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NUMERICAL APPROXIMATION OF SOME LINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY SPECIAL ADDITIVE NOISES*

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Abstract. This paper is concerned with the numerical approximation of some linear stochastic partial differential equations with additive noises. A special representation of the noise is considered, and it is compared with general representations of noises in the infinite dimensional setting. Convergence analysis and error estimates are presented for the numerical solution based on the standard finite difference and finite element methods. The effects of the noises on the accuracy of the approximations are illustrated. Results of the numerical experiments are provided.

Key words. stochastic partial differential equation, additive noise, finite difference method, finite element method, convergence, error estimate

AMS subject classifications. 65M65, 65C30, 35R60, 60H15

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1. Introduction.

In recent years, it has been increasingly acceptable to adopt SDE models as an essential component in the analysis of complex phenomena such as wave propagation [19], climate change [22], turbulence [21, 24], and phase transition [9, 16, 18]. The initial value and boundary value problems of stochastic partial differential equations (SPDEs) have been studied theoretically in, for example, [5, 6, 8, 10, 33]. Various numerical methods and approximation schemes for SDEs have also been developed, analyzed, and tested [1, 2, 4, 7, 12, 13, 14, 15, 20, 25, 27, 29, 28, 31, 34, 35].

For a given physical system, many different stochastic perturbations may be considered. Generically speaking, noise may enter the physical system either as temporal fluctuations of internal degrees of freedom or as random variations of some external control parameters; internal randomness often reflects itself in *additive* noise terms, while external fluctuations gives rise to *multiplicative* noise terms [18]. The main aim of this paper is to study the properties of some standard numerical approximations to the linear SPDEs for the random field $u = u(x,t)$ driven by an additive noise:

$$du = Au\,dt + dW, \quad x \in \Omega, \quad t > 0. \tag{1.1}$$

Here, Ω is a bounded spatial domain and A is a linear second order elliptic operator with deterministic coefficients, which is defined on a space of functions satisfying certain boundary conditions. W represents an infinite dimensional Brownian motion. We also consider the related time-independent equation

$$-Au = g + \dot{W}, \quad x \in \Omega, \tag{1.2}$$

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where g is a given deterministic function and $\dot{W}$ denote a one-parameter family noise. The additive noises may appear in various forms, ranging from the space time white noise to colored noises generated by some infinite dimensional Brownian motion with a prescribed covariance operator [6, 28]. Once the equation is reformulated into a weak form [5], the usual Galerkin finite element methods can be constructed and also analyzed using standard techniques. A priori error estimates of the numerical solution depend on the regularity of the solutions of the original SPDE. Such regularity results are often much harder to establish than their deterministic counterpart [5, 33]. In fact, if dW corresponds to the Brownian white noise, then the regularity estimates are usually very weak, and they lead to very low order error estimates [1, 7, 13]. On the other hand, if the noise is more regular, then it becomes possible to get higher order of error estimates for the numerical solution. In recent years, studies of models with colored noises and their numerical approximation have started to receive more attention; see [28] for an example of physical application and the recent works [26, 14] for works related to stochastic ordinary differential equations (SODEs) and the time discretization. In the present work, we provide the connections between the discrete realizations of noises in different formulations of some SPDEs. Moreover, we illustrate how the error analysis of the standard finite element and finite difference approximations depends on the noises used in the model and the approximation. In order to present a simple analysis, in this paper we focus on the case $\Omega = (0,1)$ and $Au = u_{xx} - bu$ with the homogeneous Dirichlet boundary condition and b being a deterministic coefficient, though much of our results can be readily extended to higher spatial dimensions and more general second order elliptic operators. For most of the discussion, we also try to present our results in simple finite element terminology that is familiar to people working on the numerical approximations of deterministic PDEs so that it is easy to be understood even for readers who are not necessarily experts on SDEs.

The paper is organized as follows. We first describe the various forms of the noises and their discrete representations. Next, we discuss some convergence results for standard finite element and finite difference approximations. The models used are one dimensional linear stochastic elliptic and parabolic equations, and the results are established for noises given in general forms, which include the spatial or space time white noises as special cases. Then numerical results are presented to support the theoretical analysis. Finally, some concluding remarks are given. The details of the proofs are provided in the appendix.

2. The representation of random noises. To study the accuracy of the discrete approximations, it is useful to first consider the properties of the noises which drive the stochastic equations and the discrete representations of the noises.

Following [1], we *regularize* the noise through discretization. Let $\{x_i = ih\}_0^n$ be a partition of $[0,1]$ with $h = 1/n$. We begin with $\dot{W}(x)$ being the standard one-parameter family Brownian white noise that satisfies

$$E(\dot{W}(x) \cdot \dot{W}(x')) = \delta(x - x'), \tag{2.1}$$

where δ denote the usual Dirac δ-function and E the expectation. A piecewise constant approximation of the one-parameter white noise is given by [1]

$$\frac{d\widehat{W}_n(x)}{dx} = c_n \sum_{j=1}^{n} \eta_j \chi_j(x), \tag{2.2}$$

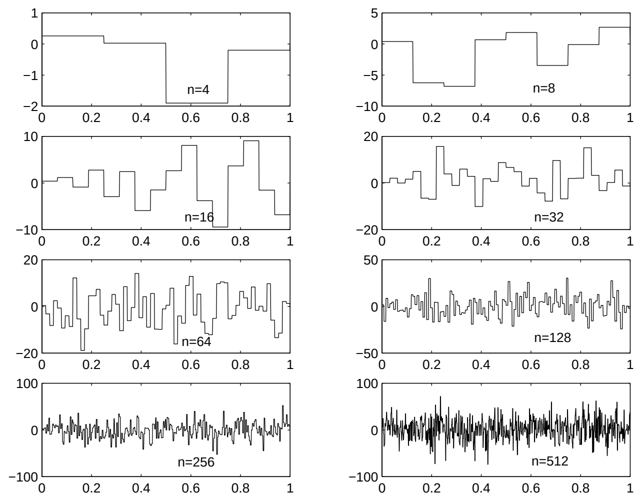

FIG. 2.1. *Piecewise constant approximation for the noise* $d\widehat{W}_n(x)/dx = (1/\sqrt{h})\sum_{j=1}^{n}\eta_j\chi_j(x)$.

where $c_n = h^{-1/2} = \sqrt{n}$ and, for $j = 1, 2, \dots, N$, $\eta_j \in N(0,1)$ is independently and identically distributed (iid),

$$\sqrt{h}\eta_j = \int_{x_j}^{x_{j+1}} dW(x)\,, \quad \text{and} \quad \chi_j(x) = \begin{cases} 1, & x_j \le x < x_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

The discrete analogue of (2.1) for the piecewise constant approximation is given by

$$E\left(\frac{d\widehat{W}_n(x)}{dx}\cdot\frac{d\widehat{W}_n(x')}{dx}\right) = \begin{cases} h^{-1} & \text{if } x_j \le x,\ x' < x_{j+1} \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\lim_{n\to\infty} E\left(\frac{d\widehat{W}_n(x)}{dx}\cdot\frac{d\widehat{W}_n(x')}{dx}\right) = \delta(x-x').$$

In Figure 2.1, some sample realizations of the piecewise constant approximation of one-parameter white noise are illustrated for various values of n. (The random numbers are generated using MATLAB.) We note that similar discussions can be easily generalized to the space time two-parameter family white noises.

2.1. Noises in abstract forms. The SPDEs driven by the white noise often have poor regularity estimates. In the physical world, to take into account the short and long range correlations of the stochastic effects, both white noise and colored noises may be considered. There are many situations where colored noises model the reality more closely, and there are also instances where the important stochastic effects are the noises acting on a few selected frequencies.

In general, we may use an abstract formulation of the infinite dimensional noise:

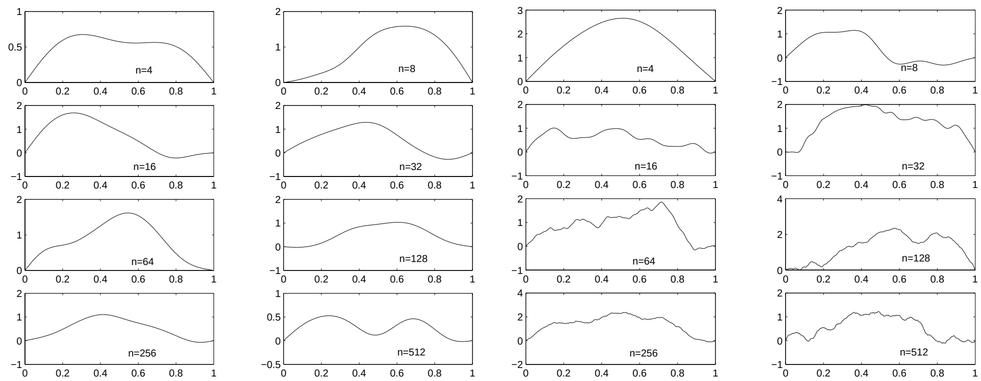

FIG. 2.2. *Noises by Fourier modes* $\sum_{k=1}^{n} \sigma_k \eta_k \sqrt{2} \sin k\pi x$ *with* $\sigma_k = \frac{1}{2^k}$ *(left) and* $\sigma_k = \frac{1}{k^{3/2}}$ *(right).*

$$\dot{W}(x) = \sum_{k=1}^{\infty} \sigma_k \eta_k \psi_k(x), \tag{2.3}$$

where the random variable $\eta_k \sim N(0,1)$ is iid for any k, the deterministic functions $\{\psi_k(x)\}$ form an orthonormal basis of $L_2(0,1)$ or its subspace, and the coefficients $\{\sigma_k\}$ are to be chosen to ascertain the convergence of the series in the mean square sense with respect to some suitable norms.

One of the examples is given by the Fourier modes $\psi_k(x) = \sqrt{2} \sin k\pi x$ which forms a basis of $H_0^1(0,1)$. According to the different decay rates of the coefficients, the noises may display quite different pictures. The pictures in Figure 2.2 and the left two columns of Figure 2.3 provide sample realizations of noises having forms (2.3) in the Fourier basis with coefficients $\sigma_k = 2^{-k}$, $k^{-3/2}$, and $k^{-1/2}$, respectively. Clearly, the realizations give trajectories that look smoother than the ones for the white noise. It can also be seen that the faster the coefficients σ_k decay, the smoother the noise trajectory dW_n/dx looks, which reflects stronger spatial correlation since the noises are heavily concentrated near a few low frequencies. On the other hand, if the coefficients decay sufficiently slowly, then the trajectory can clearly resemble that of a white noise away from the boundary. In fact, it is well known that for spatially uncorrelated white noises, their Fourier coefficients are independent of the frequencies, and they stay at a constant value.

In the analysis and numerical examples given in later sections, the noises given in terms of the Fourier modes are used. The Fourier modes provide one of many possible representations of noises where the smoothness of the noise trajectories are related to the decay of the coefficients in the representation. Another illustrative example is to define the noise in terms of the lowest order wavelet basis. We include the discussion here for comparison. Let ψ be the *wavelet function* and ϕ be the *scaling function* [32]. Let j denote the *dilation index* and k denote the *translation index*, and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. The discrete noise formulated in the wavelet basis is given as

$$\dot{W}_J(x) = c\gamma\phi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}\eta_{jk}\psi_{j,k}(x). \tag{2.4}$$

Here, J is the highest level to be considered, and $\gamma, \eta_{jk} \in N(0,1)$ are iid. In the

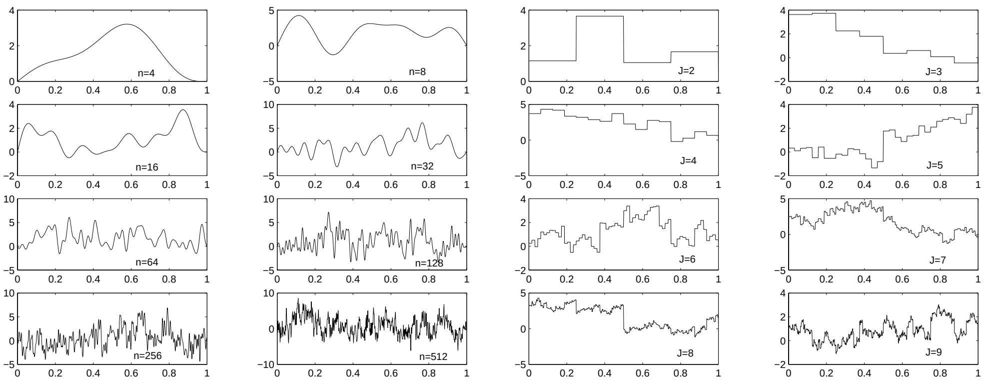

FIG. 2.3. *Noises by* $\sum_{k=1}^{n} \frac{1}{k^{1/2}}\eta_k\sqrt{2}\sin k\pi x$ *(left) and* $\gamma\phi(x)+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\frac{1}{2^j}\eta_{jk}\psi_{j,k}(x)$ *(right).*

simplest case, we may take the *Haar* wavelet

$$\psi(x)=\begin{cases}1, & 0\le x<1/2,\\ -1, & 1/2\le x<1,\\ 0 & \text{otherwise}\end{cases}\qquad \text{and}\qquad \phi(x)=\begin{cases}1, & 0\le x<1,\\ 0 & \text{otherwise.}\end{cases}$$

The right two columns of Figure 2.3 show sample realizations of noises taking the form (2.4) with $c=1$, $d_{j,k}=2^{-j}$. The correlation of the noise (2.4) $E(\frac{dW_J(x)}{dx}\cdot\frac{dW_J(x')}{dx})$ is given by

$$E\left(\frac{dW_J(x)}{dx}\cdot\frac{dW_J(x')}{dx}\right)=c^2\phi(x)\phi(x')+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}d_{j,k}^2\psi_{j,k}(x)\psi_{j,k}(x')\ . \tag{2.5}$$

If in (2.2) $n=2^J$ and $h=2^{-J}$, then the piecewise constant approximation of the white noise may also be represented using the wavelet Haar basis. In fact, let $\chi_k(x)$ be characteristic function of interval $[kh,(k+1)h]$; then

$$\frac{d\widehat{W}_n(x)}{dx}=\frac{1}{\sqrt{h}}\sum_{k=0}^{2^J-1}\eta_k\chi_k(x)=\gamma\phi(x)+\sum_{j=0}^{J-1}\sum_{l=0}^{2^j-1}\gamma_{j,l}\psi_{j,l}(x).$$

Here, $\gamma=2^{-J/2}\sum_{k=0}^{2^J-1}\eta_k\sim N(0,1)$ and

$$\gamma_{j,l}=2^{(j-J)/2}\left(\sum_{k=l2^{J-j}}^{(l+1)2^{J-j}-1}(-1)^{[k/2^{J-j-1}]}\eta_k\right)\sim N(0,1)$$

are iid. Corresponding to (2.5), $c=d_{j,k}=1$ so that (2.5) leads again to (2.1). Naturally, when higher order wavelets are used [32, 30], we may expect to have discrete noises that are smoother spatially than the ones represented by the Haar basis when the high frequency coefficients enjoy fast decay properties. Comparing with Fourier modes, wavelet functions may also have compact support; thus, on the one hand, the noises in wavelet basis can closely resemble spatially uncorrelated white noises, while on the other hand they can also be used conveniently to simulate noise more concentrated on certain frequencies as well as certain spatial regions.

In summary, different forms to represent the various noises are discussed in this section. Similar discussion can be carried out in more than one space dimension and for noises parameterized by both time and space variables. Such discussions are relevant to the numerical study of SDEs as the solutions of the stochastic equations that use noises with better regularity become more regular themselves and thus may allow higher order numerical approximations.

3. Numerical method and error analysis. In [1], approximations of SPDEs with the additive space time white noise term discretized by the piecewise constant random process have been studied. Here, we follow roughly the same route, though more general types of noises are used. We show how the accuracy is affected by the correlation or the *smoothness* of the noises.

We divide the discussion into two parts, starting with the simplest one dimensional elliptic equation (boundary value problem of a SODE) and then moving to a parabolic equation in one space dimension and in time (initial boundary value problem of a SPDE). In the set-up of the problems, noises represented in general basis are used, but in the analysis we specialize in using the Fourier modes as the basis of choice to simplify the discussion.

3.1. One dimensional elliptic equation with noise. We now consider the SDE (1.2); that is,

$$\begin{cases} -\Delta u(x) + bu(x) = g(x) + \dot{W}(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $\dot{W}(x)$ denotes the noise, $g(x)$ is a given deterministic term, and $b = b(x)$ is a given deterministic coefficient.

As in [1], we may first replace $\dot{W}(x)$ by a finite dimensional noise $\dot{W}_n(x)$ and let u_n denote the solution of the corresponding equation. We then numerically approximate the equation associated with $\dot{W}_n(x)$ and let u_n^h denote the numerical solution.

If the noise $\dot{W}(x)$ in (3.1) is the white noise, $\dot{W}_n(x)$ is the piecewise constant approximation (2.2), and the Galerkin finite element method with piecewise constant basis is applied to (3.1), the error estimate is given by [1]

$$\begin{aligned} &E\|u - u_n\|_{L_2} \le C\ h, \\ &E\|u_n - u_n^h\|_{L_2} \le C\ h^{3/2}, \\ &E\|u - u_n^h\|_{L_2} \le C\ h. \end{aligned}$$

Due to the poor regularity of the solution, it is seen that, even with higher order finite elements, the order of error estimates does not improve. With colored noises, the order of approximation may increase with better regularity on the solution and the use of higher order elements. As an illustration, we consider the following noise:

$$\dot{W}(x) = \sum_{k=1}^{\infty} \sigma_k \eta_k \psi_k(x), \tag{3.2}$$

where $\{\eta_k\}$ are random variables satisfying

$$\eta_k \sim N(0,1) \ \text{ and } \ cov(\eta_k, \eta_l) = E(\eta_k \eta_l) = q_{kl},$$

with $\{\sigma_k\}$ to be chosen.

Let $\{\sigma_k^n\}_{k=1}^\infty$ approach $\{\sigma_k\}_{k=1}^\infty$ as $n \to \infty$ in some appropriate sense; then an approximation of $\dot{W}(x)$ is

$$\dot{W}_n(x) = \sum_{k=1}^{\infty} \sqrt{2}\sigma_k^n \eta_k \psi_k(x) \sin k\pi x.$$

The definition of noise term leads to the following stochastic integral for $f \in L_2(0,1)$:

$$S = \int_0^1 f(x)dW(x) = \sum_{k=1}^{\infty} \sigma_k f_k \eta_k,$$

$$S_n = \int_0^1 f(x)dW_n(x) = \sum_{k=1}^{\infty} \sigma_k^n f_k \eta_k,$$

where $f_k = \int_0^1 f(x)\psi_k(x)dx$. That is, S and S_n are random variables having the distribution

$$S \sim N\left(0, \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sigma_k \sigma_l f_k f_l q_{kl}\right),$$

$$S_n \sim N\left(0, \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sigma_k^n \sigma_l^n f_k f_l q_{kl}\right),$$

provided the double sum is convergent.

For convenience, we introduce the following notation:

$$\overrightarrow{\sigma^n} = (\sigma_1^n, \sigma_2^n, \dots, \sigma_k^n, \dots)^T,$$
$$\vec{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_k, \dots)^T$$

are infinite column vectors. For two vectors $\overrightarrow{\sigma^n}$ and $\overrightarrow{f}$, we use $\overrightarrow{\sigma^n f}$ to denote the componentwise product

$$\overrightarrow{\sigma^n f} = (\sigma_1^n f_1, \sigma_2^n f_2, \dots, \sigma_k^n f_k, \dots)^T.$$

Let Q be the covariance matrix of random fields $\{\eta_k\}$, namely, Q is the infinite matrix (operator) with entries $Q = (q_{kl})_{k,l=1}^\infty$. For an integer s, let Q_s be the infinite matrix with entries $Q_s = ((kl)^s q_{kl})_{k,l=1}^\infty$. It is easy to see both Q and Q_s are positive semidefinite. Define the weighted semi-inner products of the vectors $\vec{\sigma}$ and $\vec{\delta}$ as

$$\langle \vec{\sigma}, \vec{\delta} \rangle_Q = \vec{\sigma}^T \cdot Q \cdot \vec{\delta} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sigma_k \delta_l q_{kl},$$

$$\langle \vec{\sigma}, \vec{\delta} \rangle_{Q_s} = \vec{\sigma}^T \cdot Q_s \cdot \vec{\delta} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sigma_k \delta_l (kl)^s q_{kl}.$$

The seminorms induced by the above semi-inner products are

$$\|\vec{\sigma}\|_Q^2 = \langle \vec{\sigma}, \vec{\sigma} \rangle_Q \quad \text{and} \quad \|\vec{\sigma}\|_{Q_s}^2 = \langle \vec{\sigma}, \vec{\sigma} \rangle_{Q_s}.$$

Note that $Q_0 = Q$. Using the above notation,

$$S \sim N\left(0, \|\overrightarrow{\sigma f}\|_Q^2\right), \qquad S_n \sim N\left(0, \|\overrightarrow{\sigma^n f}\|_Q^2\right).$$

The difference between S and S_n is given by

$$E|S - S_n|^2 = E|\sum_{k=1}^{\infty}(\sigma_k^n - \sigma_k) f_k \eta_k|^2 = \left\|\overrightarrow{\sigma f} - \overrightarrow{\sigma^n f}\right\|_Q^2.$$

Equation (3.1) can be written in a weak form or an integral form. Both forms are equivalent as shown in [3]. In fact, the solution of (3.1) is a stochastic process $u = u(x)$ which satisfies the weak formulation

$$-\int_0^1 u(x)\Delta\phi(x)dx + \int_0^1 bu(x)\phi(x)dx = \int_0^1 g(x)\phi(x)dx + \int_0^1 \phi(x)dW(x) \tag{3.3}$$

for $\phi \in C^2(0,1) \cap C_0(0,1)$. The integral form is

$$u(x) + \int_0^1 b\,k(x,y)\,u(y)dy = \int_0^1 k(x,y)g(y)dy + \int_0^1 k(x,y)dW(y). \tag{3.4}$$

Here, $k(x,y) = x \wedge y - xy$ is the Green's function associated with the elliptic equation $-\Delta v(x) = \phi(x), v(0) = v(1) = 0$ so that $v(x) = \int_0^1 k(x,y)\phi(y)dy$. ($x \wedge y$ means the smaller one of x and y.) In the present investigation, it is assumed the coefficient b is small enough so that $\lambda^2 = \int_0^1\int_0^1 b^2 k^2(x,y)dxdy < 1$. We note that this condition is primarily needed in the case of $b < 0$; such a restriction can be lifted for $b > 0$, and the conclusions given later remain valid.

We now substitute $dW(y)$ by $dW_n(y)$ in (3.4) to obtain the following equation:

$$u_n(x) + \int_0^1 b\,k(x,y)\,u_n(y)dy = \int_0^1 k(x,y)g(y)dy + \int_0^1 k(x,y)dW_n(y). \tag{3.5}$$

Thus, $u_n(x)$ satisfy the two-point boundary value problem

$$-\Delta u_n(x) + bu_n(x) = g(x) + \dot{W}_n(x), \qquad u_n(0) = u_n(1) = 0. \tag{3.6}$$

The following theorem shows that u_n indeed approximates u, the solution of (3.4). In order to illustrate the higher order of convergence for more regular noises, we specialize our discussion to the choice of $\{\psi_k(x) = \sqrt{2}\sin k\pi x\}$, that is, noises represented by the Fourier modes.

THEOREM 3.1. *For* $\dot{W}_n(x) = \sum_{k=1}^{\infty}\sigma_k^n \eta_k \psi_k(x)$ *and* $\psi_k(x) = \sqrt{2}\sin k\pi x$, *if* u_n *and* u *are the solutions of* (3.5) *and* (3.4), *respectively, then, for some constant* $C > 0$,

$$E\|u - u_n\|_{L_2} \le \frac{C}{1-\lambda}\left\|\overrightarrow{\sigma^n} - \vec{\sigma}\right\|_{Q_{-1}},$$

where $\lambda < 1$ *is defined as before.*

Proof. Let $e_n(x) = u(x) - u_n(x)$ and

$$F(x) = \int_0^1 k(x,y)dW(y) - \int_0^1 k(x,y)dW_n(y).$$

Subtracting (3.5) from (3.4), we have

$$e_n(x) = -\int_0^1 b\, k(x,y)\, e_n(y)dy + F(x).$$

By Hölder's inequality, it is easy to show that

$$\int_0^1 e_n^2(x)dx \le \lambda^2 \int_0^1 e_n^2(y)dy + 2\lambda \left(\int_0^1 F^2(x)dx\right)^{1/2} \left(\int_0^1 e_n^2(y)dy\right)^{1/2} + \int_0^1 F^2(x)dx,$$

where $\lambda^2 = \int_0^1 \int_0^1 b^2 k^2(x,y)dxdy$ and it is assumed that $\lambda < 1$. Taking expectations on both sides, letting $\hat{e}_n = E(\int_0^1 e_n^2(x)dx)$ and $\hat{G}_n = E(\int_0^1 F^2(x)dx)$ and using the Burkholder–Gundy-type inequality $(EX)^2 \le E(X^2)$, we get

$$\hat{e}_n(1-\lambda^2) - 2\lambda\sqrt{\hat{e}_n}\sqrt{\hat{G}_n} - \hat{G}_n \le 0. \tag{3.7}$$

This implies

$$\sqrt{\hat{e}_n} \le \sqrt{\hat{G}_n}(1-\lambda). \tag{3.8}$$

Now let us estimate $\hat{G}_n$.

$$\begin{aligned}\hat{G}_n &= E\left(\int_0^1 F^2(x)dx\right) = \int_0^1 E\left(\sum_{k=1}^{\infty}(\sigma_k^n - \sigma_k)f_k(x)\eta_k\right)^2 dx \\ &= \int_0^1 \left\|\overrightarrow{\sigma f(x)} - \overrightarrow{\sigma^n f(x)}\right\|_Q^2 dx,\end{aligned}$$

where $\overrightarrow{f(x)} = (f_1(x), f_2(x), \dots, f_k(x), \dots)^T$ and $f_k(x) = \int_0^1 k(x,y)\psi_k(y)dy$. Since $k(x,y) = x \wedge y - xy$, direct calculation gives that, for any $x \in [0,1]$,

$$|f_k(x)| = \left|\int_0^1 k(x,y)\psi_k(y)dy\right| = \left|\int_0^1 k(x,y)\sqrt{2}\sin k\pi y dy\right| \le \frac{c}{k},$$

which implies that, for $x \in [0,1]$,

$$\left\|\overrightarrow{\sigma f(x)} - \overrightarrow{\sigma^n f(x)}\right\|_Q \le C\left\|\overrightarrow{\sigma} - \overrightarrow{\sigma^n}\right\|_{Q_{-1}}$$

for some constant $C > 0$. Hence,

$$\hat{G}_n \le C\left\|\vec{\sigma} - \overrightarrow{\sigma^n}\right\|_{Q_{-1}}^2.$$

Combining the above inequality with (3.8), we get

$$E\|u-u_n\|_{L_2} \le \sqrt{E\|u-u_n\|_{L_2}^2} = \sqrt{\hat{e}_n} \le \frac{C}{1-\lambda}\left\|\overrightarrow{\sigma^n} - \vec{\sigma}\right\|_{Q_{-1}}.$$

This proves the theorem. □

We now state a bound on $\dot{W}_n(x)$ in the following lemma.

LEMMA 3.1. *For $\dot{W}_n(x) = \sum_{k=1}^{\infty} \sigma_k^n \eta_k \psi_k(x)$ and $\psi_k(x) = \sqrt{2}\sin k\pi x$, if $s \geq 0$ is an integer, then*

$$E\|\dot{W}_n\|_{H^s} \leq C \left(\sum_{k=1}^{\infty} (\sigma_k^n k^s)^2 \right)^{1/2},$$

provided that the right-hand side is convergent.

Proof. First,

$$\frac{d^s}{dx^s}\left(\frac{dW_n}{dx}\right) = \sum_{k=1}^{\infty} \sqrt{2}\sigma_k^n \eta_k (k\pi)^s \sin\left(s\frac{\pi}{2} + k\pi x\right).$$

Since $\{\sin(s\frac{\pi}{2} + k\pi x)\}$ are orthogonal on $[0,1]$, we have

$$\begin{aligned} E\left\|\frac{d^s}{dx^s}\left(\frac{dW_n}{dx}\right)\right\|_{L_2}^2 &= E\int_0^1 \left(\sum_{k=1}^{\infty} \sqrt{2}\sigma_k^n \eta_k (k\pi)^s \sin\left(s\frac{\pi}{2} + k\pi x\right)\right)^2 dx \\ &= E\sum_{k=1}^{\infty} (\sigma_k^n)^2 \eta_k^2 (k\pi)^{2s} \leq c\sum_{k=1}^{\infty} (\sigma_k^n \cdot k^s)^2 \end{aligned}$$

for some constant $c > 0$. The above inequality also implies that, for any $r \leq s$,

$$E\left\|\frac{d^r}{dx^r}\left(\frac{dW_n}{dx}\right)\right\|_{L_2}^2 \leq E\left\|\frac{d^s}{dx^s}\left(\frac{dW_n}{dx}\right)\right\|_{L_2}^2.$$

Hence,

$$E\|\dot{W}_n\|_{H^s} \leq \sqrt{E\|\dot{W}_n\|_{H^s}^2} \leq C\left(\sum_{k=1}^{\infty} (\sigma_k^n k^s)^2\right)^{1/2}$$

for some constant $C > 0$. ☐

Concerning the above lemma, we note that similar lower bound can also be established. Moreover, the results may be established for the case $s < 0$ as well.

We now consider a standard finite element approximation of u_n. From the weak formulation (3.3), u_n satisfies

$$\int_0^1 u_n' \phi'(x)dx + b\int_0^1 u_n(x)\phi(x)dx = \int_0^1 g(x)\phi(x)dx + \int_0^1 \phi(x)dW_n(x) \tag{3.9}$$

for $\phi(x) \in H_0^1(0,1)$. By the Lax–Milgram theorem, there exists a unique solution $u_n \in H_0^1(0,1)$ to (3.9). For convenience, we consider the same partition of $[0,1]$: $0 = x_1 < x_2 < \cdots < x_{n+1} = 1$ with $x_i = (i-1)h$ and $h = 1/n$. If $V_0^h(0,1)$ denotes the finite element subspace of $H_0^1(0,1)$, and $\{\phi_j(x)\}_{j=1}^N$ forms a basis of $V_0^h(0,1)$, the finite element solution of (3.9) is $u_n^h \in V_0^h(0.1)$ that satisfies (3.9) for all $\phi(x) \in V_0^h(0,1)$. Thus, $u_n^h(x) = \sum_{l=1}^N u_l \phi_l(x)$ satisfies the following linear system for $j = 1, 2, \ldots, N$:

$$\begin{aligned} &\sum_{l=1}^N u_l \int_0^1 \phi_l'(x)\phi_j'(x) + b\sum_{l=1}^N u_l \int_0^1 \phi_l(x)\phi_j(x)dx \\ &= \int_0^1 g(x)\phi_j(x)dx + \sum_{k=1}^{\infty} \sigma_k^n \eta_k \int_0^1 \phi_j(x)\psi_k(x)dx, \end{aligned} \tag{3.10}$$

where $\eta_k \in N(0,1)$. The solution u_n^h is clearly well defined.

The following lemma gives the standard finite element error estimates of (3.9) in the pathwise sense.

LEMMA 3.2. *If $V_0^h(0,1)$ contain all piecewise polynomials of degree r in $H_0^1(0,1)$, and $u_n \in H_0^1(0,1) \cap H^{r+1}(0,1)$, then*

$$\|u_n - u_n^h\|_{L^2} + h|u_n - u_n^h|_{H^1} \le Ch^{r+1}\|u_n\|_{H^{r+1}} \le Ch^{r+1}\|g + \dot{W}_n\|_{H^{r-1}} \tag{3.11}$$

for some constant $C > 0$. □

Furthermore, combining Theorem 3.1 and Lemma 3.2, an estimate on $E(\|u - u_n^h\|_{L_2})$ follows from the triangle inequality.

THEOREM 3.2. *Let u and u_n^h be the solution of* (3.3) *and* (3.10), *respectively; if the hypothesis in Lemma* 3.2 *is satisfied, then the error estimate is*

$$\begin{aligned} E\|u - u_n^h\|_{L_2} &\le C\left\{\left\|\overrightarrow{\sigma^n} - \vec{\sigma}\right\|_{Q_{-1}} + h^{r+1}\|g\|_{H^{r-1}} + h^{r+1}E\|\dot{W}_n\|_{H^{r-1}}\right\} \\ &\le C\left\{\left\|\overrightarrow{\sigma^n} - \vec{\sigma}\right\|_{Q_{-1}} + h^{r+1}\|g\|_{H^{r-1}} + h^{r+1}\left[\sum_{k=1}^{\infty}(\sigma_k^n k^{r-1})^2\right]^{1/2}\right\} \end{aligned} \tag{3.12}$$

for some generic constant $C > 0$. □

Numerical examples are given in a later section to provide an illustration of the specific order of error estimates one can get based on the above theorem.

Remark 3.1. The same idea can be applied to two dimensional elliptic equations in a rectangular domain, namely, by representing the two dimensional noise as the combinations of the tensor products of $\psi_k(x)$, similar to how Theorem 3.2 can be obtained.

3.2. Parabolic equation in one spatial dimension. Let $\frac{\partial^2 W}{\partial t \partial x}$ denote a space time noise term and g be a deterministic function; we now consider the linear stochastic equations of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) + bu(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x) + g(t,x), \ t > 0, \\ u(0,x) = u_0(x), \ 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, \ t \ge 0, \end{cases} \tag{3.13}$$

where the coefficient b, for simplicity, is assumed to be a constant.

The weak formulation of (3.13) is

$$\begin{aligned} &\int_0^1 u(t,x)\phi(x)dx - \int_0^t \int_0^1 u(s,x)\frac{d^2\phi}{dx^2}dxds + \int_0^t \int_0^1 bu(s,x)\phi(x)dxds \\ &= \int_0^1 u_0(x)\phi(x)dx + \int_0^t \int_0^1 \phi(x)dW(s,x) + \int_0^t \int_0^1 g(s,x)\phi(x)dxds \end{aligned} \tag{3.14}$$

for $\phi \in C^2[0,1] \cap C_0[0,1]$. The integral formulation of (3.13) is

$$\begin{aligned} u(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y)bu(x,y)dyds &= \int_0^1 G_t(x,y)u_0(y)dy \\ + \int_0^t \int_0^1 G_{t-s}(x,y)dW(s,y) &+ \int_0^t \int_0^1 G_{t-s}(x,y)g(s,y)dyds, \end{aligned} \tag{3.15}$$

where $G_t(x,y) = 2\sum_{m=1}^{\infty} \sin m\pi x \sin m\pi y e^{-(m\pi)^2 t}$ is the fundamental solution of

$$v_t(t,x) - v_{xx}(t,x) = 0, \quad v(0,x) = \phi(x), \quad v(t,0) = v(t,1) = 0,$$

so that $v(t,x) = \int_0^1 G_t(x,y)\phi(y)dy$.

Using the same idea as that in the previous section, we represent the noise as

$$\frac{\partial^2 W}{\partial t \partial x} = \sum_{k=1}^{\infty} \sigma_k(t)\dot{\eta_k}(t)\psi_k(x), \tag{3.16}$$

where $\sigma_k(t)$ is a continuous function, $\dot{\eta_k}(t)$ is the derivative of standard Wiener process, and $\psi_k(x) = \sqrt{2}\sin k\pi x$. Now define a partition of $[0,T]\times[0,1]$ by rectangles $[t_i, t_{i+1}] \times [x_j, x_{j+1}]$ for $i = 1,2,\dots,I$ and $j = 1,2,\dots,n$, where $t_i = (i-1)\Delta t$, $x_j = (j-1)h$, $\Delta t = T/I$, and $h = 1/n$. A sequence of noise which approximates the noise is defined as

$$\frac{\partial^2 W_n}{\partial t \partial x} = \sum_{k=1}^{\infty} \sigma_k^n(t)\psi_k(x) \sum_{i=1}^{I} \frac{1}{\sqrt{\Delta t}} \eta_{ki}\chi_i(t), \tag{3.17}$$

where $\chi_i(t)$ is the characteristic function for the ith time subinterval and

$$\eta_{ki} = \frac{1}{\sqrt{\Delta t}} \int_{t_i}^{t_{i+1}} d\eta_k(t) \ \sim \ N(0,1).$$

Replacing $\sigma_k(t)$ by $\sigma_k^n(t)$, we get the discretization in the x-direction, and replacing $\dot{\eta_k}(t)$ by $\sum_{i=1}^{I} \frac{1}{\sqrt{\Delta t}}\eta_{ki}\chi_i(t)$ we get the discretization in the t-direction. Then $\frac{\partial^2 W_n}{\partial t \partial x}$ is substituted for $\frac{\partial^2 W}{\partial t \partial x}$ in (3.15) to get the following equation:

$$\begin{aligned} u_n(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y) b u_n(s,y) dy ds = \int_0^1 G_t(x,y) u_0(y) dy \\ + \int_0^t \int_0^1 G_{t-s}(x,y) dW_n(s,y) + \int_0^t \int_0^1 G_{t-s}(x,y) g(s,y) dy ds; \end{aligned} \tag{3.18}$$

that is, u_n is the solution of the equation

$$\begin{cases} \frac{\partial u_n}{\partial t}(t,x) - \frac{\partial^2 u_n}{\partial x^2}(t,x) + b u_n(t,x) = \frac{\partial^2 W_n}{\partial t \partial x}(t,x) + g(t,x), \ t > 0, \\ u_n(0,x) = u_0(x), \ 0 \le x \le 1, \\ u_n(t,0) = u_n(t,1) = 0, \ t \ge 0. \end{cases} \tag{3.19}$$

Now we assume that

$$\int_0^T \int_0^1 \int_0^t \int_0^1 G_{t-s}^2(x,y) b^2 dy ds dx dt = \bar{\lambda}^2 < 1.$$

Then, under proper assumptions on $\{\sigma_k(t)\}$ and $\{\sigma_k^n(t)\}$, u_n approximates u, the solution of (3.15), as illustrated in the next theorem.

THEOREM 3.3. *Let* $\{\sigma_k(t)\}$ *and its derivative be uniformly bounded by*

$$|\sigma_k(t)| \le \beta_k, \quad |\sigma_k'(t)| \le \gamma_k \ \ \forall t \in [0,T],$$

and the coefficients $\{\sigma_k^n(t)\}$ *are constructed such that*

$$|\sigma_k(t) - \sigma_k^n(t)| \le \alpha_k^n, \quad |\sigma_k^n(t)| \le \beta_k^n, \quad |\sigma_k^{n\prime}(t)| \le \gamma_k^n \quad \forall t \in [0,T]$$

with positive sequences $\{\alpha_k^n\}$ *being arbitrarily chosen,* $\{\beta_k^n\}$ *and* $\{\gamma_k^n\}$ *being related to* $\{\alpha_k^n\, \beta_k\}$ *and* $\{\gamma_k\}$*. Let* $u_n(t,x)$ *and* $u(t,x)$ *be the solution of* (3.18) *and* (3.15)*, respectively; then, for some constants* $C > 0$*, independent of* Δt *and* h*,*

$$E\|u - u_n\|_{L_2}^2 \le \frac{C}{(1-\bar{\lambda})^2} \sum_{k=1}^{\infty} \left(\frac{(\alpha_k^n)^2}{2(k\pi)^2} + [k^4(\beta_k^n)^2 + (\gamma_k^n)^2](\Delta t)^2 \right), \tag{3.20}$$

provided that the infinite series are all convergent.

The proof of Theorem 3.3 is given in the appendix.

Remark 3.2. The assumption on $\bar{\lambda}$ being small is not crucial; some generalizations can be made without this assumption, for example when $b < 0$.

Now we consider the approximation of u_n. In particular, we use a finite element discretization with respect to the x variable and an implicit difference method in the t variable. Since u_n satisfies the weak formulation,

$$\begin{aligned} &\int_0^1 u_n(t,x)\phi(x)dx + \int_0^t \int_0^1 \frac{\partial u_n}{\partial x}(s,x)\frac{d\phi}{dx}(x)dxds + \int_0^t \int_0^1 bu_n(s,x)\phi(x)dxds \\ &= \int_0^1 u_0(x)\phi(x)dx + \int_0^t \int_0^1 \phi(x)dW_n(s,x) + \int_0^t \int_0^1 g(s,x)\phi(x)dxds \end{aligned} \tag{3.21}$$

for $\phi \in H_0^1(0,1)$. Meanwhile, the semidiscretization in space leads only to the following problem: find $u_n(t,\cdot) \in H_0^1(0,1)$, $t \in (0,T)$, such that

$$\int_0^1 \frac{\partial u_n}{\partial t}\phi dx + \int_0^1 \frac{\partial u_n}{\partial x}\frac{\partial \phi}{\partial x}dx + \int_0^1 bu_n\phi dx = \int_0^1 \left(g + \frac{\partial^2 W_n}{\partial t \partial x}\right)\phi dx \tag{3.22}$$

with

$$\int_0^1 u_n(0,x)\phi(x)dx = \int_0^1 u_0(x)\phi(x)dx$$

for all $\phi \in H_0^1(0,1), t \in (0,T)$.

The finite element discretization of (3.22) is to find $\bar{u}_n^h(t,\cdot) \in V_0^h(0,1)$, $t \in (0,T)$, such that

$$\int_0^1 \frac{\partial \bar{u}_n^h}{\partial t}\phi dx + \int_0^1 \frac{\partial \bar{u}_n^h}{\partial x}\frac{\partial \phi}{\partial x}dx + \int_0^1 b\bar{u}_n^h\phi dx = \int_0^1 \left(g + \frac{\partial^2 W_n}{\partial t \partial x}\right)\phi dx \tag{3.23}$$

with

$$\int_0^1 \bar{u}_n^h(0,x)\phi(x)dx = \int_0^1 u_0(x)\phi(x)dx$$

for all $\phi \in V_0^h(0,1)$, $t \in (0,T)$. Here, $V_0^h(0,1)$ denote the finite element subspace of $H_0^1(0,1)$. By using the expression

$$\bar{u}_n^h(t,x) = \sum_{l=1}^{n-1} u_l(t)\phi_l(x), \qquad t \in (0,T),$$

(3.23) leads to a system of ODEs for $u_l(t)$, $l = 1, \dots, n-1$. Using the backward-Euler method to solve this ODE system yields the following numerical scheme:

$$\sum_{l=1}^{n-1}(u_{i+1,l} - u_{i,l}) \int_0^1 \phi_l(x)\phi_j(x)dx + \Delta t \sum_{l=1}^{n-1} u_{i+1,l} \int_0^1 \phi_l'(x)\phi_j'(x)dx + b\Delta t \sum_{l=1}^{n-1} u_{i+1,l} \int_0^1 \phi_l(x)\phi_j(x)dx$$

$$= \int_{t_i}^{t_{i+1}} \int_0^1 g(s,x)\phi_j(x)dxds + \int_{t_i}^{t_{i+1}} \int_0^1 \phi_j(x)dW_n(s,x) \tag{3.24}$$

for $j = 1, 2, \dots, n-1, i = 1, 2, \dots, I$ where $u_{i,l} \approx u_l(t_i)$. Let

$$u_n^h(t_i, x) = \sum_{l=1}^{n-1} u_{i,l}\phi_l(x).$$

For simplicity, we now focus on the case of using the continuous piecewise linear finite element in the spatial discretization. The following pathwise error estimate can be found in Theorem 8.2 of [17]:

$$\|u_n(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2} \le C\sqrt{1 + \log\frac{t_m}{\Delta t}} \left(\max_{i \le m} \int_{t_{i-1}}^{t_i} \left\| \frac{\partial u_n}{\partial t}(\tau,\cdot) \right\|_{L_2} d\tau + \max_{t \le t_m} h^2 \|u_n(t,\cdot)\|_{H^2} \right). \tag{3.25}$$

The following lemma gives estimates of the terms on the right-hand side of (3.25).

LEMMA 3.3. *Let u_n be the solution of* (3.15) *with $g \in C^2([0,T] \times [0,1])$, $u_0 \in C^2[0,1]$, and $\sigma_k^n(t)$ has the bound given in Theorem* 3.3. *Let the constant b be suitably small. Then, if $\delta t \le 1/(2|b|)$, the following inequalities hold for some constant c, independent of Δt and h:*

$$E \int_{t_{i-1}}^{t_i} \left\| \frac{\partial u_n}{\partial t}(\tau,\cdot) \right\|_{L_2} d\tau \le c \left((\Delta t)^2 + \Delta t \sum_k k^2(\beta_k^n)^2 + \sum_k (\Delta t \beta_k^n)^2 \right)^{1/2} \tag{3.26}$$

and

$$E\|u_n(t,\cdot)\|_{H^2} \le c \left(1 + \frac{1}{\Delta t} \sum_k k^2(\beta_k^n)^2 \right)^{1/2} . \tag{3.27}$$

The proof of Lemma 3.3 is given in the appendix.

Combining Lemma 3.3 and inequality (3.25), we have the following theorem.

THEOREM 3.4. *Assume that the conditions in Lemma* 3.3 *hold; then*

$$E\|u_n(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2} \le c \left(1 + \log\frac{t_m}{\Delta t} \right)^{1/2} \times \left((\Delta t)^2 + \Delta t \sum_k k^2(\beta_k^n)^2 + \sum_k (\Delta t \beta_k^n)^2 + \frac{h^4}{\Delta t} \sum_k k^2(\beta_k^n)^2 \right)^{1/2}$$

for some constant c. □

The error $E\|u(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2}$ can be obtained by applying the triangle inequality to the results of Theorems 3.3 and 3.4.

Remark 3.3. Note that when applied to the case of white noise, that is, $\sigma_k(t) = 1$ for all k, we may take $\beta_k^n = \sigma_k^n = 1$, $\alpha_k^n = 0$ for $k \le N$, and $\beta_k^n = \sigma_k^n = 0$, $\alpha_k^n = 1$ for $k > N$, where $N \to \infty$ as $n \to \infty$; then, after simplification, the estimates in the above theorems give

$$E\|u(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2} \le c\left(1+\log\frac{t_m}{\Delta t}\right)^{1/2}\left\{\frac{1}{N^{1/2}} + (\Delta t)^{1/2}N^{3/2} + \frac{h^2N^{3/2}}{(\Delta)^{1/2}}\right\}$$

so that $h = O(\Delta t)^{1/2}$ and $N = O(h^{-1/2}) = O((\Delta t)^{-1/4})$ give a best order of $(\Delta t)^{1/8}$ or $h^{1/4}$, up to a logarithmic factor, for $E\|u(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2}$. This is indeed a very low order convergence estimate as was expected [1]. In the next section, however, we present a few examples with colored noises for which the above theorems allow much better estimates on the order of the approximations.

Remark 3.4. The estimate on the order of convergence in the time step size is seen to be at best $O(\sqrt{\Delta t})$, which is largely due to the fact that we restricted our attention to the case where $\{\dot\eta_k(t)\}$ in (3.16) correspond to the derivatives of the Wiener process with t being the parameter. In many physical applications, other processes may also be used [11]. One may also naturally consider more general formulation for the noise terms $\{\dot\eta_k(t)\}$ like what is used for dW/dx in (3.2). In the case where $\{\dot\eta_k\}$ are more regular in time, better error estimates may be obtained using similar techniques.

Discussions and extensions to higher space dimensions can be found in [36].

4. Numerical results for some model equations.

4.1. One dimensional elliptic equation. We now study two cases of the one dimensional elliptic equation with noise described in the previous section. We demonstrate that for different forms of coefficient $\{\sigma_k^n\}$, different rates of convergence are to be obtained.

Case 1. Let the random variables $\{\eta_k\}$ be iid, namely,

$$q_{kl} = E(\eta_k\eta_l) = \delta_{kl} = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \ne l, \end{cases} \qquad \sigma_k = \frac{1}{k^{3/2}}, \quad \sigma_k^n = \begin{cases} \sigma_k, & k \le n, \\ 0, & k > n. \end{cases}$$

Then

$$\left\|\overrightarrow{\sigma^n} - \vec\sigma\right\|_{Q_{-1}} = \left(\sum_{k=n+1}^{\infty}\left(\frac{1}{k^{3/2}}\cdot\frac{1}{k}\right)^2\right)^{1/2} \le \frac{1}{n^2}.$$

From Lemma 3.1, we have, for some generic constant $C > 0$,

$$E\|\dot W_n\|_{L_2} \le C\left(\sum_{k=1}^{\infty}(\sigma_k^n)^2\right)^{1/2} \le C\left(\sum_{k=1}^{\infty}\frac{1}{k^3}\right)^{1/2} = C.$$

In other words, $\dot W_n \in L_2(0,1)$; this means that, in Theorem 3.2, $r = 1$. If the piecewise linear finite element basis is used, and $g \in L_2(0,1)$, the following error estimate yields

$$E(\|u - u_n^h\|_{L_2}) \le C(n^{-2} + h^2\|g + \dot W_n\|_{L_2}) \le C\,h^2\ .$$

Thus, asymptotically, we have a second order convergence rate in h for the expectation of the L^2 error.

Case 2. Now let us consider using different coefficients $\{\sigma_k^n\}$ which yield high order convergence results for high order finite element spaces. Still let

$$q_{kl} = E(\eta_k \eta_l) = \delta_{kl} = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases} \qquad \sigma_k = \frac{1}{k^{7/2}}, \quad \sigma_k^n = \begin{cases} \sigma_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Then

$$\left\| \overrightarrow{\sigma^n} - \vec{\sigma} \right\|_{Q_{-1}} = \left(\sum_{k=n+1}^{\infty} \left(\frac{1}{k^{7/2}} \cdot \frac{1}{k} \right)^2 \right)^{1/2} \leq \frac{1}{n^4}.$$

From Lemma 3.1, we have

$$E\|\dot{W}_n\|_{H^2} \leq C \left(\sum_{k=1}^{\infty} (\sigma_k^n k^2)^2 \right)^{1/2} \leq C \left(\sum_{k=1}^{\infty} \left(\frac{1}{k^{7/2}} k^2 \right)^2 \right)^{1/2} = C.$$

In other words, $\dot{W}_n \in H^2(0,1)$; this means that, in Theorem 3.2, $r = 3$. If we use the cubic spline finite element basis, and assume that g is bounded in $H^2(0,1)$, the following error estimate yields

$$E(\|u - u_n^h\|_{L_2}) \leq C(n^{-4} + h^4 \|g + \dot{W}_n\|_{H^2}) \leq C\, h^4$$

for some constant C that depends only on g. Note that such a high order cannot be achieved if we have adopted a white noise [1].

The finite element method (3.10) is implemented for (3.1) with $g(x) = 2 + bx - bx^2$ and the noise $\dot{W}$ as defined in section 3. The exact solution of (3.1) is given by $u = u_d + u_s$, where u_d and u_s correspond to the deterministic and the stochastic parts. Moreover, $u_d(x) = x(1-x)$ and

$$u_s(x) = \sum_{k=1}^{\infty} \frac{\sqrt{2}\sigma_k}{b + (k\pi)^2} \eta_k \sin k\pi x \ .$$

The numerical solution is calculated for $n = 4, 8, 16, 32, 64, 128$ ($h = 1/n$ being the length of the subintervals). For each n, 10,000 runs are performed with different samples of the noise, $\|u - u_n^h\|_{L_2}$ is calculated for each sample, and the averaged value $E\|u - u_n^h\|_{L_2}$ is calculated.

For Case 1, we let $b = 0.5$, $\sigma_k = k^{-3/2}$, and we use the continuous piecewise linear finite element space. The left picture in Figure 4.1 gives the decay of error. The horizontal axis denotes $\log_{10} n$, and the vertical axis denotes $\log_{10} E\|u - u_n^h\|_{L_2}$. The slope of the error curve is nearly -2, in agreement with the theoretical result.

As for Case 2, we let $b = 0.5$, $\sigma_k = k^{-7/2}$, and we use the finite element space consisting of piecewise cubic splines. The right picture in Figure 4.1 gives the decay of error. The slope of the error curve is now nearly -4, also in agreement with the theoretical result.

4.2. Parabolic equation in one spatial dimension. Now consider a special case of parabolic equation described in the previous section. Let

$$\sigma_k(t) = \frac{\cos t}{k^3}, \qquad \sigma_k^n(t) = \begin{cases} \sigma_k(t), & k \leq n, \\ 0, & k > n, \end{cases}$$

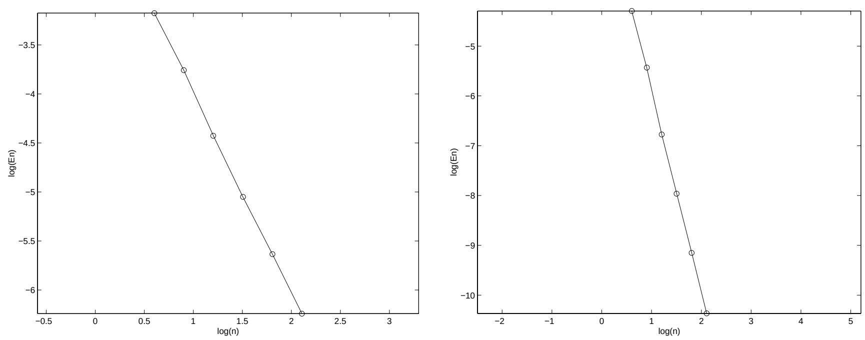

FIG. 4.1. *The error decay with* $\sigma_k = k^{-3/2}$ *and* $k^{-7/2}$.

and the upper bounds $\alpha_k^n, \beta_k^n, \gamma_k^n$ given in Theorem 3.3 can be chosen as

$$\alpha_k^n = \begin{cases} 0, & k \le n, \\ \frac{1}{k^3}, & k > n, \end{cases} \qquad \beta_k^n = \gamma_k^n = \frac{1}{k^3}.$$

Backward-Euler in time with the piecewise linear finite element in space approximation (3.24) was tested for the numerical solution of problem (3.14) with

$$g(t,x) = 10(1+b)x^2(1-x)^2e^t - 10(2-12x+12x^2)e^t .$$

We use $b = 0.5$ and $T = 1$. In the absence of noise term, the exact solution is

$$u(t,x) = u_d(t,x) = 10e^t x^2(1-x)^2 , \quad \text{with} \quad u_0(x) = 10x^2(1-x)^2 .$$

The exact value of $Eu(1, 0.5)$ is about 1.699.

In theory, using the above definitions, we have

$$\sum_{k=1}^{\infty} \frac{(\alpha_k^n)^2}{2(k\pi)^2} \le \sum_{k=n+1}^{\infty} \frac{1}{k^8} \le \frac{1}{n^7} = h^7, \tag{4.1}$$

$$\sum_{k=1}^{\infty} k^4(\beta_k^n)^2 + (\gamma_k^n)^2) \le \sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{k^3}\right) \le C, \tag{4.2}$$

$$\sum_{k=1}^{\infty} (\beta_k^n)^2 \le \sum_{k=1}^{\infty} (k\beta_k^n)^2 \le C . \tag{4.3}$$

From Theorems 3.3 and 3.4, we have

$$E\|u - u_n\|_{L_2} \le c(h^7 + (\Delta t)^2)^{1/2},$$
$$E\|u_n(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2} \le c\left(1 + \log\frac{t_m}{\Delta t}\right)^{1/2}\left((\Delta t)^{1/2} + \frac{h^2}{(\Delta t)^{1/2}}\right).$$

Hence,

$$E\|u(t_m,\cdot) - u_n^h(t_m,\cdot)\|_{L_2} \le c\left(1 + \log\frac{t_m}{\Delta t}\right)^{1/2}\left((\Delta t)^{1/2} + \frac{h^2}{(\Delta t)^{1/2}}\right). \tag{4.4}$$

TABLE 4.1
$E(u_n^h(1,0.5))$ and $E(u_n^h(1,0.5))^2$ by the backward-Euler finite element scheme.

h	Δt	$E(u_n^h(1,0.5))$	$E(u_n^h(1,0.5))^2$	$E(\eta_{n/2,I})$	$var(\eta_{n/2,I})$
.25	.25	1.5268	2.3495	.0061	.9830
.25	.125	1.6147	2.6301	-.0217	1.0141
.25	.0625	1.6599	2.7826	-.0166	1.0079
.25	.03125	1.6821	2.8586	.0083	.9908
.25	.01563	1.6976	2.9142	-.0086	.9750
.125	.25	1.5198	2.3283	.0045	.9697
.125	.125	1.6071	2.6059	-.0014	1.0097
.125	.0625	1.6529	2.7569	-.0238	.9780
.125	.03125	1.6777	2.8432	.0002	.9829
.125	.01563	1.6912	2.8910	.0006	.9687
.0625	.25	1.5193	2.3263	-.0006	1.0182
.0625	.125	1.6043	2.5963	-.0069	.9886
.0625	.0625	1.6519	2.7539	.0124	.9852
.0625	.03125	1.6758	2.8372	-.0110	.9908
.0625	.01563	1.6888	2.8825	.0069	.9962
.03125	.25	1.5198	2.3277	-.0163	.9650
.03125	.125	1.6044	2.5971	-.0217	.9527
.03125	.0625	1.6503	2.7497	-.0071	.9984
.03125	.03125	1.6731	2.8281	.0044	.9765
.03125	.01563	1.6855	2.8724	-.0101	1.0479
.01563	.25	1.5181	2.3230	-.0166	.9918
.01563	.125	1.6067	2.6041	-.0134	.9667
.01563	.0625	1.6500	2.7482	-.0114	1.0067
.01563	.03125	1.6749	2.8336	-.0170	1.0365
.01563	.01563	1.6851	2.8704	-.0067	.9872

In the actual implementation, different values of Δt and h were used. For each pair $\{\Delta t, h\}$, 10,000 runs are performed with different sample of noise, and the ensemble averages are calculated. The numerical results of $E(u_n^h(1,0.5))$ and $E(u_n^h(1,0.5))^2$ are presented in Table 4.1.

The computational results converge as Δt and h approach to 0. From the table, it can be observed that, for fixed h, the results converge faster as Δt decreases, but for fixed Δt the convergence is less transparent as h decreases. This can be explained by the error estimate (4.4), which is bounded by $(\Delta t)^{1/2} + h^2(\Delta t)^{-1/2}$. If Δt and h are of the same order, the Δt term dominates in the estimate.

The numerical accuracy is also affected by the random number generators used in the different realizations. (The particular generator used in our implementation is obtained using MATLAB.) For comparison, the last two columns of Table 4.1 list the mean and variance of $\eta_{n/2,I}$. We see that, for the relatively larger magnitude of $E(\eta_{n/2,I})$, the error of $E(u_n^h(1,0.5))$ turns out to be larger as well.

Additional numerical examples can be found in [36].

5. Conclusion. In this paper, the numerical approximations of SDEs with different noise realizations are considered. In many instances of stochastic modeling, the noises may indeed be represented in various forms, with some emphasis on the correlation in space and time, while others exhibit the correlation in frequency or spectrum. Our study indicates that the accuracy of the numerical approximation depends on the form of the underlying noise. Both rigorous error estimates and experimental results are provided in our paper.

Throughout our discussion, simple linear equations in one space dimension are used for the purpose of illustrations. We note that much of our consideration can be generalized to stochastic elliptic and parabolic equations in higher space dimensions. For the case of a simple two dimensional square domain, related discussions have been provided in [36]. By confining the theoretical analysis to the one space dimension here, some tedious technical details and complicated expressions are avoided.

Naturally, it will be very interesting to study the similar problems for nonlinear SDEs, which actually motivated the present investigation. It is hopeful that such studies may lead to a better understanding of the behaviors of the discretization error and the modeling error in conducting numerical simulations of nonlinear stochastic dynamics for practical problems [9, 18, 28].

Appendix.

Proof of Theorem 3.3.

Step 1. First, we verify the existence of such $\{\sigma_k^n(t)\}$. Since $\{\sigma_k'(t)\}$ are continuous on interval $[0,T]$, by the Weierstrass approximation theorem, for an arbitrary sequence α_k^n, where n is a fixed number, $k = 1, 2, \ldots,$ there exists a sequence of polynomial $\{P_k^n(t)\}$ such that

$$|\sigma_k'(t) - P_k^n(t)| \le \frac{\alpha_k^n}{T} \ \forall t \in [0,T] \, .$$

Let

$$\sigma_k^n(t) = \int_0^t P_k^n(s)ds + \sigma_k(0),$$

and we have

$$|\sigma_k(t) - \sigma_k^n(t)| = \left| \int_0^t (\sigma_k'(s) - P_k^n(s))ds \right| \le \alpha_k^n.$$

By the triangle inequality,

$$|\sigma_k^n(t)| \le |\sigma_k(t)| + \alpha_k^n \le \beta_k + \alpha_k^n = \beta_k^n,$$

$$|\sigma_k^{n\prime}(t)| = |P_k^n(t)| \le |\sigma_k'(t)| + \frac{\alpha_k^n}{T} \le \gamma_k + \frac{\alpha_k^n}{T} = \gamma_k^n.$$

Step 2. Let $\bar{e}_n(t,x) = u(t,x) - u_n(t,x)$ and

$$F(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)dW(s,y) - \int_0^t \int_0^1 G_{t-s}(x,y)dW_n(s,y),$$

$$\bar{e}_n = E \int_0^T \int_0^1 \bar{e}_n^2(t,x)dxdt,$$

$$\bar{F}_n = E \int_0^T \int_0^1 F^2(t,x)dxdt.$$

Subtracting (3.18) from (3.15), and applying similar manipulation as that in section 3, we get

$$E\|u-u_n\|_{L_2}^2=\bar{e}_n\le\frac{\bar{F}_n}{(1-\bar{\lambda})^2}.$$

To estimate $\bar{F}_n$, we introduce an intermediate noise form

$$\frac{\partial^2\overline{W}_n}{\partial t\partial x}=\sum_{k=1}^{\infty}\sigma_k^n(t)\dot{\eta}_k(t)\psi_k(x),$$

that is, a noise discretized only in the x-direction. Let

$$F_1(t,x)=\int_0^t\int_0^1 G_{t-s}(x,y)dW(s,y)-\int_0^t\int_0^1 G_{t-s}(x,y)d\overline{W}_n(s,y),$$

$$F_2(t,x)=\int_0^t\int_0^1 G_{t-s}(x,y)d\overline{W}_n(s,y)-\int_0^t\int_0^1 G_{t-s}(x,y)dW_n(s,y);$$

then

$$F(t,x)=F_1(t,x)+F_2(t,x),$$

$$\bar{F}_n=E\int_0^T\int_0^1 F^2(t,x)dxdt\le 2\left(E\int_0^T\int_0^1 F_1^2(t,x)dxdt+E\int_0^T\int_0^1 F_2^2(t,x)dxdt\right).$$

Taking advantage of the orthogonality of $\{\sin k\pi x\}$ on the interval $[0,1]$, we have

$$F_1(t,x)=\sum_{k=1}^{\infty}\sqrt{2}\sin k\pi x e^{-(k\pi)^2t}\int_0^t(\sigma_k(s)-\sigma_k^n(s))e^{(k\pi)^2s}d\eta_k(s).$$

Since $\eta_k(t)$ is the standard Wiener process,

$$E\int_0^T\int_0^1 F_1^2(t,x)dxdt=\sum_{k=1}^{\infty}\int_0^T e^{-2(k\pi)^2t}\left(\int_0^t e^{2(k\pi)^2s}(\sigma_k(s)-\sigma_k^n(s))^2ds\right)dt$$

$$\le\sum_{k=1}^{\infty}(\alpha_k^n)^2\int_0^T e^{-2(k\pi)^2t}\left(\int_0^t e^{2(k\pi)^2s}ds\right)dt\le C_1\sum_{k=1}^{\infty}\frac{(\alpha_k^n)^2}{2(k\pi)^2}.$$

Using

$$\eta_{ki}=\frac{1}{\sqrt{\Delta t}}\int_{t_i}^{t_{i+1}}d\eta_k(t),$$

we have

$$\begin{aligned}F_2(t,x)=\sum_{k=1}^{\infty}\psi_k e^{-(k\pi)^2t}&\left[\int_0^t e^{(k\pi)^2s}\sigma_k^n(s)d\eta_k(s)\right.\\&\left.-\int_0^t e^{(k\pi)^2s}\sigma_k^n(s)\sum_{i=1}^{I}\frac{1}{\sqrt{\Delta t}}\eta_{ki}\chi_i(s)ds\right]\\=\sum_{k=1}^{\infty}\psi_k e^{-(k\pi)^2t}&\left[\sum_{i=1}^{I_t-1}\int_{t_i}^{t_{i+1}}\left(e^{(k\pi)^2s}\sigma_k^n(s)-\frac{1}{\Delta t}\int_{t_i}^{t_{i+1}}e^{(k\pi)^2\tilde{s}}\sigma_k^n(\tilde{s})d\tilde{s}\right)d\eta_k(s)\right.\\&\left.+\int_{t_{I_t}}^t\left(e^{(k\pi)^2s}\sigma_k^n(s)-\frac{1}{t-t_{I_t}}\int_{t_{I_t}}^t e^{(k\pi)^2\tilde{s}}\sigma_k^n(\tilde{s})d\tilde{s}\right)d\eta_k(s)\right],\end{aligned}$$

where I_t is the integer such that $t_{I_t} < t \le t_{I_t+1}$. Then

$$
\begin{aligned}
&E\int_0^T\int_0^1 F_2^2(t,x)dxdt\\
&=\sum_{k=1}^{\infty}\int_0^T \frac{e^{-2(k\pi)^2 t}}{(\Delta t)^2}\left(\sum_{i=1}^{I_t-1}\int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{t_{i+1}}(e^{(k\pi)^2 s}\sigma_k^n(s)-e^{(k\pi)^2\tilde{s}}\sigma_k^n(\tilde{s}))d\tilde{s}\right)^2 ds\right.\\
&\quad\left.+\int_{t_{I_t}}^{t}(\frac{\Delta t}{t-t_{I_t}})^2\left(\int_{t_i}^{t_{i+1}}(e^{(k\pi)^2 s}\sigma_k^n(s)-e^{(k\pi)^2\tilde{s}}\sigma_k^n(\tilde{s}))d\tilde{s}\right)^2 ds\right)dt.
\end{aligned}
$$

For s, $\tilde{s}\in[t_i,t_{i+1}]$, using the smoothness assumption on $\sigma_k(t)$, we get

$$
\begin{aligned}
&|e^{(k\pi)^2 s}\sigma_k^n(s)-e^{(k\pi)^2\tilde{s}}\sigma_k^n(\tilde{s})|\\
&\le|e^{(k\pi)^2 s}-e^{(k\pi)^2\tilde{s}}|\sigma_k^n(s)+e^{(k\pi)^2\tilde{s}}|\sigma_k^n(s)-\sigma_k^n(\tilde{s})|\\
&\le(k\pi)^2 e^{(k\pi)^2 t_{i+1}}\sigma_k^n(s)\Delta t+e^{(k\pi)^2 t_{i+1}}\sigma_k^{n\prime}(\xi_i)\Delta t\\
&\le e^{(k\pi)^2 t_{i+1}}((k\pi)^2\beta_k^n+\gamma_k^n)\Delta t.
\end{aligned}
$$

Here, $t_i\le\xi_i\le t_{i+1}$. Without loss of generality, we assume $t=t_{I_t+1}$; then

$$
\begin{aligned}
&E\int_0^T\int_0^1 F_2^2(t,x)dxdt\\
&\le\sum_{k=1}^{\infty}\int_0^T e^{-2(k\pi)^2 t}\left[\sum_{i=1}^{I_t}\int_{t_i}^{t_{i+1}}\frac{1}{(\Delta t)^2}(e^{(k\pi)^2 t_{i+1}}((k\pi)^2\beta_k^n+\gamma_k^n)(\Delta t)^2)^2 ds\right]dt\\
&\le C\sum_{k=1}^{\infty}\int_0^T e^{-2(k\pi)^2 t}\left[\sum_{i=1}^{I_t}\int_{t_i}^{t_{i+1}}(e^{2(k\pi)^2 t_{i+1}}((k\pi)^4(\beta_k^n)^2+(\gamma_k^n)^2)(\Delta t)^2 ds\right]dt\\
&\le C\sum_{k=1}^{\infty}\sum_{i=1}^{I_t}\int_0^T e^{-2(k\pi)^2 t}e^{2(k\pi)^2 t_{i+1}}dt(k^4(\beta_k^n)^2+(\gamma_k^n)^2)(\Delta t)^3\\
&\le C\sum_{k=1}^{\infty}\sum_{i=1}^{I_t}(k^4(\beta_k^n)^2+(\gamma_k^n)^2)(\Delta t)^3\\
&\le C\sum_{k=1}^{\infty}(k^4(\beta_k^n)^2+(\gamma_k^n)^2)(\Delta t)^2.
\end{aligned}
$$

The last inequality comes from $\sum_{i=1}^{I_t},1 \le I = 1/\Delta t$. The theorem is now proved. ∎

Proof of Lemma 3.3. In general, by applying the same technique as in the proof of Theorem 3.1, we may first estimate

$$
\begin{aligned}
E\int_0^t\int_0^1 u_n^2(t,x)dxdt &\le \frac{c}{1-\bar{\lambda}}E\int_0^t\int_0^1[(G_t(x,y)u_0(y))^2+(G_{t-s}(x,y)g(s,y))^2]dyds\\
&\quad+\frac{c}{1-\bar{\lambda}}E\int_0^t\int_0^1\left(\int_0^t\int_0^1 G_{t-s}(x,y)dW_n(s,y)\right)^2 dxdt.
\end{aligned}
$$

Next, one may differentiate (3.18) to get

$$\begin{aligned}\frac{\partial u_n}{\partial t}(t,x) = &-\int_0^t \int_0^1 \frac{\partial}{\partial t} G_{t-s}(x,y) b u_n(s,y) dy ds + \int_0^1 \frac{\partial}{\partial t} G_t(x,y) u_0(y) dy \\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial t} G_{t-s}(x,y) g(s,y) dy ds + \int_0^t \int_0^1 \frac{\partial}{\partial t} G_{t-s}(x,y) dW_n(s,y).\end{aligned}$$

Then one may estimate $E\int_{t_{i-1}}^{t_i}\int_0^1 (\frac{\partial u_n}{\partial t}(t,x))^2 dxdt$ using the above equation. Similarly, one may estimate $E\int_0^1 (\frac{\partial^2 u_n}{\partial x^2}(t,x))^2 dxdt$.

Since we have assumed that b is a constant, we now provide a simpler estimate which, in spirit, is similar to the estimate derived from the integral formulation.

Let $g(t,x) = \sum_k g_k(t)\psi_k(x)$, $u_n(t,x) = \sum_k u_k^{(n)}(t)\psi_k(x)$, $u_0(x) = \sum_k u_k \psi_k(x)$; then

$$\frac{\partial}{\partial t} u_k^{(n)}(t) + (k^2\pi^2 + b) u_k^{(n)}(t) = g_k(t) + \frac{\sigma_k^n(t)}{\sqrt{\Delta t}} \sum_i \eta_{ki}\chi_i(t).$$

Thus, for $t \in [t_{i-1}, t_i)$,

$$\begin{aligned}u_k^{(n)}(t) = &e^{-((k\pi)^2+b)t} u_k + \int_0^t e^{-((k\pi)^2+b)(t-s)} g_k(s) ds \\ &+ \sum_{j=1}^i \int_0^t e^{-((k\pi)^2+b)(t-s)} \frac{\sigma_k^n(s)}{\sqrt{\Delta t}} \eta_{kj}\chi_j(s) ds.\end{aligned}$$

This leads to

$$\begin{aligned}u_k^{(n)}(t) = &e^{-((k\pi)^2+b)t} u_k + \int_0^t e^{-((k\pi)^2+b)(t-s)} g_k(s) ds \\ &+ \sum_{j=1}^i \frac{\eta_{kj}}{\sqrt{\Delta t}} \int_{t_{j-1}}^{t_j^*} e^{-((k\pi)^2+b)(t-s)} \sigma_k^n(s) ds,\end{aligned}$$

where $t_l^* = t_l$ for $l < i$ and $t_i^* = t$. It follows that

$$\begin{aligned}E\left[u_k^{(n)}(t)\right]^2 \le &c u_k^2 e^{-2((k\pi)^2+b)t} + cT\int_0^t e^{-2((k\pi)^2+b)(t-s)} g_k^2(s) ds \\ &+ \frac{c}{\Delta t}\sum_{j=1}^i \left(\int_{t_{j-1}}^{t_j^*} e^{-((k\pi)^2+b)(t-s)} \sigma_k^n(s) ds\right)^2\end{aligned}$$

for some constant c.

Since $u_0 \in C^2[0,1]$ and $g \in C^2([0,T]\times[0,1])$, we have, for some constant $c > 0$,

$$\sum_k (k\pi)^4 \left\{ u_k^2 e^{-2((k\pi)^2+b)t} + \int_0^t e^{-2((k\pi)^2+b)(t-s)} g_k^2(s) ds \right\} \le c.$$

Using the bounds on σ_k^n and the fact that b is a small constant, we have

$$\sum_{l=1}^{i}\left(\int_{t_{l-1}}^{t_l^*} e^{-((k\pi)^2+b)(t-s)}\sigma_k^n(s)ds\right)^2 \le c(\beta_k^n)^2\int_0^t e^{-2((k\pi)^2+b)(t-s)}ds \le \frac{c(\beta_k^n)^2}{(k\pi)^2+b}.$$

Thus, for small b, we have

$$E\|u_n(\tau,\cdot)\|_{H^2} \le (\, E\|u_n(\tau,\cdot)\|_{H^2}^2)^{1/2} \le c\left(1+\frac{1}{\Delta t}\sum_k\ k^2(\beta_k^n)^2\right)^{1/2}$$

for some constant $c > 0$. This proves the inequality (3.27).

For (3.26), we have

$$\frac{\partial}{\partial t}u_k^{(n)}(t) = -((k\pi)^2+b)u_k^{(n)}(t) + g_k(t) + \frac{\sigma_k^n(t)}{\sqrt{\Delta t}}\sum_i \eta_{ki}\chi_i(t).$$

So,

$$E\int_{t_{i-1}}^{t_i}\left(\frac{\partial}{\partial t}u_k^{(n)}(s)\right)^2 ds \le cE\int_{t_{i-1}}^{t_i}((k^2\pi^2+b)u_k^{(n)}(s))^2ds + c\int_{t_{i-1}}^{t_i}[g_k^2(s)+(\sigma_k^n(s))^2]ds.$$

Thus,

$$E\int_{t_{i-1}}^{t_i}\left\|\frac{\partial u_n}{\partial t}(\tau,\cdot)\right\|_{L_2} d\tau \le \left(\Delta t E\int_{t_{i-1}}^{t_i}\left\|\frac{\partial u_n}{\partial t}(\tau,\cdot)\right\|_{L_2}^2 d\tau\right)^{1/2} \le c\left((\Delta t)^2+\Delta t\sum_k\ k^2(\beta_k^n)^2+\sum_k(\Delta t\beta_k^n)^2\right)^{1/2}.$$

This proves (3.26). ☐

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REFERENCES

[1] E. Allen, S. Novosel, and Z. Zhang, *Finite element and difference approximation of some linear stochastic partial differential equations*, Stochastics Stochastics Rep., 64 (1998), pp. 117–142.

[2] J.F. Bennaton, *Discrete time Galerkin approximation to the nonlinear filtering solution*, J. Math. Anal. Appl., 110 (1985), pp. 364–383.

[3] R. Buckdahn and E. Pardoux, *Monotonicity methods for white noise driven quasilinear SPDEs*, in Diffusion Processes and Related Problems in Analysis, I, M. Pinsky, ed., Birkhaüser Boston, Boston, MA, 1990, pp. 219–233.

[4] P.L. Chow, J.L. Jiang, and J.L. Menaldi, *Pathwise convergence of approximation solutions to Zakai's equation in a bounded domain*, in Stochastic Partial Differential Equations and Applications, G. Da Prato and L. Tubaro, eds., Longman Scientific and Technical, Harlow, UK, 1992, pp. 111–123.
[5] G. Da Prato and L. Tubaro, *Stochastic Partial Differential Equations and Applications*, Longman Scientific and Technical, Harlow, UK, 1992.
[6] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.
[7] A. Davie and J. Gaines, *Convergence of numerical schemes for the solution of the parabolic stochastic partial differential equations*, Math. Comp., 70 (2001), pp. 121–134.
[8] D.A. Dawson, *Stochastic evolution equations*, Math. Biosci., 154 (1972), pp. 187–316.
[9] J. Deang, Q. Du, and M. Gunzburger, *Thermal fluctuations of superconducting vortices*, Phy. Rev. B, 64 (2001), pp. 52506–52510.
[10] A. Etheridge, *Stochastic Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1995.
[11] R. Fox, *Second order algorithm for the numerical integration of colored noise problems*, Phys. Rev. A(3), 43 (1991), pp. 2649–2654.
[12] J.G. Gaines, *Numerical Experiments with S(P)DE's*, in Stochastic Partial Differential Equations, London Math. Soc. Lecture Note Ser. 216, Cambridge University Press, Cambridge, UK, 1995, pp. 55–71.
[13] I. Gyongy, *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise* II, Potential Anal., 11 (1999), pp. 1–37.
[14] E. Hausenblas, *Error analysis for approximation of stochastic differential equations driven by Poisson random measures*, SIAM J. Numer. Anal., 40 (2002), pp. 87–113.
[15] D.J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numer Anal, 38 (2000), pp. 753–769.
[16] M. Ibanes, J. Garcia-Ojalvo, R. Toral, and J.M. Sancho, *Noise-induced phase separation: Mean-field results*, Phys. Rev. E(3), 60 (1999), pp. 3597–3605.
[17] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge, UK, 1994.
[18] T. Kamppeter, F.G. Mertens, E. Moro, A. Sanchez, and A.R. Bishop, *Stochastic vortex dynamics in two-dimensional easy-plane ferromagnets: Multiplicative versus additive noise*, Phys. Rev. B, 59 (1999), pp. 11349–11357.
[19] J.B. Keller, *Stochastic equations and wave propagation in random media*, Proc. Sympos. Appl. Math., 16 (1964), pp. 145–170.
[20] P. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, New York, 1992.
[21] L. Machiels, and M.O. Deville, *Numerical simulation of randomly forced turbulent flows*, J. Comput. Phys., 145 (1998), pp. 246–279.
[22] A. Majda, I. Timofeyev, and E. Eijnden, *Models for stochastic climate prediction*, Proc. Natl. Acad. Sci. USA, 96 (1996), pp. 14687–14691.
[23] G.N. Milstein, E. Platen, and H. Schurz, *Balanced implicit methods for stiff stochastic systems*, SIAM J. Numer. Anal., 35 (1998), pp. 1010–1019.
[24] E.A. Novikov, *Functionals and the random-force method in turbulence theory*, Soviet Phys. JETP, 20 (1965), pp. 1290–1294.
[25] E. Platen, *An Introduction to Numerical Methods for Stochastic Differential Equations*, Acta Numer. 8, Cambridge University Press, Cambridge, UK, 1999, pp. 197–246.
[26] P. Protter and D. Talay, *The Euler scheme for Levy driven stochastic differential equations*, Ann. Probab., 25 (1997), pp. 393–423.
[27] Y. Saito and T. Mitsui, *Stability analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal., 33 (1996), pp. 2254–2267.
[28] J. Sancho, J. Garcia-Ojalvo, and H. Guo, *Non-equilibrium Ginzburg-Landau model driven by colored noise*, Phys. D., 113 (1998), pp. 331–337.
[29] T. Shardlow, *Numerical methods for stochastic parabolic PDEs*, Numer. Funct. Anal. Optim., 20 (1999), pp. 121–145.
[30] G. Strang, *Wavelets and dilation equations: A brief introduction*, SIAM Rev., 31 (1989), pp. 614–627.
[31] D. Talay, *Simulation and numerical analysis of stochastic differential systems*, in Effective Stochastic Analysis, P. Krée and W. Wedig, eds., Springer-Verlag, Berlin, 1988.
[32] R.T. Ogden, *Essential Wavelets for Statistical Applications and Data Analysis*, Birkhäuser Boston, Boston, MA, 1997.
[33] J.B. Walsh, *An Introduction to Stochastic Partial Differential Equations*, Lecture Notes in

Math. 1180, Springer-Verlag, Berlin, 1986, pp. 265–439.

[34] M. Werner and P. Drummond, *Robust algorithms for solving stochastic partial differential equations*, J. Comput. Phys., 132 (1997), pp. 312–326.

[35] H. Yoo, *Semi-discretization of stochastic partial differential equations on* $\mathbf{R}^1$ *by a finite-difference method*, Math. Comp., 69 (2000), pp. 653–666.

[36] T. Zhang, *Numerical Approximations of Stochastic Partial Differential Equations*, M. Phil thesis, Hong Kong University of Science and Technology, Hong Kong, 2000.