

 Open access • Journal Article • DOI:10.1007/BF01385808

Numerical analysis of oscillations in nonconvex problems — [Source link](#)

Michel Chipot

Institutions: Metz

Published on: 01 Dec 1991 - Numerische Mathematik (Springer-Verlag New York, Inc.)

Topics: Finite element method, Numerical analysis and Calculus of variations

Related papers:

- [Fine phase mixtures as minimizers of energy](#)
- [Numerical approximation of the solution of variational problem with a double well potential](#)
- [Proposed experimental tests of a theory of fine microstructure and the two-well problem](#)
- [Numerical approximations in variational problems with potential wells](#)
- [Direct Methods in the Calculus of Variations](#)

Share this paper:    

View more about this paper here: <https://typeset.io/papers/numerical-analysis-of-oscillations-in-nonconvex-problems-19jvs4a50e>

**NUMERICAL ANALYSIS OF OSCILLATIONS
IN NONCONVEX PROBLEMS**

By

M. Chipot

IMA Preprint Series # 741

December 1990

NUMERICAL ANALYSIS OF OSCILLATIONS IN NONCONVEX PROBLEMS

M. Chipot

Université de Metz

Département de Mathématiques

Ile de Saulcy, 57045 METZ-CEDEX 01

(FRANCE)

1. Introduction

Let Ω be a bounded domain of \mathbf{R}^n . For the simplicity of the numerical analysis we will assume that Ω is a polygonal domain. Let $w_i \in \mathbf{R}^n$, $i = 1, \dots, k$, $k \geq 2$, and consider a function $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\varphi(w_i) = 0 \quad \forall i = 1, \dots, k, \tag{1.1}$$

$$\varphi(w) > 0 \quad \forall w \neq w_i, \quad i = 1, \dots, k. \tag{1.2}$$

In physical terms, φ is some energy that vanishes at wells w_i . Let us denote by $\partial\Omega$ the boundary of Ω and by Γ_0 some portion of $\partial\Omega$ of positive measure.

Let a be in the convex hull of the w_i 's, such that

$$a \neq w_i \quad \forall i = 1, \dots, k. \tag{1.3}$$

If V_a denotes some space of functions v such that

$$v(x) = a \cdot x \quad \text{on} \quad \Gamma_0 \tag{1.4}$$

($a \cdot x$ denotes the scalar product of a and x) we would like to consider the problem

$$\text{Inf}_{v \in V_a} \int_{\Omega} \varphi(\nabla v(x)) dx. \tag{1.5}$$

As we will see, this problem in general does not admit a minimizer, i.e., the infimum of (1.5) is not attained for some $u \in V_a$. In other words minimizing sequences of (1.5) do not converge in a strong enough topology to allow their limit points to be a minimizer. In such an event the problem seems to lack interest. However, it turns out that it arises in numerous physical settings. For instance (see [B.J.1], [B.J.2], [Er.], [F.1], [F.2], [J.K.], [K.], [Ko.], [S.]), some crystals have their deformations governed by energies φ experiencing some potential wells w_i . For such a crystal, even if no unique equilibrium is available,

there is some organized way to reach its lowest energy level. In terms of problem (1.5) the minimizing sequences have some pattern that we would like to investigate here. Note that we are considering here the so called scalar case –i.e. the w_i 's are vectors and not matrices. So, in particular hyperelasticity for ordered material seems to be beyond our scope. We will comment on that at the end of the paper. We refer also the reader to [C.C.K.] for results in this direction.

Some similar problems have been first considered by C. Collins, D. Kinderlehrer and M. Luskin in one dimension (see [C.K.L.], [C.L.1] - [C.L.4]).

One can add also to the energy functional defined in (1.5) some kind of weak coerciveness by considering instead as in [B.M.]

$$\int_{\Omega} \varphi(\nabla v(x)) + (v - a \cdot x)^2 dx. \quad (1.6)$$

In this event, the second term under the integral suffices to force the pattern of the minimizing sequences. We refer the reader to [C.C.] for a complete analysis of this problem. On the other hand, when no such a term is introduced, the oscillations of the minimizing sequences are forced, as we will see, only by the boundary condition that has to be matched.

The paper is divided as follows. In section 2 we estimate in terms of the mesh size the infimum (1.5) using Lagrange elements of type P_1 . We give also an existence result for the approximated problem. In section 3 we show that (1.5) does not admit in general a minimizer. We also analyse the Young measure associated to the problem proving in particular its uniqueness. In section 4 we investigate the pattern of the minimizing sequences. In section 5 we comment on some numerical computations for these types of problems.

2. Energy estimate on a finite element space

In what follows we will always assume

$$a = 0. \tag{2.1}$$

Indeed, there is no loss of generality in doing so. To see this, remark that if we set $v = u - a \cdot x$ then

$$\int_{\Omega} \varphi(\nabla u(x)) dx = \int_{\Omega} \varphi(\nabla v(x) + a) dx \tag{2.2}$$

thus minimize the left hand side of (2.2) on V_a reduces to minimize the right hand side of (2.2) on V_0 . We did not yet define V_a but, as we will see, for the problem that we have in mind, we just need V_a to be a space of functions equal to $a \cdot x$ on Γ_0 and containing the piecewise affine functions (by a piecewise affine function we mean a continuous function affine on simplices covering Ω). But then we are led to a problem identical to the one we had with now

$$\tilde{\varphi}(w) = \varphi(w + a)$$

i.e. with a function $\tilde{\varphi}$ having $w_i - a$ as wells, with 0 in the convex hull of these wells (since clearly a in the convex hull of the w_i 's is equivalent to 0 in the convex hull of the $w_i - a$'s).

Let τ a triangulation of Ω with simplices of diameters less than h . If K is some simplex of τ denote by $P_1(K)$ the space of polynomials of degree 1 on K . Set

$$V_h^0 = \{v : \Omega \rightarrow \mathbf{R}, \text{ continuous} \mid v|_K \in P_1(K) \quad \forall K \in \tau, v = 0 \text{ on } \Gamma_0\}.$$

($v|_K$ denotes the restriction of v to K).

Then one has:

Theorem 1: *Assume that φ is bounded on bounded subsets of \mathbf{R}^n . Then there exists a constant C such that*

$$E_h = \inf_{v \in V_h^0} \int_{\Omega} \varphi(\nabla v(x)) dx \leq C \cdot h^{1/2}. \tag{2.3}$$

Remark 1: Note that $\nabla v(x)$ and thus $\varphi(\nabla v(x))$ are constant on every simplex of τ so that no further assumption on φ is needed for the integral of (2.3) to make sense.

Proof of Theorem 1: Clearly since $a = 0$ belongs to the convex hull of the w_i 's one can find w_i 's, that we will denote by w_1, \dots, w_p , $p \geq 2$, such that

$$w_i - w_1, \quad i = 2, \dots, p \quad \text{are linearly independent} \quad (2.4)$$

and such that for some unique $\alpha_i \in (0, 1)$

$$\sum_{i=1}^p \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^p \alpha_i = 1. \quad (2.5)$$

Set

$$w_h(x) = \bigwedge_{i=1}^p w_i \cdot x + h^\alpha \quad (2.6)$$

where \wedge denotes the minimum of two functions and α is some number in $(0, 1)$. Since $\alpha \in (0, 1)$ it should be noticed that

$$h^\alpha \gg h. \quad (2.7)$$

First, remark that

$$w_h(x) \leq h^\alpha \quad \forall x. \quad (2.8)$$

Indeed if not we would have for some x

$$w_i \cdot x > 0 \quad \forall i = 1, \dots, p$$

and by (2.5)

$$\sum_{i=1}^p \alpha_i w_i \cdot x = 0$$

hence a contradiction.

Moreover, this “hat” function w_h has exactly p different slopes, i.e. for every i the set of x such that $w_h(x) = w_i \cdot x + h^\alpha$ has a non empty interior. Indeed, otherwise for some i_0 we would have

$$w_{i_0} \cdot x + h^\alpha \geq w_i \cdot x + h^\alpha \quad \forall i, \quad \forall x,$$

thus

$$\sum_{i=1}^p \alpha_i w_{i_0} \cdot x \geq \sum_{i=1}^p \alpha_i w_i \cdot x = 0 \quad \forall x.$$

It follows that

$$w_{i_0} \cdot x \geq 0 \quad \forall x$$

which is impossible unless $w_{i_0} = 0 = a$ which has been excluded.

Consider the set

$$\begin{aligned} S_h &= \{x \in \mathbf{R}^n | w_h(x) \geq 0\} \\ &= \{x \in \mathbf{R}^n | w_i \cdot x + h^\alpha \geq 0 \quad \forall i = 1, \dots, p\}. \end{aligned}$$

Clearly S_h is the intersection of p half spaces and thus a convex domain having p edges. For instance when $n = 2$, if $p = 2$ then S_h is a strip, if $p = 3$ S_h is a triangle. When $n = 3$, if $p = 2$, S_h is a strip, if $p = 3$, S_h is a cylinder with a triangular basis, if $p = 4$, S_h is a tetrahedron...

Let us denote by W the subspace of \mathbf{R}^n spanned by the w_i 's. Then, $S_h \cap W$ is a $(p - 1)$ -simplex with vertices v_0, v_1, \dots, v_{p-1} . Note that the functions $w_i \cdot x$ are constant on any subspace orthogonal to W . For any $z = (z_1, \dots, z_{p-1}) \in \mathbf{Z}^{p-1}$ set

$$w_{h,z}(x) = w_h(x - \sum_{i=1}^{p-1} z_i(v_i - v_0)) \quad (2.9)$$

Clearly, $w_{h,z}$ is a piecewise affine function, non negative only on each of the sets

$$S_{h,z} = S_h + \sum_{i=1}^{p-1} z_i(v_i - v_0). \quad (2.10)$$

Note that these subsets are disjoint or have in common a subspace parallel to W^\perp , the orthogonal of W . Set

$$u_h(x) = \bigvee_{z \in \mathbf{Z}^{p-1}} w_{h,z}(x) \quad (2.11)$$

where \bigvee denotes the supremum of functions. Then u_h is a piecewise affine function equal to $w_{h,z}$ on $S_{h,z}$. Note that for a given x the supremum in (2.11) is taken only on a finite

number of z such that $S_{h,z}$ neighbours x . Now clearly one has

$$|u_h(x)| \leq Ch^\alpha \tag{2.12}$$

for some constant C . Moreover, at every point x of in \mathbf{R}^n one has

$$\nabla u_h(x) = w_i \tag{2.13}$$

except on the edges of the function u_h . To obtain a function u_h which is affine on each simplex of τ we modify u_h in the following way. On each simplex of τ where u_h has some edge we replace it by the affine function that coincides with u_h on its vertices. Clearly, (2.12) is preserved. Now the volume of the simplices where this modification occurred is no more than

$$h \cdot (n - 1\text{-area of the edge of } u_h) \tag{2.14}$$

(the edge of u_h is cutting one simplex at a time, and any dimension of this simplex is bounded by h).

Denote by $N(h^\alpha)$ the number of $S_{h,z}$ covering Ω . By a scaling argument one has

$$\begin{aligned} n - 1\text{-area of the edge of } u_h &= C.N(h^\alpha)(h^\alpha)^{p-2} \\ &\leq C'N(1)(h^\alpha)^{p-2}/(h^\alpha)^{p-1} \\ &\leq Ch^{-\alpha}. \end{aligned}$$

Hence, by (2.14) the volume where the modification has occurred is less than

$$C.h^{1-\alpha} \tag{2.15}$$

i.e.

$$\nabla u_h = w_i \tag{2.16}$$

except on a part of n -dimensional volume $C \cdot h^{1-\alpha}$.

To match the boundary condition we modify u_h once more by replacing u_h by the affine function which coincides with

$$\text{dist}(\cdot, \partial\Omega) \wedge u_{h\nu} - \text{dist}(\cdot, \partial\Omega) \quad (2.17)$$

at the node of τ ($\text{dist}(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$). Let us still denote this last function by u_h . Clearly the gradient of u_h coincide with the previous gradient of u_h except (see (2.12)) on a neighbourhood of Γ of volume

$$C|\partial\Omega|h^\alpha \quad (2.18)$$

Thus, see (2.13) – (2.17), we have obtained a function $u_h \in V_h^0$ such that (2.13) holds except on a set S of volume

$$Ch^{1-\alpha} + C|\partial\Omega|h^\alpha$$

where, of course, u_h has a bounded gradient. Thus we have

$$\begin{aligned} \int_{\Omega} \varphi(\nabla u_h(x)) dx &= \int_{\Omega \setminus S} \varphi(\nabla u_h(x)) dx + \int_S \varphi(\nabla u_h(x)) dx \\ &\leq C(h^{1-\alpha} + h^\alpha) \end{aligned}$$

where C is some constant independent of h .

We thus obtain, since $\alpha \in (0, 1)$, and h could be assumed to be less than 1

$$E_h \leq \int_{\Omega} \varphi(\nabla u_h(x)) dx \leq 2Ch^{\min(1-\alpha, \alpha)}$$

Now, the function $\min(1 - \alpha, \alpha)$ is maximum for $\alpha = \frac{1}{2}$ which gives the result.

Remark 2: Of course, for some triangulation, the rate of $h^{1/2}$ is not sharp, but this rate holds for any φ and any triangulation. In particular we did not assume the triangulation to be quasi-uniform.

Remark 3: In the case of a quasi-uniform mesh and a slow growth of φ at infinity one can improve our estimate. Indeed assume that for some $r < 1$ one has

$$|\varphi(\xi)| \leq C|\xi|^r \quad \forall \xi \in \mathbf{R}^n. \quad (2.19)$$

Then, instead of using (2.17) to match the boundary condition we can set u_h to 0 on the nodes of $\partial\Omega$ and leave it unchanged elsewhere.

Due to (2.12) the gradient of u_h is bounded by Ch^α/h on a neighbourhood of $\partial\Omega$ of thickness h . So, by (2.19), (2.15) we obtain

$$\begin{aligned} \int_{\Omega} \varphi(\nabla u_h(x)) dx &\leq Chh^{(\alpha-1)r} + Ch^{1-\alpha} \\ &\leq Ch^{\min(1-\alpha, 1-r(1-\alpha))}. \end{aligned}$$

The function in the exponent of h is maximum when

$$1 - \alpha = 1 - r(1 - \alpha) \quad \Leftrightarrow \quad \alpha = 1 - \frac{1}{1+r}$$

which leads to

$$E_h \leq \int_{\Omega} \varphi(\nabla u_h(x)) dx \leq Ch^{\frac{1}{1+r}} \quad (2.20)$$

which is slightly better than (2.3)

Remark 4: When $r \rightarrow 0$ the rate $h^{\frac{1}{1+r}}$ is almost sharp in the sense that, in general, one cannot expect a rate better than h .

Indeed, consider for instance

$$\begin{aligned} n = 2, \quad \Omega &= (-1, +1)^2, \quad \Gamma_0 = \partial\Omega, \\ w_1 &= (-1, 0), \quad w_2 = (1, 0), \\ \varphi(\xi) &= \text{Min}_i |\xi - w_i|^2 \text{ for } |\xi - w_i| \leq 1, \quad i = 1, 2, \\ \varphi(\xi) &\geq 1 \quad \text{for } |\xi - w_1| \geq 1 \quad \text{or} \quad |\xi - w_2| \geq 1. \end{aligned}$$

Assume that our mesh is quasi-uniform. If $u_h = 0$ on $(-1, +1) \times \{1\}$ then necessarily $\frac{\partial u_h}{\partial x} = 0$ on the triangles touching this part of the boundary and one obtains

$$\begin{aligned} \int_{\Omega} \varphi(\nabla u_h(x)) dx &\geq \text{measure of these triangles} \\ &= Ch. \end{aligned}$$

and thus the rate of convergence cannot be better than h in general.

It could be useful to know that the approximated problem

$$\inf_{v \in V_h^0} \int_{\Omega} \varphi(\nabla v(x)) dx \quad (2.21)$$

admits a solution. In this spirit we can prove:

Theorem 2: *Assume that φ is continuous and that*

$$\varphi(\xi) \rightarrow +\infty \text{ when } |\xi| \rightarrow +\infty \quad (2.22)$$

then there exists $u_h \in V_h^0$ such that

$$\int_{\Omega} \varphi(\nabla u_h(x)) dx = \inf_{v \in V_h^0} \int_{\Omega} \varphi(\nabla v(x)) dx. \quad (2.23)$$

Proof: A function $v_h \in V_h^0$ is completely determined by its value at the nodes of the triangulation. Let X denote the vector of the values of v_h at the nodes not on Γ_0 . Then

$$\int_{\Omega} \varphi(\nabla v_h(x)) dx$$

becomes a continuous function F of X that we have to minimize over \mathbf{R}^N where N is the number of nodes outside Γ_0 . If we can show that

$$F(X) = \int_{\Omega} \varphi(\nabla v_h(x)) dx \rightarrow +\infty \quad (2.24)$$

when $|X| \rightarrow +\infty$ we are led to minimize F , which is continuous, over a compact set of \mathbf{R}^N and we are done. If $|X| \rightarrow +\infty$, then one of the value of v_h at some node N_i must go to $+\infty$. Then due to the 0 boundary condition one must have on some simplex

$$|\nabla v_h| \geq \frac{v_h(N_i)}{\text{dist}(N_i, \Gamma_0)}.$$

(Since we do not assume that Ω is convex, $\text{dist}(N_i, \Gamma_0)$ is here the length of the shortest path linking N_i to Γ_0 and passing through the different nodes).

Hence if we denote by K such a simplex

$$\int_{\Omega} \varphi(\nabla v_h(x)) dx \geq |K| \varphi(\nabla v_h|_K)$$

where $|K|$ denotes the measure of K and (2.24) follows by (2.22).

3. Oscillation phenomenon

In order to obtain a unique pattern for the minimizing sequences of (1.5) we have to limit the number of wells and assume that φ has at most n wells i.e.:

$$2 \leq k \leq n. \tag{3.1}$$

Recall that $a = 0$ belongs to the convex hull of the w_i 's. It will be useful to assume:

$$w_i - w_1, \quad i = 2, \dots, k \text{ are linearly independent} \tag{3.2}$$

and for some $\alpha_i \in (0, 1)$ one has

$$\sum_{i=1}^k \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^k \alpha_i = 1. \tag{3.3}$$

It results from (3.1) and the fact that a belongs to the convex hull of the w_i 's that the w_i 's span a subspace W which is of dimension strictly less than n . This fact, as we will see later, is crucial. Let us denote by W^\perp the orthogonal complement of W . We will assume that

$$\forall x \in \Omega, \exists x_0 \in \Gamma_0 \quad \text{such that } (x, x_0) \subset \Omega \text{ and } x - x_0 \in W^\perp \tag{3.4}$$

$((x, x_0)$ denotes the segment with end points x and x_0 . Note that this assumption is automatically satisfied when $\Gamma_0 = \partial\Omega$).

Under the above assumptions we can prove

Theorem 3: *Let us assume that φ is bounded on bounded subsets of \mathbf{R}^n . Let $V_0 \subset W^{1,1}(\Omega)$ be a space of functions vanishing on Γ_0 , containing the continuous piecewise affine functions. If (3.1), (3.4) hold then the problem*

$$\text{Inf}_{v \in V_0} \int_{\Omega} \varphi(\nabla v(x)) dx \quad (3.5)$$

cannot have a minimizer.

Remark 5: $W^{1,1}(\Omega)$ is the usual Sobolev space of functions in $L^1(\Omega)$ with first derivatives in $L^1(\Omega)$ —(see [G.T]). In the integral (3.5) we make the convention that

$$\int_{\Omega} \varphi(\nabla v(x)) dx = +\infty \quad (3.6)$$

if $\varphi(\nabla v(x)) \notin L^1(\Omega)$. Recall that φ is a nonnegative function and this convention avoids imposing some growth conditions on φ .

Proof of Theorem 3: Let ν_1, \dots, ν_ℓ an orthonormal basis of W^\perp . Due to (3.4) we have a Poincaré inequality of the type

$$\int_{\Omega} |v(x)| dx \leq C \int_{\Omega} |\nabla_{\nu} v(x)| dx \quad \forall v \in V_0 \quad (3.7)$$

(∇_{ν} denotes the vector of components $\frac{\partial}{\partial \nu_i}$, $|\cdot|$ the Euclidean norm).

By Theorem 1 one has

$$\text{Inf}_{v \in V_0} \int_{\Omega} \varphi(\nabla v(x)) dx = 0.$$

If for some $u \in V_0$

$$\int_{\Omega} \varphi(\nabla u(x)) dx = 0$$

then by (1.1), (1.2) one has:

$$\nabla u(x) = w_i \neq 0 \quad \text{a.e in } \Omega \quad (3.8)$$

hence since $\frac{\partial u}{\partial \nu_j} = \nabla u \cdot \nu_j$

$$\nabla_\nu u(x) = 0 \quad \text{a.e in } \Omega.$$

Then from (3.7) applied to u one deduces

$$u(x) = 0 \quad \text{a.e in } \Omega. \tag{3.9}$$

But then $\nabla u = 0$ and a contradiction with (3.8).

So, in the absence of minimizers we turn to the study of minimizing sequences. As we will see, they have a common pattern which is explained by the uniqueness of the Young measure associated to them. More precisely, recall that if $u_h \in W^{1,\infty}(\Omega)$,

$$|u_h|_\infty, \quad \|\nabla u_h\|_\infty \leq C \tag{3.10}$$

where C is some constant independent on h , then u_h defines a Young measure in the sense that there exists a subsequence of u_h , that for simplicity we will still denote by u_h , a parametrized probability measure ν_x on \mathbf{R}^n , such that for any bounded Carathéodory function $F(x, \xi)$ one has

$$\lim_{h \rightarrow 0} \int_{\Omega} F(x, \nabla u_h(x)) dx = \int_{\Omega} \int_{\mathbf{R}^n} F(x, \lambda) d\nu_x(\lambda) dx. \tag{3.11}$$

(We refer for instance to [B.], [C.K.], [K.], [K.P.₁], [K.P.₂], [D.], [Ev.], [T.], [V.] for this question, recall that $\|\cdot\|_\infty$ denotes the usual L^∞ norm-see [G.T.] for more information on Sobolev spaces).

Then we can prove:

Theorem 4: *Assume that (3.1)–(3.4) hold, φ continuous. Let u_h be a minimizing sequence of (3.5). If u_h defines a Young measure ν_x –i.e. if (3.10) holds– then one has necessarily*

$$\nu_x = \sum_{i=1}^k \alpha_i \delta_{w_i} \quad \text{a.e } x \in \Omega \tag{3.12}$$

where the α_i 's are the constants appearing in (3.3) and δ_{w_i} denotes the Dirac mass at w_i .

Proof: Up to a subsequence one can assume that one has for some $u \in W^{1,\infty}(\Omega)$

$$u_h \rightarrow u \quad \text{uniformly in } \Omega \quad (3.13)$$

$$\nabla u_h \rightharpoonup \nabla u \quad \text{in } L^\infty(\Omega) \quad \text{weak} - *. \quad (3.14)$$

((3.14) means that each component of the gradient is converging in $L^\infty(\Omega)$ weak $- *$). Let ν_x be a Young measure defined by u_h .

From (3.11) applied with $F(x, \xi) = \varphi(\xi)$ we have

$$0 = \lim_{h \rightarrow 0} \int_{\Omega} \varphi(\nabla u_h(x)) dx = \int_{\Omega} \int_{\mathbf{R}^n} \varphi(\lambda) d\nu_x(\lambda) dx.$$

It results that

$$\int_{\mathbf{R}^n} \varphi(\lambda) d\nu_x(\lambda) = 0 \quad \text{a.e in } \Omega.$$

Hence, by (1.1), (1.2) the support of ν_x has to be contained in the w_i 's for a.e x , i.e.

$$\nu_x = \sum_{i=1}^k \beta_i(x) \delta_{w_i}. \quad (3.15)$$

Applying (3.11) again we have from (3.15)

$$\begin{aligned} \int_{\Omega} |\nabla_{\nu} u_h(x)| dx &= \int_{\Omega} \left[\sum_{j=1}^{\ell} (\nabla u_h(x) \cdot \nu_j)^2 \right]^{1/2} dx \rightarrow \int_{\Omega} \int_{\mathbf{R}^n} \left[\sum_{j=1}^{\ell} (\lambda \cdot \nu_j)^2 \right]^{1/2} d\nu_x(\lambda) dx \\ &= \int_{\Omega} \sum_{i=1}^k \beta^i(x) \left[\sum_{j=1}^{\ell} (w_i \cdot \nu_j)^2 \right]^{1/2} dx = 0 \end{aligned}$$

since the ν_j 's belong to W^\perp .

Recalling (3.7) and (3.13) we obtain

$$\int_{\Omega} |u(x)| dx = \lim_{h \rightarrow 0} \int_{\Omega} |u_h(x)| dx = 0.$$

Hence $u = 0$.

Let B be a ball included in Ω .

Since the limit of any subsequence of u_h is 0, (see (3.14), we have when $h \rightarrow 0$

$$\nabla u_h \rightharpoonup 0 \text{ in } L^\infty(\Omega) \text{ weak} - *.$$

From (3.11) applied with $F(x, \xi) = X_B(x)\xi$ (this function is not bounded in ξ , but the gradient of u_h is!) we obtain

$$0 = \lim_{h \rightarrow 0} \int_B \nabla u_h(x) dx = \int_B \int_{\mathbf{R}^n} \lambda d\nu_x(\lambda) dx \quad (3.16)$$

By (3.15) this reads

$$0 = \sum_{i=1}^k \int_B \beta_i(x) dx \cdot w_i.$$

Hence, by (3.3) and the uniqueness of barycentric coordinates

$$\frac{1}{|B|} \int_B \beta_i(x) dx = \alpha_i \quad \forall i = 1, \dots, k.$$

(Recall that since ν_x is a probability measure $\sum_{i=1}^k \beta_i(x) = 1$). Since this is true for any B it results from the Lebesgue differentiation theorem that

$$\beta_i(x) = \alpha_i \quad \text{a.e in } \Omega, \quad \forall i = 1, \dots, k$$

and thus (3.15) reads

$$\nu_x = \sum_{i=1}^k \alpha_i \delta_{w_i} \quad \text{a.e in } \Omega.$$

This shows that ν_x is necessarily the homogeneous Young measure (3.12) (see [K.P.1]) and concludes the proof.

Remark 6: The construction of u_h given in section 2 provides a minimizing sequence such that (3.10) holds.

To end this section we would like to discuss the necessity of our assumptions. For instance in theorem 3, the assumption (3.1) cannot be relaxed. Indeed consider the one dimensional case and assume for instance that $\varphi: \mathbf{R} \rightarrow \mathbf{R}$ satisfies

$$\varphi(-1) = \varphi(1) = 0 \quad , \quad \varphi(\xi) > 0 \quad \forall \xi \neq \pm 1. \quad (3.17)$$

Take $\Omega = (0, 1)$, $\Gamma_0 = \partial\Omega$, $a = 0$. So, we are in the case $n = 1$, $k = 2$. Then clearly any piecewise affine function such that

$$u'(x) = \pm 1 \quad , \quad u(0) = u(1) = 0$$

satisfies

$$0 = \int_{\Omega} \varphi(u'(x)) dx = \text{Inf}_{v \in V_0} \int_{\Omega} \varphi(v'(x)) dx$$

where V_0 is any space containing the piecewise affine functions vanishing on $\partial\Omega$. So, the problem (3.5) admits infinitely many minimizers. The situation is exactly the same in higher dimensions. Consider for instance the case $n = 2$, $k = 3$. Assume that Ω is an equilateral triangle centered at 0 and of sides of length 1. Consider the “hat” function u defined on this triangle i.e. such that

$$u(0) = 1, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

and interpolates linearly between these points. Let w_1, w_2, w_3 the different gradients of this function and φ a function vanishing on these w_i 's and positive elsewhere and satisfying also the assumptions of theorems 3 and 4. Clearly u is a minimizer of

$$\int_{\Omega} \varphi(\nabla v(x)) dx$$

on $W_0^{1,1}(\Omega)$. So the problem (3.5) admits in this case a minimizer. Now, if we split Ω into equilateral triangles (not necessarily of the same size) and if on each of these triangles

we define u to be equal to 0 on the boundary and to the length side at the center, u interpolating linearly between these points then u is an other minimizer and we get that way infinitely many of them.

To see now the necessity of (3.4) we can argue for instance in dimension 2 with $\Omega = (-1, 1)^2$, $\Gamma_0 = \{-1\} \times (-1, 1)$, $w_1 = (1, 0)$, $w_2 = -w_1$. Then any function depending on x only, vanishing on Γ_0 and climbing or going down with a slope ± 1 is a minimizer.

A similar example can provide an understanding of the necessity of (3.2), (3.3) in theorem 4. Indeed assume

$$\Omega = (-1, 1)^2 \quad , \quad \Gamma_0 = \partial\Omega \quad , \quad w_1 = (1, 0) \quad , \quad w_2 = -w_1 \quad , \quad w_3 = -2w_1$$

then

$$0 = \frac{1}{2}w_1 + \frac{1}{2}w_2 \quad \text{or} \quad 0 = \frac{2}{3}w_1 + \frac{1}{3}w_3$$

and

$$\nu_x = \frac{1}{2}\delta_{w_1} + \frac{1}{2}\delta_{w_2} \quad \text{or} \quad \nu_x = \frac{2}{3}\delta_{w_1} + \frac{1}{3}\delta_{w_3}$$

are two admissible Young measures for the problem (3.5) and indeed (see the proof of Theorem 1 for some insight on this question) one can construct minimizing sequences corresponding to both of them.

4. Probabilistic analysis of oscillations

In this section we would like to analyse the behaviour of the minimizing sequences of (1.5). As we will see, under the assumptions of theorem 4, they have a common pattern. In particular in order to minimize the energy they are choosing their gradients around each of the wells with a probability which tends to be constant. We will give an estimate of this probability in terms of h when the minimizing sequence considered is in V_h^0 . First let us make precise our assumptions.

In all this section we will assume that (3.1)–(3.4) hold. Moreover, we denote by π the projection on the w_i 's (see [C.K.L.]) i.e. π denotes the function π defined by

$$\pi(\xi) = w_i$$

where w_i is the well of smallest index i such that (4.1)

$$|\xi - w_i| = \text{Min}_j |\xi - w_j|$$

($|\cdot|$ denotes the Euclidean norm of \mathbf{R}^n). The function π takes only a finite number of values -i.e. the w_k 's. Moreover, it is clearly a Borel function so that if $\xi(x)$ is any measurable function, $\pi(\xi(x))$ will be measurable.

Then, we will assume that there exists $\wedge > 0$, $p \geq 1$ such that

$$\begin{aligned} \varphi(\xi) &\geq \wedge \cdot |\xi - \pi(\xi)|^p \\ &= \wedge \cdot \text{Min}_i |\xi - w_i|^p \quad \forall \xi \in \mathbf{R}^n. \end{aligned} \tag{4.2}$$

Now, let us denote by R some positive number such that

$$R < \frac{1}{2} |w_i - w_j| \quad \forall i, j = 1, \dots, k, \quad i \neq j. \tag{4.3}$$

If $v \in W^{1,\infty}(\Omega)$ and if B is some subset of Ω we will denote by B_i^R or $B_i^R(v)$ (we will drop sometimes for convenience the dependence in v) the set

$$B_i^R = B_i^R(v) = \{x \in B \mid \nabla v(x) \in B(w_i, R)\} \tag{4.4}$$

where $B(w_i, R)$ is the ball of center w_i and radius R in \mathbf{R}^n -i.e.

$$B(w_i, R) = \{w \in \mathbf{R}^n \mid |w - w_i| < R\}.$$

If $|\cdot|$ denotes the Lebesgue measure in \mathbf{R}^n , $|B_i^R|/|B|$ represents the probability for v to have its gradient on B in $B(w_i, R)$. For convenience we introduce also the notation

$$B_{ex}^R = B \setminus \bigcup_{i=1}^k B_i^R. \tag{4.5}$$

We are first going to assume that B is a Lipschitz domain for which the inequality

$$\int_{\partial B} |v(x)| d\sigma(x) \leq C \int_B |v| + |\nabla_\nu v(x)| dx \quad \forall v \in W^{1,1}(\Omega) \quad (4.6)$$

holds (see (3.7) for the meaning of ∇_ν). It is easy to see that (4.6) holds for instance for a Lipschitz domain of Ω such that the outward normal $n(x)$ satisfies: there exists $i \in 1, \dots, \ell$ and a constant c such that

$$|n(x) \cdot \nu_i| > c \quad \sigma - \text{a.e. } x \in \partial B. \quad (4.7)$$

Recall that ν_1, \dots, ν_ℓ denotes an orthonormal basis of W^\perp .

We will set

$$E(v) = \int_\Omega \varphi(\nabla v(x)) dx. \quad (4.8)$$

Then we can prove:

Lemma 1: *Assume that (3.1)-(3.4) hold. Moreover assume that φ satisfies (4.2). If B is a Lipschitz domain such that (4.6) holds, then one has for some constant C*

$$\left| \sum_{i=1}^k |B_i^R(v)| w_i \right| \leq C \cdot E(v)^{\frac{1}{p}} \quad (4.9)$$

for any $v \in W^{1,p}(\Omega)$, v vanishing on Γ_0 .

(Recall that $|\cdot|$ denotes the Lebesgue measure or the usual Euclidean norm in \mathbf{R}^n)

Proof: Note first that

$$\begin{aligned} \sum_{i=1}^k |B_i^R| w_i &= \int_{\cup_i B_i^R} \pi(\nabla v(x)) dx \\ &= \int_B \pi(\nabla v(x)) dx - \int_{B_{e_x}^R} \pi(\nabla v(x)) dx \\ &= \int_B \pi(\nabla v(x)) - \nabla v(x) dx + \int_B \nabla v(x) dx - \int_{B_{e_x}^R} \pi(\nabla v(x)) dx \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3. \end{aligned} \quad (4.10)$$

To estimate I_1 note that by Hölder's inequality and (4.2)

$$\begin{aligned}
|I_1| &\leq \int_B |\pi(\nabla v(x)) - \nabla v(x)| dx \\
&\leq \left(\int_B |\pi(\nabla v(x)) - \nabla v(x)|^p dx \right)^{\frac{1}{p}} |B|^{1-\frac{1}{p}} \\
&\leq \wedge^{-\frac{1}{p}} |B|^{1-\frac{1}{p}} E(v)^{\frac{1}{p}}.
\end{aligned} \tag{4.11}$$

To estimate I_2 one applies the divergence theorem to get

$$I_2 = \int_B \nabla v(x) dx = \int_{\partial B} v(x) n(x) d\sigma(x)$$

where $n(x)$ denotes the outward normal to ∂B . Hence, by (4.6), (3.7):

$$\begin{aligned}
|I_2| &\leq \int_{\partial B} |v(x)| d\sigma(x) \\
&\leq \int_B |v(x)| + |\nabla_\nu v(x)| dx \\
&\leq \int_\Omega |v(x)| + |\nabla_\nu v(x)| dx \\
&\leq C \int_\Omega |\nabla_\nu v(x)| dx
\end{aligned} \tag{4.12}$$

On the other hand from (4.2) we deduce

$$\varphi(\xi) \geq \wedge \cdot \text{Min}_i |\xi - w_i|^p \geq \wedge \cdot |\xi - P_W(\xi)|^p$$

where P_W denotes the orthogonal projection on W . Due to the equality

$$\xi = P_W(\xi) + P_{W^\perp}(\xi)$$

one deduces

$$\varphi(\xi) \geq \wedge \cdot |P_{W^\perp}(\xi)|^p \quad \forall \xi \in \mathbf{R}^n. \tag{4.13}$$

Since

$$P_{W^\perp}(\nabla v(x)) = \sum_{i=1}^{\ell} (\nabla v(x) \cdot \nu_i) \cdot \nu_i$$

we obtain for almost every x

$$\varphi(\nabla v(x)) \geq \wedge \cdot |\nabla_\nu v(x)|^p$$

and integrating (see (3.6)) we obtain

$$\int_{\Omega} \varphi(\nabla v(x)) dx \geq \wedge \cdot \int_{\Omega} |\nabla_\nu v(x)|^p dx \quad \forall v \in V_0. \quad (4.14)$$

Hence by the Hölder Inequality and (4.12)

$$\begin{aligned} |I_2| &\leq C |\Omega|^{1-\frac{1}{p}} \left(\int_{\Omega} |\nabla_\nu v(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq C \wedge^{-\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} E(v)^{\frac{1}{p}}. \end{aligned} \quad (4.15)$$

To estimate I_3 , note that

$$\begin{aligned} |I_3| &\leq \int_{B_{e_x}^R} |\pi(\nabla v(x))| dx \\ &\leq \text{Max}_i |w_i| |B_{e_x}^R|. \end{aligned}$$

Now, clearly

$$\begin{aligned} R |B_{e_x}^R| &\leq \int_{B_{e_x}^R} |\pi(\nabla v(x)) - \nabla v(x)| dx \\ &\leq \int_B |\pi(\nabla v(x)) - \nabla v(x)| dx. \end{aligned}$$

Hence, using Hölder's inequality

$$|B_{e_x}^R| \leq \frac{1}{R} |B|^{1-\frac{1}{p}} \wedge^{-\frac{1}{p}} E(v)^{\frac{1}{p}} \quad (4.16)$$

and thus

$$|I_3| \leq \text{Max}_i |w_i| \frac{1}{R} |B|^{1-\frac{1}{p}} \wedge^{-\frac{1}{p}} E(v)^{\frac{1}{p}} \quad (4.17)$$

Combining (4.10), (4.11), (4.15), (4.17) we obtain clearly

$$\left| \sum_{i=1}^k |B_i^R(v)| w_i \right| \leq C E(v)^{\frac{1}{p}}$$

which is (4.9).

As a consequence we have:

Theorem 5: *Assume that (3.1)-(3.4) hold. Moreover assume that φ satisfies (4.2). If B is a Lipschitz domain such that (4.6) holds then for some constant C*

$$\left| \alpha_i - \frac{|B_i^R(v)|}{|B|} \right| \leq C \cdot E(v)^{\frac{1}{p}} \quad \forall i = 1, \dots, k \quad (4.18)$$

for any $v \in W^{1,p}(\Omega)$, v vanishing on Γ_0 .

Proof: Let M denote the $n + 1 \times k$ matrix

$$M = \begin{bmatrix} w_1 & \dots & w_k \\ 1 & \dots & 1 \end{bmatrix}. \quad (4.19)$$

Since the vectors $w_i - w_1$ are linearly independent this matrix has rank k . In particular, the system

$$My = b \quad (4.20)$$

has at most one solution which, when it exists, is given by

$$y = (M^T M)^{-1} M^T b$$

(M^T denotes the transpose of M) and one has

$$|y| \leq \|(M^T M)^{-1} M^T\| |b| \quad (4.21)$$

where $\| \cdot \|$ denotes the matrix norm corresponding to the Euclidean norm.

Since

$$\sum_{i=1}^k \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^k \alpha_i = 1$$

one has

$$\begin{aligned} \sum_{i=1}^k (\alpha_i |B| - |B_i^R|) w_i &= - \sum_{i=1}^k |B_i^R| w_i \\ \sum_{i=1}^k (\alpha_i |B| - |B_i^R|) &= |B_{e_x}^R|. \end{aligned} \quad (4.22)$$

Hence, the vector with entries $\alpha_i|B| - |B_i^R|$ satisfies (4.20) for b given by the right hand side of (4.22). The result follows then by combining (4.21), (4.9) and (4.16).

As a consequence of this result we see that for any minimizing sequence v_h of (3.5) one has

$$\frac{|B_i^R(v_h)|}{|B|} \rightarrow \alpha_i \quad \forall i = 1, \dots, k$$

at a rate proportional to $E(v_h)^{1/p}$. In particular if one sets

$$E_h = \inf_{v \in V_h^0} \varphi(\nabla v(x)) dx \quad (4.23)$$

one has:

COROLLARY 1: *Under the assumption of Theorem 5, if B is a Lipschitz domain such that (4.6) holds and if $v_h \in V_h^0$ is such that*

$$E(v_h) \leq 2E_h$$

then there exists a constant C such that

$$\left| \alpha_i - \frac{|B_i^R(v_h)|}{|B|} \right| \leq Ch^{\frac{1}{2p}} \quad \forall 1, \dots, k. \quad (4.24)$$

Proof: This results from Theorem 5 and (2.3).

Remark 7: Note that this estimate holds in particular for any minimizer $u_h \in V_h^0$ of (2.23). In the case where some low growth condition on φ is assumed then the rate $h^{\frac{1}{2p}}$ can be improved (see Remark 3).

We now assume that B is any Lipschitz domain of Ω and we use the same notation than before in particular for B_i^R . Then we have:

Lemma 2: *Assume that (3.1)-(3.4) hold. Moreover assume that φ satisfies (4.2) with $p > 1$. If B is a Lipschitz domain included in Ω then there exists a constant C such that:*

$$\left| \sum_{i=1}^k |B_i^R(v)| w_i \right| \leq C \cdot (E(v))^{\frac{1}{p}} + E(v)^{\frac{1}{pq}} \cdot \|v\|_{1,p,B}^{\frac{1}{p}} \quad (4.25)$$

for any $v \in W^{1,p}(\Omega)$, v vanishing on Γ_0 .

(q denotes the conjugate exponent of p , $\|v\|_{1,p,B} = (\int_B |v(x)|^p + |\nabla v(x)|^p dx)^{\frac{1}{p}}$).

Proof: One proceeds as in Lemma 1 -i.e. one writes (4.10). There is no change in the estimate of I_1 , I_3 , and (4.11) and (4.17) hold. To estimate I_2 one notices as before that

$$|I_2| = \left| \int_{\partial B} v(x)n(x) d\sigma(x) \right| \leq \int_{\partial B} |v(x)| d\sigma(x). \quad (4.26)$$

Now, for any function in $W^{1,1}(B)$ one has for some constant C

$$\int_{\partial B} |v(x)| d\sigma(x) \leq C \int_B |v(x)| + |\nabla v(x)| dx.$$

Applying this inequality to $v = v^p$ we get

$$\int_{\partial B} |v(x)|^p d\sigma(x) \leq C \int_B |v|^p + (p-1)|v|^{p-1}|\nabla v| dx. \quad (4.27)$$

Now, by Hölder's inequality

$$\int_B |v|^{p-1}|\nabla v| dx \leq \left(\int_B |v|^p dx \right)^{\frac{1}{q}} \left(\int_B |\nabla v|^p dx \right)^{\frac{1}{p}}$$

where q is the conjugate exponent of p -i.e. $\frac{1}{p} + \frac{1}{q} = 1$. From (4.27) we deduce

$$\int_{\partial B} |v(x)|^p d\sigma(x) \leq C \left(\int_B |v|^p dx \right)^{\frac{1}{q}} \cdot \left(\int_B |v|^p + |\nabla v|^p dx \right)^{\frac{1}{p}}$$

for some constant C .

Recalling (4.26) and using Hölder's inequality we obtain

$$|I_2| \leq \int_{\partial B} |v(x)| d\sigma(x) \leq |\partial B|^{1-\frac{1}{p}} \left(\int_{\partial B} |v(x)|^p d\sigma(x) \right)^{\frac{1}{p}}$$

($|\partial B|$ denotes the $n-1$ dimensional Hausdorff measure of ∂B). Hence for some positive constant C

$$|I_2| \leq C \left(\int_B |v|^p dx \right)^{\frac{1}{pq}} \|v\|_{1,p,B}^{\frac{1}{p}}. \quad (4.28)$$

Now, due to (3.4) we have a Poincaré Inequality of the type

$$\int_{\Omega} |v|^p dx \leq C \int_{\Omega} |\nabla_{\nu} v(x)|^p dx$$

for any $v \in W^{1,p}(\Omega)$, v vanishing on Γ_0 .

Hence by (4.14)

$$\int_B |v|^p dx \leq \int_{\Omega} |v|^p dx \leq C \cdot \int_{\Omega} \varphi(\nabla v(x)) dx$$

and (4.23) becomes

$$|I_2| \leq C \cdot E(v)^{\frac{1}{pq}} \|v\|_{1,p,B}^{\frac{1}{p}}.$$

One concludes then as in lemma 1.

As a consequence we can prove

THEOREM 6: *Assume that (3.1)-(3.4) hold. Moreover assume that φ satisfies (4.2) with $p > 1$. Then if B is a Lipschitz domain included in Ω one has for some constant C*

$$\left| \alpha_i - \frac{|B_i^R(v)|}{|B|} \right| \leq C \cdot (E(v)^{\frac{1}{pq}} + E(v)^{\frac{1}{p}}) \quad \forall i = 1, \dots, k \quad (4.29)$$

for any $v \in W^{1,p}(\Omega)$, v vanishing on Γ_0 .

Proof: Denote by $\|\cdot\|_{1,p}$ the usual norm on $W^{1,p}(\Omega)$. One argues exactly as in Theorem 5 but one replaces now (4.9) by (4.25). Noting first that

$$\|v\|_{1,p,B} \leq \|v\|_{1,p}$$

one obtains

$$\left| \sum_{i=1}^k |B_i^R(v)| w_i \right| \leq C(E(v)^{\frac{1}{p}} + \|v\|_{1,p}^{\frac{1}{p}} E(v)^{\frac{1}{q}}) \quad (4.30)$$

Now, for any $v \in W^{1,p}$, vanishing on Γ_0 one has

$$\int_{\Omega} |v|^p dx \leq C \int_{\Omega} |\nabla v|^p dx$$

and thus

$$\begin{aligned}
|v|_{1,p} &\leq C \left(\int_{\Omega} |\nabla v(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq C \left(\int_{\Omega} |\nabla v - \pi(\nabla v)|^p dx \right)^{\frac{1}{p}} + C \left(\int_{\Omega} |\pi(\nabla v)|^p dx \right)^{\frac{1}{p}} \\
&\leq C_1 E(v)^{\frac{1}{p}} + C_2
\end{aligned}$$

where C_1 , and C_2 are constants independent of v . (Recall that $|\pi(\nabla v)| \leq \text{Max}_i |w_i|$).

Thus

$$|v|_{1,p}^{\frac{1}{p}} \leq C_1 E(v)^{\frac{1}{p^2}} + C_2$$

for some other constants C_1, C_2 and (4.30) becomes

$$\left| \sum_{i=1}^k |B_i^R(v)| w_i \right| \leq C(E(v)^{\frac{1}{pq}} + E(v)^{\frac{1}{p}} + E(v)^{\frac{1}{p^2} + \frac{1}{pq}})$$

Hence (4.29) follows, since $\frac{1}{p} + \frac{1}{q} = 1$.

As a consequence we see that any minimizing sequence v_h of (3.5) is such that

$$\frac{|B_i^R(v_h)|}{|B|} \rightarrow \alpha_i \quad \forall i, \dots, k$$

at a rate proportional to $E(v_h)^{\frac{1}{pq}}$ ($\frac{1}{pq} < \frac{1}{p}$).

In particular one can show

COROLLARY 2: *Under the assumptions of Theorem 6, if B is a Lipschitz domain included in Ω and if $v_h \in V_h^0$ is such that*

$$E(v_h) \leq 2E_h$$

then there exists a constant C such that

$$\left| \alpha_i - \frac{|B_i^R(v_h)|}{|B|} \right| \leq Ch^{\frac{1}{2pq}} \quad \forall i = 1, \dots, k. \quad (4.31)$$

Proof: This is an immediate consequence of (2.3) and Theorem 6. Of course we assume here that h is small – i.e. $h < 1$.

Remark 8: This estimate holds in particular for any minimizer $u_h \in V_h^0$ of (2.23). See also remark 7.

Remark 9: To see how minimizing sequences have their gradients that approach the Young measure ν_x defined by (3.12) one could evaluate both against some function $F(x, \cdot)$ and estimate the difference. We refer the reader to [C.C.] for this kind of results.

5. Numerical results and concluding remarks

Let us assume in this section that Ω is the square $(-1, 1)^2$, $\Gamma_0 = \partial\Omega$, and consider the case of two wells

$$w_1 = (-1, 1) \quad , \quad w_2 = (1, -1).$$

Our results apply since $n = 2$. Moreover set

$$\varphi(\xi) = |\xi - w_1|^2 + |\xi - w_2|^2.$$

It is not difficult to check that these data fit our previous analysis. Since

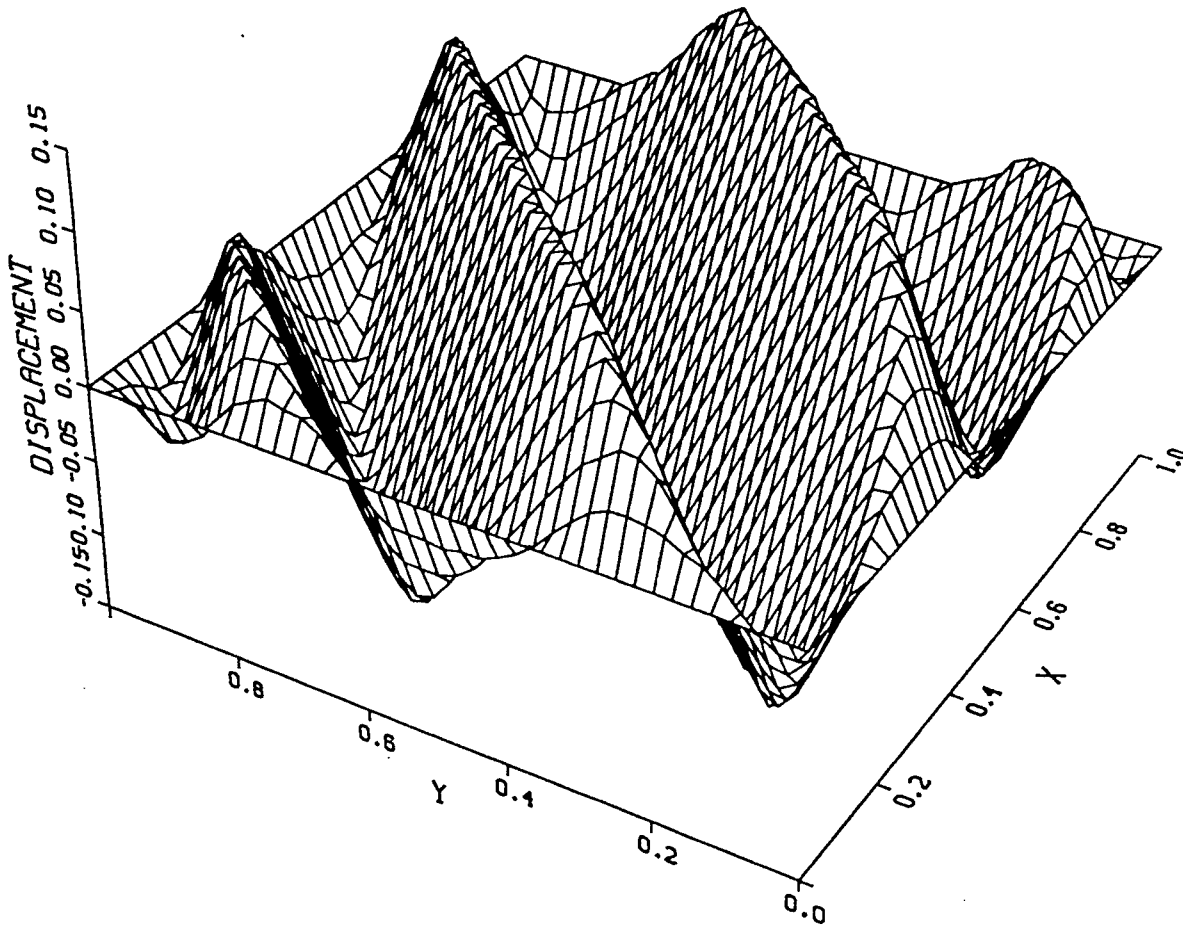
$$0 = \frac{1}{2}w_1 + \frac{1}{2}w_2$$

the minimizing sequences of (1.5) are choosing their gradient alternatively around w_1 and w_2 with a probability which tends to $\frac{1}{2}$ (see Theorem 6 and Corollary 2). This can be observed on the figure below.

In fact, in order to minimize the energy, the computer uses almost exactly the technique developed in Theorem 1. Indeed, with two wells the function u_h built in Theorem 1 looks as the one produced by the computer. Of course, the oscillations of the minimizing sequences are stopped at the level of the grid.

Our numerical codes use standard iteration algorithms. We start with a random distribution of initial values of the function close to 0 and we decrease the energy using

a conjugate gradient method. An example of the final result is plotted on the following figure.



MESH 40×40 (FINAL)

Computational Result for $h = 1/40$.

As we mentioned earlier the most interesting examples of applications of our results occur in the study of deformations of ordered materials such as crystals. In this case the potential wells are matrices. Moreover, due to the frame indifference, these wells are infinite in number. Also deformations have to satisfy the compatibility condition which

makes the problem more involved. For some results in this direction and complements we refer the reader to [C.C.K.] and [Co.].

Acknowledgements: This work was done during one visit at the University of Minnesota. I would like to thank the I.M.A. and the School of Mathematics who are providing such a good working atmosphere. This research has been supported there by the NSF and ASOFR through DMS 87-18881.

I thank also my collaborators C. Collins and D. Kinderlehrer for several fruitful discussions on the subject.

REFERENCES

[B.] J. M. Ball : A version of the fundamental theorem for Young measures.

In *Partial Differential Equations and Continuum Models of Phase Transitions*, Lecture Notes in Physics # 344, M. Rascle, D. Serre and M. Slemrod, eds., Springer-Verlag, (1989), p. 207-215.

[B.J.₁] J. M. Ball and R. D. James : Fine phase mixtures as minimizers of energy.

Arch. Rational Mech. Anal., 100, (1987), p. 13-52.

[B.J.₂] J. M. Ball and R. D. James : Experimental tests of a theory of fine microstructures.

(1988), preprint.

[B.M.] J. M. Ball and F. Murat : $W^{1,p}$ quasiconvexity and variational problems for multiple integrals.

J. Funct. Anal., 58, (1984), p. 255-253.

- [C.] M. Chipot : Hyperelasticity for crystals.
 European Journ. of Appl. Math., 1, (1990), p. 113-129.
- [C.C.] M. Chipot and C. Collins : Numerical approximation in variational problems
 with potential wells.
 (1990), preprint. (To appear).
- [C.C.K.] M. Chipot, C. Collins and D. Kinderlehrer : (In preparation).
- [C.K.] M. Chipot and D. Kinderlehrer : Equilibrium configurations of crystals.
 Arch. Rational Mech. Anal., 103, (1988), p. 237-277.
- [Co.] C. Collins : Thesis, University of Minnesota, 1990.
- [C.K.L.] C. Collins, D. Kinderlehrer and M. Luskin : Numerical approximation of the
 solution of a variational problem with a double well potential.
 IMA preprint # 605, (1990). (To appear in SIAM J. Numer. Anal.).
- [C.L.₁] C. Collins and M. Luskin : The computation of the austenitic-martensitic
 phase transition.
 In *Partial Differential Equations and Continuum Models of Phase Transitions*, Lecture Notes in Physics # 344, M. Rascle, D. Serre and M. Slemrod, eds., Springer-Verlag, Berlin, New York, (1989), p. 34–50.
- [C.L.₂] C. Collins and M. Luskin : Computational results for phase transitions in
 shape memory materials.
 In *Smart Materials, Structures, and Mathematical Issues*, C. Rogers, ed.
 Technomic Publishing Co., Lancaster, Pennsylvania, (1989), p. 198–215.
- [C.L.₃] C. Collins and M. Luskin : Numerical modeling of the microstructure of crystals
 with symmetry-related variants.
 In *Proceedings of the ARO US-Japan Workshop on Smart/Intelligent*

Materials, Technomic Publishing Co., Lancaster, Pennsylvania, (to appear).

[C.L.₄] C. Collins and M. Luskin : Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem.
Preprint, (1990).

[D.] B. Dacorogna : *Weak continuity and weak lower semicontinuity of nonlinear functionals*.
Springer Lecture Notes # 922, (1982).

[Er.] J. L. Ericksen : Some constrained elastic crystals.
In *Material Instabilities in Continuum Mechanics and Related Problems*,
J. M. Ball, ed., Oxford University Press, Oxford, (1987), p. 119-137.

[Ev.] L. C. Evans : *Weak Convergence Methods for Nonlinear Partial Differential Equations*.
A.M.S. Regional Conference Series in Mathematics # 74, (1989).

[F.₁] I. Fonseca : Variational methods for elastic crystals.
Arch. Rat. Mech. Anal., 97, (1985), p. 189-220.

[F.₂] I. Fonseca : The lower quasiconvex envelope of the stored energy function for an elastic crystal.
J. Math. Pures et Appl., 67, (1988), p. 175-195.

[Fr.] D. A. French : On the convergence of finite element approximations of a relaxed variational problem.
SIAM J. Numer. Anal., (1990), p. 419-436.

[G.T.] D. Gilbard and N.S. Trudinger : *Elliptic Partial Differential Equations of Second Order*.

Springer Verlag, (1985).

[J.1] R. James : Basic principles for the improvement of shape-memory and related materials.

In *Smart Materials, Structures, and Mathematical Issues*, C. Rogers, ed., Technomic Publishing Co. Lancaster, PA, (1989).

[J.2] R. D. James : Microstructure and weak convergence.

In *Material Instabilities in Continuum Mechanics and Related Problems*, J. M. Ball, ed., Oxford University Press, Oxford , (1987), p. 175-196.

[J. K.] R. D. James and D. Kinderlehrer : Theory of diffusionless phase transitions.

In *Partial Differential Equations and Continuum Models of Phase Transitions*, Lecture Notes in Physics # 344, M. Rascle, D.Serre and M. Slemrod, eds., Springer-Verlag, Berlin, New York, (1989), p. 51-84

[K.] D. Kinderlehrer : Remarks about equilibrium configurations of crystals.

In *Material Instabilities in Continuum Mechanics and Related Problems*, J. M. Ball ed., Oxford University Press, Oxford, (1987), p. 217-242.

[K. P.1] D. Kinderlehrer and P. Pedregal : Analysis of Young measures (Preprint).

[K. P.2] D. Kinderlehrer and P. Pedregal : Weak convergence of integrands and the Young measure representation (Preprint).

[Ko.] R. Kohn : The relationship between linear and nonlinear variational models of coherent phase transitions.

In *Proceedings of the Seventh Army Conference on Applied Mathematics and Computing*, West Point, June 1989.

[P.] P. Pedregal : Thesis, University of Minnesota, 1989.

[S.] S. A. Silling : Phase changes induced by deformation in isothermal elastic crystals.

(1988), (preprint).

[T.] L. Tartar : Compensated compactness and application to partial differential equations

In *Nonlinear analysis and mechanics: Heriot-Watt Symp IV*, R.J. Knops ed., Pitman, (1979), 136–212.

[V.] M. Valadier : Young measures.

CIME Course (1989), preprint.

Recent IMA Preprints

#	Author/s	Title
662	Geneviève Raugel and George R. Sell	Navier-Stokes equations in thin 3d domains: Global regularity of solutions I
663	Emanuel Parzen	Time series, statistics, and information
664	Andrew Majda and Kevin Lamb	Simplified equations for low Mach number combustion with strong heat release
665	Ju. S. Il'yashenko	Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation
666	James F. Reineck	Continuation to gradient flows
667	Mohamed Sami Elbially	Simultaneous binary collisions in the collinear N–body problem
668	John A. Jacquez and Carl P. Simon	Aids: The epidemiological significance of two different mean rates of partner-change
669	Carl P. Simon and John A. Jacquez	Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations
670	Matthew Stafford	Markov partitions for expanding maps of the circle
671	Ciprian Foias and Edriss S. Titi	Determining nodes, finite difference schemes and inertial manifolds
672	M.W. Smiley	Global attractors and approximate inertial manifolds for abstract dissipative equations
673	M.W. Smiley	On the existence of smooth breathers for nonlinear wave equations
674	Hitay Özbay and Janos Turi	Robust stabilization of systems governed by singular integro-differential equations
675	Mary Silber and Edgar Knobloch	Hopf bifurcation on a square lattice
676	Christophe Golé	Ghost circles for twist maps
677	Christophe Golé	Ghost tori for monotone maps
678	Christophe Golé	Monotone maps of $T^n \times R^n$ and their periodic orbits
679	E.G. Kalnins and W. Miller, Jr.	Hypergeometric expansions of Heun polynomials
680	Victor A. Pliss and George R. Sell	Perturbations of attractors of differential equations
681	Avner Friedman and Peter Knabner	A transport model with micro- and macro-structure
682	E.G. Kalnins and W. Miller, Jr.	A note on group contractions and radar ambiguity functions
683	George R. Sell	References on dynamical systems
684	Shui-Nee Chow, Kening Lu and George R. Sell	Smoothness of inertial manifolds
685	Shui-Nee Chow, Xiao-Biao Lin and Kening Lu	Smooth invariant foliations in infinite dimensional spaces
686	Kening Lu	A Hartman–Grobman theorem for scalar reaction-diffusion equations
687	Christophe Golé and Glen R. Hall	Poincaré's proof of Poincaré's last geometric theorem
688	Mario Taboada	Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations
689	Peter Rejto and Mario Taboada	Weighted resolvent estimates for Volterra operators on unbounded intervals
690	Joel D. Avrin	Some examples of temperature bounds and concentration decay for a model of solid fuel combustion
691	Susan Friedlander and Misha M. Vishik	Lax pair formulation for the Euler equation
692	H. Scott Dumas	Ergodization rates for linear flow on the torus
693	A. Eden, A.J. Milani and B. Nicolaenko	Finite dimensional exponential attractors for semilinear wave equations with damping
694	A. Eden, C. Foias, B. Nicolaenko & R. Temam	Inertial sets for dissipative evolution equations
695	A. Eden, C. Foias, B. Nicolaenko & R. Temam	Hölder continuity for the inverse of Mañé's projection
696	Michel Chipot and Charles Collins	Numerical approximations in variational problems with potential wells
697	Huanan Yang	Nonlinear wave analysis and convergence of MUSCL schemes
698	László Gerencsér and Zsuzsanna Vágó	A strong approximation theorem for estimator processes in continuous time
699	László Gerencsér	Multiple integrals with respect to L -mixing processes
700	David Kinderlehrer and Pablo Pedregal	Weak convergence of integrands and the Young measure representation
701	Bo Deng	Symbolic dynamics for chaotic systems
702	P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero	Mathematical problems for miscible, incompressible fluids with Korteweg stresses
703	Charles Collins and Mitchell Luskin	Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
704	Peter Gritzmann and Victor Klee	Computational complexity of inner and outer j -radii of polytopes in finite-dimensional normed spaces
705	A. Ronald Gallant and George Tauchen	A nonparametric approach to nonlinear time series analysis: estimation and simulation

- 706 **H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging
- 707 **M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems
- 708 **Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems
- 709 **Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films
- 710 **Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels
- 711 **Wenxiong Liu**, Blow-up behavior for semilinear heat equations: multi-dimensional case
- 712 **Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems
- 713 **Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems
- 714 **Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation
- 715 **Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling
- 716 **C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter rings
- 717 **W. Balsler, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations
- 718 **Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations
- 719 **Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II
- 720 **Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without shifts
- 721 **Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries
- 722 **C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I
- 723 **Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations
- 724 **Adelina Georgescu**, Models of asymptotic approximation
- 725 **A. Makagon and H.Salehi**, On bounded and harmonizable solutions on infinite order arma systems
- 726 **San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws
- 727 **J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure
- 728 **KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves
- 729 **J.L. Ericksen**, Local bifurcation theory for thermoelastic Bravais lattices
- 730 **Mario Taboada and Yuncheng You**, Some stability results for perturbed semilinear parabolic equations
- 731 **A.J. Lawrance**, Local and deletion influence
- 732 **Bogdan Vernescu**, Convergence results for the homogenization of flow in fractured porous media
- 733 **Xinfu Chen and Avner Friedman**, Mathematical modeling of semiconductor lasers
- 734 **Yongzhi Xu**, Scattering of acoustic wave by obstacle in stratified medium
- 735 **Songmu Zheng**, Global existence for a thermodynamically consistent model of phase field type
- 736 **Heinrich Freistühler and E. Bruce Pitman**, A numerical study of a rotationally degenerate hyperbolic system part I: the Riemann problem
- 737 **Epifanio G. Virga**, New variational problems in the statics of liquid crystals
- 738 **Yoshikazu Giga and Shun'ichi Goto**, Geometric evolution of phase-boundaries
- 739 **Ling Ma**, Large time study of finite element methods for 2D Navier–Stokes equations
- 740 **Mitchell Luskin and Ling Ma**, Analysis of the finite element approximation of microstructure in micromagnetics
- 741 **M. Chipot**, Numerical analysis of oscillations in nonconvex problems
- 742 **J. Carrillo and M. Chipot**, The dam problem with leaky boundary conditions
- 743 **Eduard Harabetian and Robert Pego**, Efficient hybrid shock capturing schemes
- 744 **B.L.J. Braaksma**, Multisummability and Stokes multipliers of linear meromorphic differential equations
- 745 **Tae Il Jeon and Tze-Chien Sun**, A central limit theorem for non-linear vector functionals of vector Gaussian processes
- 746 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
- 747 **Mario Taboada and Yuncheng You**, Invariant manifolds for retarded semilinear wave equations
- 748 **Peter Rejto and Mario Taboada**, Unique solvability of nonlinear Volterra equations in weighted spaces
- 749 **Hi Jun Choe**, Holder regularity for the gradient of solutions of certain singular parabolic equations
- 750 **Jack D. Dockery**, Existence of standing pulse solutions for an excitable activator-inhibitory system
- 751 **Jack D. Dockery and Roger Lui**, Existence of travelling wave solutions for a bistable evolutionary ecology model
- 752 **Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo**, Singular perturbation problems with a compact support semilinear term
- 753 **Emad A. Fatemi**, Numerical schemes for constrained minimization problems
- 754 **Y. Kuang and H.L. Smith**, Slowly oscillating periodic solutions of autonomous state-dependent delay equations