NUMERICAL APPROACH TO THE WAITING TIME FOR THE ONE-DIMENSIONAL POROUS MEDIUM EQUATION

By

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Abstract. We consider the nonlinear degenerate diffusion equation. The most striking manifestation of the nonlinearity and degeneracy is an appearance of interfaces. Under some condition imposed on the initial function, the interfaces do not move on some time interval $[0, t^*]$. In this paper, from numerical points of view, we try to determine the value of t^*, which is called the *waiting time*.

1. Introduction. From numerical points of view, we are concerned with the time when isentropic flow of an ideal gas through a one-dimensional homogeneous porous medium begins to move. Such a flow is described by the equation

$$u_t = (u^m)_{xx}, \qquad x \in \mathbb{R}^1, \quad t > 0, \tag{1.1}$$

where $u = u(x, t)$ represents the density of the gas and $m > 1$ is a constant. (1.1) is called the *porous medium equation*, which is also known as a simple model in the fields of the grand water, population dynamics, and radiative heat transfer problems (see [3], [13] and the references therein). In this paper we shall consider the initial value problem for (1.1) with

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^1, \tag{1.2}$$

where u_0 is a nonnegative function. The existence and uniqueness of the weak solution of (1.1)–(1.2) are shown by Oleinik, Kalashnikov, and Yui-Lin [16]. They also derive several properties of the solution. One of the important properties is the finite speed of propagation; if $S(0)$ is compact in $\mathbb{R}^1$, so is $S(t)$ for all $t > 0$, where $S(t) = \operatorname{supp} u(\cdot, t)$. The behavior of $S(t)$ is studied by many authors ([7], [10], [12], and [17]). Kalashnikov [10] shows that there exist functions $\zeta_i(t)$ $(i = 1, 2)$ satisfying $S(t) = [\zeta_1(t), \zeta_2(t)]$ for all $t \geq 0$.

The function $\zeta_1(t)$ (resp. $\zeta_2(t)$) is called the *left (*resp. *right) interface.* It is shown in [12] that there exist some constants t_i^* $(i = 1, 2)$ satisfying

$$\zeta_i(t) = \zeta_i(0) \qquad \text{if} \quad t \in [0, t_i^*],$$
$$(-1)^i \zeta_i(t) < (-1)^i \zeta_i(\tau) \quad \text{if} \quad t_i^* \leq t < \tau.$$

When t_i^* is positive, we call t_i^* the *waiting time* of the interface ζ_i, which indicates the time when the interface $\zeta_i(t)$ begins to move. In the following, we focus our attention on the waiting time of the left interface $\zeta_1(t)$, and simplify the notations by putting $\zeta(t) = \zeta_1(t)$ and $t^* = t_1^*$. Without loss of generality, we may assume $\zeta(0) = 0$; that is, supp $u_0 = [0, a]$ for some $a > 0$.

Aronson [2] shows the following example which has a waiting time. Let the initial function be given by

$$u_0(x) = \begin{cases} \sin^{2/(m-1)} x, & \text{if} \ 0 \leq x \leq \pi, \\ 0, & \text{otherwise,} \end{cases}$$

then $t^* = (m-1)/2m(m+1)$. In general, Knerr [12] proves that

$$t^* > 0 \quad \text{if} \quad u_0(x) \leq cx^{2/(m-1)} \ \text{on} \ x \in [0, \delta],$$
$$t^* = 0 \quad \text{if} \quad u_0(x) \geq cx^\gamma \ \text{on} \ x \in [0, \delta]$$

for some constants $c > 0$, $\delta > 0$ and $\gamma < 2/(m-1)$. Vázquez [19] has gone even further. Let $M(x) = \int_0^x u_0(\xi)d\xi$. Then

$$t^* > 0 \qquad \text{if and only if} \qquad \sup_{x>0} \frac{M(x)}{x^{(m+1)/(m-1)}} < \infty. \qquad (1.3)$$

The upper and lower bounds of the waiting time are obtained as follows.

THEOREM 1.1 (Aronson, Caffarelli, and Kamin [4]). If

$$u_0^{m-1}(x) = \alpha x^2 + o(x^2) \quad \text{as} \ x \downarrow 0 \quad \text{and} \quad u_0^{m-1}(x) \leq \beta x^2 \ \text{on} \ x \geq 0 \qquad (1.4)$$

hold for some constants $\alpha > 0$ and $\beta > 0$, then

$$(m-1)/2m(m+1)\beta \leq t^* \leq (m-1)/2m(m+1)\alpha. \qquad (1.5)$$

In particular, if $\alpha = \beta$ then $t^* = (m-1)/2m(m+1)\alpha$.

Some physical background of the waiting time is stated by Lacey, Ockendon, and Tayler [13]. Kath and Cohen [11] study the waiting time when $m-1$ is sufficiently small and obtain an estimate for t^* up to terms which are $o(m-1)$. The determination of the waiting time is also important to study the regularity of the interface. Caffarelli and Friedman [7] show $\zeta \in C^1([0, t^*) \cup (t^*, \infty))$. The regularity of $\zeta(t)$ at $t = t^*$ are obtained by Aronson, Caffarelli, and Kamin [4] as follows (see also [5] and [19]). If $(m-1)/2m(m+1)\beta < t^* < (m-1)/2m(m+1)\alpha$, then $\zeta \notin C^1$. If $t^* = (m-1)/2m(m+1)\beta$, then $\alpha = \beta$ and $\zeta \in C^1$. Here α and β are the constants in Theorem 1.1.

To illustrate the results stated above, Aronson, Caffarelli, and Kamin [4] considered the following example:

$$u_0^{m-1}(x) = \begin{cases} (1-\theta)\sin^2 x + \theta\sin^4 x, & \text{if } 0 \le x \le \pi, \\ 0, & \text{otherwise,} \end{cases} \tag{1.6}$$

where $\theta \in [0,1]$ is a parameter. Applying Theorem 1.1 to (1.6), they show $t^* = (m-1)/2m(m+1)(1-\theta)$ for $0 \le \theta \le \frac{1}{4}$; however, the waiting time for $\frac{1}{4} < \theta \le 1$ cannot be explicitly determined. The waiting time is also studied by Alikakos [1], Chipot and Sideris [8], Lacey [14], and Vázquez [19]. However, they do not explicitly determine the waiting time for $\theta \in (\frac{1}{4}, 1]$.

In numerical points of view, there are some numerical methods to estimate the waiting time t^*. Tomoeda and Mimura [18], Mimura, Nakaki, and Tomoeda [15], Gurtin, Mac-Camy, and Socolovsky [9], and Bertsch and Dal Passo [6] introduce interface tracking algorithms to (1.1)–(1.2), and show the numerical simulations in the case of (1.6) with $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$. However, especially in the case of $\theta = \frac{1}{2}$, the numerical waiting time cannot be clearly estimated (see Fig. 1). Therefore, we need a numerical method to determine the waiting time.

In this paper we try to determine the waiting time t^* from a numerical point of view, and we obtain

$$(m-1)/2m(m+1)\beta \le T_h^* \le (m-1)/2m(m+1)\alpha, \tag{1.7}$$

where T_h^* is the numerical waiting time determined by the scheme in Sec. 3, and α and β are the constants in Theorem 1.1.

We briefly explain our idea, which plays an important role in the approximation of waiting time. The initial function $u_0(x)$ is approximated by the piecewise-linear interpolation in usual difference scheme. This implies that the derivative of the numerical initial function is discontinuous at the interface, and then the numerical interface initially moves, even if the waiting time is positive. To avoid this numerical inconsistency, we transform (1.1)–(1.2) into an another problem (2.4)–(2.7), in which the solution blows up at the waiting time (see Theorem 2.3). Thus our numerical scheme is reduced to the approximations to the blow-up time of the solution of (2.4)–(2.7). Unfortunately, we do not succeed in proving the convergence of the scheme. However, as is shown in Sec. 4, our scheme gives good numerical approximations.

2. A Blow-Up Problem. Let us assume that $u_0(x) = O(x^p)$ $(x \downarrow 0)$ for some constant $p > 0$. Then (1.3) yields that $t^* > 0$ if and only if $p \ge 2/(m-1)$, which implies

$$\lim_{x \downarrow 0} u_0(x)/x^{2/(m-1)} \quad \text{exists.} \tag{2.1}$$

We define $f(x,t)$ by

$$f(x,t) = u(x,t)/x^{2/(m-1)}. \tag{2.2}$$

Then one can expect that $\sup f(\cdot, t)$ is finite if $t < t^*$, and that $\sup f(\cdot, t)$ blows up as $t \uparrow t^*$, which shall be proved later in Theorem 2.3.

Since supp $u(\cdot, t)$ is compact for $t \geq 0$, there exists a constant $L(\geq a > 0)$ such that

$$u(x, t) = 0 \quad \text{for } (x, t) \in [L, \infty) \times [0, t^*]. \tag{2.3}$$

By using this constant L, we consider the following blow-up problem, which is obtained from (1.1)–(1.2) and (2.2).

$$f_t = x^2 (f^m)_{xx} + \frac{4m}{m-1} x (f^m)_x + \frac{2m(m+1)}{(m-1)^2} f^m, \qquad 0 < x < L, \quad t > 0, \tag{2.4}$$

$$f_t(0, t) = \frac{2m(m+1)}{(m-1)^2} f^m(0, t), \qquad t > 0, \tag{2.5}$$

$$f(L, t) = 0, \qquad t > 0, \tag{2.6}$$

$$f(x, 0) = f_0(x) \equiv u_0(x)/x^{2/(m-1)}, \qquad 0 \leq x \leq L. \tag{2.7}$$

Throughout this paper we impose

CONDITION 2.1. A continuous function $f_0(x)$ defined on $[0, L]$ satisfies

$$f_0 > 0 \quad \text{on } (0, a) \qquad \text{and} \qquad f_0 = 0 \quad \text{on } [a, L] \tag{2.8}$$

for some constant $a \in (0, L)$, and $x^{2m/(m-1)} f_0^m(x)$ is Lipschitz continuous on $[0, L]$.

DEFINITION 2.2. A continuous function $f(x, t)$ defined on $(x, t) \in [0, L] \times [0, T^*)$ is said to be a weak solution of (2.4)–(2.7) if a function

$$u(x, t) = \begin{cases} x^{2/(m-1)} f(x, t), & \text{if } x \in [0, L], \\ 0, & \text{otherwise} \end{cases}$$

is the weak solution of (1.1)–(1.2) on $0 < t < T^*$.

The existence and uniqueness of the weak solution of (1.1)–(1.2) is obtained when $u_0^m \in \text{Lip}(\mathbb{R}^1)$ (see [16]). Hence Condition 2.1 implies that the unique weak solution of (2.4)–(2.7) exists.

Under this definition we have

THEOREM 2.3. Let

$$T^* = \sup \{ t \geq 0; \ \sup \{ f(x, t); \ 0 \leq x \leq L \} < \infty \}, \tag{2.9}$$

where f is the weak solution of (2.4)–(2.7). Then $t^* = T^*$ holds.

Proof. Let t_0 be an arbitrary nonnegative number satisfying $t_0 < T^*$. Then $f(x, t_0) \leq C$ $(0 \leq x \leq L)$ holds for some constant $C > 0$. Since

$$M(x, t_0) \equiv \int_0^x u(\xi, t_0) \, d\xi = \int_0^x \xi^{2/(m-1)} f(\xi, t_0) \, d\xi \leq \frac{C(m-1)}{m+1} x^{(m+1)/(m-1)},$$

it follows from (1.3) that $t_0 < t^*$. Hence we have $T^* \leq t^*$.

On the other hand, let $t_1 < t^*$. Then, from Lemma 1.1 in [4], for any $\tau \in (0, t^*)$, there exists a constant $C_0(\tau) > 0$ satisfying

$$\frac{u^{m-1}(x, t)}{x^2} \leq \frac{C_0(\tau)}{t^* - t} \qquad \text{on } x > 0, \quad t \in [\tau, t_1].$$

Putting $t = \tau = t_1$ and $C_1 = C_0(t_1)/(t^* - t_1)$, we have

$$\frac{u^{m-1}(x, t_1)}{x^2} \leq C_1 \qquad \text{on } x > 0.$$

Thus it follows from (2.2) that $f(x, t_1) \leq C_1^{1/(m-1)}$, which means $t_1 < T^*$. Hence we have $t^* \leq T^*$, and the proof is complete. $\qquad\square$

REMARK 2.4. Since $t^* < \infty$ (see [12]), we find that the solution of (2.4)–(2.7) always blows up in finite time. We call T^* the *blow-up time* of the problem (2.4)–(2.7).

From Theorem 2.3, to know the value of waiting time t^*, it suffices to compute the blow-up time of $f(x, t)$. In the following section, we introduce a numerical scheme to the blow-up problem.

3. Numerical Scheme. We use a set of irregular nodal points $\{x_j\}_{0 \leq j \leq N}$ satisfying

$$0 = x_0 < x_1 < \cdots < x_j < \cdots < x_N = L, \tag{3.1}$$

where $N \in \mathbb{N}$. We denote by f_j^n the numerical approximations to $f(x_j, t_n)$, where $\{t_n\}_{n \geq 0}$ is an increasing sequence which is determined later. Our difference scheme is written in the following form: For $n \geq 0$,

$$\frac{f_j^{n+1} - f_j^n}{t_{n+1} - t_n} = x_j^2 \frac{2}{x_{j+1} - x_{j-1}} \left\{ \frac{(f_{j+1}^n)^m - (f_j^n)^m}{x_{j+1} - x_j} - \frac{(f_j^n)^m - (f_{j-1}^n)^m}{x_j - x_{j-1}} \right\} \tag{3.2}$$

$$+ \frac{4m}{m-1} x_j \frac{(f_{j+1}^n)^m - (f_j^n)^m}{x_{j+1} - x_j}$$

$$+ \frac{\Phi(t_{n+1} - t_n; f_j^n) - f_j^n}{t_{n+1} - t_n} \quad (1 \leq j \leq N - 1),$$

$$f_0^{n+1} = \Phi(t_{n+1} - t_n; f_0^n), \tag{3.3}$$

$$f_N^{n+1} = 0, \tag{3.4}$$

$$t_0 = 0 \quad \text{and} \quad f_j^0 = f_0(x_j) \quad (0 \leq j \leq N), \tag{3.5}$$

where

$$\Phi(t; z) = \begin{cases} \left(z^{1-m} - \frac{2m(m+1)}{m-1} t \right)^{1/(1-m)}, & \text{if } z > 0, \\ 0, & \text{if } z = 0 \end{cases} \tag{3.6}$$

is a solution of

$$\frac{d}{dt} \Phi(t; z) = \frac{2m(m+1)}{(m-1)^2} \Phi(t; z)^m \quad (t > 0), \qquad \Phi(0; z) = z. \tag{3.7}$$

We determine a time step $\Delta t_n \equiv t_{n+1} - t_n > 0$ satisfying

$$\Delta t_n \leq \Delta t_n^* \tag{3.8}$$

$$\equiv \min_{1 \leq j \leq N} \frac{(m-1) h_j h_{j-1} (h_j + h_{j-1})}{4 m x_j \gamma_j^n h_{j-1} (h_j + h_{j-1}) + 2(m-1) x_j^2 (h_{j-1} \gamma_j^n + h_j \gamma_{j-1}^n)}$$

and

$$\Delta t_n < \Delta t_n^{**} \equiv \min_{0 \leq j \leq N} \frac{m-1}{2m(m+1)(f_j^n)^{m-1}}, \tag{3.9}$$

where $h_j = x_{j+1} - x_j$ and

$$\gamma_j^n = \begin{cases} \dfrac{(f_{j+1}^n)^m - (f_j^n)^m}{f_{j+1}^n - f_j^n}, & \text{if } f_{j+1}^n \neq f_j^n, \\ m(f_j^n)^{m-1}, & \text{if } f_{j+1}^n = f_j^n. \end{cases} \tag{3.10}$$

To show that the numerical solution blows up (see Theorem 3.2), we impose

$$\liminf_{n \to \infty} \frac{\Delta t_n}{\min\{\Delta t_n^*, \Delta t_n^{**}\}} > 0. \tag{3.11}$$

We define a numerical blow-up time T_h^* by

$$T_h^* = \lim_{n \to \infty} t_n. \tag{3.12}$$

In view of Theorem 2.3, we call T_h^* the *numerical waiting time* of (1.1)–(1.2).

We note that the third term of the right hand side of (3.2) approximates that of (2.4); that is,

$$\frac{\Phi(t_{n+1} - t_n; f_j^n) - f_j^n}{t_{n+1} - t_n} = \frac{\Phi(t_{n+1} - t_n; f_j^n) - \Phi(0; f_j^n)}{t_{n+1} - t_n}$$

$$\approx \frac{d}{dt}\Phi(0; f_j^n) = \frac{2m(m+1)}{(m-1)^2}\Phi(0; f_j^n)^m = \frac{2m(m+1)}{(m-1)^2}(f_j^n)^m.$$

We impose the conditions (3.8) and (3.9) on Δt_n. The former is the usual condition in construction of difference scheme for $f_t = x^2(f^m)_{xx} + \frac{4m}{m-1}x(f^m)_x$. The latter guarantees the existence of the solution of (3.7) on the interval $[0, \Delta t_n^{**})$.

We obtain the following basic inequality:

THEOREM 3.1. *Under* (3.8)–(3.9), *the estimate*

$$0 \le f_j^n \le \Phi(t_n; \|f_0\|_\infty) \tag{3.13}$$

holds for all $0 \le j \le N$ *and* $n \ge 0$, *where* $\| \cdot \|_\infty = \| \cdot \|_{L^\infty([0,L])}$.

Proof. It is clear that (3.13) holds for $j = 0$ and $j = N$. So we shall show (3.13) for $1 \le j \le N - 1$. The proof shall be done by induction on $n(\ge 0)$. For $n = 0$, (3.13) holds by (2.8) and (3.5). Assume that (3.13) holds for some $n \ge 0$. We shall show

$$0 \le f_j^{n+1} \le \Phi(t_{n+1}; \|f_0\|_\infty) \quad \text{for } 1 \le j \le N - 1. \tag{3.14}$$

It follows from (3.2) that

$$f_j^{n+1} = \left\{\theta_j^n f_j^n + \Delta t_n A_j^n\right\} + \left\{(1 - \theta_j^n)f_j^n + \Delta t_n B_j^n\right\} \tag{3.15}$$
$$+ \left\{\Phi(\Delta t_n; f_j^n) - f_j^n\right\},$$

where $\theta_j^n \in [0, 1]$ is some constant determined later and

$$A_j^n = x_j^2 \frac{2}{h_j + h_{j-1}} \left\{\frac{(f_{j+1}^n)^m - (f_j^n)^m}{h_j} - \frac{(f_j^n)^m - (f_{j-1}^n)^m}{h_{j-1}}\right\},$$

$$B_j^n = \frac{4m}{m-1}x_j \frac{(f_{j+1}^n)^m - (f_j^n)^m}{h_j}.$$

By using (3.10), we have

$$\theta_j^n f_j^n + \Delta t_n A_j^n = \Delta t_n p_j^n f_{j+1}^n + \left(\theta_j^n - \Delta t_n(p_j^n + q_j^n)\right)f_j^n + \Delta t_n q_j^n f_{j-1}^n,$$

$$(1 - \theta_j^n)f_j^n + \Delta t_n B_j^n = \Delta t_n r_j^n f_{j+1}^n + \left((1 - \theta_j^n) - \Delta t_n r_j^n\right)f_j^n,$$

where p_j^n, q_j^n, and r_j^n are nonnegative numbers given by

$$p_j^n = x_j^2 \frac{2}{h_j + h_{j-1}} \frac{\gamma_j^n}{h_j}, \quad q_j^n = x_j^2 \frac{2}{h_j + h_{j-1}} \frac{\gamma_{j-1}^n}{h_{j-1}}, \quad \text{and} \quad r_j^n = \frac{4m}{m-1} x_j \frac{\gamma_j^n}{h_j},$$

respectively. Let $\theta_j^n = (p_j^n + q_j^n)/(p_j^n + q_j^n + r_j^n)$. Then we have from (3.8)

$$\theta_j^n - \Delta t_n(p_j^n + q_j^n) \geq 0 \quad \text{and} \quad (1 - \theta_j^n) - \Delta t_n r_j^n \geq 0,$$

which implies

$$\theta_j^n \min\{f_{j+1}^n, f_j^n, f_{j-1}^n\} \leq \theta_j^n f_j^n + \Delta t_n A_j^n \leq \theta_j^n \max\{f_{j+1}^n, f_j^n, f_{j-1}^n\}, \tag{3.16}$$

$$(1 - \theta_j^n) \min\{f_{j+1}^n, f_j^n\} \leq (1 - \theta_j^n) f_j^n + \Delta t_n B_j^n \leq (1 - \theta_j^n) \max\{f_{j+1}^n, f_j^n\}. \tag{3.17}$$

On the other hand, it follows from (3.6) and (3.9) that

$$0 \leq \Phi(\Delta t_n; f_j^n) - f_j^n \leq \Phi(\Delta t_n; \max_{0 \leq j \leq N} f_j^n) - \max_{0 \leq j \leq N} f_j^n. \tag{3.18}$$

From (3.15)–(3.18), we obtain

$$\min_{0 \leq j \leq N} f_j^n \leq f_j^{n+1} \leq \Phi(\Delta t_n; \max_{0 \leq j \leq N} f_j^n).$$

Hence, by the inductive hypothesis,

$$0 \leq f_j^{n+1} \leq \Phi(\Delta t_n; \Phi(t_n; \|f_0\|_\infty)) = \Phi(t_{n+1}; \|f_0\|_\infty),$$

which implies (3.14). Hence our induction on n is complete, which gives the proof. $\square$
The above theorem shows a comparison result in the numerical scheme. We may expect that the comparison result similar to (3.13) holds for the solution of (2.4). Unfortunately we are unable to prove it.

We now show that this numerical solution always blows up under some conditions imposed on the nodal points.

THEOREM 3.2. Under (3.8)–(3.9) and (3.11), assume

$$x_1 < a \quad \text{and} \quad \frac{x_1}{x_2 - x_1} < \frac{m(m+1)}{(m-1)(3m-1)}, \tag{3.19}$$

where a is the constant in Condition 2.1. Then

$$\lim_{n \to \infty} \max_{0 \leq j \leq N} f_j^n = \infty. \tag{3.20}$$

Proof. Since $f_j^n \geq 0$ ($0 \leq j \leq N$, $n \geq 0$) holds by Theorem 3.1, it follows from (3.2) that

$$\frac{f_1^{n+1} - f_1^n}{\Delta t_n} \geq -\left[x_1^2 \frac{2}{x_2 - x_0} \left\{\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right\} + \frac{4m}{m-1} x_1 \frac{1}{x_2 - x_1}\right] (f_1^n)^m$$

$$+ \frac{\Phi(\Delta t_n; f_1^n) - f_1^n}{\Delta t_n}$$

$$= -2 \frac{x_1}{x_2 - x_1} \frac{3m-1}{m-1} (f_1^n)^m + \frac{\Phi(\Delta t_n; f_1^n) - f_1^n}{\Delta t_n}.$$

Using (3.7), we have

$$\frac{\Phi(\Delta t_n; f_1^n) - f_1^n}{\Delta t_n} = \Phi_t(\xi_n; f_1^n) = \frac{2m(m+1)}{(m-1)^2} \Phi(\xi_n; f_1^n)^m \geq \frac{2m(m+1)}{(m-1)^2} (f_1^n)^m$$

for some $\xi_n \in (0, \Delta t_n)$. Hence we obtain

$$\frac{f_1^{n+1} - f_1^n}{\Delta t_n} \geq p(f_1^n)^m, \tag{3.21}$$

where

$$p = \frac{2(3m-1)}{m-1} \left\{ \frac{m(m+1)}{(m-1)(3m-1)} - \frac{x_1}{x_2 - x_1} \right\}.$$

We note that p is a positive constant by (3.19).

Suppose that (3.20) does not hold. Then there exists a constant $C_1 > 0$ and a sequence of integers $\{n_k\}_{k \geq 0}$ satisfying

$$n_0 < n_1 < \cdots < n_k < \cdots \qquad \text{and} \qquad f_j^{n_k} \leq C_1 \quad (0 \leq j \leq N, \; k \geq 0). \tag{3.22}$$

From (3.8)–(3.11) and (3.22), we can easily obtain

$$\Delta t_{n_k} \geq C_2 \quad (k \geq 0) \tag{3.23}$$

for some constant $C_2 > 0$. Since $\{f_1^{n_k}\}_{k \geq 0}$ is a bounded and increasing sequence from (3.21) and (3.22), $f_1^{n_k}$ converges to some constant $\overline{f}_1 (\geq f_1^0)$ as $k \to \infty$. In view of (3.21) and (3.23),

$$f_1^{n_k+1} - f_1^{n_k} \geq pC_2(f_1^{n_k})^m \quad (k \geq 0).$$

Letting $k \to \infty$, we have

$$0 \geq pC_2\overline{f}_1{}^m \geq pC_2(f_1^0)^m.$$

On the other hand, $f_1^0 > 0$ holds by (2.8) and (3.19). This is a contradiction, which gives the proof. $\qquad \square$

By using Theorem 3.1, let us show the estimate (1.7) stated in Sec. 1.

THEOREM 3.3. Assume (1.4) in Theorem 1.1. Let the same assumptions as in Theorem 3.1 be satisfied. Then

$$(m-1)/2m(m+1)\beta \leq T_h^* \leq (m-1)/2m(m+1)\alpha \tag{1.7}$$

holds, where α and β are the constants defined by (1.4).

Proof. Since (1.4) implies that $f_0(0) = \alpha^{1/(m-1)}$, we have from (3.3)

$$\Phi(t_n; \alpha^{1/(m-1)}) = f_0^n \quad (n \geq 0).$$

It follows from (3.6) that $\Phi(t; \alpha^{1/(m-1)})$ is bounded if $t < (m-1)/2m(m+1)\alpha$. Hence $t_n < (m-1)/2m(m+1)\alpha$ holds for $n \geq 0$, which yields $T_h^* \leq (m-1)/2m(m+1)\alpha$.

On the other hand, it follows from (1.4) that $\|f_0\|_\infty \leq \beta^{1/(m-1)}$. By Theorems 3.1 and 3.2,

$$\Phi(t_n; \beta^{1/(m-1)}) \geq \max_{0 \leq j \leq N} f_j^n \to \infty \quad \text{as} \quad n \to \infty.$$

Thus we have $T_h^* \geq (m-1)/2m(m+1)\beta$, and the proof is complete. $\qquad \square$

4. Numerical simulations. Now let us show some numerical simulations for $m = 2$ and the initial function (1.6). Throughout this section, we use

$$T_h^* = \min \left\{ t_n; \max_{0 \leq j \leq N} f_j^n > 10^{50} \right\}, \tag{4.1}$$

instead of (3.12). The constant L which satisfies (2.3) is taken as $L = 5$, because (1.6) and (2.3) implies $L \geq \pi$. The nodal points $\{x_j\}_{0 \leq j \leq N}$, which satisfy (3.19), are determined by $x_0 = 0$, $x_{j+1} = x_j + h_j$ $(0 \leq j \leq N - 2)$ and $x_N = L$, where

$$h_j = \begin{cases} 2^{-12}, & \text{if } 0 \leq x_j < 0.01, \\ 2^{-11}, & \text{if } 0.01 \leq x_j < 0.1, \\ 2^{-10}, & \text{if } 0.1 \leq x_j < 1, \\ 2^{-9}, & \text{if } 1 \leq x_j \leq L. \end{cases} \tag{4.2}$$

In order to obtain an accurate numerical waiting time T_h^*, we need a lot of nodal points $\{x_j\}$ near the origin $x = 0$, because the numerical solution f_j^n blows up at those points. This is the reason why we use the irregular mesh points (3.1).

Figure 1 shows the numerical interfaces $\zeta^h(t)$ by the interface tracking scheme by Mimura, Nakaki, and Tomoeda [15], and the numerical waiting times T_h^* by the present scheme (3.2)–(3.5) for $\theta = 0, \frac{1}{2}, 1$. The convergence of these numerical interfaces is proved in [15]. Comparing the numerical interfaces and numerical waiting times, we can say that our numerical waiting times are reliable.

When the initial function is given by (1.6), it seems that the numerical waiting time T_h^* decreases as the mesh width decreases. To demonstrate this, we choose

$$h_j = \begin{cases} 2^{N-12}, & \text{if } 0 \leq x_j < 0.01, \\ 2^{N-11}, & \text{if } 0.01 \leq x_j < 0.1, \\ 2^{N-10}, & \text{if } 0.1 \leq x_j < 1, \\ 2^{N-9}, & \text{if } 1 \leq x_j \leq L. \end{cases} \tag{4.3}$$

and compute the numerical waiting time $T_h^* = T_h^*(N)$. In Fig. 2, we show $E_N \equiv (T_h^*(N) - T_h^*(0)) / T_h^*(0)$ with $N = 1, 2, \cdots, 5$ and $\theta = \frac{1}{2}, 1$. One can find that E_N is nonnegative and decreases as N decreases. Figure 3 displays the upper and lower bounds of the waiting time given by (1.5) in Theorem 1.1, and the numerical waiting times T_h^* by the present scheme (3.2)–(3.5) for $\theta = 0, 0.1, 0.2, \ldots, 1$. We use the mesh points generated by (4.2). Aronson [3] conjectures that the interface $\zeta(t)$ is not smooth at $t = t^*$ when $\frac{1}{4} < \theta \leq 1$. Our simulation suggests that the conjecture is true, because, for $\frac{1}{4} < \theta \leq 1$, it seems that $t^* < (m-1)/2m(m+1)\alpha$, which implies $\zeta \notin C^1$ (see Sec. 1). However, we are not able to give the mathematical proof.
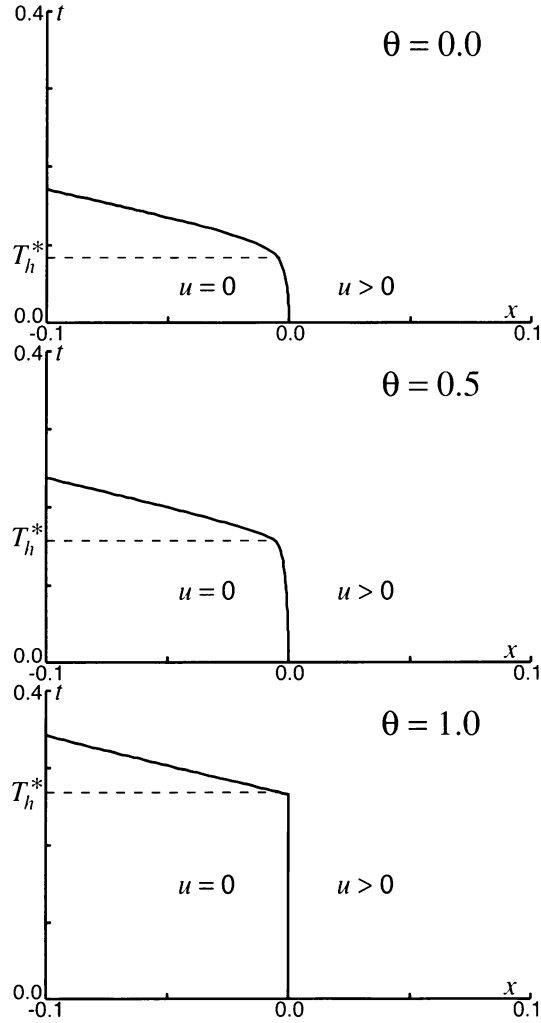
FIG. 1. Numerical interface $\zeta^h(t)$ by the interface tracking scheme [15] with the mesh width $\Delta x = 0.01$, and numerical waiting time T_h^* by the present scheme. The initial function is given by (1.6).
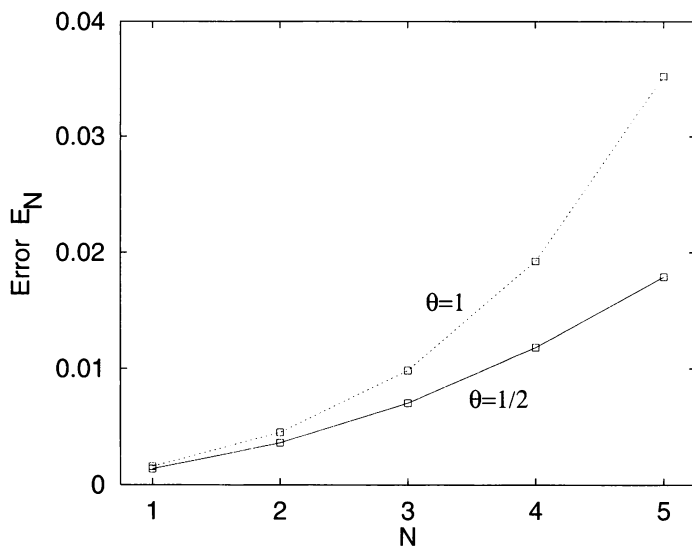
FIG. 2. Relative error E_N with $N = 1, 2, \cdots, 5$ and $\theta = \frac{1}{2}, 1$.
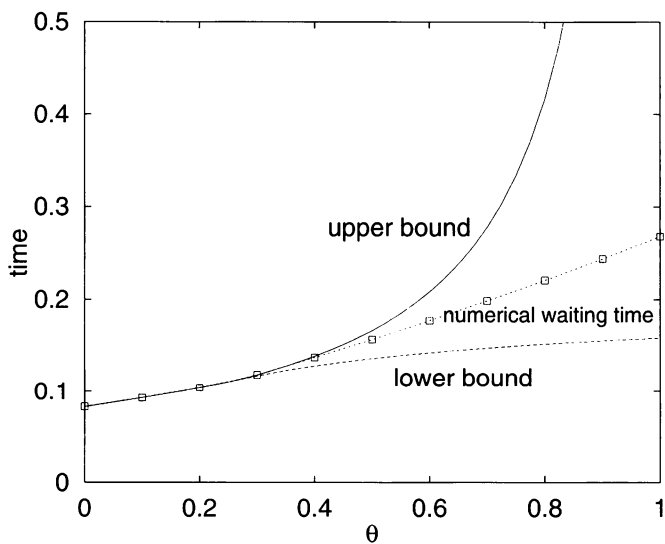

FIG. 3. Upper and lower bounds of waiting time in Theorem 1.1, and numerical waiting time by the present scheme. The initial function is given by (1.6).

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