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Numerical Analysis of Peirce's Cantilever Test for the Bending Rigidity of Textiles

Abstract
This paper deals with the numerical analysis of a test for the bending rigidity of textiles as proposed by Peirce. The mathematical model treats textile product as an elastica which is subject to large deflections. The results of the numerical calculations discussed in this paper are presented on the relevant graphs. The optimal conditions of Peirce's test were also considered, in order to obtain the results of measurements most sensitive to changes of the input parameters. The results of the calculations are compared with the practical implementation of this test as commonly applied in textile laboratories.

Key words: *bending, textiles, bending length, bending rigidity, numerical analysis, elastica.*

Introduction

In most cases, authors who deal with the bending effect of textiles take advantage of Peirce's theory as presented in the classic work [1], which contains the theoretical fundamentals on which most of today's methods for the static measurement of bending rigidity are based.

Peirce considered the large deflection of thin beams using the basic formula which describes the pure bending theory of an elastic beam bending within the limit of linear strain. This can be solved by using the Bernoulli-Euler law given by

$$1/\rho = M/C \quad (1)$$

where:
- ρ - the radius of curvature of the deflected curve,
- M - the moment occurring at any section within the beam, and C is the bending rigidity.

Assuming a constant value of the bending rigidity C = const., we obtaain the information that the bending moment is directly proportional to the curvature

$$\kappa = 1/\rho \qquad M = C\,\kappa \quad (2)$$

Peirce proposed a simple test for describing bending rigidity, the scheme of which is shown in Figure 1. The starting point for this test is the measurement of the cantilever length l of a textile sample with one edge fixed on the platform and deformed under its own weight as a cantilever. As soon as the straight line connecting the edge of the platform and the leading edge of the fabric makes an angle of $\theta = 43°$ to the horizontal, the cantilever length l is measured.

Next, Peirce introduced in his considerations the concept of bending length D defined as

$$D = l \cdot f(\theta) \quad (3)$$

where for the sample, as in Figure 1, the function $f(\theta)$ can be assumed in the form

$$f(\theta) = \frac{1}{2}\left(\frac{\cos\frac{\theta}{2}}{\mathrm{tg}\,\theta}\right)^{\frac{1}{3}} \quad (4)$$

Hence

$$D = \frac{l}{2}\left(\frac{\cos\frac{\theta}{2}}{\mathrm{tg}\,\theta}\right)^{\frac{1}{3}} \quad (5)$$

The fabric bending rigidity C is calculated by the formula

$$C = D^3 q \quad (6)$$

where q is the fabric weight per unit area.

The choice of inclination angle of he chord θ =43° is primarily based on the ease of calculating the bending length as half of the cantilever length. For θ =43° we have

$$\cos(\theta/2)/\mathrm{tg}(\theta) \approx 0.998 \approx 1.$$

Thus $D = l/2$ and bending rigidity is defined in a simple way as

$$C = (l/2)^3\, q \quad (7)$$

Finally, the bending length is a certain measure of bending rigidity.

$$D = (C/q)^{1/3} \quad (8)$$

Numerical Analysis of Peirce's Test

In this paper, it is assumed that during the run of bending effect the flat strip of the fabric will be represented as its longitudinal section. The mathematical model will be described as a flat deflection curve; this will be treated as a heavy elastica, as shown in Figure 2. It is assumed that the particular longitudinal sections do not act on each other by internal forces (plane stress). Furthermore, the constancy of properties along the whole width of the bending strip is assumed.

Therefore, instead of studying the strip of fabric, the numerical analysis will be con-

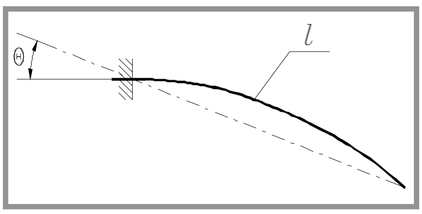

***Figure 1.** Peirce's cantilever tester.*

cerned with deflections of heavy elastica of a given bending rigidity and appropriate linear weight. Furthermore, it will be assumed that the elastica is inextensible. It should also be pointed out that the assumption of inextensibility is somewhat limiting. However, this assumption is often made in large-deflection analysis.

Each point of the centre line of elastica is defined by a curvilinear coordinate s measured along the elastica passes to the point *x(s)*, *y(s)* in a fixed coordinate system. Internal forces occurring at any cross-section within the elastica are reduced to the following components: horizontal force $F_x(s)$, vertical force $F_y(s)$ and the couple of forces *M(s)*.

The equilibrium of an infinitesimal section of elastica ds (Figure 3), the assumption of inextensibility and physical law (2) taking into consideration

$$\kappa = d\alpha/ds$$

lead to the following system of nonlinear first-order differential equations (9) which describe the elastica's bending behaviour.

$$\begin{aligned} &\frac{d\alpha}{ds} = \frac{M}{C}, \\ &\frac{dM}{ds} = F_x \sin\alpha + F_y \cos\alpha, \\ &\frac{dF_x}{ds} = 0, \quad \frac{dF_y}{ds} = q, \\ &\frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \sin\alpha. \end{aligned} \tag{9}$$

Equations (9) are the differential equations of heavy elastica in which the unknowns are the functions of variable *s*: F_x, F_y, M, α, x, y. The system of equations is completed by boundary conditions that are usually connected with the two ends of the elastica.

In the case of the fabric sample fixed as in Figure 4, the boundary conditions are as follows:

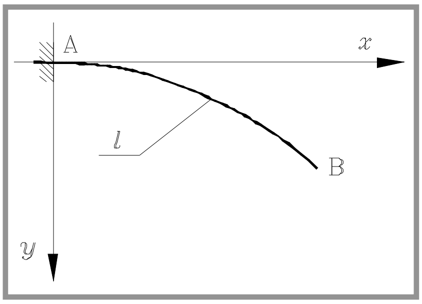

Figure 4. *Elastica in fixed coordinate system.*

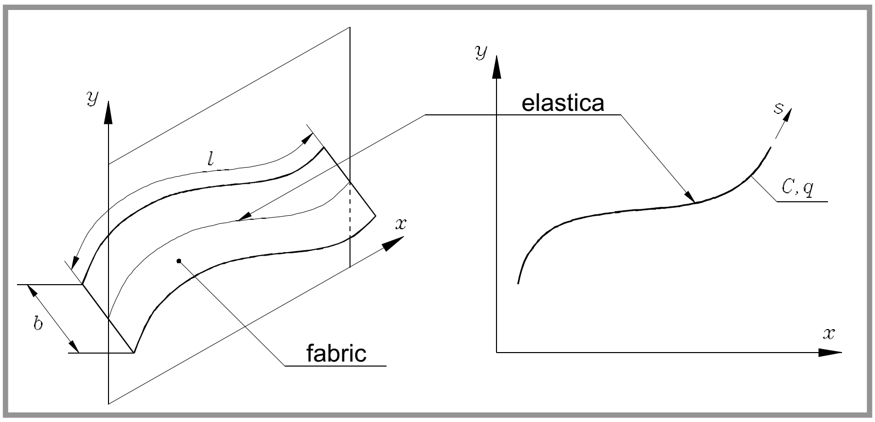

Figure 2. *The model of fabric approximate to elastica.*

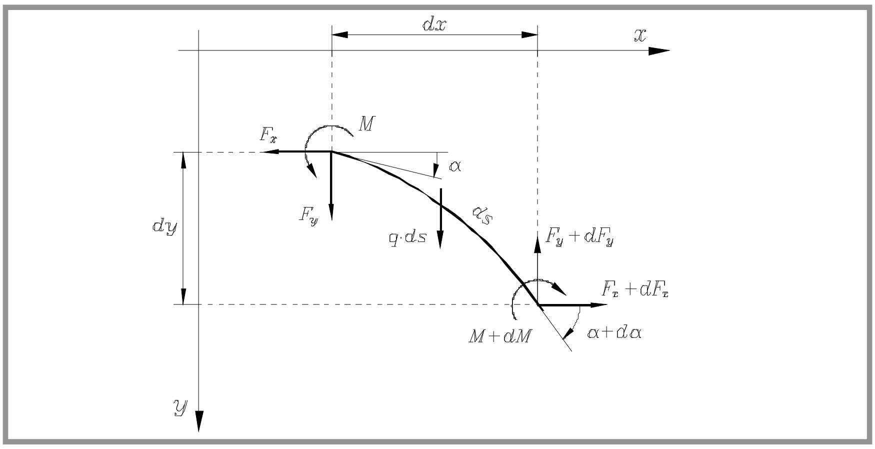

Figure 3. *Infinitesimal section of elastica showing forces acting on it.*

- end A fixed: $x^A = 0$, $y^A = 0$, $\alpha^A = 0$
- end B free: $F_x^B = 0$, $F_y^B = 0$, $M^B = 0$.

Owing to $F_x^B = 0$, $F_y^B = 0$, and $dF_x/ds = 0$, $dF_y/ds = q$, equations (9) can be expressed in simple form because the horizontal force is missing and the vertical force is described by very simple expression.

$$\begin{aligned} &F_x = 0 \\ &F_y = q(s-l) \end{aligned} \tag{10}$$

For the above-mentioned example, a system of four first-order differential equations was obtained.

$$\begin{aligned} &\frac{d\alpha}{ds} = \frac{M}{C}, \quad \frac{dM}{ds} = q(s-l)\cos\alpha \\ &\frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \sin\alpha \end{aligned} \tag{11}$$

Equations (11) and two algebraic equations (10) describe the fabric bending behaviour. The fabric, treated as an elastica, is fixed as in Peirce's cantilever test.

Before solving the differential, equations (11) were transformed to dimensionless form. For that purpose an additional parameter was defined. This parameter combines bending rigidity *C* with linear weight *q* according to formula (8). Parameter *D* is the bending length proposed by Peirce. All variables occurring in equations (11) were reduced to dimensionless form according to the formulae

$$\begin{aligned} &\bar{s} = \frac{s}{D}, \quad \bar{l} = \frac{l}{D} \\ &\overline{M} = \frac{M}{D^2 q}, \quad \overline{F}_y = \frac{F_y}{Dq} \\ &\bar{x} = \frac{x}{D}, \quad \bar{y} = \frac{y}{D} \end{aligned} \tag{12}$$

Then equations (11) become

$$\begin{aligned} &\frac{d\alpha}{d\bar{s}} = \overline{M}, \quad \frac{d\overline{M}}{d\bar{s}} = \left(\bar{s} - \bar{l}\right)\cos\alpha \\ &\frac{d\bar{x}}{d\bar{s}} = \cos\alpha, \quad \frac{d\bar{y}}{d\bar{s}} = \sin\alpha \end{aligned} \tag{13}$$

Finally, equations (13) were applied for calculation. Using parameter *D* brings about the situation that, in equations (13), it is not necessary to directly apply the values of *C* and *q*, but only their quotient.

The solution of the elastica equations with specified boundary conditions by

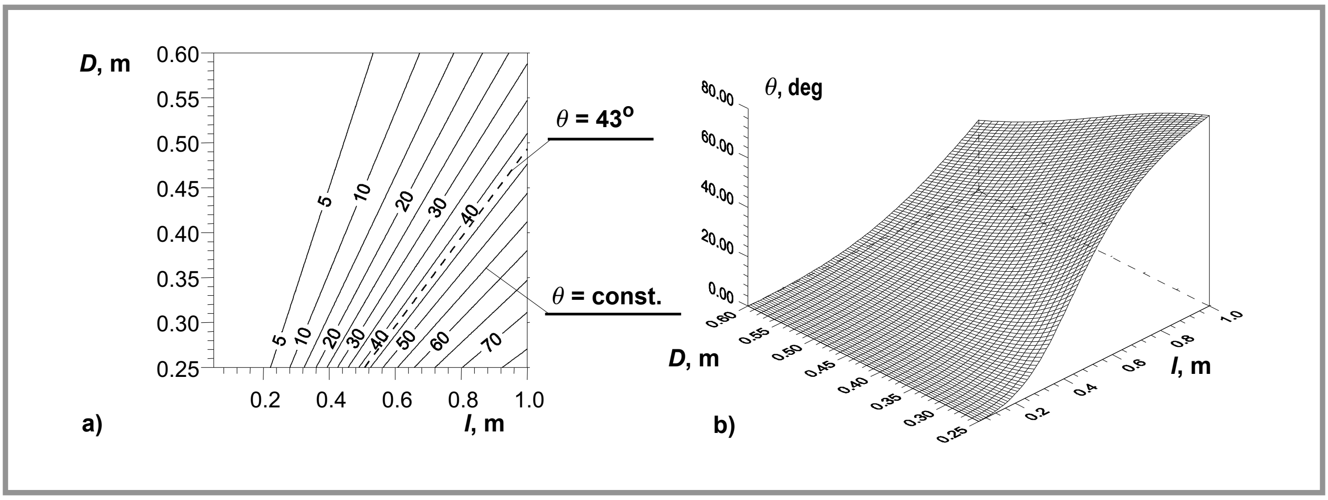

Figure 5. Graph of the angle θ (l, D) in the form of contour lines (a) and 3D-graph (b).

means of elementary functions is not generally possible. Only for weightless elastica (q=0) are the solutions given with the help of elliptic integrals. This can be found (among other things) in work [2].

For that reason, the elastica equations were solved using numerical methods. In other words, solutions are sought in the form of numerical values of unknown functions in individual points of elastica. The assumed number of points of division for elastica can affect the accuracy of the calculations.

Numerical Solution of Differential Equations of Elastica Using Shooting Method

Differential equations (11) or their dimensionless form (13) present a system of four ordinary differential equations with four unknowns which are the functions of curvilinear coordinate s. At the initial point of integration range (clamped end A) for s=0, only three values α, x, y are known, whereas the value of the bending moment M is unknown. For that reason, it is not possible to directly use the method which solves the initial value problem (for example the Runge-Kutta method).

The shooting method was used to solve the system of differential equations (13). This method is described (among other things) in work [3]. Thanks to the shooting method, we can find the missing boundary conditions at the clamped end of the elastica (for $\bar{s}$=0). In this case, the value of bending moment $\bar{M}$ is missing at the starting point.

The boundary conditions for each of the two ends of elastica are as follows.

Initial point A ($\bar{s}$=0):
known $\bar{x}^A = 0$, $\bar{y}^A = 0$, $\alpha^A = 0$
unknown $\bar{M}^A = ?$

Final point B ($\bar{s} = \bar{l}$):
known $\bar{M}^B = 0$,
unknown $\bar{x}^B = ?$, $\bar{y}^B = ?$, $\alpha^B = ?$

The unknown value of bending moment $\bar{M}^A$ at the initial point A yields a so-called initial vector for the shooting method which in this case has one component. It should be near to a possible accurate solution.

The bending moment $\bar{M}$ is the function of s and $\bar{M}^A$.

$$\bar{M} = \bar{M}(s, \bar{M}^A)$$

for $s = l$, $\bar{M} = \bar{M}(l, \bar{M}^A) = \bar{M}^B$

Additionally, in the shooting method one nonlinear equation is solved which relates to the end B.

$$\bar{M}(l, \bar{M}^A) = 0 \qquad (14)$$

To solve equation (14) the Newton-Raphson method has been applied.

Because in Peirce's test an angle of chord (Figure 1) is an essential parameter, the presentation of the results of the numerical calculations was limited only to this angle θ, which was obtained by means of the coordinates x^B, y^B of end B according to the formula

$$\theta = \text{arc tg}(y^B/x^B) \qquad (15)$$

The calculations were carried out for the following range of l and D:

$$0.05m \le l \le 1.0m$$
$$0.25m \le D \le 0.6m$$

In this way, a table of the values of angle θ for various l and D is obtained. The graph of the angle θ as a function of two variables $\theta = \theta(l, D)$ is shown in Figure 5 in the form of contour lines (a) and in the form of a 3D-graph (b). Additionally, in Figure 5a, the contour line for θ = 43° is denoted by a thick dashed line. Measurements are most often made at the angle of 43°.

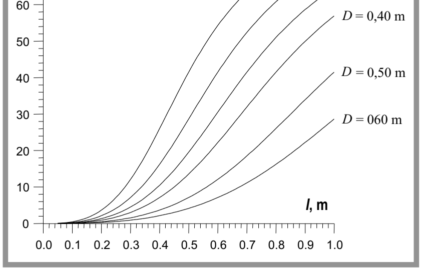

Figure 6. Graph of function θ (l).

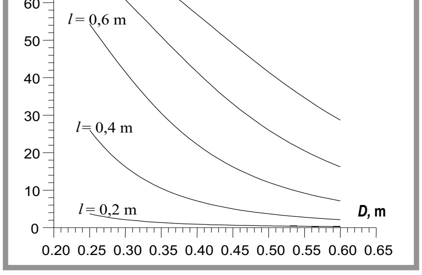

Figure 7. Graph of function θ (D).

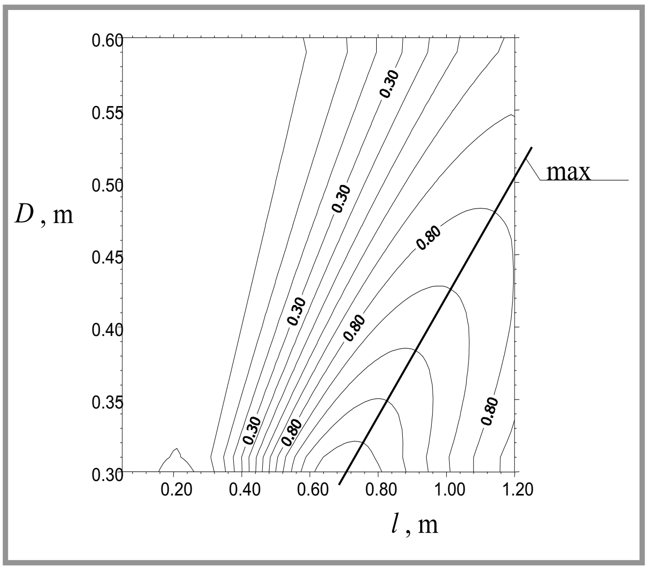

Figure 8. Graph of $\Delta\theta$ *in the form of contour lines.*

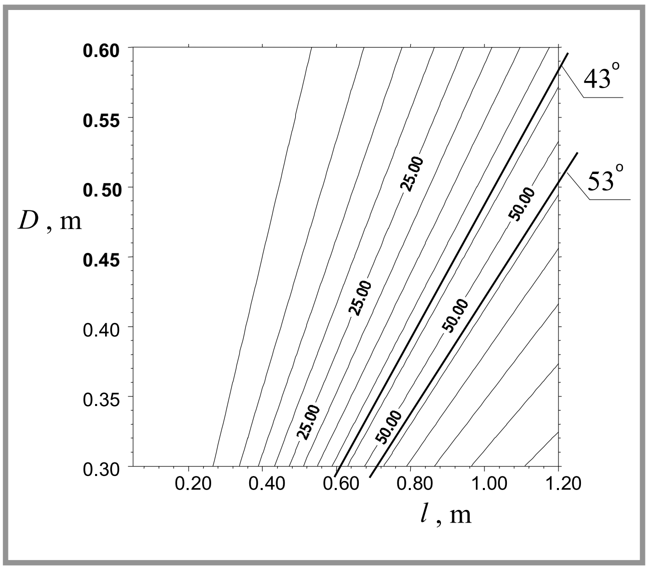

Figure 9. Graph of the angle θ *with the line of the most sensitivity* θ*=53°.*

As we can conclude from Figure 5, the bending angle of the textile sample increases along with an increase in length l or when its bending length D decreases (bending rigidity). Figure 6 shows a graph of the function $\theta(l)$ for several selected bending lengths D; Figure 7 shows a graph of the function $\theta(D)$ for several selected lengths l of samples.

No experimental test is effective and reliable when its results are sensitive to changes in the values of the input parameters. In the case of Peirce's test, this means that if the change in bending rigidity C or the length l of sample do not clearly affect bending angle θ, then the bending rigidity cannot be estimated by this method.

The change in length Δl and bending length ΔD produce a change in the bending angle $\Delta\theta$. Because the angle θ is a function of two variables, the differential $\Delta\theta$ can be numerically calculated at individual points of the integration range, as it has the values of the bending angle θ at these points.

$$\Delta\theta = \frac{\partial\theta}{\partial l}\Delta l + \frac{\partial\theta}{\partial D}\Delta D \qquad (16)$$

Of most interest is the place where the function $\Delta\theta$ achieves its maximum, because in this test the greatest possible increment of the angle θ is expected. The partial derivatives in equation (16) were calculated by the finite differences method using values of the angle θ. By fixed relative increments

$$\varepsilon_l = \Delta l / l$$
$$\varepsilon_D = \Delta D / D,$$

the increment $\Delta\theta$ is a function of the length l and bending length D. The maximum value of the function $\Delta\theta$ denotes the maximum sensitivity of the angle θ to the changes in l and D. The graph of $\Delta\theta$ for the assumed $\varepsilon_l=\varepsilon_p=0.01$ is shown in Figure 8, in which the maximum of the function $\Delta\theta$ by thick line is marked. As a result of the comparison of graphs of the angle θ and its increments $\Delta\theta$, it was found that maximum sensitivity of the angle θ occurs for the value θ=53°; this is greater by about 10 degrees than the value θ=43° proposed by Peirce. This angle is marked in Figure 9 by a thick line. At this angle, the measurement is most sensitive to changes in the parameters l and D.

Conclusions

Numerical analysis of the mathematical model of Peirce's test proved to be effective. The shooting method applied for solving the boundary value problem was sufficiently fast and stable. However, the disadvantage of this method for solving the equation of elastica is the limitation of its usage for relatively large bending rigidity. Below a certain limiting value (about 0.18 m for bending length), the shooting method may be divergent.

On the basis of numerical analysis, it was found that measurement is the most sensitive to changes in the parameters l and D for the angle θ=43°. Small changes of l and D, of the order of 1%, cause a maximum change in the increment of the bending angle $\Delta\theta$.

The findings obtained differ by about 10° from Peirce's recommendation regarding measurements by bending angle θ=43°. However, this difference is relatively small and does not call into question the implemented tests of the bending rigidity of textiles. It should also be pointed out that the choice of inclination angle of chord θ=43° by Peirce is primarily based on the ease of calculation of bending length D.

On the basis of Figures 8 and 9, it should also be noted that the increment $\Delta\theta$ rapidly decreases for small values of the angle θ. Therefore if it is necessary to change the angle of measurement 53° it is better to increase its value, because then the high sensitivity of the obtained results will be maintained.

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