$; $1, \ldots, n_2$; $n_2+1, \ldots, n_2+m_2$ index the points of $\dot{\beta}_1, \dot{\alpha}_1, \dot{\beta}_2, \dot{\alpha}_2$ respectively. Let $(f_1)_i$ denote the i th component of the vector $(\dot{f}_1)$, and likewise for the other vectors. By the linearity from II we can write for an arbitrary vector H_1 on $\dot{\partial} R_1$,

$$\left[\dot{T}_1 (H_1)\right]_i = \sum_{j=1}^{n_1+m_1} (A_1)_{ij} \quad (H_1)_j \quad , \quad i = 1, \ldots, n_2.$$

The matrix coefficients $(A_1)_{ij}$ are given by

$$(A_1)_{ij} = \left[\dot{T}_1 (D^j)\right]_i \quad ; \; i = 1, \ldots n_2; \; j = 1, \ldots, n_1 + m_1 \; ;$$

where D^j is the vector on R_1 with components

$$(D^j)_k = \begin{cases} 1 & , k = j, \\ 0 & k \neq j. \end{cases}$$

<u>Lemma 2.8</u> If II holds and $(A_1)ij$, $(A_2)ij$ are the matrices of the operators $\dot{T}_1$ and $\dot{T}_2$, then the numerical problem (2.2) or (2.2') may be written as the following set of simultaneous linear equations.

$$(G_1)_i - \sum_{j=1}^{n_2} (A_2)_{ij} \quad (G_2)_j = \sum_{j=n_2+1}^{n_2+m_2} (A_2)_{ij} \quad (f_2)_j \quad , \; i = 1, \ldots, n_1 \; ;$$

$$(G_2)_i - \sum_{j=1}^{n_1} (A_1)_{ij} \quad (G_1)_j = \sum_{j=n_1+1}^{n_1+m_1} (A_1)_{ij} \quad (f_1)_j \quad , \; i = 1, \ldots, n_2.$$

<u>Proof:</u> We have just written the vector equations in component form. ||

We will see in thr examples that condition IVB may be quite difficult to prove, no matter how intuitively obvious it may seem. This will be especially true in case $\dot{T}_1$ and $\dot{T}_2$ come from finite-difference approximations with non-constant coefficients. However, in practical use of the method one can act upon one's intuition concerning the validity of IVB. One of the advantages of the method is that the convergence factors Q_1, Q_2, of IVA for a particular partition are easily found, as we will see below.

Lemma 2.9 $\qquad Q_1 = \max_{i \in \dot\beta_2} \left\{ \sum_{j \in \dot\beta_1} |(A_1)_{ij}| \right\}.$

Proof: Consider all G_1 defined on $\dot\beta_1$; $= 0$ on $\dot\alpha_1$.

Then $\{\dot T_1 (G_1)\}_i = \sum_{j \in \dot\beta_1} (A_1)_{ij} \quad (G_1)_j, \; i \in \dot\beta_2, \; j \in \dot\beta_1$.

It is a trivial result that the uniform norm on G_1 induces the above operator norm for the matrix $(A_1)_{ij}$. That is, $\| T_1 (G_1)\| \dot\beta_2 \leq Q_1 \| G_1 \| \dot\beta_1$, and no smaller number than Q_1 will suffice.

Lemma 2.10 If all values in our problem are real, and $\dot T_1$ satisfies the minimum - maximum principle, that is

$$\min_{\partial R_1} H_1 \leq \dot T_1 (H_1) \text{ on } \dot\beta_2 \leq \max_{\partial R_1} H_1 ,$$

then the coefficients $(A_1)_{ij}$, $i \in \dot\beta_2$, $j \in \dot\beta_1$, are non-negative.

If for this reason, or any other reason, these coefficients are real and non-negative, then $Q_1 = \| \dot T_1 \; (G_1) \| \dot\beta_2$ where $G_1 \equiv 1$ on $\dot\beta_1$, $\equiv 0$ on $\dot\alpha_1$.

Proof: Recall $(A_1)_{ij} = (\dot T_1 (D^j))_i$, where D^j has value 1 at the j th point on ∂R_1, value 0 elsewhere on ∂R_1. Thus, by the minimum - maximum principle $0 \leq (A_1)_{ij} \leq 1$.

Now given $(A_1)_{i,j} \geq 0$, then by Lemma 2.9,

$$Q_1 = \max_{i \in \beta_2} \left\{ \sum_{j \in \beta_1} (A_1)_{i,j} \right\} = \max_{i \in \beta_2} \left\{ \dot{T}_1 (Q_1) \right\} \qquad ||$$

As a result of Lemma 2.9, if the operators $\dot{T}_1$, $\dot{T}_2$ are employed in matrix form then Q_1, Q_2 are obtained easily from the coefficients. In fact, the so called numerical Schwarz alternating procedure becomes the usual Gauss-Seidel iterative solution of a set of simultaneous linear equations with diagonally dominant matrix. If $\dot{T}_1$, $\dot{T}_2$ are not employed in matrix form, but lemma 2.10 applies, then Q_1, Q_2 can be found by solving only one boundary value problem for $\dot{T}_1$ and one for $\dot{T}_2$.

3. An Example With Two Overlapping Circles

Laplace's Equation. All of our examples will concern Laplace's equation on the plane. We, therefore, state the general problem. A function $u(x,y)$ is called harmonic in an open set if at each point it satisfies Laplace's equation $\Delta u = 0$ and is continuous.

Consider the Dirichlet problem. Let D be a finite open connected plane set, whose boundary Γ consists of a finite number of Jordan curves. Let f be a bounded function prescribed on Γ, continuous with the exception of at most a finite number of points. Find a function $u(x,y)$ which is harmonic and bounded on D, continuous on the closure $D \cup \Gamma$, except at the points of discontinuity of f, and which has the value of f at each point of Γ.

Uniqueness is guaranteed by the Phragmén-Lindeloff form of the strong maximum principle. If $u(x,y)$ is harmonic and bounded on D, and $\lim u(x,y) \leq M$ at each boundary point with the exception of at most a finite number of points, then $u(x,y) \leq M$ on D, and $u(x,y) = M$ at a point of D only when $u \equiv M$.

For our purposes we need never consider boundary data more general than piecewise continuous functions, that is, functions on Γ which are continuous, except at a finite number of points, but which have left and right sided limits at every point.

For a circle Γ of radius a, with polar coordinates (r,θ), let $f(\psi)$ be given piecewise continuous boundary data. Then the solution to the Dirichlet problem is given by the Poisson integral

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\psi-\theta)} f(\psi)\, d\psi,$$

for $r < a$. We define $u(r,\theta) = f(\theta)$ for $r = a$. We denote this $P(f(\psi))(r,\theta)$, or sometimes just $P(f)$.

Let α be an arc of the circle Γ, with end points P and Q. Let $\beta = \Gamma - \alpha$. Let $w(r,\theta)$ be the Dirichlet solution for $f(\psi) \equiv 1$ on β, $\equiv 0$ on α. In other words, $w(r,\theta)$ is the harmonic measure at (r,θ) of the arc β. Then it is well known that the level curves for the harmonic measure of β are the circles through its end points. Along a circle of this family which meets β at an interior angle of ϕ radians, $w(r,\theta)$ has the value $1 - \phi/\pi$. As an immediate result we have the following:

<u>Lemma 1.1</u> Let α consist of a finite number of closed, disjoint arcs of positive length on Γ. Let β be the open intervals $\Gamma - \alpha$. Let γ be any Jordan curve lying wholly interior to Γ, except that one or both of its endpoints may meet Γ at points of α, having tangents there, and making non-zero angles of intersection with respect to the arcs of β. Then the harmonic measure $w(r,\theta)$ of β obeys

$$\| w(r,\theta) \|_\gamma = q < 1.$$

Lemma 3.2 For $g(\psi)$ piecewise continuous boundary data prescribed on Γ, such that $g(\psi) \equiv 0$ on the arc α above, then

$$\| F(g) \|_{\gamma} \leq q \; \| g \|_{\beta} \; .$$

where q is the number < 1 above.

Proof: This follows immediately from 3.1 and the maximum principle, or from the integral representation. The above Lemma states the "$q < 1$ property" for the operator F, the arc β, and the interior arc γ. Notice that for simple cases it is quite easy to estimate q graphically. See [3] for the proof of this property on a general region.

The polygonal approximation for one circle. Let $\dot{\Gamma}$ be a finite grid of points on the circle Γ. That is, let ψ_i $(i = 1, \ldots, n)$ be a partition of the circle, with the ψ_i ordered in the direction of increasing angle ψ. Then for a boundary function $h(\psi)$ we have the vector $\dot{h}$, with components $\dot{h}_i = h(\psi_i)$. We wish a numerical approximation to the operator $F(h)\,(r,\theta)$ in terms of $\dot{h}$ only. Our first question is to attempt to approximate h. Let $\hat{h}(\psi)$ be the polygonal function equal $\dot{h}$ at the grid points, and defined by linear interpolation between the grid points. Notice also that $\| \hat{h} \|_{\Gamma} = \| \dot{h} \|_{\dot{\Gamma}}$. Thus we would have for $g \equiv 0$ on α that

$$\| F(\hat{g}) \|_{\gamma} \leq q \; \| \hat{g} \|_{\beta} = q \; \| \dot{g} \|_{\dot{\beta}} \; .$$

in other words, the $q < 1$ property would be maintained for the approximation operator $P(\hat{\ })$. We show now that $P(\hat{\ })$ is a useful numerical approximation.

Let $t_i(\psi)$ be defined for $i = 1, \ldots, n$ by

$$t_i(\psi) = \begin{cases} \dfrac{\psi - \psi_{i-1}}{\psi_i - \psi_{i-1}}, & \text{for } \psi_{i-1} \le \psi \le \psi_i \,; \\[2ex] \dfrac{\psi_{i+1} - \psi}{\psi_{i+1} - \psi_i}, & \text{for } \psi_i \le \psi \le \psi_{i+1} \,; \\[2ex] 0, & \text{for all other } \psi \text{ on } \Gamma \,; \end{cases}$$

with the understanding that $i-1 = n$ for $i = 1$, and $i + 1 = 1$ for $i = n$. Then $t_i(\psi)$ is the triangular function between ψ_{i-1} and ψ_{i+1} with a peak of height 1 at ψ_i. Then $\hat{h}(\psi)$ may be written

$$\hat{h}(\psi) = \sum_{i=1}^{n} \dot{h}_i \, t_i(\psi).$$

At a point (r,θ) interior to the circle, the Dirichlet solution for $\hat{h}(\psi)$ is given by

$$P(\hat{h}(\psi))(r,\theta) = \sum_{i=1}^{n} \dot{h}_i \, P(t_i(\psi))(r,\theta)$$
$$= \sum_{i=1}^{n} \dot{h}_i \, A_i(r,\theta) \,,$$

where for a fixed point (r,θ) the ith coefficient $A_i(r,\theta)$ is given by the integrals

$$A_i(r,\theta) = \int_{\psi_{i-1}}^{\psi_i} \frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\psi - \theta)} \; \frac{\psi - \psi_{i-1}}{\psi_i - \psi_{i-1}} \, d\psi$$

(3.1)

$$+ \int_{\psi_i}^{\psi_{i+1}} \frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\psi - \theta)} \; \frac{\psi_{i+1} - \psi}{\psi_{i+1} - \psi_i} \, d\psi .$$

Perhaps these can be integrated directly, or at least they can be integrated easily by higher order correct numerical quadratures, to arbitrary accuracy. In either case we get not exactly $A_i(r,\theta)$, but instead $B_i(r,\theta)$. Let $\epsilon(r,\theta)$ be the error

$$\epsilon(r,\theta) = \sum_{i=1}^{n} \left| A_i(r,\theta) - B_i(r,\theta) \right| .$$

<u>Defn 2.1</u> For an arbitrary vector H on $\dot{\Gamma}$ define

$$T(H)(r,\theta) = \sum_{i=1}^{n} B_i(r,\theta)\, H_i .$$

<u>Lemma 2.2.</u> If $|\partial^2 h/\partial\psi^2| < M$ on Γ, then the error of the numerical approximation at a point (r,θ) is bounded by

$$\left| P(h)(r,\theta) - T(\dot{h})(r,\theta) \right| \leq M \,\|\delta\psi\|^2 + \epsilon(r,\theta)\, \|\dot{h}\|_{\dot{\Gamma}} ,$$

where $\|\delta\psi\|$ is the maximum angular interval of the partition.

<u>Proof:</u>

$$\left| P(h) - T(\dot{h}) \right| \leq \left| P(h) - P(\hat{h}) \right| + \left| P(\hat{h}) - T(\dot{h}) \right| .$$

Now $\left| P(\hat{h}) - T(\dot{h}) \right| = \left| \sum_{i=1}^{n} \dot{h}_i \left[A_i(r,\theta) - B_i(r,\theta) \right] \right|$

$$\leq \|\dot{h}\|_{\dot{\Gamma}}\, \epsilon(r,\theta).$$

And $\left| P(h) - P(\hat{h}) \right| \leq \|h - \hat{h}\|_{\Gamma}$ by the maximum principle.

By Taylor's series with remainder we can easily show that

$$\left| h(\psi) - \hat{h}(\psi) \right| \leq M \|\delta\psi\|^2, \text{ or } \|h - \hat{h}\|_{\Gamma} \leq M \|\delta\psi\|^2 . \ \|$$

We show that the $q < 1$ property holds true for our approximation.

__Lemma 3.4__ Let α, β, γ, q be as in lemma 3.1. Consider any partition $\dot{\Gamma}$ of Γ which includes all end points of the arcs of α. Then for any vector G defined on $\dot{\beta}$, and $\equiv 0$ on $\dot{\alpha}$, and any point (r,θ) on γ

$$\left| T\,(G)(r,\theta) \right| \quad \leq \quad \left[q + \epsilon(r,\theta) \right] \quad \| G \|_{\dot{\beta}} .$$

__Proof:__

$$\left| T\,(G)(r,\theta) \right| = \left| \sum_{i \in \dot{\beta}} G_i \quad B_i\,(r,\theta) \right|$$

$$\leq \| G \|_{\dot{\beta}} \sum_{i \in \dot{\beta}} \left| B_i(r,\theta) \right| .$$

But $\displaystyle\sum_{i \in \dot{\beta}} \left| B_i(r,\theta) \right| \leq \epsilon\,(r,\theta) + \sum_{i \in \dot{\beta}} \left| A_i\,(r,\theta) \right|$.

Since $t_i(\psi) \geq 0$, $A_i\,(r,\theta) = P(t_i(\psi))\,(r,\theta) \geq 0$.

Thus
$$\sum_{i \in \dot{\beta}} \left| A_i(r,\theta) \right| = P\left[\sum_{i \in \dot{\beta}} t_i(\psi) \right](r,\theta)$$

$$\leqq P\,(\hat{G})\,(r,\theta)$$

$$\leq q \quad \| G \|_{\beta} = q.$$

where G is a vector $\equiv 0$ on $\dot{\alpha}$, $\equiv 1$ on $\dot{\beta}$. ||

__The third order correct approximation.__ Consider a partition $\dot{\Gamma}$ of the circle Γ formed in the following way. First take a partition of Γ such that all the end points of the arcs of α are included as before. This gives us the grid points with even index. Then add the midpoints of each of these partition intervals. This gives us the grid points with odd index. Then the total number of points n is even, and we have $n/2$ pairs of adjacent equal intervals on Γ. For an arbitrary vector H on $\dot{\Gamma}$ let $\hat{H}(\psi)$ be obtained by fitting a parabola through the three values of H on each pair of intervals. That is, for i odd, and

$\delta_i \psi = \psi_i - \psi_{i-1} = \psi_{i+1} - \psi_i$, and $\psi = \psi_i + t \ \delta_i \psi$, $-1 \le t \le 1$,

then

$$\hat{H}(\psi) = \left(\frac{t^2}{2} - \frac{t}{2}\right) H_{i-1} + (1 - t^2) H_i + \left(\frac{t^2}{2} + \frac{t}{2}\right) H_{i+1} .$$

Now

$$\hat{H}(\psi) \le \max\left[|H_{i-1}|, |H_i|, |H_{i+1}|\right]\left[\left|\frac{t^2}{2} - \frac{t}{2}\right| + |1 - t^2| + \left|\frac{t^2}{2} + \frac{t}{2}\right|\right] .$$

Consider the last factor for $0 \le t \le 1$,

$$\left|\frac{t^2}{2} - \frac{t}{2}\right| + |1 - t^2| + \left|\frac{t^2}{2} + \frac{t}{2}\right| = 1 + t - t^2 \le 5/4 .$$

By symmetry the factor is also $\le 5/4$ for $-1 \le t \le 0$.

Therefore, for $G \equiv 0$ on α, and (r,θ) on γ, we have

$$|P(\hat{G})(r,\theta)| \le q \ \|\hat{G}\|_\beta \le \frac{5}{4} q \ \|G\|_{\dot\beta} .$$

If $\left|\frac{\partial^3 h}{\partial \psi^3}\right| \le M$ on Γ, then one easily shows by Taylor's series with remainder that

$$|h(\psi) - \hat{h}(\psi)| \le \frac{M}{2} (\delta_i \psi)^3 .$$

The operator $P(\hat{\ })$ can be given in matrix form. For a vector H on $\dot\Gamma$,

$$P(\hat{H})(r,\theta) = \sum_{i=1}^{n} A_i(r,\theta) \ H_i ,$$

$$A_i(r,\theta) = P(\widehat{D^i})(r,\theta),$$

where D^i is the vector on $\dot{\Gamma}$ with 1 for its ith component and zero for its other components. If i is odd, $A_i(r,\theta)$ equals

$$(3.2)\quad \left\{ \frac{1}{2\pi}\int_{\psi_{i-1}}^{\psi_{i+1}} \frac{a^2 - r^2}{a^2 + r^2 - 2\,ar\cos(\psi-\theta)} \left[1-\left(\frac{\psi-\psi_i}{\psi_i-\psi_{i-1}}\right)^2\right] d\psi \right.$$

If i is even, $A_i\,(r,\theta)$ equals

$$\frac{1}{4\pi}\int_{\psi_{i-2}}^{\psi_i} \frac{a^2 - r^2}{a^2 + r^2 - 2\,ar\cos(\psi-\theta)} \left[\left(\frac{\psi-\psi_{i-1}}{\psi_i-\psi_{i-1}}\right)^2 + \left(\frac{\psi-\psi_{i-1}}{\psi_i-\psi_{i-1}}\right)\right] d\psi$$

(3.3)

$$+\ \frac{1}{4\pi}\int_{\psi_i}^{\psi_{i+2}} \frac{a^2 - r^2}{a^2 + r^2 - 2\,ar\cos(\psi-\theta)} \left[\left(\frac{\psi-\psi_{i+1}}{\psi_{i+1}-\psi_i}\right)^2 - \left(\frac{\psi-\psi_{i+1}}{\psi_{i+2}-\psi_i}\right)\right] d\psi$$

Again we get not exactly $A_i(r,\theta)$, but instead $B_i(r,\theta)$. Let the error $\epsilon\,(r,\theta)$ be defined as previously. Again we define the approximation operator by

$$T(H)\,(r,\theta) = \sum_{i=1}^{n} B_i\,(r,\theta)\,H_i\,.$$

Then the following lemmas follow analogously to lemmas 3.3 and 3.4.

<u>Lemma 3.5</u> If $\left|\frac{\partial^3 h}{\partial \psi^3}\right| \leq 2M$ on Γ, then the error of the numerical approximation at a point (r,θ) is bounded by

$$\left|P(h)(r,\theta) - T(\dot{h})\,(r,\theta)\right| \leq M\,\|\delta\psi\|^3 + \epsilon\,(r,\theta)\,\|\dot{h}\|_{\dot{\Gamma}}\,,$$

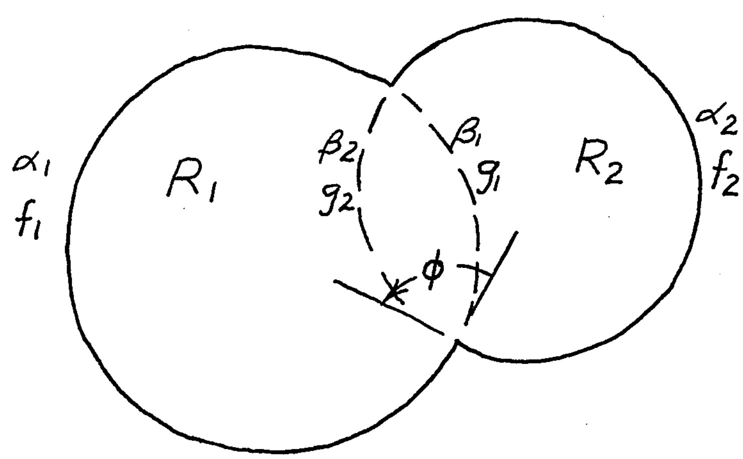
where $\|\delta\psi\|$ is the maximum angular interval of the partition $\dot{\Gamma}$.

<u>Lemma 3.6</u> Let α, β, γ, q be as in Lemma 3.1. Let $\dot{\Gamma}$ be a partition of Γ as described above. Notice that each end point of the arcs of α is a grid point with even index. If G is any vector defined on $\dot{\beta}$, and $\equiv 0$ on $\dot{\alpha}$, and (r,θ) is any point on γ, then

$$\left| T(G)(r,\theta) \right| \leq \left[\frac{5}{4} q + \epsilon(r,\theta) \right] \| G \|_{\dot{\beta}}.$$

<u>Notice</u> The coefficient evaluation error $\epsilon(r,\theta)$ enters in so simply that, henceforth, we ignore it, and assume that $B_1(r,\theta) = A_1(r,\theta)$, and therefore, T is the operator $P(\wedge)$ or $P(\frown)$.

<u>The problem for two overlapping circles.</u> Let R_1, R_2 be two closed circular regions, neither wholly containing the other, whose interiors intersect. Let $R = R_1 \cup R_2$ and let α_1, α_2, β_1, β_2 be as described in the abstract case. Let f be a boundary function on ∂R with Dirichlet solution u on R for Laplace's equation. Let f_1, f_2, g_1, g_2, P_1, P_2 be defined as in the abstract case. We include a diagram below.



Let ϕ be the interior angle of intersection of β_1 and β_2. Then on the arc β_2 interior to R_1 the harmonic measure of β_1 is a constant $1 - \phi/\pi$. Therefore, if $\phi > \pi/5$ we will employ the third order correct approximation, if not we will employ the polygonal approximation. We make this precise.

Case A. $\phi > \pi/5$, and u has continuous third order partial derivatives on the closure of R. Let $\dot{\partial} R_1$ be a partition of ∂R_1 as described for the third order correct approximation, with ∂R_1, α_1, β_1, β_2, $1 - \phi/\pi$ corresponding to Γ, α, β, γ, q. Let $\dot{\partial} R_2$ be the analogous partition of ∂R_2, with ∂R_2, α_2, β_2, β_1, $1 - \phi/\pi$ corresponding to Γ, α, β, γ, q. Let (r, θ) be polar coordinates on R_1. Then the points of have coordinates (r_i, θ_i) on R_1^0, $i = 1, \ldots, n_2$. Let the matrix $(A_1)_{i,j}$ be given by

$$(A_1)_{i,j} = A_j(r_i, \theta_i)$$

where $A_j(r, \theta)$ is defined by formulas (3.2) and (3.3). Let the operator $\dot{T}_1$ be defined by

$$\left[\dot{T}_1 (U_1) \right]_i = \sum_{j=1}^{n_1 + m_1} (A_1)_{i,j} (U_1)_j \quad , \quad i = 1, \ldots, n_2,$$

for an arbitrary vector U_1 on $\dot{\partial} R_1$. Similarly, define the operator $\dot{T}_2$ with matrix $(A_2)_{i,j}$.

Case B. The Case A does not apply, but u does have continuous second order partial derivatives on the closure of R. Let $\dot{\partial R_1}$ $\dot{\partial R_2}$ be partitions of ∂R_1, ∂R_2 which includes the two intersection points. Let $\dot{T}_1$ and $\dot{T}_2$ be defined as in Case A, except that we use formula (3.1) for the coefficients $(A_1)_{i,j}$ and $(A_2)_{i,j}$.

<u>Theorem 3.1</u> For Case A or for Case B the formal hypotheses I, II, III, IVA, and IVB are satisfied.

<u>Proof of I:</u> This is clear.

<u>Proof of II:</u> This is clear from the matrix representation of $\dot{T}_1$ and $\dot{T}_2$.

<u>Proof of III:</u> For Case A u has continuous and, therefore, bounded third partials on the closure of R. Therefore, $|\partial^3(g_1 + h_1)/\partial\psi^3|$ on ∂R_1 is bounded by some number $2M_1$. By lemma 3.5 we have

$$\| \dot{P}_1(f_1 + g_1) - \dot{T}_1(\dot{f}_1 + \dot{g}_1)\|_{\dot{\beta}_2} \leq M_1 \|\delta\psi\|^3 .$$

Similarly, we get

$$\| \dot{P}_2(f_2 + g_2) - \dot{T}_2(\dot{f}_2 + \dot{g}_2)\|_{\dot{\beta}_1} \leq M_2 \|\delta\psi\|^3 .$$

For Case B we get that $|\partial^2(g_1 + h_1)/\partial\psi^2|$ on ∂R_1 is bounded by some number M_1. By lemma 3.3

$$\| \dot{P}_1(f_1 + g_1) - \dot{T}_1(\dot{f}_1 + \dot{g}_1)\|_{\dot{\beta}_2} \leq M_1 \|\delta\psi\|^2 .$$

and similarly

$$\| \dot{P}_2(f_2 + g_2) - \dot{T}_2(\dot{f}_2 + \dot{g}_2)\|_{\dot{\beta}_2} \leq M_2 \quad \|\delta \Psi\|^2 \; .$$

<u>Proof of IV, XVI</u>: For Case A, $q_1 = q_2 = \frac{5}{4} (1 - \phi / \pi)$ by lemma 3.6. For Case B, $q_1 = q_2 = (1 - \phi / \quad)$ by lemma 3.4. $\|$

Therefore, the solution G_1, G_2 of the numerical problem (2.2) converges to the differential solution g_1, g_2 on $\beta_1 + \beta_2$. Since we employ $\dot{T}_1$, $\dot{T}_2$ in matrix form we will actually be solving the set of simultaneous linear equations of lemma 2.8. We consider the number of calculations involved. First, we must evaluate the approximately $(1/\|\delta \Psi\|)^2$ coefficients.

For Case A the error is

$$\| G_1^n - \dot{g}_1 \| \leq \| G_1^n - G_1 \| \quad + \| G_1 - \dot{g}_1 \| \; .$$

The second term is $0\left[\| \delta \Psi \|^3 \right]$. To make the first term of the same order of accuracy requires approximately $n \cong 3 \log (1/\|\delta \Psi\|)$ steps. Each step requires $0(1/\|\delta\Psi\|^2)^2$ calculations. In summary we have that the total calculations, besides the evaluation of coefficients, is $0\left[(1/\|\delta \Psi\|)^2 \log (1/\|\delta \Psi\|) \right]$; the error is $0\left[\|\delta \Psi\|^3\right]$.

For Case B the work is approximately the same, but the error is $0\left[\|\delta\Psi\|^2\right]$.

It is clear that the method with these approximation operators can be extended to other regions and other Dirichlet problems where the solution is given in closed integral form, and where the $q < 1$ property holds.

4. The Laplace Difference Equation on Two Overlapping Rectangles

We first state some results for the Laplace difference equation on a somewhat general region.

Let $R = R^o \cup \partial R$, where R^o is an open connected set in the plane, bounded by the curve ∂R made up of closed polygons with sides either parallel to or at 45^o to the coordinate axes. Assume there exists a sequence $\{\delta x_\alpha\}$, $\delta x_\alpha \to 0$, such that for each α each corner of the polygons of ∂R falls on one of the lattice points $(m\,\delta x_\alpha\ ,\ n\,\delta x_\alpha\ )$. Consider the lattice points $(m\,\delta x,\ n\,\delta x)$ for a fixed δx of this sequence. Let $\dot{R}$, $\dot{R}^o$, $\dot{\partial} R$ be the lattice points of R, R^o, ∂R respectively. Then notice that each interior lattice point $(m,n) \in \dot{R}^o$ has all four of its lattice neighbors $(m-1,\ n)$, $(m+1,\ n)$, $(m,n-1)$, $(m,\ n+1)$ in $\dot{R}$.

Let $u(x,y)$ be the solution on R of the Dirichlet problem for Laplace's equation, for a given boundary function f on ∂R. Let $w_{m,n}$ be the solution of the corresponding Dirichlet problem for the Laplace difference equation

$$\begin{cases} w_{m,n} = 1/4\,(w_{m-1,n} + w_{m+1,n} + w_{m,n-1} + w_{m,n+1}), & (m\,\delta x,\ n\,\delta x) \in \dot{R}^o\ , \\ w_{m,n} = f\,(m\,\delta x,\ n\,\delta x)\ , & (m\,\delta x,\ n\,\delta x)\ \dot{\partial} R. \end{cases}$$

Uniqueness, existence, etc. are well known for this problem, for example see [6]. In particular, we will later use the weak minimum - maximum principle: If $w_{m,n}$ satisfies the Laplace difference equation on $\dot{R}^o$, then $w_{m,n}$ takes on its maximum and its minimum on $\dot{\partial} R$.

Let $w(x,y)$ be the bilinear interpolation of the lattice function $w_{m,n}$. Notice that $w(x,y)$ also takes on its maximum and its minimum on $\partial \dot{R}$. It is well known [5] that under quite general conditions $w(x,y)$ converges to $u(x,y)$ as $\delta x \to 0$. In particular, if f is such that u has continuous fourth partial derivatives on the closure of R,

$$|u(x,y) - w(x,y)| < M(\delta x)^2 \text{ on } R,$$

where M is a constant proportional to the bound for the second and fourth partials of u.

In the abstract notation of section 2 we would denote the restriction of f to $\partial \dot{R}$ by $\dot{f}$, and

$$u(x,y) = F(f)(x,y),$$
$$w(x,y) = T(\dot{f})(x,y).$$

Then we have

$$\| F(f) - T(\dot{f}) \|_R < M(\delta x)^2.$$

We consider Laplace's difference equation on a rectangle. Let $\dot{R}$ be the set of points $(m\delta x, n\delta x)$ with m and n integers and with $0 \leq m \leq M$, $0 \leq n \leq N$. We prove the $Q < 1$ property for lines intersecting $\dot{R}$ at right angles to its boundaries.

<u>Lemma 4.1</u> Let $\dot{f}$ be given boundary data on $\partial \dot{R}$ such that $\dot{f} \leq 1$ for $m < m_0$, and $\dot{f} = 0$ for $m \geq m_0$. Let $w_{m,n}$ be the Dirichlet solution for $\dot{f}$ on $\dot{R}$. If $m_0/N \geq (1/2)^p$, then $w_{m,n} \leq 1 - (1/2)^p$ on and to the right of the vertical line $m = m_0$; that is, for $m_0 \leq m \leq M$, and $(m,n) \in \dot{R}$.

<u>Proof</u>: <u>The Case P = 1.</u> First consider $m_0 = M/2$. Consider $\dot{f} \equiv 1$, $m < m_0$; $\dot{f} = 1/2$, $m = m_0$; $\dot{f} \equiv 0$, $m > m_0$. By symmetry the solution $w_{m,n}$ must have value 1/2 along the center line $m = m_0$.

Then on the right half rectangle $m_0 \leq m \leq M$, $w_{m,n}$ has boundary values 1/2 along its left side $m = m_0$, and 0 along its other sides. By the maximum principle $w_{m,n} \leq 1/2$ for $m_0 \leq m \leq M$.

By the maximum principle, if we lower $\dot{f}$, we can only lower its solution w, thus if $\dot{f} \leq 1$ for $m < m_0$, and $\dot{f} \equiv 0$ for $m \geq m_0$, then its solution has $w_{m,n} \leq 1/2$ for $m \geq m_0$.

Now consider $m_0 > M/2$. Let $M' = 2m_0$. On the extended rectangle $0 \leq m \leq M'$, $0 \leq n \leq N$, let $\dot{f}' = \dot{f}$ for $m < M$, $\dot{f}' = 0$ for $M \leq m \leq M'$. Let w' be the solution for $\dot{f}'$. Then $w'_{M,n} \geq 0$ for $n = 0, \ldots\ldots, N$; and $w' = \dot{f}' = \dot{f} = w$ on the other three sides of $\partial^* R$. By the maximum principle $w_{m,n} \leq w'_{m,n}$ on $\dot{R}$. But by the previous result, $w'_{m,n} \leq 1/2$ for $m_0 \leq m \leq M'$; thus $w_{m,n} \leq 1/2$ for $m_0 \leq m \leq M$.

<u>The Case P + 1</u>: We assume case P. Now let $m_0/M \geq (1/2)^{P+1}$. We are given the boundary function $\dot{f}$ with $\dot{f} \leq 1$ for $m < m_0$, and $\dot{f} = 0$ for $m \geq m_0$. Let $w_{m,n}$ be the solution for this $\dot{f}$. Now $2m_0/M \geq (1/2)^P$, thus by case P we have $w_{m,n} \leq 1 - (1/2)^P$ for $2m_0 \leq m \leq M$. Now on the smaller rectangle $0 \leq m \leq 2m_0$, let $w'_{m,n} = w_{m,n} - (1-(1/2)^P)$. Then $w'_{m,n}$ has the boundary value $\dot{f}'$ on this rectangle, where $\dot{f}' \leq 0$ for $m_0 \leq m \leq 2m_0$, and $f \leq 1 - (1-(1/2)^P) = (1/2)^P$ for $0 \leq m < m_0$. Let $w''_{m,n}$ be the solution with boundary value $\dot{f}''$ on this smaller

rectangle, where $\dot{f}'' = (1/2)^P$ for $0 \le m < m_0$, and $\dot{f}'' = 0$ for $m_0 \le m \le 2m_0$. Then by the maximum principle $w'_{m,n} \le w''_{m,n}$ on this rectangle. But by case 1 we have $w''_{m,n} \le (1/2)^{P+1}$ for $m_0 \le m \le 2m_0$. Thus, $w_{m,n} = 1 - (1/2)^P + w'_{m,n} \le 1 - (1/2)^P + (1/2)^{P+1}$ $= 1 - (1/2)^{P+1}$, for $m_0 \le m \le 2m_0$.

Thus, case P + 1 is proved. By finite induction the theorem is proved for every positive integer P. $\|$

The above argument employed only symmetry and the minimum-maximum principle, and works for other finite difference equations. For example, consider the backward difference approximation for the heat equation on the rectangular cylinder which consists of the lattice points

$$\dot{D} = \{(m\,\delta x,\ n\,\delta x,\ i\,\delta t) : 0 \le m \le M,\ 0 \le n \le N,\ 0 \le i \le T\}$$

$$\partial\dot{D} = \{(m\,\delta x,\ n\,\delta x,\ i\,\delta t) : m = 0, \text{ or } m = M, \text{ or } n = 0, \text{ or } n = N, \text{ or } i = 0, \text{ and } (m,n,i) \text{ is } \in \dot{D}.\}$$

With the change in notation from $\dot{R}$ to $\dot{D}$, and from $w_{m,n,i}$ to $w_{m,n}$, the above lemma holds true exactly as stated and proved.

Notice that since the lattice function $w_{m,n}$ is $\le 1 - (1/2)^P$ on and to the right of $m = m_0$, its bilinear interpolation $u(x,y)$ is likewise $\le 1 - (1/2)^P$ for $x \ge m_0\,\delta x$.

We show that the Laplace difference equation approximation can be used for the numerical Schwarz alternating procedure on two rectangles.

<u>Theorem 4.2</u> Let R_1, R_2 be two closed rectangular regions, neither wholly containing the other, whose interiors intersect, and whose sides are parallel to the x-y axes. Let $R = R_1 \cup R_2$, and let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be defined as in the abstract case.

Suppose that for some δx we can place a square mesh of size δx over R such that the eight corners of R_1 and R_2 fall on lattice points. Let $\dot{R}_1$, $\dot{R}_2$ be the lattice points on R_1, R_2 respectively. Then $\dot{R} = \dot{R}_1 \cup \dot{R}_2$, and we have $\partial \dot{R}_1$, $\partial \dot{R}_2$, $\dot{\alpha}_1, \dot{\alpha}_2, \dot{\beta}_1, \dot{\beta}_2$ are the lattice points on ∂R_1, ∂R_2, $\alpha_1, \alpha_2, \beta_1, \beta_2$ respectively.

Let $T_1(H_1) = u_{m,n}$ map a given vector H_1 on $\partial \dot{R}_1$ into the solution $u_{m,n}$ on $\dot{R}_1$ for Laplace's difference equation. Let $\dot{T}_1(H_1)$ be $u_{m,n}$ restricted to $\dot{\beta}_2$. Similarly define the operator $\dot{T}_2$.

Let f be a boundary function on ∂R, whose Dirichlet solution u on R for Laplace's equation has bounded fourth order partial derivatives. Let f_1, f_2, g_1, g_2, F_1, F_2, $\dot{f}_1$, $\dot{f}_2$, $\dot{g}_1$, $\dot{g}_2$ be defined as in the abstract case.

Then the formal hypotheses I, II, III, IVA, and IVB are satisfied.

<u>Proof of I</u>: This is clear.

Proof of II. The solution of Laplace's difference equation depends linearly on the boundary data. Thus, the operators $\dot{T}_1$, $\dot{T}_2$ are linear operators.

Proof of III. $u - u_1 = F_1(g_1 + f_1)$ has bounded fourth partials on R_1. Thus we have

$$\left|\dot{F}_1(g_1+f_1) - \dot{T}_1(\dot{g}_1+\dot{f}_1)\right| \leq M_1(\delta x)^2 \text{ on } \dot{\beta}_2,$$

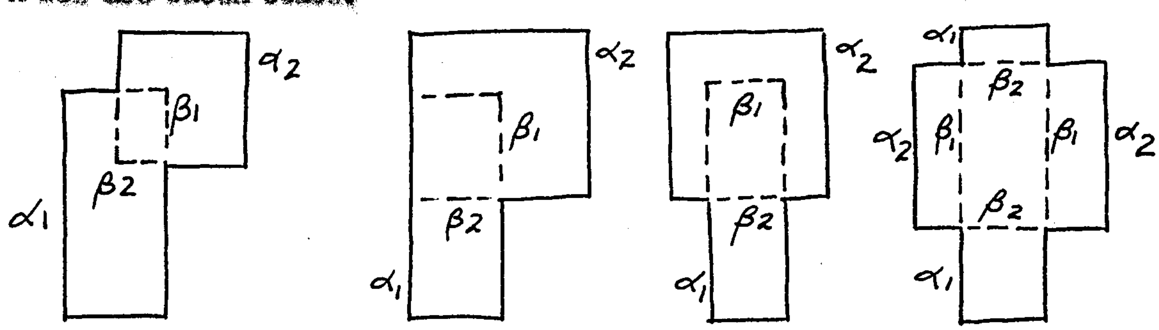
where M_1 is a constant proportional to this bound. Likewise we have

$$\left|\dot{F}_2(g_2+f_2) - \dot{T}_2(\dot{g}_2+\dot{f}_2)\right| \leq M_2(\delta x)^2 \text{ on } \dot{\beta}_1.$$

Further we can take a sequence of permissible meshes for which $\delta x \to 0$, for example, by halving δx successively.

Proof of IVA, IVB: Let the corners of R_1 be the points (x_1,y_1), (X_1,y_1), (x_1,Y_1), (X_1, Y_1); corresponding to the indices (m_1,n_1), (m_1+M_1, n_1), (m_1, n_1+N_1), (m_1+M_1, n_1+N_1) for some particular mesh size δx. Let the corners of R_2 be the points (x_2, y_2) (X_2, y_2), (x_2, Y_2), (X_2, Y_2) corresponding to the indices (m_2, n_2), (m_2+M_2, n_2), (m_2, n_2+N_2), (m_2+M_2, n_2+N_2) for this same mesh.

There are many different manners in which R_1 and R_2 can overlap. A few are shown below.



We consider only the first configuration, in which $x_1 < x_2 < X_1 < X_2$ and $y_1 < y_2 < Y_1 < Y_2$. The proofs for the other configurations are similar. Now let $G_2 = 1$ on $\dot{\beta}_2 \equiv 0$ on $\dot{\alpha}_2$. There exists a first positive integer P for which $(X_1-x_2)/(X_2-x_2) \geq (1/2)^P$. Since $\dot{\beta}_2$ lies entirely to the left of X_1 by the previous Lemma the solution $v_{m,n}$ on $\dot{R}_2$ for the boundary data G_2 is $\leq 1 - (1/2)^P$ at every point of $\dot{R}_2$ on and to the right of $x = X_1$. In particular, $\dot{T}_2(G_2) = v_{\cdot} \leq 1 - (1/2)^P$ on the vertical portion of $\dot{\beta}_1$. Likewise, there exists a first positive integer r for which $(Y_1 - y_2)/(Y_2 - y_2) \geq (1/2)^r$; and we get $\dot{T}_2(G_2) < 1 - (1/2)^r$ on the horizontal portion of also. We then have

$$\| \dot{T}_2 (G_2) \|_{\dot{\beta}_1} \leq 1 - (1/2)^{\max (p,r)} .$$

As was pointed out in Lemma (2.10), because of the linearity of T_2 and the maximum principle, we, therefore, have

$$\| \dot{T}_2 (G_2) \|_{\dot{\beta}_1} \leq 1 - (1/2)^{\max (p,r)} \| G_2 \|_{\dot{\beta}_2} .$$

for any vector G_2 defined on $\dot{\beta}_2$, $= 0$ on $\dot{\alpha}_2$. In other words, we identify $q_2 = 1 - (1/2)^{\max (p,r)}$, which number is independent of the mesh size δx. Similarly we get a $q_1 < 1$. ||

Therefore, the solution G_1, G_2 of the numerical problem (2.2) converges to the differential solution $\dot{g}_1$, $\dot{g}_2$ on $\dot{\beta}_1$, $\dot{\beta}_2$. However, in the present case hypotheses I and III were unnecessary, for clearly, if we let $u_{m,n}$ be the solution to the Laplace difference equation on $\dot{R}$, for the boundary data $\dot{f}$ on $\partial\dot{R}$, then u's values on $\dot{\beta}_1$, $\dot{\beta}_2$ satisfy the equation (2.2). Thus, G_1, G_2 just take on the values of

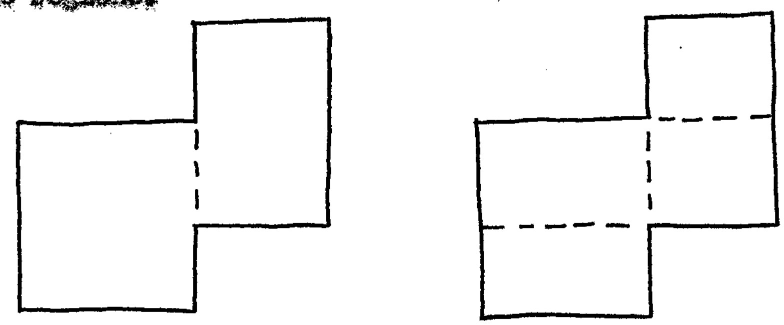
$v_{m,n}$ on $\dot{\beta}_1$, $\dot{\beta}_2$ respectively. It is clear that the present example is just a simple case of the method II of Sims and Roberts [6]. However, that article gives no consideration to the rate of convergence, stability against error, etc. It merely assumes that the difference equation can be solved exactly on each of the regions at each step. By reason of bounded increasing sequences this procedure was shown to eventually converge.

Saltzer [7] has given the matrix in explicit form for the solution of the Laplace difference equation on the rectangle. That is, the solution for boundary data $\dot{f}$ on ∂R is given by

$$v_{m,n} = \sum_{j \in \partial R} A_{m,n,j} \, (\dot{f})_j$$

However, to evaluate each coefficient $A_{m,n,j}$ requires $0(1/\delta x)$ operations, involving sin, cosh, etc., so 0 is a fairly large number. For the present application we would need approximately $(1/\delta x)^2$ coefficients. Thus $0[(1/\delta x)^3]$ operations are required just to evaluate the coefficients. The number of iterations required to evaluate G_1, G_2 to within an error of ϵ is $n \cong (\log 1/\epsilon) / \log (1/q_1 q_2)$. If $\dot{T}_1$, $\dot{T}_2$ are given in matrix form the calculations per iteration are approximately $0[(1\delta x)^2]$. Thus, besides the evaluation of the coefficients, the calculations to evaluate g_1, g_2 to within $0[(\delta x)^2]$ would be $0[(1/\delta x)^2 \log (1/\delta x)]$.

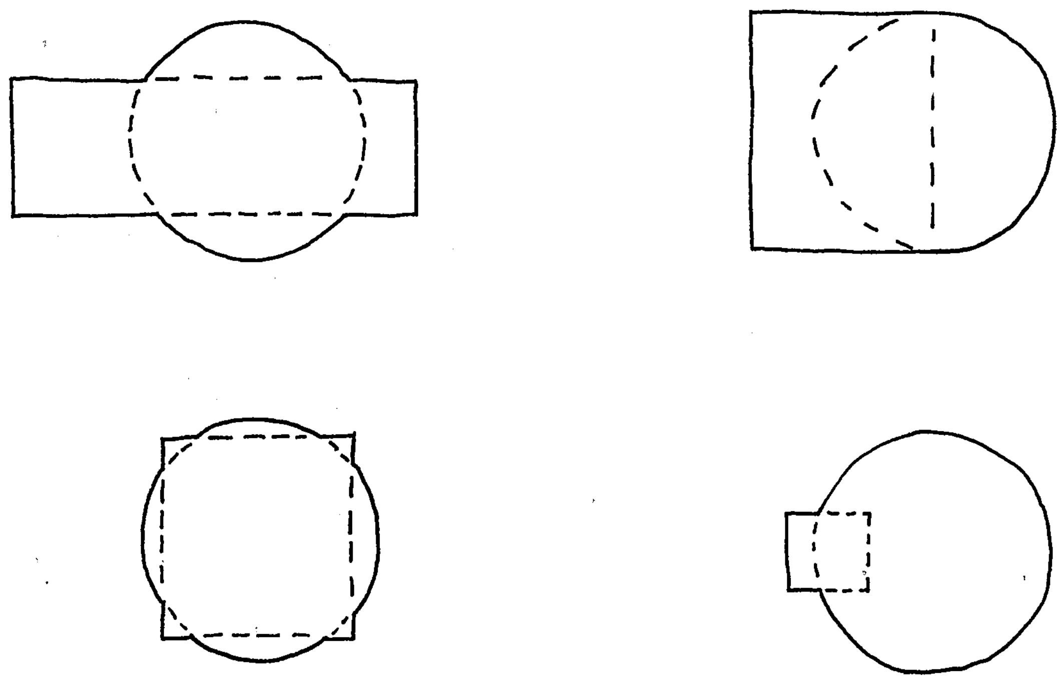
In the same article Saltzer gives an abridged block method for the solution of Laplace's difference equation on two adjacent rectangles, shown below at the left. Using the above coefficients Saltzer derives a set of equations for the unknown values on the common side. Although, these equations are diagonally dominant, the dominance definitely decreases as δx decreases; the equations will require excessive computation for solution, and excessive accuracy in the evaluation of the coefficients. It would be preferable to consider the region as the three overlapping rectangles shown below at the right, applying the obvious extension of the alternating procedure to three regions.



At each step we could use the alternating direction procedure on $\dot{R}_1$, and $\dot{R}_2$, as will be illustrated in the next example. However, the present example, in which all the corners and sides of $R = R_1 \cup R_2$ fall on the single lattice region $\dot{R} = \dot{R}_1 \cup \dot{R}_2$, is largely an academic exercise, for it would involve less work to apply the alternating direction procedure directly to $\dot{R}$.

5. An Example With a Rectangle and a Circle

Let R_1 be a closed circular region. Let R_2 be a closed rectangular region whose corners fall on some square lattice of lattice size δx. Assume R_1° and R_2° overlap. Examples of possible interest are shown below



Let $R = R_1 \cup R_2$ and let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be defined as in the abstract case. Upon R_1 we will employ the polygonal approximation $T_1(R_1)\ (r,\theta) \cong T_1(\hat{R}_1)\ (r,\theta)$ developed in section two (in most cases we could actually employ the third order correct approximation or even a fourth order correct approximation). Let every end point of the arcs of α_1 be included in the partition $\dot{\partial} R_1$ of ∂R_1.

On R_2 we employ as our approximation the solution to the Laplace difference equation with bilinear interpolation, as described at the beginning of section four. At each step the numerical alternating procedure will present us with new boundary data H_2 on $\partial^{*} R_2$. We will solve for the lattice solution $w_{i,j}$ at each point of $\dot{R}_2$ and interpolate to get $w(x_k, y_k)$, $k = 1, \ldots, n_1$, at the n_1 points of $\dot{\beta}_1$. In other words

$$\left[\dot{T}_2 (H_2)\right]_k = w(x_k, y_k).$$

On R_1^0, the points of $\dot{\beta}_2$ have polar coordinates (r_p, θ_p), $p = 1, \ldots, n_2$. By formula (3.1) we calculate the coefficients $(A_1)_{pk} = (A_1)_k (r_p \; \theta_p)$. At each step the numerical alternating procedure will present us with new boundary data (H_1) on $\partial^{*} R_1$ and we will calculate

$$\left[\dot{T}_1 (H_1)\right]_p = \sum_{k=1}^{n_1+m_1} (A_1)_{pk} \; (H_1)_k \text{ for } p = 1, \ldots, n_2.$$

Let f be a boundary function on ∂R, whose Dirichlet solution u has continuous fourth partial derivatives on the closure of R. Let f_1, f_2, g_1, g_2, P_1, P_2 be defined as in the abstract case

Theorem 5.1 Under these conditions the formal hypotheses I, II, III, IVA, IVB are satisfied.

Proof of I: This is clear.

Proof of II: This is clear.

Proof of III: Because u has continuous second partials, $\frac{\partial^2}{\partial \psi^2}(f_1+g_1)$ is continuous and bounded on the circle ∂R_1. Therefore, by lemma 3.3

$$\|\dot{F}_1(f_1+g) - \dot{T}_1(\dot{f}_1+\dot{g}_1)\|_{\dot{\beta}_2} \leq M_1 \|\delta \varphi\|^2,$$

where M_1 is this bound, and $\|\delta \varphi\|$ is the maximum angular interval of the partition $\dot{\partial} R_1$. Likewise in section four we saw that

$$\|\dot{F}_2(f_2+g_2) - \dot{T}_2(\dot{f}_2+\dot{g}_2)\|_{\dot{\beta}_1} \leq \|u(x,y) - v(x,y)\|_{R_2}$$
$$\leq M_2(\delta x)^2.$$

Proof of IVA and IVB:

We have $\|\dot{T}_1(G_1)\|_{\dot{\beta}_2} \leq q_1 \|G_1\|_{\dot{\beta}_1}$ for all $G_1 \equiv 0$ on $\dot{\alpha}_1$, where $q_1 < 1$. For certain special cases our lemma 4.1 would give us a $q_2 < 1$ for the operator $\dot{T}_2$. But we have not proved this for the general case; we have only the maximum principle

$$\|\dot{T}_2(G_2)\|_{\dot{\beta}_1} < 1 \quad \|G_2\|_{\dot{\beta}_2}.$$

Therefore, let $q_2 = 1$. ||

At each step in the numerical Schwarz alternating procedure we will solve the Laplace difference equation on $\dot{R}_2$ by means of the alternating direction procedure. For details about the alternating direction procedure see [8]. We can summarize the results as follows: For an arbitrary vector $z_{i,j}$ on the interior points $\dot{R}_2^0$, define the N_2 norm by

$$N_2(z_{i,j}) = \left[\sum_{(i\delta x,\, j\delta x) \in \dot{R}_2^0} |z_{i,j}|^2 \right]^{1/2}.$$

Recall that

$$\|z_{i,j}\|_{\dot{R}_2^0} = \text{maximum } |z_{i,j}| \text{ for } (i\delta x,\, j\delta x) \in \dot{R}_2^0.$$

Then we have the comparison between the norms

$$\|Z_{i,j}\|_{\dot{R}_2^o} \leq N_2(Z_{i,j}) \leq n^{1/2}\|Z_{i,j}\|_{\dot{R}_2^o},$$

where n is the number of points in $\dot{R}_2^o$. Now let $T_2(H_2) = w_{i,j}$ be the solution on $\dot{R}_2$ of the Laplace difference equation for arbitrary data H_2 on $\partial\dot{R}_2$. Let $x_{i,j}$ equal H_2 on $\partial\dot{R}_2$ and be an initial guess at $w_{i,j}$ on $\dot{R}_2^o$. Proceeding iteratively by the alternating direction procedure we can reduce the N_2 error by any factor e^{-Q}; that is, we end up with an approximate solution $y_{i,j}$ such that

$$N_2(w_{i,j} - y_{i,j}) \leq e^{-Q} N_2(w_{i,j} - x_{i,j}).$$

The number of calculations involved is

$$W \leq Q \quad C \quad (1/\delta x)^2 \log (1/\delta x),$$

where C is a constant depending upon the dimensions of the region R_2. In terms of our uniform norm, using the comparison above, we have

$$\|(w_{i,j} - y_{i,j})\|_{\dot{R}_2^o} \leq e^{-Q} n^{1/2} \|(w_{i,j} - x_{i,j})\|_{\dot{R}_2^o}.$$

Therefore, to reduce the error in the uniform norm by a factor e^{-K} we must set

$$e^{-K} = e^{-Q} n^{1/2},$$

or

$$Q = K + \log (n^{1/2}).$$

However, if we assume that our problem has been normalized so that the longest side of our rectangle R_2 is ≤ 1, then $n < (1/\delta x)^2$ and we have

$$Q \leq K + \log (1/\delta x).$$

Noting that the errors are zero on $\partial \dot{R}_2$ we have

$$\|(w_{i,j} - y_{i,j})\|_{\dot{R}_2} \leq e^{-K} \|(w_{i,j} - x_{i,j})\|_{\dot{R}_2};$$

and the number of calculations required is

$$N \leq \left[K + \log(1/\delta x)\right] \quad C \quad (1/\delta x)^2 \log(1/\delta x).$$

Recall that we would wish to calculate

$$\left\{ G_1^0 \text{ arbitrary on } \dot{\beta}_1, \right.$$

$$\begin{cases} G_2^n = \dot{T}_1 (G_1^{n-1} + \dot{f}_1) & \text{on } \dot{\beta}_2, \\ G_1^n = \dot{T}_2 (G_2^n + \dot{f}_2) & \text{on } \dot{\beta}_1, \qquad n = 1, 2, \ldots . \end{cases}$$

Instead we are forced to calculate $\dot{T}_2$ only approximately. We pick a fixed number $\|\epsilon\| > 0$. At each step we will maintain this accuracy for $\dot{T}_2$.

$$\left\{ \text{Let } J_1^0 = G_1^0 \text{ be arbitrary on } \dot{\beta}_1 . \right.$$

For n = 1:

Let $J_2^1 = \dot{T}_1 (J_1^0 + \dot{f}_1)$ on $\dot{\beta}_2$.
Let $x_{i,j}^1$ be an arbitrary guess at $T_2(J_2^1 + \dot{f}_2)$ on $\dot{R}_2^0$.
We then iterate by the alternating direction procedure until we have

$$y_{i,j}^1 = T_2 (J_2^1 + \dot{f}_1) + \epsilon^1_{i,j} \text{ on } \dot{R}_2 ,$$

where $\|\epsilon^1_{i,j}\|_{\dot{R}_2} \leq \|\epsilon\|$. Interpolating bilinearly we get

$$J_1^1 = \dot{T}_2 (J_2^1 + \dot{f}_2) + \epsilon^1 \text{ on } \dot{\beta}_1 ,$$

where $\|\epsilon^1\| \leq \|\epsilon\|$.

For $n = 2, 3, \ldots$ our initial guess is automatic:

Let $J_2^n = \dot{T}_1 (J_1^{n-1} + \dot{f}_1)$ on β_2.

Let $x_{i,j}^n = y_{i,j}^{n-1}$ on $\dot{R}_2^o$.

We then iterate until we have

$$y_{i,j}^n = T_2(J_2^n + \dot{f}_2) + \epsilon_{i,j}^n \quad \text{on } \dot{R}_2,$$

where $\|\epsilon_{i,j}^n\|_{\dot{R}_2} \leq \|\epsilon\|$. Interpolating bilinearly we get

$$J_1^n = \dot{T}_2(J_2^n + \dot{f}_2) + \epsilon^n \quad \text{on } \beta_1,$$

where $\|\epsilon^n\| \leq \|\epsilon\|$.

We investigate the work at each step. Notice that for $n = 2, 3, \ldots$

$$T_2 (J_2^n + \dot{f}_2) - x_{i,j}^n = T_2 (J_2^n - J_2^{n-1}) + \epsilon_{i,j}^{n-1} \quad \text{on } \dot{R}_2^o.$$

Therefore, by the maximum principle

$$\| T_2 (J_2^n + f_2) - x_{i,j}^n \| \leq \| J_2^n - J_2^{n-1} \| + \|\epsilon\|$$

$$\leq \| J_2^n - G_2^n \| + \| G_2^n - G_2 \| + \| G_2 - G_2^{n-1} \| + \| G_2^{n-1} - J_2^{n-1} \| + \|\epsilon\|,$$

or applying lemmas 2.6 and 2.7,

$$\leq \frac{2\|\epsilon\|}{1 - q_1} + q_1^n \| G_1^o - G_1 \| + \frac{2\|\epsilon\|}{1 - q_1} + \|\epsilon\|,$$

which is certainly

$$\leq \frac{5\|\epsilon\|}{1 - q_1} + \| G_1^o - G_1 \|.$$

But we can easily find a bound for the initial error $\| G_1^0 - G_1 \|$. Likewise, we can easily find a bound for the error of the initial guess $x^1_{1,j}$ of step 1. Thus, we get a constant B such that

$$\| T_2 (J_2^n - f_2) - x^n_{1,j} \| \le B, \text{ for } n = 1, 2, 3, \ldots .$$

Therefore, to maintain the accuracy $\|\epsilon\|$ at each step it is sufficient (in fact much more than sufficient for larger n) to choose K such that

$$B e^{-K} = \|\epsilon\| ,$$

or

$$K = \log (1/\|\epsilon\|) + \log B.$$

We consider the total error. Recall $q_1 < 1$, and $q_2 = 1$. By lemma 2.4 and part II of theorem 5.1 we have

$$\left.\begin{array}{l} \| \dot{g}_1 - G_1 \| \\ \| \dot{g}_2 - G_2 \| \end{array}\right\} \le \frac{1}{1 - q_1} (M_1 \quad \| \delta \psi \|^2 + M_2 (\delta x)^2 .$$

By lemma 2.6

$$\left.\begin{array}{l} \| G_1^n - G_1 \| \\ \| G_2^n - G_2 \| \end{array}\right\} \le (q_1)^n \quad \| G_1^0 - G_1 \| .$$

We actually calculate J_1^n and J_2^n; by lemma 2.7

$$\left.\begin{array}{l} \| J_1^n - G_1^n \| \\ \| J_2^n - G_1^n \| \end{array}\right\} \le \frac{2}{1 - q_1} \|\epsilon\|$$

It will be most efficient to make all error sources approximately equal. Let $\|\delta\psi\| \cong (\delta x)$. Let $(q_1)^n \cong (\delta x)^2$, therefore, $n \cong 2 \log (1/\delta x)$. Let $\|\epsilon\| \cong (\delta x)^2$, therefore $K \cong 2 \log (1/\delta x)$ and $Q \cong K + \log (1/\delta x) \cong 3 \log (1/\delta x)$. The errors are then of the order

$$\left.\begin{array}{l} \| \dot{s}_1 - s_1^n \|_{\dot{\beta}_1} \\ \| \dot{s}_2 - s_2^n \|_{\dot{\beta}_2} \end{array}\right\} \leq 0 \left[(\delta x)^2 \right].$$

The work involved consists of firstly, the evaluation of the approximately $0\left[(1/\delta x)^2\right]$ coefficients $(A_1)i,j$; secondly, approximately $3 \log (1/\delta x)$ steps, each involving $W \leq 0 \log (1/\delta x) (1/\delta x)^2$ $\log (1/\delta x)$ calculations. In summary we have

$$\text{total calculations} \leq 0 \left\{ \left[\log (1/\delta x) \right]^3 (1/ x)^2 \right\},$$

$$\text{absolute error} \leq 0 \left[(\delta x)^2 \right].$$

REFERENCES

1. Schwarz, H.A., *Gesammelte Mathematische Abhandlungen*, Vol. 2, Berlin, Springer, 1890, pp. 133-134.

2. Nevanlinna, R., *Uber das alternierende Verfahren von Schwarz*, Journal fur die reine und angewandte Mathematik, Vol. 180 (1939) pp. 121-130.

3. Kantorovich, L.V., and Krylov, V.I., *Approximate Methods of higher analysis*, trans. by C.D. Benster, New York, Interscience Publishers, 1958, pp. 616-639.

4. Diaz, J.B., and Birkhoff, G., *Non-linear network problems*, Quart. of Appl. Math., Vol. 13 (1956) pp. 431-443.

5. Forsythe, G.E., and Wasow, W.R., *Finite difference methods for partial differential equations*, New York, John Wiley and Sons, 1960, section 23.1.

6. Diaz, J.B., and Roberts, R.C., *On the numerical solution of the Dirichlet problem for Laplace's difference equation*, Quart. of Appl. Math. Vol. 9 (1952) pp. 355-360.

7. Saltzer, C., *An abridged block method for the solution of the Dirichlet problem for Laplace's difference equation*, Journal of Math. and Phys. Vol. 32 (1953) p. 63.

8. Douglas, J. Jr., and Rachford, H.H. Jr., *On the numerical solution of heat conduction problems in two and three space variables*, Trans. Amer. Math. Soc. Vol. 82 (1956) p. 436.