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Numerical Approximation Methods for Elliptic Boundary Value Problems

Finite and Boundary Elements

 Springer

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