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# NUMERICAL ANALYSIS OF THE GENERAL BIHARMONIC PROBLEM BY THE FINITE ELEMENT METHOD

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## INTRODUCTION

The present paper deals with solving the general biharmonic problem by the finite element method using curved triangular finite $C^1$ – elements introduced in [15]. Till now curved triangular $C^1$ – elements have been analysed only for such second and fourth order elliptic boundary value problem where boundary conditions do not imply a boundary bilinear form in the variational formulation of the problem. The effect of numerical integration in the case of the fourth order problems has been studied only in the case of the Dirichlet problem (see [3], [6], [8], [12]–[17]). In this paper the general form of the bilinear form is considered and the effect of numerical integration is analysed in the case of mixed boundary conditions (1.2)–(1.4).

In Section 1 the general nonhomogeneous biharmonic problem is described and a weak solution is defined. In Section 2 the finite element spaces are defined using curved triangular finite $C^1$ – elements from [15] and Bell's elements [2]. The discrete and completely discrete problems are formulated together with an abstract error theorem which is a modification of similar theorems (see [3], [8], [16], [17]). In the last section the effect of numerical integration and sufficient conditions for the uniform $V_{0h}$ – ellipticity are studied. The results presented are generalizations of the similar results introduced in [3], [15]–[17].

The notation in this paper is the following. Let $\Omega$ be a bounded domain in the $x, y$-plane with sufficiently smooth boundary $\Gamma$. Let $k \geqq 0$ be an integer. The symbol $W_p^{(k)}(\Omega)$ denotes the Sobolev space

$$W_p^{(k)}(\Omega) = \{v \in L_p(\Omega) : D^\alpha v \in L_p(\Omega)\ |\alpha| \leqq k\},$$

where $D^\alpha v$ is the multiindex notation for derivatives, i.e., $\alpha = (\alpha_1, \alpha_2) \in N^2$, $|\alpha| = \alpha_1 + \alpha_2$, $D^\alpha v = \partial^{|\alpha|} v / \partial x^{\alpha_1}\, \partial y^{\alpha_2}$.

The norm and seminorm are defined in $W_p^{(k)}(\Omega)$ by

$$\|v\|_{k,p,\Omega}^p = \sum_{j=0}^{k} |v|_{j,p,\Omega}^p , \quad |v|_{j,p,\Omega}^p = \sum_{|\alpha|=j} \iint_\Omega |D^\alpha v|^p \, dx \, dy$$

for $1 \leqq p \leqq \infty$, with the standard modification for $p = +\infty$. We denote $H^k(\Omega) = W_2^{(k)}(\Omega)$, $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$ and $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$.

For the sake of brevity we also use the symbol $v_x$ for $\partial v/\partial x$, $v_{xy}$ for $\partial^2 v/\partial x \, \partial y$ etc. The normal derivative $v_n$ and the tangential derivative $v_s$ are defined by

$$v_n = v_x n_x + v_y n_y , \quad v_s = v_x(-n_y) + v_y n_x ,$$

where $n_x$, $n_y$ are the direction cosines of the outward normal $n$ to $\Gamma$.

$C$ will denote a generic constant, i.e. $C$ will not denote necessarily the same constant in any two places.

## 1. SETTING OF THE BOUNDARY VALUE PROBLEM

Let $\Omega$ be a bounded, simply connected domain in the $x$, $y$-plane representing the shape of a thin plate, $\Gamma$ its boundary. Let $\Gamma$ consists of three mutually disjoint parts

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 ,$$

where each $\Gamma_i$ $(i = 1, 2, 3)$ is either empty or possesses a positive measure and does not contain isolated points.

Let us consider the following problem of bending of a thin elastic plate:

$$\Delta^2 u = f \quad \text{in} \quad \Omega , \tag{1.1}$$

$$u = g_0 , \quad u_n = g_1 \quad \text{on} \quad \Gamma_1 , \tag{1.2}$$

$$u = g_2 , \quad Mu + k_1 u_n = m_1 \quad \text{on} \quad \Gamma_2 , \tag{1.3}$$

$$Nu + k_0 u = m_0 , \quad Mu + k_1 u_n = m_1 \quad \text{on} \quad \Gamma_3 , \tag{1.4}$$

where $f$, $g_i$, $k_i \geqq 0$, $m_i$, $(i = 0, 1)$, $g_2$ are sufficiently smooth prescribed functions (their smoothness will be specified later). Further,

$$Mu = \mu \Delta u + (1 - \mu)(u_{xx} n_x^2 + 2u_{xy} n_x n_y + u_{yy} n_y^2) , \tag{1.5}$$

$$Nu = (-\Delta u)_n + (1 - \mu)[u_{xx} n_x n_y - n_{xy}(n_x^2 - n_y^2) - u_{yy} n_x n_y]_s , \tag{1.6}$$

where $\mu = \text{const}$ $(0 \leqq \mu < 1/2)$ is the Poisson ratio of the plate material.

In this paper we consider (1.2)–(1.4) under the assumption that at least one of the following six conditions holds:

$1^0$ mes $(\Gamma_1) > 0$;

$2^0$ mes $(\Gamma_2) > 0$, $\Gamma_2$ is not part of a straight line;

$3^0$ mes $(\Gamma_2) > 0$, $\Gamma_2$ is part of a straight line and there exists $\Gamma_2' \subset \Gamma_2$ such that mes $(\Gamma_2') > 0$ and $k_1 \geqq k_{1c} > 0$ on $\Gamma_2'$ $(k_{1c} = \text{const})$;

$4^0$ mes $(\Gamma_2) > 0$, $\Gamma_2$ is part of a straight line, $k_1 = 0$ on $\Gamma_2$, and there exists $\Gamma_3^* \subset \Gamma_3$ such that mes $(\Gamma_3^*) > 0$, $k_0 \geqq k_{0c} > 0$ on $\Gamma_3^*$ $(k_{0c} = \text{const})$ and $\Gamma_3^*$ is not part of the straight line containing $\Gamma_2$;

$5^0$ there exists a subset $\Gamma_3' \subset \Gamma_3$ such that mes $(\Gamma_3') > 0$, $\Gamma_3'$ is not part of a straight line and $k_0 \geqq k_{0c} > 0$ on $\Gamma_3'$ $(k_{0c} = \text{const})$;

$6^0$ there exist subsets $\Gamma_3'' \subset \Gamma_3$, $\Gamma_3^{**} \subset \Gamma_3$ such that mes $(\Gamma_3'') > 0$, mes $(\Gamma_3^{**}) > 0$, $\Gamma_3''$ is a part of a straight line, $\Gamma_3^{**}$ is not a union of segments which are perpendicular to the straight line containing $\Gamma_3''$ and $k_0 \geqq k_{0c} > 0$ on $\Gamma_3''$, $k_1 \geqq k_{1c} > 0$ on $\Gamma_3^{**}$ $(k_{0c} = \text{const}, k_{1c} = \text{const})$.

Let us define the space

(1.7)
$$V_0 = \{v \in H^2(\Omega) : v = 0 \text{ on } \Gamma_1 \cup \Gamma_2, v_n = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\}$$

and the set

(1.8)
$$V_g = \{v \in H^2(\Omega) : v = g_0, v_n = g_1 \text{ on } \Gamma_1, v = g_2 \text{ on } \Gamma_2 \text{ in the sense of traces}\}.$$

The variational formulation of the problem (1.1)–(1.4) can be written as follows: Find $u \in V_g$ such that

(1.9)
$$a(u, v) = l(v) \quad \forall v \in V_0 .$$

with

(1.10)
$$a(u, v) = a^\Omega(u, v) + a^\Gamma(u, v),$$

(1.11)
$$a^\Omega(u, v) = \sum_{|\alpha|,|\beta|=2} \iint_\Omega a_{\alpha\beta} D^\alpha u D^\beta v \, dx \, dy,$$

where $a_{\alpha\beta}$ are constant coefficients (for their definition see [10, p. 365]),

(1.12)
$$a^\Gamma(u, v) = \int_{\Gamma_2} k_1 u_n v_n \, ds + \int_{\Gamma_3} (k_0 uv + k_1 u_n v_n) \, ds$$

and

(1.13)
$$l(v) = l^\Omega(v) + l^\Gamma(v),$$

(1.14)
$$l^\Omega(v) = \iint_\Omega fv \, dx \, dy,$$

(1.15)
$$l^\Gamma(v) = \int_{\Gamma_2} m_1 v_n \, ds + \int_{\Gamma_3} (m_0 v + m_1 v_n) \, ds .$$

It follows from [4, Lemma 3.1] that problem (1.9) has a unique solution. We shall solve problem (1.9) by the finite element method.

## 2. SETTING OF THE DISCRETE PROBLEMS

Let $\Omega_h$ be a finite element approximation of the domain $\Omega$ generated by a triangulation $\tau_h$ (for details see [15], [17]). The symbols $\Gamma_h$ and $\Gamma_{hi}$ $(i = 1, 2, 3)$ will denote the boundary of $\Omega_h$ and the parts of $\Gamma_h$ approximating $\Gamma_i$, respectively.

With every triangulation $\tau_h$ we associate three parameters $h$, $\bar{h}$ and $\vartheta$ defined by

$$h = \max_{T \in \tau_h} h_T, \quad \bar{h} = \min_{T \in \tau_h} h_T, \quad \vartheta = \min_{T \in \tau_h} \vartheta_T, \tag{2.1}$$

where $h_T$ and $\vartheta_T$ are the length of the greatest side and the smallest angle, respectively, of the triangle with straight sides which has the same vertices as the triangle $T$. We restrict ourselves to triangulations $\tau_h$ satisfying

$$\vartheta \geqq \vartheta_0, \quad \vartheta_0 = \text{const} > 0, \tag{2.2}$$

$$\bar{h} \geqq C_0 h, \quad C_0 = \text{const} > 0. \tag{2.3}$$

Let $W_h^2$ denote the finite element subspace of the space $C^1(\Omega_h)$ consisting of functions which we obtain by piecing together curved triangular finite $C^1$-elements [15] with Bell's elements [2].

Let $V_{0h}$ be a subspace of $W_2$ defined by

$$V_{0h} = \{v \in W_h^2 : v = 0 \quad \text{on} \quad \Gamma_{h1} \cup \Gamma_{h2}, v_{n_h} = 0 \text{ on } \Gamma_{h1}\}, \tag{2.4}$$

where $n_h$ is the outward normal to $\Gamma_h$.

Let $V_{gh}$ be the subset of $W_h^2$ consisting of those functions which at the nodal points lying on $\Gamma_h$ satisfy the boundary conditions (1.2) and the stable boundary condition (1.3) and all consequences of those conditions containing at most second derivatives (for details see [5], [16], [17]).

Let $n_i$ $(i = 1, 2, 3)$ be the degrees of curved sides $c_h$ (of the curved triangles) which $\Gamma_h$ consists of. If we set

$$n_1 = 3, \quad n_2 = 5, \quad n_3 = 3. \tag{2.5}$$

then (see [16], [17])

$$v, w \in V_{gh} \Rightarrow v - w \in V_{0h}. \tag{2.6}$$

Now we can formulate the discrete problem corresponding to problem (1.9): Find $u_h \in V_{gh}$ such that

$$\tilde{a}_h(u_h, v) = \tilde{l}_h(v) \quad \forall v \in V_{0h}, \tag{2.7}$$

where the bilinear form $\tilde{a}_h(v, w)$ is defined by

$$\tilde{a}_h(v, w) = \tilde{a}_h^{\Omega}(v, w) + \tilde{a}_h^{\Gamma}(v, w), \tag{2.8}$$

$$\tilde{a}_h^{\Omega}(v, w) = \sum_{|\alpha|, |\beta| = 2} \iint_{\Omega_h} a_{\alpha\beta} D^\alpha v D^\beta w \, dx \, dy, \tag{2.9}$$

(2.10) $$\tilde{a}_h^\Gamma(v, w) = \int_{\Gamma_{h2}} k_{1h} v_{n_h} w_{n_h} \, ds + \int_{\Gamma_{h3}} (k_{0h} v w + k_{1h} v_{n_2} w_{n_2}) \, ds$$

and the linear form $\tilde{l}_h(w)$ is defined by

(2.11) $$\tilde{l}_h(w) = \tilde{l}_h^\Omega(w) + \tilde{l}_h^\Gamma(w),$$

(2.12) $$\tilde{l}_h^\Omega(w) = \iint_{\Omega_h} \tilde{f} w \, dx \, dy,$$

(2.13) $$\tilde{l}_h^\Gamma(w) = \int_{\Gamma_{h2}} m_{1h} w_{n_h} \, ds + \int_{\Gamma_{h3}} (m_{0h} w + m_{1h} w_{n_h}) \, ds.$$

The symbol $\tilde{f}$ in (2.12) denotes a continuous extension of the function $f$ onto a domain $\tilde{\Omega} \supset \tilde{\Omega}_h$ $(h < \tilde{h})$ and will be specified in (2.21); $k_{ih}$, $m_{ih}$ $(i = 0, 1)$ are functions obtained by "transferring" the functions $k_i$, $m_i$ from $\Gamma$ onto $\Gamma_h$ (the definition of transferring functions from $\Gamma$ onto $\Gamma_h$ is the same as in [16], thus we refer to [16, p. 130]).

Using quadrature formulae with integrating points lying in $\bar{\Omega}$ we replace the forms $\tilde{a}_h^\Omega(v, w)$ and $\tilde{l}_h^\Omega(w)$ by the forms $a_h^\Omega(v, w)$ and $l_h^\Omega(w)$, respectively (for their definition see Section 3). Further, computing numerically the integrals on the right-hand sides of (2.10) and (2.13) for each $c_h \subset \Gamma_{h2} \cup \Gamma_{h3}$ we obtain the forms $a_h^\Gamma(v, w)$ and $l_h^\Gamma(v)$, respectively (see Section 3).

Now we solve the following problem instead of problem (2.7): Find $u_h \in V_{gh}$ such that

(2.14) $$a_h(u_h, v) = l_h(v) \quad \forall v \in V_{0h},$$

where the forms $a_h(v, w)$ and $l_h(w)$ are defined by

(2.15) $$a_h(v, w) = a_h^\Omega(v, w) + a_h^\Gamma(v, w)$$

and

(2.16) $$l_h(w) = l_h^\Omega(w) + l_h^\Gamma(w).$$

**Theorem 2.1.** *Let a family of discrete problems (2.14) be given. Let us assume that there exists a constant $\gamma > 0$ independent of $h$, satisfying for $h < \tilde{h}$ the inequality*

(2.17) $$\gamma \|v\|_{2,\Omega_h}^2 \leqq a_h(v, v) \quad \forall v \in V_{0h}.$$

*Then for $h < \tilde{h}$ problem (2.14) has a unique solution $u_h$. If $\tilde{u} \in H^4(\tilde{\Omega})$ is a continuous extension of the solution $u$ of (1.1)–(1.4) onto a domain $\tilde{\Omega} \supset \Omega_h$ then for $h < \tilde{h}$ we have*

(2.18) $$\|\tilde{u} - u_h\|_{2,\Omega_h} \leqq C \left\{ \inf_{v \in V_{gh}} \left[ \|\tilde{u} - v\|_{2,\Omega_h} + \sup_{w \in V_{0h}} \frac{|\tilde{a}_h(v, w) - a_h(v, w)|}{\|w\|_{2,\Omega_h}} \right] + \right.$$
$$\left. + \sup_{w \in V_{0h}} \frac{|\tilde{l}_h(w) - l_h(w)|}{\|w\|_{2,\Omega_h}} + \sup_{w \in V_{0h}} \frac{|l^\Gamma(w) - \tilde{l}_h^\Gamma(w)|}{\|w\|_{2,\Omega_h}} \right\},$$

*where $C$ is a constant independent of both $\tilde{u}$ and $h$, and $\tilde{l}^{\Gamma}(w)$ is defined by* (2.23).

Proof. Assumption (2.17) implies that for $h < \bar{h}$ every problem (2.14) has a unique solution $u_h$. Using (2.6) and (2.17) we obtain similarly as in the proof of [16, Theorem 1]

$$(2.19)\quad \|\tilde{u} - u_h\|_{2,\Omega_h} \leqq C\left\{\inf_{v \in V_{gh}}\left[\|\tilde{u} - v\|_{2,\Omega_h} + \sup_{w \in V_{0h}} \frac{|\tilde{a}_h(v, w) - a_h(v, w)|}{\|w\|_{2,\Omega_h}}\right] + \right.$$
$$\left. + \sup_{w \in V_{0h}} \frac{|\tilde{a}_h(\tilde{u}, w) - l_h(w)|}{\|w\|_{2,\Omega_h}}\right\},$$

$C$ being a constant independent of $\tilde{u}$ and $h$.

Using the identity [10, (23, 22)] we obtain the following identity for $\tilde{u} \in H^4(\tilde{\Omega})$ and $w \in V_{0h}$:

$$(2.20)\quad \tilde{a}_h^{\Omega}(\tilde{u}, w) = \iint_{\Omega_h} w\Delta^2\tilde{u}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_{h2}} w_{n_h} M_h\tilde{u}\,\mathrm{d}s + \int_{\Gamma_{h3}} (wN_h\tilde{u} + w_{n_h}M_h\tilde{u})\,\mathrm{d}s\,,$$

where the operators $M_h$ and $N_h$ are defined by relations similar to (1.5), (1.6). The only difference consists in replacing $n$, $n_x$, $n_y$ and $s$ by $n_h$, $n_{hx}$, $n_{hy}$ and $s_h$, respectively. Setting

$$(2.21)\quad \tilde{f} = \Delta^2\tilde{u}\,,$$

adding $\tilde{a}_h^{\Gamma}(u, w)$ to both sides of (2.20) and using (2.12) (2.13) we can easily obtain for $\tilde{u} \in H^4(\tilde{\Omega})$ the identity

$$(2.22)\quad \tilde{a}_h(\tilde{u}, w) = \tilde{l}_h(w) + \tilde{l}^{\Gamma}(w) - \tilde{l}_h^{\Gamma}(w)\quad \forall w \in V_{0h}\,,$$

where

$$(2.23)\quad \tilde{l}^{\Gamma}(w) = \int_{\Gamma h_2} \tilde{m}_{1h} w_{n_h}\,\mathrm{d}s + \int_{\Gamma_{h3}} (\tilde{m}_{0h} w + \tilde{m}_{1h} w_{n_h})\,\mathrm{d}s\,,$$

$$(2.24)\quad \tilde{m}_{0h} = N_h\tilde{u} + k_{0h}\,\tilde{u},\quad \tilde{m}_{1h} = M_h\tilde{u} + k_{1h}\tilde{u}_{n_h}\,.$$

Inserting (2.22) into (2.19) we get (2.18). Theorem 2.1 is proved.

There are three sources of errors in solving (1.1)–(1.4) by the finite element method:

i) the error of interpolation (the first term on the right-hand side of (2.18)). In estimating the interpolation error we shall use [15, Theorem 5] and a similar theorem for Bell's elements [2, p. 819]. In accordance with the assumptions of these theorems we shall assume that $\tilde{u} \in H^5(\tilde{\Omega})$. Let $v_I$ be the function from $W_h^2$ which interpolates $\tilde{u}$ (i.e. the parameters uniquely determining $v_I$ are the function values and derivatives of $\tilde{u}$ at the corresponding nodal points). Then $v_I \in V_{gh}$ and we have

$$(2.25)\quad \inf_{v \in V_{gh}} \|\tilde{u} - v\|_{2,\Omega_h} \leqq \|\tilde{u} - v_I\|_{2,\Omega_h} \leqq Ch^3\|\tilde{u}\|_{2,\Omega_h}\,;$$

ii) the error of approximation of the boundary (the last term on the right-hand side of (2.18)). In estimating the boundary approximation error it suffices to modify slightly the proofs of inequalities [17, (109) and (110)]. We obtain

$$|\bar{l}^{\Gamma}(w) - \bar{l}_h^{\Gamma}(w)| \leqq Ch^3 \|w\|_{2,\Omega_h} \quad \forall w \in V_{0h}\,; \tag{2.26}$$

iii) the error of numerical integration (the second and third terms on the right-hand side of (2.18)). It will be analysed in Section 3.

## 3. THE EFFECT OF NUMERICAL INTEGRATION

Before studying the effect of numerical integration we introduce some lemmas.

**Lemma 3.1.** (see [3, (4.1.42)]). *Let $D$ be an open bounded subset of $E_N$. Let $\varphi \in W_q^{(k)}(D)$ $(1 \leqq q \leqq \infty)$, $w \in W_\infty^{(k)}(D)$. Then the function $\varphi w$ belongs to the space $W_q^{(k)}(D)$ and*

$$|\varphi w|_{k,q,D} \leqq C \sum_{j=0}^{k} |\varphi|_{k-j,q,D}\, |w|_{j,\infty,D}\,, \tag{3.1}$$

*where $C$ is a constant depending only on the integers $k$ and $N$.*

**Lemma 3.2.** (see [14]). *Let $D$ be an open bounded subset of $E_N$. Let $k$ be a given integer. There exists a constant $C$ independent of $p \in P_N(k)$ such that*

$$|p|_{j,D} \leqq C|p|_{i,D} \quad 0 \leqq i \leqq j \quad \forall p \in P_N(k)\,, \tag{3.2}$$

$$|p|_{j,\infty,D} \leqq C|p|_{j,D} \qquad j \geqq 0 \quad \forall p \in P_N(k)\,, \tag{3.3)}$$

*$P_N(k)$ being the space of all polynomials in $N$ variables of degree not greater than $k$.*

**Lemma 3.3.** *Let $D$ be an open bounded subset of $E_N$ with a Lipschitz-continuous boundary. Let $k$, $m$ be given integers and $u \in H^{k+m+1}(D)$. Let $\Pi$ be the orthogonal projection in the space $H^k(D)$ onto the subspace $P_N(k)$, i.e.*

$$(u - \Pi u, p)_{k,D} = 0 \quad \forall p \in P_N(k)\,. \tag{3.4}$$

*Then there exists a constant $C$ such that*

$$|u - \Pi u|_{i,D} \leqq C \sum_{j=k+1}^{k+1+m} |u|_{j,D}\,, \quad 0 \leqq i \leqq k + 1 + m\,. \tag{3.5}$$

Proof. For a given $v \in H^{k+m+1}(D)$ we define the linear functional

$$f(u) = (u - \Pi u, v)_{k+m+1,D} \quad \forall u \in H^{k+m+1}(D)\,. \tag{3.6}$$

As $\|\Pi u\|_{k,D} \leqq \|u\|_{k,D}$, the linear functional (3.6) is continuous with the norm less than or equal to $2\|v\|_{k+m+1,D}$ on the one hand, and is vanishing on $P_N(k)$ on the other hand. Therefore, using the Bramble - Hilbert lemma in the form introduced in [7] we obtain

$$|f(u)| \leqq C\|v\|_{k+m+1,D} \sum_{j=k+1}^{k+m+1} |u|_{j,D}, \tag{3.7}$$

where $C$ is a constant independent of $u$, $v$.

Choosing $v = u - \Pi u$ we get from (3.6) and (3.7) the inequality

$$\|u - \Pi u\|_{k+m+1,D} \leqq C \sum_{j=k+1}^{k+m+1} |u|_{j,D}$$

from which (3.5) follows. Lemma 3.3 is proved.

First we shall analyse the effect of numerical integration in the domain $\Omega_h$. The theory is a generalization of Ženíšek's results [15]. Let us have a numerical quadrature scheme over the unit triangle $T_0$

$$\iint_{T_0} F^*(\xi, \eta)\, d\xi\, d\eta \sim \sum_{i=1}^{I} \omega_i^* F^*(B_i^*), \tag{3.8}$$

where $\omega_i^*$ are the coefficients and $B_i^*$ the integration points of the formula. (We can use conical product formulae which are known for arbitrary degree of precision — see [11]). Using the theorem on the transformation of multiple integrals and the transformation [15, (23)] of a curved triangle $T$ onto the unit triangle $T_0$ we obtain in the same way as in [15].

$$\iint_T F(x, y)\, dx\, dy \sim \sum_{i=1}^{I_T} \omega_{i,T} F(B_{i,T}), \tag{3.9}$$

where $\omega_{i,T}$ and $B_{i,T}$ are defined by [15, (125)].

Through the paper we assume that for $h < \tilde{h}$ the integration points $B_i$ lie in the set $\bar{\Omega}$. Then

$$\tilde{f}(B_i) = f(B_i). \tag{3.10}$$

Let us approximate the bilinear form (2.9) and the linear form (2.12) by means of (3.9), i.e. let us define the forms

$$a_h^{\Omega}(v, w) = \sum_{T \in \tau_h} \sum_{i=1}^{I_T} \omega_{i,T} \sum_{|\alpha|,|\beta|=2} (a_{\alpha\beta} D^\alpha v D^\beta w)(B_{i,T}), \tag{3.11}$$

$$l_h^{\Omega}(w) = \sum_{T \in \tau_h} \sum_{i=1}^{I_T} \omega_{i,T}(fw)(B_{i,T}). \tag{3.12}$$

With respect to (3.10) we write $f$ instead of $\tilde{f}$ in (3.12).

Let us define the error functionals by

$$E_T(F) = \iint_T F(x, y)\,\mathrm{d}x\,\mathrm{d}y - \sum_{i=1}^{I_T} \omega_{i,T}\, F(B_{i,T}), \tag{3.13}$$

$$E^*(F^*) = \iint_{T_0} F^*(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta - \sum_{i=1}^{I_T} \omega_i^*\, F^*(B_i^*). \tag{3.14}$$

In the same way as [15, (150)] was derived we obtain

$$E_T\Big(\sum_{|\alpha|,|\beta|=2} a_{\alpha\beta} D^\alpha v D^\beta w\Big) = E^*\Big(\sum_{\substack{|\alpha|,|\beta|=2\\ 1\le|\gamma|,|\delta|\le 2}} a_{\alpha\beta} b_{\alpha\gamma}^* b_{\beta\delta}^* J^* D^\gamma v^* D^\delta w^*\Big), \tag{3.15}$$

where $J^*$ is the Jacobian of the transformation [15, (23)] and where the coefficients $b_{\alpha\gamma}^*$ have the property (see [15, Lemma 3]),

$$D^\mu(b_{\alpha\gamma}^* b_{\beta\delta}^* J^*) = O(h_T^{2-|\gamma|-|\delta|+|\mu|}). \tag{3.16}$$

The proof of the following two theorems is a generalization of the proof of [15, Th. 7, Th. 8] for the case of $C^1$-elements. In [15] the prescribed homogeneous Dirichlet boundary conditions are utilized in the proof; here we consider more general boundary conditions (see (1.2)–(1.4)), thus the proofs must be modified.

**Theorem 3.1.** *Let*

$$E^*(\varphi^*) = 0 \quad \forall \varphi^* \in P_2(2N^* - 4), \tag{3.17}$$

*where $N^* = 4 + n$ for curved $C^1$-elements and $N^* = 5$ for interior elements and $n$ is the degree of the curved side $c_h$ of the boundary triangle, i.e. $n = n_i$ if $c_h \subset \Gamma_{hi}$. Then*

$$\Big|E_T\Big(\sum_{|\alpha|,|\beta|=2} a_{\alpha\beta} D^\alpha v D^\beta w\Big)\Big| \leqq C h_T \|v\|_{2,T}\, \|w\|_{2,T}, \tag{3.18}$$

*where $C$ is a constant independent of $h_T$, $v$ and $w$.*

**Theorem 3.2.** *Let $r \geqq 2$ be a given integer and let*

$$E^*(\psi^*) = 0 \quad \forall \psi^* \in P_2(r + N^* - 3), \tag{3.19}$$

*where $N^*$ is the same as in Theorem 3.1. Then*

$$\Big|E_T\Big(\sum_{|\alpha|,|\beta|=2} a_{\alpha\beta} D^\alpha v D^\beta w\Big)\Big| \leqq C h_T^r \|v\|_{r+2,T}\, \|w\|_{2,T}, \tag{3.20}$$

where $C$ is a constant independent of $h_T$, $v$ and $w$.

Proof of Theorems 3.1 and 3.2. A typical term on the right-hand side of (3.15) is of the form

$$E^*(c^* D^\gamma v^* D^\delta w^*) \tag{3.21}$$

with $|\gamma| = |\delta| = 2$ for the interior elements and with $1 \leqq |\gamma|, |\delta| \leqq 2$ for the boundary elements. The function $c^*$ is given by

$$(3.22) \qquad c^* = a_{\alpha\beta} b^*_{\alpha\gamma} b^*_{\beta\delta} J^* .$$

Since $v^*, w^* \in P_2(N^*)$ we have $D^\gamma v^* \in P_2(N^* - |\gamma|)$, $D^\delta w^* \in P_2(N^* - |\delta|)$. Now we distinguish two cases:

1) Estimating (3.21) for the interior elements and the boundary elements with $1 \leqq |\gamma| \leqq 2$ and $|\delta| = 2$ is the same as in [15, pp. 370–372]. Thus we omit it.

2) We shall estimate (2.21) for the boundary elements with $1 \leqq |\gamma| \leqq 2$ and $|\delta| = 1$. In the proofs we modify the method of orthogonal projections introduced in [3]. Let us consider the form

$$E^*(\varphi^* u^*) \quad \forall \varphi^* \in W_\infty^{(s)}(T_0), \quad \forall u^* \in P_2(N^* - |\delta|),$$

where in the case of Theorem 3.1

$$(3.23) \qquad s = N^* + |\delta| - 3$$

and in the case of Theorem 3.2

$$(3.24) \qquad s = r + |\delta| - 2 .$$

Let $u^* \in P_2(N^* - |\delta|)$ and let $\Pi$ be the orthogonal projection in the space $L_2(T_0)$ onto the subspace $P_2(0)$, i.e.

$$(u^* - \Pi u^*, p)_{0,T_0} = 0 \quad \forall p \in P_2(0) .$$

The following identity holds (according to (3.14))

$$(3.25) \qquad E^*(\varphi^* u^*) = E^*(\varphi^*(u^* - \Pi u^*)) + E^*(\varphi^* \Pi u^*) .$$

Now we consider the form

$$(3.26) \qquad E^*(\varphi^*(u^* - \Pi u^*)) \quad \forall \varphi^* \in W_\infty^{(s)}(T_0), \quad \forall u^* \in P_2(N^* - |\delta|) .$$

We have, according to (3.14) and (3.26),

$$(3.27) \qquad |E^*(\varphi^*(u^* - \Pi u^*))| \leqq C|\varphi^*|_{0,\infty,T_0} |u^* - \Pi u^*|_{0,\infty,T_0} .$$

Using (3.3) and Lemma 3.3 we obtain the inequality

$$|u^* - \Pi u^*|_{0,\infty,T_0} \leqq |u^* - \Pi u^*|_{0,T_0} \leqq C|u^*|_{1,T_0} .$$

Thus we get from (3.27)

$$|E^*(\varphi^*(u^* - \Pi u^*))| \leqq C|\varphi^*|_{s,\infty,T_0} |u^*|_{1,T_0} .$$

From here and from (3.17), (3.19), (3.23), (3.24) we obtain by means of the Bramble-Hilbert lemma (see [1] or [2])

$$(3.28)\qquad \left|E^*(\varphi^*(u^* - \Pi u^*))\right| \leqq C\left|\varphi^*\right|_{s,\infty,T_0}\left|u^*\right|_{1,T_0} \quad \forall \varphi^* \in W_\infty^{(s)}(T_0)\,, \quad \forall u^* \in P_2(N^* - |\delta|)\,.$$

Let us consider the form

$$(3.29)\qquad E^*(\varphi^* \Pi u^*) \quad \forall \varphi^* \in W_\infty^{(s+|\delta|)}(T_0)\,, \quad \forall u^* \in P_2(0)\,.$$

Using the inequalities $\left|\Pi u^*\right|_{0,T_0} \leqq \left|u^*\right|_{0,T_0}$, $\left|\varphi^*\right|_{0,\infty,T_0} \leqq \left\|\varphi^*\right\|_{s+|\delta|,\infty,T_0}$ and $2|\delta| < N^*$ we obtain in a similar way as above that

$$(3.30)\qquad \left|E^*(\varphi^* \Pi u^*)\right| \leqq C\left|\varphi^*\right|_{s+|\delta|,\infty,T_0}\left|u^*\right|_{0,T_0} \quad \forall \varphi^* \in W_\infty^{(s+|\delta|)}(T_0)\,, \quad \forall u^* \in P_2(N^* - |\delta|)\,.$$

According to (3.16), (3.22) we have

$$(3.31)\qquad \left|c^*\right|_{k,\infty,T_0} \leqq C h_T^{2-|\gamma|-|\delta|+k}\,.$$

Let us set $\varphi^* = c^* D^\gamma v^*$. Using Lemmas 3.1 and 3.2 and (3.31) we obtain

$$(3.32)\qquad \left|\varphi^*\right|_{k,\infty,T_0} \leqq C h_T^{2-|\gamma|-|\delta|+k} \sum_{j=0}^{k} h_T^{-j}\left|v^*\right|_{j+|\gamma|,T_0}\,.$$

Let us set $u^* = D^\delta w^*$. Inserting (3.32) into (3.28) and (3.30) we obtain, according to (3.25),

$$(3.33)\qquad \left|E^*(c^* D^\gamma v^* D^\delta w^*)\right| \leqq C\{h_T^{2-|\gamma|-|\delta|+s} \sum_{j=0}^{s} h_T^{-j}\left|v^*\right|_{j+|\gamma|,T_0}\left|w^*\right|_{1+|\delta|,T_0} + h_T^{2-|\gamma|+s} \sum_{j=0}^{s+|\delta|} h_T^{-j}\left|v^*\right|_{j+|\gamma|,T_0}\left|w^*\right|_{|\delta|,T_0}\}\,.$$

The theorem on transformation of multiple integrals and [15, Theorem 2] imply the estimate

$$(3.34)\qquad \left|w^*\right|_{k,T_0} \leqq C h_T^{k-1}\left\|w\right\|_{k,T}\,.$$

Using (3.34) we obtain from (3.33) (because $|\delta| = 1$)

$$(3.35)\qquad \left|E^*(c^* D^\gamma v^* D^\delta w^*)\right| \leqq C\{h_T^{2-|\gamma|+s} \sum_{j=0}^{s} h_T^{-j}\left|v^*\right|_{j+|\gamma|,T_0} + h_T^{1-|\gamma|+s+|\delta|} \sum_{j=0}^{s+|\delta|} h_T^{-j}\left|v^*\right|_{j+|\gamma|,T_0}\}\left\|w\right\|_{2,T}\,.$$

First we prove Theorem 3.1. By using (3.2) and (3.34) it is easy to see that the first term in brackets on the right-hand side of (3.35) can be estimated by $h_T\|v\|_{2,T}$. As to the second term we shall distinguish two cases:

i) If $|\gamma| = 1$ then using (3.34) and (3.2) we obtain for sufficiently small $h$

$$\sum_{j=0}^{s+|\delta|} h_T^{-j} |v^*|_{j+|\gamma|,T_0} \leqq C h_T^{1-s-|\delta|} \|v\|_{2,T} . \tag{3.36}$$

ii) If $|\gamma| = 2$ then $|v^*|_{s+|\delta|+|\gamma|,T_0} = |v^*|_{N^*+1,T_0} = 0$, according to (3.23), and we obtain

$$\sum_{j=0}^{s+|\delta|} h_T^{-j} |v^*|_{j+|\gamma|,T_0} \leqq C h_T^{1-s} \|v\|_{2,T} . \tag{3.37}$$

The inequalities (3.35)–(3.37) imply (3.18). Theorem 3.1 is proved.

Now we prove Theorem 3.2. According to (3.34) we have

$$|v^*|_{j+|\gamma|,T_0} \leqq C h_T^{j+|\gamma|-1} \|v\|_{j+2,T} . \tag{3.38}$$

Inserting (3.38) into (3.35) and using (3.24) we obtain (3.20).

Remark. It is not difficult to generalize the proof of Theorems 3.1 and 3.2 for $C^m$-elements. We should define the orthogonal projections $\Pi$ in the spaces $H^{|\delta|-1}(T_0)$ onto the subspaces $P_2(|\delta| - 1)$ and we should prove (3.18), (3.20) in a similar way as above for $1 \leqq |\delta| \leqq m$.

With respect to (2.12) and (3.12) we derive

$$\tilde{l}_h^{\Omega}(w) - l_h^{\Omega}(w) = \sum_{T \in \tau_h} E_T(\tilde{f} w) . \tag{3.39}$$

The relation [15, (144)], i.e. $E_T(F) = E^*(F^* J^*)$, implies

$$E_T(\tilde{f} w) = E^*(\tilde{f}^* w^* J^*)$$

with $\tilde{f}^*(\xi, \eta) = \tilde{f}(x^*(\xi, \eta), y^*(\xi, \eta))$, where

$$x = x^*(\xi, \eta) , \quad y = y^*(\xi, \eta) \tag{3.40}$$

is the transformation [15, (23)].

**Theorem 3.3.** *Let the assumptions of Theorem 3.2. be satisfied and $\tilde{f} \in H^r(\tilde{\Omega})$. Then*

$$|\tilde{l}_h^{\Omega}(w) - l_h^{\Omega}(w)| \leqq C h^r \|\tilde{f}\|_{r,\tilde{\Omega}} \|w\|_{2,\Omega_h} \tag{3.41}$$

*where $C$ is a constant independent of $h$, $\tilde{f}$ and $w$.*

The proof of Theorem 3.3 is similar to that of [3, Theorem 4.1.5] or [5, Věta 6.3]; therefore it is omitted.

Now we shall analyze the effect of numerical integration on the boundary $\Gamma_h$. Let us have a numerical quadrature scheme over the segment $I = [0, 1]$

$$\int_0^1 G^*(t)\, dt \sim \sum_{j=1}^{J} \omega_j G^*(t_j) , \tag{3.42}$$

where $\omega_j$ are the coefficients and $t_j$ are the integration points of the formula. According to the definition of the line integral we have

$$(3.43)\qquad \int_{c_h} G(x, y)\,\mathrm{d}s = \int_0^1 G(\varphi^*(t), \psi^*(t))\,\varrho^*(t)\,\mathrm{d}t = \int_0^1 G^*(t)\,\varrho^*(t)\,\mathrm{d}t\,,$$

where

$$(3.44)\qquad \varphi^*(t) = x^*(1 - t, t)\,, \quad \psi^*(t) = y^*(1 - t, t)\,,$$

$x^*(\xi, \eta)$, $y^*(\xi, \eta)$ being the functions from (3.40), and

$$(3.45)\qquad x = \varphi^*(t)\,, \quad y = \psi^*(t)\,, \quad t \in [0, 1]$$

are the parametric equations of the curved side $c_h$ of the boundary triangle $T^*$. The function $\varrho^*(t)$ is defined by

$$(3.46)\qquad \varrho^*(t) = ([\varphi^{*\prime}(t)]^2 + [\psi^{*\prime}(t)]^2)^{1/2}\,.$$

The relations (3.42) and (3.43) imply

$$(3.47)\qquad \int_{c_h} G(x, y)\,\mathrm{d}s \sim \sum_{j=1}^{J} \omega_{j,c_h}\, G(B_{j,c_h})$$

with $\omega_{j,c_h} = \omega_j\, \varrho^*(t_j)$, $B_{j,c_h} = (\varphi^*(t_j), \psi^*(t_j))$.

Let us approximate the bilinear form (2.10) and the linear form (2.13) by means of (3.47), i.e. let us define the forms

$$(3.48)\qquad a_h^\Gamma(v, w) = \sum_{c_h \subset \Gamma_{h2}} \sum_{j=1}^{J} \omega_{j,c_h}(k_{1h} v_{n_h} w_{n_h})\,(B_{j,c_h}) + \sum_{c_h \subset \Gamma_{h3}} \sum_{j=1}^{J} \omega_{j,c_h}(k_{0h} v w + k_{1h} v_{n_h} w_{n_h})\,(B_{j,c_h})\,,$$

$$(3.49)\qquad l_h^\Gamma(w) = \sum_{c_h \subset \Gamma_{h2}} \sum_{j=1}^{J} \omega_{j,c_h}(m_{1_h} w_{n_h})\,(B_{j,c_h}) + \sum_{c_h \subset \Gamma_{h3}} \sum_{j=1}^{J} \omega_{j,c_h}(m_{0h} w + m_{1h} w_{n_h})\,(B_{j,c_h})\,.$$

With the quadrature schemes (3.42), (3.47) we associate error functionals

$$(3.50)\qquad E_{c_h}(G) = \int_{c_h} G(x, y)\,\mathrm{d}s - \sum_{j=1}^{J} \omega_{j,c_h}\, G(B_{j,c_h})\,,$$

$$(3.51)\qquad E_I(G^*) = \int_0^1 G^*(t)\,\mathrm{d}t - \sum_{j=1}^{J} \omega_j\, G^*(t_j)\,.$$

According to (3.42)–(3.47), we have

$$(3.52)\qquad E_{c_h}(G) = E_I(G^*\varrho^*)\,.$$

Taking into account (2.10), (3.48) and (3.50), we can write

$$(3.53)\qquad \tilde{a}_h^\Gamma(v, w) - a_h^\Gamma(v, w) = \sum_{c_h \subset \Gamma_{h2}\cup\Gamma_{h3}} E_{c_h}(k_{1h}v_{n_h}w_{n_h}) + \sum_{c_h\subset\Gamma_{h3}} E_{c_h}(k_{0h}vw)\,.$$

In what follows it will be convenient to use the following notation:

$$(3.54)\qquad \{f\}(t) = f(x^*(1-t, t),\ y^*(1-t, t))\,,$$

$$(3.55)\qquad \{D^\gamma x^*\}(t) = (D^\gamma x^*)(1-t, t)\,,\quad \{D^\gamma y^*\}(t) = (D^\gamma y^*)(1-t, t)\,,$$

$$(3.56)\qquad \{w_{n_h}\}(t) = w_{n_h}(x^*(1-t, t),\ y^*(1-t, t))\,,$$

$$(3.57)\qquad \{D^\gamma w^*\}(t) = (D^\gamma w^*)(1-t, t)\,,$$

where $w^*(\xi, \eta) = w(x^*(\xi, \eta), y^*(\xi, \eta))$. It should be noted that the definition of curved triangular $C^m$-elements (see [15]) implies that $\{w\}$ and $\{D^\gamma w^*\}$ ($|\gamma| = 1$) are polynomials of the fifth and fourth degree, respectively.

With respect to (3.52) we have

$$(3.58)\qquad E_{c_h}(k_{0h}vw) = E_I(k_{0h}^*\,\varrho^*\{v\}\,\{w\})\,,$$

$$(3.59)\qquad E_{c_h}(k_{1h}v_{n_h}w_{n_h}) = E_I(k_{1h}^*\,\varrho^*\{v_{n_h}\}\,\{w_{n_h}\})\,,$$

where, according to the definition of transferring functions $k_i$ from $\Gamma$ onto $\Gamma_h$ (see [16, p. 130]),

$$(3.60)\qquad k_{ih}^*(t) = k_i(\varphi(s_2 + \bar{s}_{32}t),\ \psi(s_2 + \bar{s}_{32}t))\,,\quad i = 0, 1\,.$$

Here $x = \varphi(s)$, $y = \psi(s)$, $s_2 \leqq s \leqq s_3$, is the parametric representation of the arc $c \subset \Gamma$ which is approximated by the arc $c_h \subset \Gamma_h$. $P_2$ and $P_3$ are the end points of both arcs $c$, $c_h$ and $\bar{s}_{32} = s_3 - s_2$.

Now we derive (3.62). We have

$$(3.61)\qquad w_{n_h} = w_x n_{hx} + w_y n_{hy}\,,$$

where, according to (3.45), $n_{hx} = \psi^{*\prime}(t)/\varrho^*(t)$, $n_{hy} = -\varphi^{*\prime}(t)/\varrho^*(t)$. Further, $w(x, y) = w^*(\xi^*(x, y), \eta^*(x, y))$, where $\xi = \xi^*(x, y)$, $\eta = \eta^*(x, y)$ is the inverse transformation to the transformation (3.40). Using the rule of differentiation of a composite function and the relations [15, (13)] we obtain from (3.61)

$$(3.62)\qquad \{w_{n_h}\} = \sum_{|\gamma|=1} \{c_\gamma\}\,\{D^\gamma w^*\}\,,$$

where we denote

$$(3.63)\qquad \{c_{1,0}\} = [\{x_\eta^*\}\,\varphi^{*\prime} + \{y_\eta^*\}\,\psi^{*\prime}]/(\{J^*\}\,\varrho^*)\,,$$

$$(3.64)\qquad \{c_{0,1}\} = -[\{x_\xi^*\}\,\varphi^{*\prime} + \{y_\xi^*\}\,\psi^{*\prime}]/(\{J^*\}\,\varrho^*)\,.$$

Inserting (3.62) into (3.59) we obtain

$$(3.65)\qquad E_{c_h}(k_{1h}v_{n_h}w_{n_h}) = E_I(k_{1h}^* \varrho^* \sum_{|\gamma|,|\delta|=1} \{c_\gamma\}\{c_\delta\}\{D^\gamma v^*\}\{D^\delta w^*\}).$$

In [16] the following estimates are proved:

$$(3.66)\qquad |\varphi^{*(j)}(t)| \leqq Ch_T^j, \quad |\psi^{*(j)}(t)| \leqq Ch_T^j \quad (j = 1, 2, \ldots)$$

$$(3.67)\qquad Ch_T \leqq \varrho^*(t), \quad |\varrho^{*(j)}(t)| \leqq C^* h_T^{j+1} \quad (j = 0, 1, \ldots)$$

where the constant $C$ depends only on $\Gamma$ and the constant $C^*$ only on $\Gamma$ and $j$. Using (3.66), (3.67), the rule of differentiation of a composite function and [15, (24), (25)] we obtain from (3.63), (3.64)

$$(3.68)\qquad d^k(\{c_\gamma\}\{c_\delta\}\varrho^*)/dt^k = O(h_T^{k-1}) \quad (k = 1, 2, \ldots).$$

**Lemma 3.4.** *The inequality*

$$(3.69)\qquad |\{D^\gamma w^*\}|_{k,I} \leqq Ch_T^{k+|\gamma|-1/2} |||w|||_{k+|\gamma|,c_h} \quad (k, |\gamma| = 0, 1 \ldots)$$

*holds, where C is a constant independent of $h_T$ and w and we denote*

$$|||w|||^2_{k+|\gamma|,c_h} = \sum_{|\alpha| \leqq k+|\gamma|} \int_{c_h} (D^\alpha w)^2 \, ds.$$

Proof. According to the rule of differentiation of a composite function we have for $|\gamma| > 0$

$$(3.70)\qquad \frac{\partial^{\gamma_1+\gamma_2} w^*}{\partial \xi^{\gamma_1} \partial \eta^{\gamma_2}} = \frac{\partial^{\gamma_1+\gamma_2} w}{\partial x^{\gamma_1+\gamma_2}} \left(\frac{\partial x^*}{\partial \xi}\right)^{\gamma_1} \left(\frac{\partial x^*}{\partial \eta}\right)^{\gamma_2} + \ldots + \frac{\partial w}{\partial y} \frac{\partial^{\gamma_1+\gamma_2} y^*}{\partial \xi^{\gamma_1} \partial \eta^{\gamma_2}}.$$

Thus we can write with respect to (3.54) and (3.55)

$$\int_0^1 \{D^\gamma w^*\}^2 \varrho^* \, dt =$$

$$= \int_0^1 \left[\left\{\frac{\partial^{\gamma_1+\gamma_2} w}{\partial x^{\gamma_1+\gamma_2}}\right\} \left\{\frac{\partial x^*}{\partial \xi}\right\}^{\gamma_1} \left\{\frac{\partial y^*}{\partial \eta}\right\}^{\gamma_2} + \ldots + \left\{\frac{\partial w}{\partial y}\right\} \left\{\frac{\partial^{\gamma_1+\gamma_2} y^*}{\partial \xi^{\gamma_1} \partial \eta^{\gamma_2}}\right\}\right]^2 \varrho^* \, dt.$$

Using the Cauchy inequality, (3.67) and [15, (25)] we obtain

$$C_1 h_T \int_0^1 \{D^\gamma w^*\}^2 \, dt \leqq C_2 h_T^{2|\gamma|} \int_0^1 \Big[\sum_{1 \leqq |\alpha| \leqq |\gamma|} \{D^\alpha w\}^2\Big] \varrho^* \, dt,$$

from where and [16, Lemma 3] we deduce

$$(3.71)\qquad |\{D^\gamma w^*\}|_{0,I} \leqq Ch_T^{|\gamma|-1/2} |||w|||_{|\gamma|,c_h}, \quad |\gamma| = 0, 1, \ldots.$$

According to the rule of differentiation of a composite function we have

$$\frac{\mathrm{d}^k(\{D^\gamma w^*\})}{\mathrm{d}t^k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left\{\frac{\partial^{k+|\gamma|} w^*}{\partial \xi^{k-j+\gamma_1} \partial \eta^{j+\gamma_2}}\right\}$$

and using the Cauchy inequality we obtain

$$(3.72) \qquad |\{D^\gamma w^*\}|^2_{k,I} \leqq C \sum_{j=0}^{k} \int_0^1 \left\{\frac{\partial^{k+|\gamma|} w^*}{\partial \xi^{k-j+\gamma_1} \partial \eta^{j+\gamma_2}}\right\}^2 \mathrm{d}t\,,$$

where $C$ is a constant dependent on $k$. The inequalities (3.71) and (3.72) imply (3.69). Lemma 3.4 is proved.

**Theorem 3.4.** *Let the parts $\Gamma_2$, $\Gamma_3$ of the boundary $\Gamma$ of $\Omega$ be of class $C^6$. Let the functions $k_i \in C^5(U)$, $(i = 0, 1)$, where $U$ is a domain containing $\Gamma_2 \cup \Gamma_3$. Let*

$$(3.73) \qquad E_I(\sigma) = 0 \quad \forall \sigma \in P_1(8)\,.$$

*Then*

$$(3.74) \qquad |E_{c_h}(k_{0h} v w)| \quad \leqq C h_T |||v|||_{1,c_h} \, |||w|||_{1,c_h}\,,$$

$$(3.75) \qquad |E_{c_h}(k_{1h} v_{n_h} w_{n_h})| \leqq C h_T |||v|||_{1,c_h} \, |||w|||_{1,c_h}\,,$$

*where $C$ is a constant independent of $h_T$, $v$ and $w$.*

**Theorem 3.5.** *Let $r \geqq 1$ be a given integer. Let the parts $\Gamma_2$, $\Gamma_3$ of the boundary $\Gamma$ of $\Omega$ be of class $C^{r+1}$ and the functions $k_0$, $k_1 \in C^r(U)$, where $U$ is a domain containing $\Gamma_2 \cup \Gamma_3$. Let*

$$(3.76) \qquad E_I(\sigma) = 0 \quad \forall \sigma \in P_1(r+3)\,.$$

*Then*

$$(3.77) \qquad |E_{c_h}(k_{0h} v w)| \quad \leqq C h_T^r |||v|||_{r+1,c_h} \, |||w|||_{1,c_h}\,,$$

$$(3.78) \qquad |E_{c_h}(k_{1h} v_{n_h} w_{n_h})| \leqq C h_T^r |||v|||_{r+1,c_h} \, |||w|||_{1,c_h}\,,$$

*where $C$ is a constant independent of $h_T$, $v$ and $w$.*

Proof of Theorems 3.4 and 3.5. First we shall prove (3.75) and (3.78). A typical term on the right-hand side of (3.65) is of the form

$$(3.79) \qquad E_I(c^*\{D^\gamma v^*\}\,\{D^\delta w^*\})$$

with $|\gamma| = |\delta| = 1$. The function $c^*$ is given by

$$(3.80) \qquad c^* = k^*_{1h}\, \varrho^*\{c_\gamma\}\,\{c_\delta\}\,.$$

We have $\{D^\gamma v^*\}$, $\{D^\delta w^*\} \in P_1(4)$. Let us consider the form

$$(3.81) \qquad E_I(\sigma^* u^*) \quad \forall \sigma^*\; W^{(s)}_\infty(I)\,, \quad \forall u^* \in P_1(4)\,,$$

where in the case of Theorem 3.4

$$s = 5 \tag{3.82}$$

and in the case of Theorem 3.5.

$$s = r\,. \tag{3.83}$$

According to (3.51) and (3.81) we can write

$$\left|E_I(\sigma^* u^*)\right| \leqq C\left|\sigma^*\right|_{0,\infty,I} \left|u^*\right|_{0,\infty,I}$$

and using Lemma 3.2 we obtain

$$\left|E_I(\sigma^* u^*)\right| \leqq C\left\|\sigma^*\right\|_{s,\infty,I} \left|u^*\right|_{0,I}\,.$$

According to the Bramble - Hilbert lemma and (3.73), (3.82) and (3.76), (3.83), respectively, we conclude that

$$\left|E_I(\sigma^* u^*)\right| \leqq C\left|\sigma^*\right|_{s,\infty,I} \left|u^*\right|_{0,I}\,. \tag{3.84}$$

The relation (3.60) gives

$$k_{ih}^{*(j)}(t) = \bar{s}_{32}^{j}\left[\frac{\partial^j k_i}{\partial x^j}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^j + \ldots + \frac{\partial k_i}{\partial y}\frac{\mathrm{d}^j\psi}{\mathrm{d}s^j}\right], \quad j = 0, \ldots, s,\ i = 0, 1\,. \tag{3.85}$$

The assumptions of Theorem 3.4 and 3.5 about $\Gamma_2, \Gamma_3$ and $k_i$ together with (3.85) imply $k_{ih} \in C^s(I)$. Thus, the relations (3.85) and the estimate $\bar{s}_{32} \leqq Ch_T$ (see [16]) give

$$\left|k_{ih}^*\right|_{j,\infty,I} \leqq Ch_T^j\,, \quad j = 0, 1, \ldots, s\,, \tag{3.86}$$

where $C$ is a constant depending on $k_i$ and $\Gamma$. Now we use Lemma 3.1 which together with (3.68), (3.80) and (3.86) gives

$$\left|c^*\right|_{k,\infty,I} \leqq Ch_T^{k-1}\,. \tag{3.87}$$

Let us set $\sigma^* = c^*\{D^\gamma v^*\}$, $u^* = \{D^\delta w^*\}$ and estimate $\left|\sigma^*\right|_{s,\infty,I}$ by means of Lemmas 3.1, 3.2 and (3.87), and $\left|u^*\right|_{0,I}$ by means of Lemma 3.4. Then we obtain from (3.84)

$$\left|E_I(c^*\{D^\gamma v^*\}\{D^\delta w^*\})\right| \leqq Ch_T^{s-1/2}\sum_{j=0}^{s} h_T^{-j}\left|\{D^\gamma v^*\}\right|_{j,I} \left\|\left|w\right|\right\|_{1,c_h}\,. \tag{3.88}$$

In the proof of (3.75) we use the relation $\left|\{D^\gamma v^*\}\right|_{s,I} = 0$. Then according to Lemmas 3.2 and 3.4 we have

$$\sum_{j=0}^{s} h_T^{-j}\left|\{D^\gamma v^*\}\right|_{j,I} \leqq Ch_T^{3/2-s}\left\|\left|v\right|\right\|_{1,c_h}\,. \tag{3.89}$$

The inequalities (3.88) and (3.89) together with (3.65) and (3.80) imply (3.75).

Now we prove (3.78). Again using Lemma 3.4 and (3.83) we obtain

$$(3.90)\qquad \sum_{j=0}^{s} h_T^{-j}|\{D^\gamma v^*\}|_{j,I} \leqq C h_T^{1/2} |||v|||_{r+1,c_h}$$

and the inequalities (3.88) and (3.90) together with (3.65) and (3.83) imply (3.78).

Now we prove (3.74) and (3.77). Let us consider the form

$$E_I(\sigma^* u^*) \quad \forall \sigma^* \in W_\infty^{(s)}(I)\,, \quad \forall u^* \in P_1(5)\,,$$

where in the case of Theorem 3.4

$$(3.91)\qquad s = 4$$

and in the case of Theorem 3.5

$$(3.92)\qquad s = r - 1\,.$$

Let $u^* \in P_1(5)$ and let $\Pi$ be the orthogonal projection in the space $L_2(I)$ onto the subspace $P_1(0)$, i.e.

$$(u^* - \Pi u^*, p)_{0,I} = 0 \quad \forall p \in P_1(0)\,.$$

We can write

$$(3.93)\qquad E_I(\sigma^* u^*) = E_I(\sigma^*(u^* - \Pi u^*)) + E_I(\sigma^* \Pi u^*)$$

and in a similar way as in the proofs of Theorems 3.1 and 3.2 we estimate

$$(3.94)\qquad |E_I(\sigma^*(u^* - \Pi u^*))| \leqq C|\sigma^*|_{s,\infty,I}\, |u^*|_{1,I} \quad \forall \sigma^* \in W_\infty^{(s)}(I)\,, \quad \forall u^* \in P_1(5)\,,$$

$$(3.95)\qquad |E_I(\sigma^* \Pi u^*)| \leqq C|\sigma^*|_{s+1,\infty,I}\, |u^*|_{0,I} \quad \forall \sigma^* \in W_\infty^{(s+1)}(I)\,, \quad \forall u^* \in P_1(5)\,.$$

Let us set $\sigma^* = k_{0h}^* \varrho^*\{v\}$. Then, using Lemma 3.1 and (3.3), we obtain

$$|\sigma^*|_{k,\infty,I} \leqq C \sum_{j=0}^{k} |k_{0h}^* \varrho^*|_{k-j,\infty,I}\, |\{v\}|_{j,I}$$

and using (3.67) and (3.86) we derive that

$$(3.96)\qquad |\sigma^*|_{k,\infty,I} \leqq C h_T^{k+1} \sum_{j=0}^{k} h_T^{-j} |\{v\}|_{j,I}\,,$$

where $C$ is a constant depending on $k_0$ and $\Gamma_3$.

Let us set $u^* = \{w\}$. Inserting (3.96) into (3.94) and (3.95) and using (3.93), Lemma 3.4 and the inequality $|||w|||_{0,c_h} \leqq |||w|||_{1,c_h}$ we obtain

$$(3.97)\qquad |E_I(k_{0h}^* \varrho^*\{v\}\,\{w\})| \leqq C h_T^{s+3/2} \Big[\sum_{j=0}^{s} h_T^{-j} |\{v\}|_{j,I} + \sum_{j=0}^{s+1} h_T^{-j} |\{v\}|_{j,I}\Big]\, |||w|||_{1,c_h}\,.$$

Inserting the inequality

$$\sum_{j=0}^{k} h_T^{-j} |\{v\}|_{j,I} \leqq C h_T^{1/2-k} |||v|||_{1,c_h} \quad (k = s,\, k = s + 1)$$

into (3.97) we get (3.74) and Theorem 3.4 is proved. The assertion (3.77) of Theorem 3.5 follows from (3.97), Lemma 3.4 and (3.92). Theorems 3.4 and 3.5 are completely proved.

We can write with respect to (2.13) and (3.49)

$$\tilde{l}_h^\Gamma(w) - l_h^\Gamma(w) = \sum_{c_h \subset \Gamma_{h2} \cup \Gamma_{h3}} E_{c_h}(m_{1h} w_{nh}) + \sum_{c_h \subset \Gamma_{h3}} E_{c_h}(m_{0h} w) .$$

Using (3.51), (3.54) we have

$$E_{c_h}(m_{0h} w) = E_I(m_{0h}^* \, \varrho^* \{w\})$$

and

$$E_{c_h}(m_{1h} w_{nh}) = E_I(m_{1h}^* \, \varrho^* \sum_{|\gamma|=1} \{c_\gamma\} \{D^\gamma w^*\}) ,$$

where $m_{ih}^*(t) = m_i(\varphi(s_2 + \bar{s}_{32} t), \psi(s_2 + \bar{s}_{32} t)) \quad (i = 0, 1)$.

**Theorem 3.6.** *Let $r \geqq 3$ be a given integer. Let $\Gamma$ be of class $C^{r+1}$ and $m_0, m_1 \in C^r(U)$, where $U$ is a domain containing $\Gamma_2 \cup \Gamma_3$. Let*

$$E_I(\sigma) = 0 \quad \forall \sigma \in P_1(r + 3) .$$

*Then*

$$|\tilde{l}_h^\Gamma(w) - l_h^\Gamma(w)| \leqq C h^r \|w\|_{2,\Omega_h} , \tag{3.98}$$

*where $C$ is a constant independent of $h$ and $w$.*

The proof of Theorem 3.6 is similar to that of Theorem 3.5. Thus it is omitted (for details see [5]).

It remains to establish the validity of (2.17) which expresses the uniform $V_{0h}$-ellipticity of the bilinear forms $a_h(v, w)$ $(h < \tilde{h})$.

**Theorem 3.7.** *Let the assumptions of Theorems 3.1 and 3.4 be satisfied. Let $\Gamma$ be of class $C^4$. Let (2.3) and (2.5) be satisfied. Then the inequality (2.17) holds for sufficiently small $h$.*

Proof. First we establish the validity of the inequality

$$\tilde{a}_h(v, v) \geqq K \|v\|_{2,\Omega_h}^2 \quad \forall v \in V_{0h} , \quad h < \tilde{h} , \tag{3.99}$$

where $K$ is a constant independent of $v$ and $h$. We shall consider the cases $1^0 - 6^0$ introduced in Section 1.

In the cases $1^0$ and $2^0$ inequality (3.99) follows from considerations introduced in [17, proof (99)] and from the inequality $\tilde{a}_h(v, v) \geqq (1 - \mu) |v|_{2,\Omega_h}^2$.

In the case $3^0$ we have

$$\tilde{a}_h(v, v) \geqq C_1 \left[ \int_{\Gamma'_{2h}} v_n^2 \, ds + |v|_{2,\Omega_h}^2 \right] .$$

To prove (3.99) it suffices to use the inequality

$$\|v\|^2_{2,\Omega_h} \leqq C\left[\int_{\Gamma'_{2h}} v_n^2 \,\mathrm{d}s + |v|^2_{2,\Omega_h}\right] \quad \forall v \in V_{0h},$$

the proof of which is a simple modification of proofs of [17, Theorems 1, 2] because $\Gamma'_{2h} = \Gamma'_2$.

In the case $4^0$ we have

$$\tilde{a}_h(v, v) \geqq C_2\left[\int_{\Gamma^*_{3h}} v^2 \,\mathrm{d}s + |v|^2_{2,\Omega_h}\right].$$

To prove (3.99) it suffices to use the inequality

$$\|v\|^2_{2,\Omega_h} \leqq C\left[\int_{\Gamma_{2h}\cup\Gamma^*_{3h}} v^2 \,\mathrm{d}s + |v|^2_{2,\Omega_h}\right] \quad \forall v \in W^2_h(\supset V_{0h}),$$

which follows from [17, Corollary 3]. In the case $5^0$ the proof of (3.99) follows the same lines.

In the case $6^0$ we have

$$\tilde{a}_h(v, v) \geqq C_3\left[\int_{\Gamma''_{3h}} v^3 \,\mathrm{d}s + \int_{\Gamma^{**}_{3h}} v^2_{n_h} \,\mathrm{d}s + |v|^2_{2,\Omega_h}\right].$$

The proof of (3.99) now follow from the inequality

$$\text{(3.100)} \qquad \|v\|^2_{2,\Omega_h} \leqq C\left[\int_{\Gamma''_{3h}} v^2 \,\mathrm{d}s + \int_{\Gamma^{**}_{3h}} v^2_{n_h} \,\mathrm{d}s + |v|^2_{2,\Omega_h}\right] \quad \forall v \in W^2_h,$$

which is proved in [5]. The inequality (3.99) is completely proved.

Theorem 3.1 implies

$$\text{(3.101)} \qquad -\sum_{T\in\tau_h} \left|E_T\left(\sum_{|\alpha|,|\beta|=2} a_{\alpha\beta} D^\alpha v D^\beta v\right)\right| \geqq -Ch\|v\|^2_{2,\Omega_h}.$$

Theorem 3.4 together with the discrete form of the trace theorem with a constant $C$ independent of $h$ (see [16, Lemma 4]) give

$$\text{(3.102)} \qquad -\sum_{c_h\subset\Gamma_{h2}\cup\Gamma_{h3}} |E_{c_h}(k_{1h} v^2_{n_h})| - \sum_{c_h\subset\Gamma_{h3}} |E_{c_h}(k_{0h} v^2)| \geqq -Ch\|v\|^2_{2,\Omega_h}.$$

Relations [15, (145)], (3.53), (3.99), (3.101) and (3.102) imply

$$a_h(v, v) \geqq (K - 2Ch)\,\|v\|^2_{2,\Omega_h}.$$

Let us choose $h_1 = K/4C$. Then inequality (2.17) is satisfied with $\gamma = K/2$ for $h < \min(\tilde{h}, h_1)$. Theorem 3.7 is proved.

The main result of the paper is formulated in the following theorem where the results of Sections 2–3 are summarized.

**Theorem 3.8.** *Let the extension $\tilde{u}$ of $u$, the solution of the problem (1.1)–(1.4) to the domain $\tilde{\Omega} \supset \Omega_h$, satisfy*

$$\tilde{u} \in H^5(\tilde{\Omega}), \quad \tilde{f} \equiv \Delta^2 \tilde{u} \in H^3(\tilde{\Omega}).$$

*Let $g_0, g_1 \in C^2(U_1)$, $g_2 \in C^2(U_2)$, $k_i \in C^5(U_{23})$, $m_i \in C^3(U_{23})$ $(i = 0, 1)$, where $U_j$ is a domain containing $\Gamma_j$ $(j = 1, 2)$ and $(j = 1, 2)$ and $U_{23}$ is a domain containing $\Gamma_2 \cup \Gamma_3$. Let the part $\Gamma_1$* of *$\Gamma$ be of class $C^4$ and the parts $\Gamma_2, \Gamma_3$ of $\Gamma$ of class $C^6$. Let (2.3) and (2.5) be satisfied. Let*

$$E^*(\varphi^*) = 0 \quad \forall \varphi^* \in P_2(2N^* - 4)$$

*with $N^* = 5$ for Bell's elements and $N^* = 4 + n$ for curved triangular $C^1$-elements from [15]. Let*

$$E_I(\sigma) = 0 \quad \forall \sigma \in P_1(8).$$

*Then for sufficiently small $h$ the solution $u_h$ of the discrete problem (2.14) exists and is unique and the following estimate holds:*

$$\|\tilde{u} - u_h\|_{2,\Omega_h} \leqq Ch^3[\|\tilde{u}\|_{5,\tilde{\Omega}} + \|\tilde{f}\|_{3,\tilde{\Omega}} + 2], \tag{3.103}$$

*where $C$ is a constant independent of $h$ and $\tilde{u}$.*

Proof. According to Theorem 3.7, the assumptions of Theorem 3.8 imply for sufficiently small $h$ the inequality (2.17). Thus the solution $u_h$ of (2.14) exists and is unique.

As $P_2(2N^* - 4) \supset P_2(r + N^* - 3)$ for $r \leqq N^* - 1$, the assumption (3.19) holds with $r = 3$. Using (2.9), (3.11), (3.13) and Theorem 3.2 we obtain

$$|\tilde{a}_h^\Omega(v, w) - a_h^\Omega(v, w)| \leqq Ch^3 \sum_{T \in \tau_h} \|v\|_{5,T} \|w\|_{2,T}. \tag{3.104}$$

As $P_1(8) \supset P_1(r + 3)$ for $r \leqq 5$, the assumption (3.76) of Theorem 3.5 holds with $r = 3$. Using (3.53) and Theorem 3.5 we obtain

$$|\tilde{a}_h^\Gamma(v, w) - a_h^\Gamma(v, w)| \leqq Ch^3 \sum_{c_h \subset \Gamma_{h2} \cup \Gamma_{h3}} |||v|||_{4,c_h} |||w|||_{1,c_h} \tag{3.105}$$

Let $v_I$ be the function from $W_h^2$ interpolating $\tilde{u}$ (see Section 2). Then, according to [15, Theorem 5] and [5, (8.49)],

$$\|v_I\|_{5,T} \leqq \|\tilde{u}\|_{5,T} + \|v_I - \tilde{u}\|_{5,T} \leqq C\|\tilde{u}\|_{5,T}, \tag{3.106}$$

$$|||v_I|||_{4,c_h} \leqq |||\tilde{u}|||_{4,c_h} + |||\tilde{u} - v_I|||_{4,c_h} \leqq C(|||\tilde{u}|||_{4,c_h} + \|\tilde{u}\|_{5,T}). \tag{3.107}$$

As $v_I \in V_{gh}$ we obtain from (3.104), (3.105), (3.106), (3.107), the Cauchy inequality and a modified trace theorem with a constant independent of $h$ (see [16, Lemma 4])

$$\text{(3.108)} \qquad \inf_{v \in V_{gh}} \sup_{w \in V_{0h}} \frac{|\tilde{a}_h(v, w) - a_h(v, w)|}{\|w\|_{2,\Omega_h}} \leqq Ch^3 \|\tilde{u}\|_{5,\Omega_h} .$$

Inserting (2.25), (2.26), (3.41), (3.98) and (3.108) into (2.18) we obtain (3.103). Theorem 3.8 is proved.

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## Souhrn

# NUMERICKÁ ANALÝZA OBECNÉHO BIHARMONICKÉHO PROBLÉMU METODOU KONEČNÝCH PRVKŮ

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Zakřivené trojúhelníkové $C^m$ – prvky zavedené v [12]–[14] ($m = 0$) a [6], [8], [15] ($m > 0$) byly doposud analyzovány pouze pro řešení takových eliptických okrajových problémů řádu $2(m + 1)$ metodou konečných prvků, kde okrajové podmínky neimplikují hraniční bilineární formu ve variační formulaci problému, přičemž účinek numerické integrace byl studován pouze v případě Dirichletových problémů (viz [3], [6], [8], [12]–[17]).

V článku jsou zakřivené trojúhelníkové $C^1$-prvky použity k řešení biharmonické rovnice metodou konečných prvků již s obecnými okrajovými podmínkami implikujícími obecný tvar bilineární i lineární formy problému a účinek numerické integrace je studován jak v oblasti $\Omega_h$, tak na hranici $\Gamma_h$.

Bilineární a lineární forma diskrétního problému (2.14) je definována pomocí kvadraturních formulí. Je-li stupeň přesnosti kvadraturních formulí $2N^* - 4$ pro zakřivené trojúhelníkové $C^1$ – prvky, 6 pro Bellovy prvky, 8 pro integraci hraniční bilineární formy (3.48) a hranice $\Gamma$ je dostatečně hladká (viz větu 3.8), pak existuje právě jedno řešení diskrétního problému (2.14) a rychlost konvergence k přesnému řešení problému (1.1)–(1.4) je $O(h^3)$ v normě prostoru $H^2(\Omega_h)$.



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