8-2015

Numerical Approximation of Stochastic Differential Equations Driven by Levy Motion with Infinitely Many Jumps

Ernest Jum
University of Tennessee - Knoxville, ejum@vols.utk.edu

To the Graduate Council:

I am submitting herewith a dissertation written by Ernest Jum entitled "Numerical Approximation of Stochastic Differential Equations Driven by Levy Motion with Infinitely Many Jumps." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Jan Rosinski, Major Professor

We have read this dissertation and recommend its acceptance:

Xia Chen, Vasileios Maroulas, Hamparsum Bozdogan

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Numerical Approximation of Stochastic Differential Equations Driven by Levy Motion with Infinitely Many Jumps

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Ernest Jum

August 2015

Dedication

This work is dedicated to my beloved wife Delphine, my beautiful daughters Arianna and Jeslyn, my parents Martin and Anna Jum, my siblings Clement, Samuel, Glory and Vera. Their prayers, love and support have been the backbone of my success. And finally to my cousin and wife Isaiah and Gladys.

Acknowledgments

I am extremely grateful and indebted to my mentor Dr. Jan Rosinski for directing the progress of this dissertation. His unwavering support and encouragement are herein commended. In addition to directing my work, he furnished me with an enormous amount of fatherly advice, and also exposed me to many opportunities for growth at professional gatherings.

My unequivocal appreciation also goes to Dr. Xia Chen, Dr. Vasileios Maroulas and Dr. Hamparsum Bozdogan for serving in my final examination committee, and being a constant source of encouragement to me throughout my Ph.D. program. The faculty and staff of the Department of Mathematics at the University of Tennessee provided a very cordial environment. In particular, my gratitude goes to Dr. Balram Rajput for his words of encouragement and advice, and, Dr. Joan Lind for all her useful advice and guidance on my job applications. I also want to mention Ms. Pam Amentrout for her endless hours of help and dedication to all the graduate students in the Department of Mathematics, and to me in particular.

My colleagues and friends Liguo Wang, Eddie Tu, Tyler Massaro, Andrew Marchese, Kang Kai, Khoa Dinh, and Brian Allen also deserve special mention. I also want to thank Craig Collins for his useful comments on my MATLAB codes. In addition, I want to thank Dr. Kei Kobayashi, Dr. Xing Fei, and Dr. Joseph White for their friendly advice in times of difficulties.

Words cannot express my sense of indebtedness to my parents Martin and Anna Jum, my siblings Clement, Samuel, Glory and Vera. My appreciation is also extended to my cousin and his wife Isaiah and Gladys Jam for making it possible for me to complete my college education through their financial and moral support.

Saving the best for the last, my source of inspiration throughout this journey has been the trio that has filled my life with joy and hope. I lovingly acknowledge the help accorded to me by my wife Delphine and our two daughters Arianna and Jeslyn. Their kindness and devotion spurred me to work hard, and I am elated to see their patience finally rewarded.

"God does not play dice" Albert Einstein

Abstract

In this dissertation, we consider the problem of simulation of stochastic differential equations driven by pure jump Levy processes with infinite jump activity. Examples include, the class of stochastic differential equations driven by stable and tempered stable Levy processes, which are suited for modeling of a wide range of heavy tail phenomena. We replace the small jump part of the driving Levy process by a suitable Brownian motion, as proposed by Asmussen and Rosinski, which results in a jump-diffusion equation. We obtain L^p [the space of measurable functions with a finite p-norm], for p greater than or equal to 2, and weak error estimates for the error resulting from this step. Combining this with numerical schemes for jump diffusion equations, we provide a good approximation method for the original stochastic differential equation that can also be implemented numerically. We complement these results with concrete error estimates and simulation.

Table of Contents

List of Tables

List of Figures

Notation

$\mathbb{R}^d$	Euclidean space
$\mathbb{R}^+$	The set of nonnegative real numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}_0^d$	The Euclidean space minus the origin
$\\|.\\|$	Appropriate norm on a given space
C_b^n	The Space of n times continuously differentiable functions with bounded derivatives
$\mathrm{C}^{1,n}$	The Space of functions which are continuously differentiable with respect to the first variable, and n times continuously differentiable with respect to the second variable
C_p^n	The Space of n times continuously differentiable functions with polynomial growth
L^p	The Space of measurable functions with a finite p^{th} norm
$\langle ., . \rangle$	Inner product

Chapter 1

Introduction

1.1 Preview

Stochastic differential equation (SDE) models and in particular SDEs driven by Lévy processes, play a prominent role in a wide range of applications, including biology for modeling the spreading of diseases [19], in genetics [13], for the movement patterns of various animals [30], for various phenomenons in physics [38] and in financial mathematics [10].

Throughout this dissertation, $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F})_{t\geq 0}, \mathbb{P})$ will always denote a filtered probability space satisfying the usual hypothesis of right-continuity and completeness. Here, we are interested in the *strong* and *weak numerical approximation* of the stochastic process $\{X(t) : t \in [0, T]\}$, which is the solution to the SDE

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t h(X(s-))dZ(s),\ t \in [0, T], \tag{1.1}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are Lipschitz functions, Z is a d–dimensional *pure jump Lévy process* with *infinite jump activity*, which is independent of the d–dimensional Brownian motion W (see Chapter 2 for details on (1.1)).

Given that only a small class of (1.1) admit closed form solutions, it is important to construct its discrete time approximations. In this sequel, we consider the discrete time approximation Y of X, the solution of (1.1) constructed on a time discretization $(t)_\Delta$ with maximum step size $\Delta \in (0, \Delta_0)$, where, $\Delta_0 \in (0, 1)$. If the *jump times* of the driving process Z are not included in the time discretization $(t)_\Delta$ then such a discretization is called *regular*. On the other hand, if the jump times are included in the discretization, then resulting discretization is called *jump-adapted*. Consequently, a discrete-time approximation constructed on a regular time discretization is called a *regular scheme* while that constructed on jump-adapted time discretization is called a *jump-adapted scheme*, (see Chapter 3 for a general overview of the different kinds of time discretization).

For any numerical scheme, it is important to investigate the rate of convergence. The two common modes of convergence that exists in literature are–*strong* and *weak* convergence.

Definition 1.1.1. *A discrete time approximation Y constructed on a time-discretization $(t)_\Delta$ with maximum step size $\Delta > 0$, converge with strong order γ at time T to the solution X of a given SDE, if there exists a positive constant C, independent of Δ, and a finite number $\Delta_0 \in (0, 1)$, such that*

$$\mathbb{E}\left[\|X(T)) - Y(T)\|^2\right] \leq C\Delta^{2\gamma}, \tag{1.2}$$

for all $\Delta \in (0, \Delta_0)$.

Definition 1.1.2. *A discrete-time approximation Y constructed on a time discretization $(t)_\Delta$ with maximum step size $\Delta > 0$, converge with weak order β at time T to the solution X of a given SDE, if for a smooth enough function g, there exists a positive* *constant C, independent of Δ, and a finite number $\Delta_0 \in (0,1)$, such that*

$$|\mathbb{E}\left[g(X(T)) - g(Y(T))\right]| \leq C\Delta^{\beta}, \tag{1.3}$$

for all $\Delta \in (0, \Delta_0)$.

When $b = 0$ and $\sigma = 0$, (1.1) reduces to

$$X(t) = x + \int_0^t h(X(s-))dZ(s),\ t \in [0,T]. \tag{1.4}$$

The most common numerical scheme for approximating (1.4) is the Euler scheme on an equally spaced time discretization $(t)_\Delta$: $0 = t_0 < t_1 < t_N = T$, of the interval $[0,T]$, which is given by

$$Y_{n+1} = Y_n + h(Y_n)\left(Z_{n+1} - Z_n\right), \tag{1.5}$$

where $Y_n \stackrel{def}{=} Y(t_n)$ and $Z_n \stackrel{def}{=} Z(t_n), n = 1,2,3,\cdots N,$ where, N is some positive integer.

Remark 1.1.1. *1) When Z is a Brownian motion B in (1.5), the increments of B, i.e., $B_{n+1} - B_n$ are i.i.d. random variables that are normally distributed with mean zero and variance $t_{n+1} - t_n$. Thus one can simulate the increments exactly.*

2) There is no closed formula for simulating the increments of Lévy processes in the multi-dimensional setting, with the exception of the multivariate Brownian motion. This makes it more difficult to simulate a path of X using (1.5). Thus ones should employ approximate methods.

1.2 Historicity

The case when Z in (1.4) is a Brownian motion, has been extensively studied in literature. See e.g., [20] for a detailed treatment of numerical approximations of this case. The literature on weak numerical approximations of (1.4) is sparse, and even more scarce for strong numerical approximations. Protter and Talay in [27] were one of the first to consider a discrete-time approximation of the solution of (1.4). Under some smoothness conditions on h and the function g in (1.3), together with the assumption that the increments of Z can be simulated exactly, they studied the weak approximation of the Euler scheme. Jacod et al. in [16], considered the approximate Euler method for (1.4). Approximate in the sense that the increments of Z are approximated by i.i.d random variables. They studied the weak convergence of their method.

Rubenthaler in [34] then went on to approximate Z with a suitable compound Poisson process. By using the jump times of the compound Poisson process as discretization points, he studied the weak approximation of (1.4). His approximation was very "rough" when the driving process Z had a very strong jump activity. That is, when the Lévy measure has a strong singularity at the origin.

Bruti and Platen in [26] via the *stochastic Taylor expansion*, developed strong and weak numerical schemes of any desired order for the following SDE

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) + \int_0^t \int_E G(X(s-), z)N(ds, dz), \tag{1.6}$$

where, $x \in \mathbb{R}^n$, b and σ are as in (1.1), and, $G : \mathbb{R}^{n\times d} \to \mathbb{R}^d$ is Lipschitz continuous matrix-valued function. B is a standard Brownian motion which is independent of the Poisson random measure N, and $E \subset \mathbb{R}_0^d$, such that $\nu(E) < \infty$. That is (1.6) is an SDE driven by a Brownian motion and a compound Poisson process which we call a *jump-diffusion* SDE. The authors of [8] have also constructed Runge-Kutta schemes for (1.6), and have established the strong order of convergence of their schemes.

Mordecki et al. in [25] proposed and studied an adaptive (jump-adapted) method for (1.6). They considered the weak approximation, and in order to control errors that may arise from large jumps, they proposed to simulate all the jumps of the compound Poisson process. Thus, their method cannot be extended to the case where Z has infinite jump activity.

Kohatsu-Higa and Tankov in [21], in order to remedy the situation in [34] considered (1.4) in the case when Z has infinite jump activity. They studied the weak approximation of (1.4). They are the first to employ the idea of Asmussen and Rosiński [3] (see also [9]), to replace small jumps of Z by a Brownian motion with appropriate variance. Here, they proposed a jump-adapted approximation scheme where at any time when there is a jump larger than ϵ, the jump size and the change in the approximating system is computed. Between two jumps larger than ϵ, they used the approximate solution of the continuous SDE driven by the approximating Brownian motion. This approximation is constructed as a random perturbation of the corresponding deterministic ODE, which is obtained by completely removing all small jumps. In their one-dimensional approximation, they required that the function $1/h(x)$ be locally integrable which is not always the case, or the ODE could be approximated using a Runge-Kutta type approach which generates another source of error that needs to be analyzed. Their order of convergence of the weak approximation is given in terms of the intensity of the jumps larger than ϵ in magnitude.

Other works in the area of strong approximations include [14] where, the strong convergence of the Euler scheme for (1.4) in the one dimensional case is given in terms of the Wasserstein distance. The authors in [39, 37, 11], have also considered the weak approximation of (1.4).

1.3 Motivation

As earlier mentioned, when one speaks of an SDE driven by a Lévy process, one is generally referring to (1.1) when $b = 0$ and $\sigma = 0$, i.e., (1.4). This clearly, is not the most general model. Most of the work in the numerical approximation of (1.4) is on weak approximation of the Euler scheme. The literature on the strong (mean-squared) approximation is still very scarce. Moreover, as mentioned before, much of the work is limited to (1.4). With this in mind, I am motivated to consider (1.1) with, $b \neq 0, \sigma \neq 0$ and $h \neq 0$, and develop both strong and weak numerical schemes for (1.1) of any desired order in the light of [7, 26]. Thus, my interest is to extend the results in [7, 26], to include a larger class of SDEs, i.e., SDEs driven by Lévy processes with infinite jump activity. It is worth mentioning that the random perturbation method employed in [21] no longer works when $b \neq 0$ and $\sigma \neq 0$.

That said, we use the idea of Asmussen and Rosiński [3] (see also [9]), and in the spirit of Kuhatsu-Higa and Tankov [21], we approximate the small jumps of Z by an appropriate Brownian motion which is independent of Z. We then obtain a jump-diffusion SDE. The resulting SDE consists of a drift term, a Gaussian part and a compound Poisson part. We show that the resulting jump-diffusion SDE is a good approximation of the original SDE in the strong and weak sense, by giving $L_p, p \geq 2$ error estimates and weak error estimates. With this kind of approximation, we are able to give numerical approximations of (1.1) of any desired order.

The rest of this dissertation is organized as follow. In Chapter two, we give a general overview of Lévy processes, followed by two methods of simulating Lévy processes, namely, the random walk approximation and series representation, and finally, we give an overview of SDEs driven by Lévy processes. Since we are interested in numerical schemes of any desired order, in Chapter three, we give an overview of strong and weak numerical schemes of (1.6). The main results in this dissertation are in Chapter four. Here, we give the formulation of the model we are interested in, and then, state and proof our main results. Finally, in Chapter five we give numerical experiments to demonstrate our method with concluding remarks, and further directions.

Chapter 2

An Overview of Lévy Processes and SDEs Driven by Lévy Processes

For a collection of unique knowledge on the subject of Lévy processes see [4]. This text gives a survey on the theory and application Lévy processes. Texts that have given a systematic exposition of this theory include Bertoin [5] and Sato [36]. A detailed exposition of SDEs driven by Lévy process can be found in [1]. This chapter is organized as follows. In the Section 2.1, we give a formal definition of Lévy processes. Section 2.2 is concerned with an overview of two techniques of simulating Lévy processes. These include, the Random walk approximation and series representation, which gives us an intelligent way of simulating sample paths of a Lévy process. Finally, in Section 2.3, we give an overview of stochastic differential equations driven by Lévy processes.

2.1 Lévy Processes

2.1.1 Definition

Definition 2.1.1. *A d-dimensional adapted stochastic process $Z = \{Z(t) : t \geq 0\}$ on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$, is called a Lévy process if the following conditions are satisfied.*

(1) $Z(0) = 0$ a.s.,

(2) Z is a.s. càdlàg (right-continuous with left limits),

(3) Z has independent increments, i.e., for all $n \in \mathbb{N}$, and all sequences $0 = t_0 \leq t_1 \leq \cdots \leq t_n < \infty$, $Z(t_1) - Z(t_0), \cdots Z(t_n) - Z(t_{n-1})$ are independent,

(4) Z has stationary increments, i.e., for all $0 \leq s < t < \infty$, $Z(t) - Z(s)$ has the same distribution as $Z(t - s)$,

(5) Z is stochastically continuous, i.e., for all $A > 0$, $\lim_{s\to t} \mathbb{P}(\|(Z(t) - Z(s)\| > A) = 0$.

2.1.2 Infinitely Divisible Distributions

The primary tool used in the analysis of distributions of Lévy processes is its characteristic function (or Fourier transform).

Definition 2.1.2. *The characteristic function $\hat{\mu}(z)$ of a probability measure on $\mathbb{R}^d$ is*

$$\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \mu(dx), \quad z \in \mathbb{R}^d. \tag{2.1}$$

The characteristic function of the distribution $\mathbb{P}_X$ of a random variable X on $\mathbb{R}^d$ is denoted by $\hat{\mathbb{P}}_X(z)$. That is

$$\hat{\mathbb{P}}_X(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \mathbb{P}_X(dx) = \mathbb{E}\left[e^{i\langle z,X\rangle}\right]. \tag{2.2}$$

Another crucial notion in the study of Lévy processes is that of infinitely divisible distributions. We denote by μ^n the n-fold convolution of a probability measure μ with itself, i.e.,

$$\mu^n = \mu \star \mu \star \cdots \star \mu, \; n \text{ times}. \tag{2.3}$$

Definition 2.1.3. *A probability measure μ on $\mathbb{R}^d$ is* infinitely divisible *if for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$.*

Remark 2.1.1. *Since the convolution is expressed by a the product of its characteristic functions, μ is infinitely divisible if and only if, for each n, an n^{th} root of the characteristic function $\hat{\mu}(z)$ can be chosen in such a way that it is the characteristic function of some probability measure.*

Example 2.1.1. *If $\{Z(t) : t \geq 0\}$ is a Lévy process on $\mathbb{R}^d$, then, for every $t \geq 0$, the distribution of $Z(t)$ is infinitely divisible. Indeed, let $t_k = kt/n, k = 0, 1, \cdots, n$. Let $\mu = \mathbb{P}_{Z(t)}$ and $\mu_n = \mathbb{P}_{Z(t_k)-Z(t_{k-1})}$. Since*

$$Z(t) = (Z(t_1) - Z(t_0)) + (Z(t_2) - Z(t_1)) + \cdots + (Z(t_n) - Z(t_{n-1})) \tag{2.4}$$

is the sum of n independent and identically distributed random variables by property three in Definition 2.1.1, it follows that $\mu = \mu_n^n$.

This example shines some light on the close relationship between Lévy processes and infinitely divisible distributions. The one-to-one correspondence between the two is specified by the following result.

Theorem 2.1 ([36], Theorem 7.10, pp. 35)**.** *(i) If $\{Z(t) : t \geq 0\}$ is a Lévy process in law on $\mathbb{R}^d$, then, for $t \geq 0$, $\mathbb{P}_{Z(t)}$ is infinitely divisible and letting $\mathbb{P}_{Z(1)} = \mu$ we have $\mathbb{P}_{Z(t)} = \mu^t$.*

(ii) Conversely, if μ is an infinitely divisible distribution on $\mathbb{R}^d$, then there is a Lévy process in Law $\{Z(t) : t \geq 0\}$ such that $\mathbb{P}_{Z(1)} = \mu$.

The following is the famous *Lévy-Khintchine formula* which gives a representation of the characteristic function of all infinitely divisible random variables.

Theorem 2.2 ([36], Theorem 8.2, pp. 38)**.** *(i) If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then*

$$\hat{\mu}(z) = \exp\left[\frac{1}{2}\langle z, \Sigma z\rangle + i\langle a, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{\{\|x\|\leq 1\}}\right)\nu(dz)\right] \tag{2.5}$$

where Σ is a symmetric nonnegative definite $d\times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} \min\left(1, \|z\|^2\right)\nu(dz) < \infty, \tag{2.6}$$

and $a \in \mathbb{R}^d$.

(ii) The representation of $\hat{\mu}$ in (2.5) by a, Σ and ν is unique.

(iii) Conversely, if Σ is a symmetric nonnegative definite $d\times d$ matrix, ν is a measure satisfying (2.6), and $a \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (2.5).

Definition 2.1.4. *The triplet (a, Σ, ν) in Theorem 2.2 is called the* generating triplet *or the Lévy triplet of μ. Σ and the ν are called, respectively, the Gaussian covariance matrix and the Lévy measure of μ. When $\Sigma = 0$, μ is called purely non-Gaussian (or a pure jump Lévy process).*

The following proposition gives an explicit form of the characteristic function of the law of a Lévy process.

Proposition 2.1.1 (Characteristic function of a Lévy process)**.** *For any Lévy process $\{Z(t) : t \geq 0\}$ in $\mathbb{R}^d$, there is a unique Lévy triplet (a, Σ, ν) such that for every $t \geq 0$,*

$$\mathbb{E}\left[e^{i\langle u, Z(t)\rangle}\right] = \exp(tC(u)), \tag{2.7}$$

where

$$C(u) = i\langle a, u\rangle - \frac{1}{2}\langle u, \Sigma u\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u, Z\rangle} - 1 - i\langle u, z\rangle \mathbb{1}_{\{\|z\|\leq 1\}}\right) \nu(dz). \tag{2.8}$$

2.1.3 The Lévy-Itô Decomposition

Here, we state the celebrated Lévy-Itô decomposition, which describes the structure of the sample path of a Lévy process. It expresses the sample path of a Lévy process as a sum of four independent parts—the drift, the Gaussian part, the "small jump" part and the "large jump" part. In order to state this result, one needs the notion of the Poisson random measure which we shall abbreviate PRM.

Definition 2.1.5. *Let $S \subset \mathbb{R}^d$ be bounded, and $B(S) = \{B \cap S : B \in \mathcal{B}(\mathbb{R}^d)\}$, and let $B_0(S)$ denote all bounded sets in $B(S)$. A stochastic process $\{N(A)\}_{A \in B_0(S)}$ is said to be a PRM on S with intensity measure m if*

1. $N(\emptyset) = 0$

2. *For all pairwise disjoint sets $\{A_i\}$ in $B_0(S)$, such that $\cup A_i$ is bounded, $\{N(A_i), i = 1, 2, \cdots\}$ are independent and*

$$N\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} N(A_i) \ a.s. \tag{2.9}$$

3. *For all $A \in B_0(S), N(A)$ has a Poisson distribution with parameter $m(A)$. i.e.,*

$$\mathbb{P}(N(A) = k) = \frac{m(A)^k}{k!} e^{-m(A)}, \ k = 0, 1, 2, \cdots \tag{2.10}$$

Definition 2.1.6. *The compensated PRM denote by $\widetilde{N}(A)$ is defined as $\widetilde{N}(A) = N(A) - \mathbb{E}[N(A)]$.*

A PRM N is usually written as

$$N(A) = \sum_i \delta_{V_i}(A) \tag{2.11}$$

where δ denotes the Dirac measure and $\{V_i\}$ is a sequence of random variables taking values in S, i.e., $V_i : \Omega \to S, i \in \mathbb{N}$. In this case, N is written symbolically as

$$N = \sum_i \delta_{V_i}, \tag{2.12}$$

In such a situation, a PRM is called a Poisson Point Process (PPP). In this sequel, we shall denote by

$$\Delta Z(t) = Z(t) - Z(t-), \tag{2.13}$$

the jump of the Lévy process $Z(t)$ at time t. Much of the analytic difficulty in dealing with Lévy processes arise from the fact that, the number of jumps on a finite interval can be infinite. That is,

$$\sum_{0 \le s \le t} \|\Delta Z(s)\| = \infty \text{ a.s..} \tag{2.14}$$

This difficulty is overcome by making use of the fact that

$$\sum_{0 \le s \le t} \|\Delta Z(s)\|^2 < \infty \text{ a.s.} \tag{2.15}$$

Rather than examining $\Delta Z(t)$ itself, it is more profitable to count jumps of a specific size. More precisely, let $0 \le t < \infty$ and for each $A \in \mathcal{B}([0, \infty) \times \mathbb{R}_0^d)$, define

$$\begin{aligned} N(A) = N(t, A) &= \#\{0 \leq s \leq t :, \Delta Z(s) \neq 0, (s, \Delta Z(s)) \in A\} \\ &= \sum_{0 \leq s \leq t} \mathbb{1}_A(\Delta Z(s)) \qquad (2.16) \\ &= \sum_{\{0 \leq s \leq t,\ \Delta Z(s) \neq 0\}} \delta_{(s,\ \Delta Z(s))}(A). \end{aligned}$$

$N(A)$ counts the number of jumps of a particular size that belong to the set A. Before stating the Lévy-Itô decomposition of a Lévy process, we state the following result concerning càdlàg functions.

Theorem 2.3 ([1], Theorem 2.9.1, pp. 140)**.** *If $f : [0, \infty) \to \mathbb{R}^d$ is càdlàg, then for all ϵ and $T > 0$, the set*

$$\{t \in [0, T] : \|\Delta f(t)\| > \epsilon\} \tag{2.17}$$

is finite. Hence $\{t \in [0, T] : \|\Delta f(t)\| \neq 0\}$ is at most countable.

Theorem 2.4 (Lévy-Itô decomposition, [36], Theorem 19.2, pp.120)**.** *Let $\{Z(t) : t \geq 0\}$ be a Lévy process with Lévy triplet (a, Σ, ν)*

1. *Let $N(A)$ for every $A \in \mathcal{B}([0, \infty) \times \mathbb{R}^d)$ be as in (2.16). Then $N(A)$ is a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity Leb $\otimes\, \nu$.*

2. *The process $\{Z^1(t) : t \geq 0\}$ defined by*

$$Z^1(t) = \lim_{\epsilon \searrow 0} \sum_{0 \leq s \leq t} \left(\Delta Z(s) \mathbb{1}_{\{\|\Delta Z(s)\| > \epsilon\}} - t \int_{\epsilon < x \leq 1} z \nu(dz) \right) \tag{2.18}$$

is a Lévy process with Lévy triplet $(a_1, 0, \nu)$, where

$$a_1 = \int_{\|z\| > 1} \nu(dz). \tag{2.19}$$

The limit on the right-hand side of (2.18) exists a.s.

3. $Z^2(t) = Z(t) - Z^1(t)$ *is a Brownian motion with drift. Thus it has the Lévy triplet* $(\Sigma, 0, a - a_1)$*. Moreover, the process* $\{Z^2(t) : t \geq 0\}$ *is independent of the PRM* N*.*

4. The Lévy process $\{Z(t) : t \geq 0\}$ *can be written as*

$$Z(t) = Z^2(t) + \int_0^t \int_{0<\|z\|\leq 1} z\tilde{N}(dz, ds) + \int_0^t \int_{\|z\|>1} zN(dz, ds), \, a.s., t \geq 0. \quad (2.20)$$

By this theorem, we can interpret a Lévy process as a Brownian motion with drift, a "small jump" part given by the second integral in (2.20) and a "large jump" part given by the last integral in (2.20).

2.1.4 Some Examples of Lévy Processes

Example 2.1.2 (Brownian motion). *A stochastic process* $B = \{B(t) : t \geq 0\}$ *in* $\mathbb{R}$ *is called a Brownian motion with variance* σ^2 *if* $B(1)$ *follows a normal distribution with mean zero and variance* σ^2*, and the sample paths of* $\{B(t) : t \geq 0\}$ *are continuous a.s. In fact, this is the only Lévy process with continuous sample paths.*

Example 2.1.3 (The Poisson process). *The Poisson process of intensity* $\lambda > 0$*, is a Lévy process* $\{N(t) : t \geq 0\}$ *taking values in* $\mathbb{N} \cup \{0\}$ *where each* $N(t)$ *follows a Poisson distribution with parameter* λt*.*

Example 2.1.4 (The compound Poisson process). *Let* $\{Y(n) : n \in \mathbb{N}\}$ *be a sequence of i.i.d. random variables in* $\mathbb{R}^d$ *with common distribution* μ_Y*. Let* N *be the Poisson process in Example 2.1.3 which is independent of all* $Y(n)$*. The compound Poisson process* Z *is defined as*

$$Z(t) = \sum_{k=1}^{N(t)} Y(k) \quad (2.21)$$

for each $t \geq 0$*.*

Example 2.1.5 (Jump-diffusion process)**.** *Let B be given by Example 2.1.2, and Z by Example 2.1.4, then the process $J = \{J(t) : t \geq 0\}$ given by*

$$J(t) = B(t) + Z(t) \tag{2.22}$$

is Lévy process known as a jump-diffusion process.

Example 2.1.6 (α-stable Lévy process)**.** *A Lévy process $Z = \{Z(t) : t \geq 0\}$ in $\mathbb{R}^d$ with Lévy triplet $(a, 0, \nu)$ is said to be an α-stable process with $0 < \alpha < 2$ if there exists a finite measure λ on $\mathbb{S}$ the unit sphere of $\mathbb{R}^d$, such that*

$$\nu(A) = \int_{\mathbb{S}} \int_0^{\infty} \mathbb{1}_A(ru) r^{-\alpha-1} dr \lambda(u).$$

The Lévy measure ν has an explicit form in the one dimensional case which is given by

$$\nu(z) = c_1 |z|^{-\alpha-1} \mathbb{1}_{\{z<0\}} + c_2 z^{-\alpha-1} \mathbb{1}_{\{z>0\}}$$

where c_1 and c_2 are non-negative constants such that $c_1 + c_2 > 0$. For a comprehensive introduction of the subject of stable processes, see [35]. For the simulation of these processes, see [18].

Example 2.1.7 (Tempered α-stable process)**.** *A Lévy process $\{Z(t) : t \geq 0\}$ with Lévy triplet $(a, 0, \nu)$ is said to be* tempered α-stable if the Lévy measure ν is given by

$$\nu(dz) = \|z\|^{-\alpha-1} g(z) dz$$

where $(-1)^n D^n g(z) \geq 0$ for $z > 0$, $(-1)^n D^n g(z) \geq 0$ for $z < 0$, $\alpha \in (0, 2)$, $n \in \mathbb{N} \cup \{0\}$ and $\lim_{\|z\| \to \infty}$g(z)=0. Here, $D^n g(z)$ denotes the n^{th} derivative of g. For a detailed treatment of tempering stable processes, see [32].

2.2 An Overview on Simulation of Lévy Processes Using Random Walks and Series Representations

The triumph of Lévy processes cannot be understood without the progress of computers. The simulation of Brownian motion and compound Poisson processes are discussed in many textbooks, see e.g., the monographs [10] and [2]. Unlike Brownian motion, the simulation of general Lévy processes is not very straight forward. Let $Z = \{Z(t) : t \geq 0\}$ be a Lévy process in $\mathbb{R}^d$ with characteristic triplet (a, Σ, ν). By the Lévy-Itô decomposition (Theorem 2.4), the sample paths of Z can be decomposed as follows:

$$Z(t) = at + B_\Sigma(t) + \int_0^t \int_{0<\|z\|\leq 1} z\widetilde{N}(dz, ds) + \int_0^t \int_{\|z\|>1} zN(dz, ds). \tag{2.23}$$

Thus the problem of simulating a Lévy process boils down to that of simulation of a Brownian motion with drift, the small jump part and the a compound Poisson process. When the Lévy measure ν is finite, then one needs only simulate a jump-diffusion process. The main difficulty in simulating the Lévy process is when ν is infinite. In this case, the sample paths of $Z = \{Z(t) : t \geq 0\}$ has infinitely many jumps in any finite interval $[0, T]$. Thus exact simulation of such processes is impossible. A process close to the original one is generated instead. Here, we examine two methods of accomplishing this task, namely, the random walk approximation and series approximation of Lévy processes.

2.2.1 Random Walk Approximation

Because of the property of stationary independent increments, the problem of simulating a discrete skeleton $\{Z_n\}$, where $Z_n \stackrel{def}{=} Z(t_n), t_n = n\Delta$, of a Lévy process is equivalent to the problem of a random variable generation from a specific infinitely divisible distribution. Suppose $\{Z(t) : t \geq 0\}$ is a Lévy process determined by (2.23), with $\Sigma \equiv 0$. On a fixed time interval $[0, T]$, with $n \in \mathbb{N}$, put $\Delta = T/n$. Generate the increments $\Delta Z_j = Z(j\Delta) - Z((j-1)\Delta)$ as i.i.d. random variables with common distribution $\mathbb{P}_\Delta(A) = \mathbb{P}(Z(\Delta) \in A)$, $j = 1, 2, \cdots, n-1$. Let

$$Z^\Delta(t) = \begin{cases} 0 & \text{if } 0 \leq t < \Delta \\ \sum_{k=1}^{j} \Delta Z_k & \text{if } j\Delta \leq t < (j+1)\Delta. \end{cases} \tag{2.24}$$

The process $\{Z^\Delta(t) : t \geq 0\}$ is a random walk approximation to $\{Z(t) : t \geq 0\}$.

Example 2.2.1 (Discretized trajectory for a symmetric α-stable process)**.** *Let $Z = \{Z(t) : t \geq 0\}$ be a symmetric α-stable process with index of stability $\alpha \in (0, 2)$. Let $\{t_i\}_{i=1}^n$ be a discretization of the time interval $[0, T]$. We have the following algorithm for simulating Z (see [10], Algorithm 6.6, pp. 180)*

Algorithm 1

- *Simulate n independent random variables γ_i, uniformly distributed on $(-\pi/2, \pi/2)$, and n independent standard exponential random variables W_i.*
- *Compute ΔZ_i, for $i = 1, 2, \cdots, n$ using*

$$\Delta Z_i = (t_i - t_{i-1})^{1/\alpha} \frac{\sin \alpha\gamma_i}{(\cos \gamma_i)^{1/\alpha}} \left(\frac{\cos((1-\alpha)\gamma_i)}{W_i} \right)^{(1-\alpha)/\alpha} \tag{2.25}$$

  with $t_0 = 0$.
- *The discretized trajectory of Z is given by $Z(t_i) = \sum_{k=1}^{j} \Delta Z_i$*

See the graphs in Figures 2.1–2.3 for typical sample paths of a symmetric α-stable process. Figure 2.1 resembles the graph of a compound Poisson process. When α is large, the behavior of the sample path is determined by small jumps (see in Figure 2.3). In this case the sample path resembles that of a Brownian motion. Figure

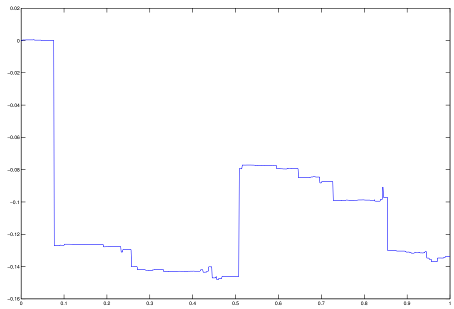

Figure 2.1: Simulated trajectory of an α-stable process with $\alpha = 0.5$.

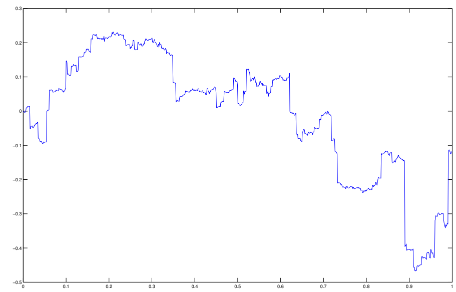

Figure 2.2: Simulated trajectory of an α-stable process with $\alpha = 1$.

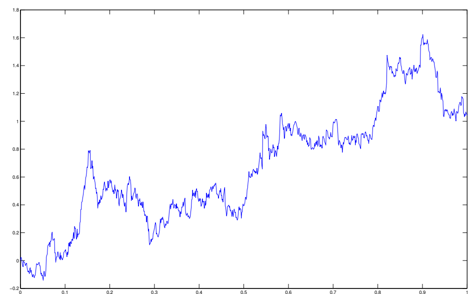

Figure 2.3: Simulated trajectory of an α-stable process with $\alpha = 1.7$.

2.2 corresponds to the graph of a Cauchy process which is between the two cases. For more details on random walk approximation of Lévy process see Section 6.2 of [10].

Remark 2.2.1. *(1) We observe that in order to implement the random walk approximation method of simulation, one needs to know the law of the increment of the Lévy process. For most Lévy processes, the law of the increments is not known explicitly.*

(2) Contrary to the one-dimensional case, no closed formulae are available for simulating the increments of Lévy processes in the multi-dimensional setting, with the exception of the multivariate Brownian motion. This makes it more difficult to simulate a path of the general Lévy process. Thus one should use approximate methods such as the series representations which we overview below.

2.2.2 Series Representation

Here, we state a general result due to Rosiński [31], that allows one to construct series representation of Lévy processes. It turns out that series representations is a more intelligent way of approximating a Lévy process by compound Poisson process. Let $\{Z(t) : t \geq 0\}$ be a pure jump Lévy process, i.e.,

$$Z(t) = at + \int_0^t \int_{\|z\|\leq 1} z\widetilde{N}(dz, ds) + \int_0^t \int_{\|z\|>1} zN(dz, ds) \tag{2.26}$$

where $a \in \mathbb{R}^d$ and N is a PRM on $[0, 1] \times \mathbb{R}^d_0$. Let $\{V_i\}_{i\geq 1}$ be an i.i.d. sequence of random elements in a measurable space S with common distribution F. Assume that $\{V_i\}_{i\geq 1}$ is independent of $\{\Gamma_i\}_{i\geq 1}$ of partial sums of standard exponential random variables. Let

$$H : (0, \infty) \times S \to \mathbb{R}^d \tag{2.27}$$

be a measurable function such that for each $v \in S$

$$r \to \|H(r, v)\|, \tag{2.28}$$

in noninecreasing. Let $\{U_i\}_{i\geq 1}$ be denote i.i.d. sequence of uniform random variables in $[0, 1]$, which is independent of $\{\Gamma_i\}_{i\geq 1}$ and $\{V_i\}_{i\geq 1}$. Let $B \in \mathcal{B}(\mathbb{R}^d)$, and define measures on $\mathbb{R}^d$ by

$$\sigma(r, B) = \mathbb{P}(H(r, V_i) \in B),\ r > 0,\ \text{and}\ \nu(B) = \int_0^\infty \sigma(r, B)dr. \tag{2.29}$$

Put

$$A(s) = \int_0^s \int_{\|z\|\leq 1} z\sigma(r, dz)dr,\ s \geq 0. \tag{2.30}$$

Theorem 2.5 (Rosiński, [31], Theorem 4.1). *(A) The series* $\sum_{i=1}^{\infty} H(\Gamma_i, V_i)$ *converges a.s. if and only if*

(i) ν *is a Lévy measure on* $\mathbb{R}_0^d$ *and*

(ii) $a \overset{def}{=} \lim_{s\to\infty} A(s)$ *in* $\mathbb{R}^d$.

If (i) and (ii) are satisfied, the the law of $\sum_{i=1}^{\infty} H(\Gamma_i, V_i)$ *is infinitely divisible with characteristic function* $\phi(u)$ *given by*

$$\phi(u) = \exp\left[i\langle a, u\rangle + \int_{\mathbb{R}_0^d} \left(e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle \mathbb{1}_{\|z\|\le 1} \right) \nu(dz) \right]. \tag{2.31}$$

(B) If only (i) holds, then $\sum_{i=1}^{\infty} [H(\Gamma_i, V_i) - c_i]$ *converges a.s. for* $c_i = A(i) - A(i-1)$*. In this case, the characteristic function of the law of* $\sum_{i=1}^{\infty} [H(\Gamma_i, V_i) - c_i]$ *is given by (2.31) with* $a = 0$.

The series

$$at + \sum_{i=1}^{\infty} [H(\Gamma_i, V_i)\mathbb{1}_{U_i \le t} - tc_i] \tag{2.32}$$

converges a.s. uniformly on $[0, 1]$ to a Lévy process $\{Z(t) : t \ge 0\}$ with Lévy triplet $(a, 0, \nu)$, (see [31], Theorem 5.1).

Remark 2.2.2. *The process*

$$Y^{\tau}(t) = \sum_{\{i:\Gamma_i \le \tau\}} [H(\Gamma_i, V_i)\mathbb{1}_{U_i \le t} - tA(\tau)] \tag{2.33}$$

is a compound Poisson process with Lévy triplet $(0, \nu_\tau, 0)$ *where*

$$\nu_\tau(B) = \int_0^{\tau} \sigma(r, B) dr \tag{2.34}$$

See the proof of Theorem 4.1 in [31]. Thus, series representations of Lévy processes obtained using Theorem 2.5 can be viewed as compound Poisson approximations, *though, the transformations applied to the initial Lévy measure may be more complex than simply deleting small jumps.*

The following examples are series representation of the symmetric α-stable and exponentially tempered α-stable processes respectively.

Example 2.2.2 (Symmetric α-stable process)**.** *Let $\{V_j\}_{j\geq 1}$ be an i.i.d. sequence such that $\mathbb{P}(V_j = \pm 1) = 1/2$. Then*

$$Z(t) = T^{1/\alpha} c_\alpha \sum_{j=1}^{\infty} V_j \Gamma_j^{-1/\alpha} \mathbb{1}_{\{U_j \leq t\}}, \ 0 \leq t \leq T, \tag{2.35}$$

represents an α-stable process with Lévy triplet $(0, \nu, 0)$, where $0 < \alpha < 2$,

$$c_\alpha = \begin{cases} |\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}|^{-1/\alpha} \ \text{when } \alpha \neq 1 \\ c_1 = \pi/2, \end{cases} \tag{2.36}$$

and

$$\nu(dz) = (|z|^{-1-\alpha} \mathbb{1}_{\{z<0\}} + z^{-1-\alpha} \mathbb{1}_{\{z>0\}}) dz. \tag{2.37}$$

Next, we give a series representation of a tempered α-stable process. A detailed treatment of series representation of tempered stable processes is due to Rosiński [31], in particular, Theorem 5.1 and Theorem 5.3 of this reference. Here, we give an example which is suitable for our purposes, (see Example 4.3 of [33]).

Example 2.2.3 (Exponentially tempered stable process)**.** *Suppose that*

$$\nu(dz) = \kappa(|z|^{-1-\alpha} e^{-\lambda|z|} \mathbb{1}_{\{z<0\}} + z^{-1-\alpha} e^{-\lambda z} \mathbb{1}_{\{z>0\}}) dz$$

where $\lambda > 0$. Let $\{V_j\}$, $\{\eta_j\}$ and $\{\xi_j\}$ be i.i.d sequences such that $\mathbb{P}(V_j = \pm 1) = 1/2$, η_j are exponentially distributed with rate one, and $\{\xi_j\}$ are uniformly distributed on $(0, 1)$. Assume all random sequences $\{\Gamma_j\}$, $\{U_j\}$, $\{V_j\}$, $\{\eta_j\}$ and $\{\xi_j\}$ are independent *of each other. Then*

$$Z(t) = \sum_{j=1}^{\infty} V_j \left(\left(\frac{\alpha \Gamma_j}{2\kappa T} \right)^{-1/\alpha} \wedge \eta_j \xi_j^{1/\alpha} \right) \mathbb{1}_{\{U_j \le t\}}, \quad 0 \le t \le T \tag{2.38}$$

represents a tempered α-stable process with Lévy measure ν. From Remark 2.2.2,

$$Z^\tau(t) = \sum_{\{j:\Gamma_j \le \tau\}} V_j \left(\left(\frac{\alpha \Gamma_j}{2\kappa T} \right)^{-1/\alpha} \wedge \eta_j \xi_j^{1/\alpha} \right) \mathbb{1}_{\{U_j \le t\}} \tag{2.39}$$

is a compound Poisson process.

Example 2.2.4 (An algorithm for simulating a tempered α stable process using series representation)**.** *Fix a number τ depending on the required precision. This number is proportional to the average number of terms in the series and it determines the truncation level i.e., jumps smaller than $\tau^{-1/\alpha}$ are truncated.*

Algorithm 2: Simulating a tempered α-stable process

- *Initialize $j := 0$*
- *REPEAT WHILE $\sum_{n=1}^{j} T_n < \tau$*
- *Set $j = j + 1$*
- *Simulate T_j and η_j: standard exponential*
- *Simulate U_j and ξ_j: uniform on $[0, 1]$*
- *Simulate V_j taking values 1 or -1 with probability $1/2$*

The trajectory of $\{Z(t) : t \ge 0\}$ is given by

$$Z(t) = \sum_{n=1}^{j} V_j \left(\left(\frac{\alpha \Gamma_j}{2\kappa T} \right)^{-1/\alpha} \wedge \eta_j \xi_j^{1/\alpha} \right) \mathbb{1}_{\{U_j \le t\}} \text{ where } \Gamma_j = \sum_{k=1}^{j} T_k.$$

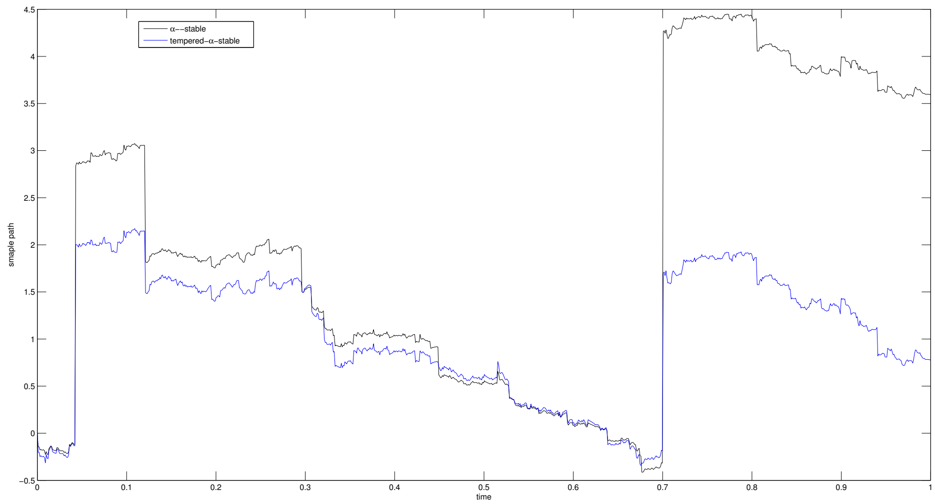

Figure 2.4: Simulated trajectories of an α-stable and a tempered α-stable process using series representation. Here, $\alpha = 1.3$, and precision $\tau = 1000$ and N=10,000 iterations.

2.3 SDEs Driven by Lévy Processes

In this section we give an overview of SDEs driven by Lévy processes. Let $\{B(t) : t \geq 0\}$ be a d-dimensional Brownian motion, and N an independent PRM on $\mathbb{R}^+ \times \mathbb{R}_0^d$ with associated compensator $\widetilde{N}$, and intensity measure ν, a Lévy measure. We also assume that B and N are independent. We consider the following stochastic differential equation

$$\begin{aligned} X(t) = x &+ \int_0^t b(X(s-))ds + \int_0^t \sigma(X(s-))dB(s) \\ &+ \int_0^t \int_{\|z\| \leq c} F(X(s-), z)\widetilde{N}(ds, dz) \\ &+ \int_0^t \int_{\|z\| > c} G(X(s-), z)N(ds, dz) \end{aligned} \tag{2.40}$$

where $x \in \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$, $F : \mathbb{R}^{n \times d} \to \mathbb{R}^n$ and $G : \mathbb{R}^{n \times d} \to \mathbb{R}^n$. The parameter $c \in (0, \infty)$ is known as the "cut-off" parameter and determines the "small" and "large" jumps. The solution to (2.40) when it exists, will be an n-dimensional stochastic process $X = \{X(t) : t \geq 0\}$.

Definition 2.3.1. *By a solution of the equation (2.40), we mean an n-dimensional càdlàg, $\mathcal{F}_t$-adapted stochastic process $X = \{X(t) : t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ such that (2.40) is satisfied.*

Since B and N are specified in advance, any solution of (2.40) is called a *strong solution* in literature. There is also the notion of *weak solution*. Here, we are interested in the strong solution. In addition, we require that the solution of (2.40) be unique. There exists various notions of uniqueness available, the strongest of which is the following.

Definition 2.3.2 (Path-wise uniqueness)**.** *The solution to (2.40) is said to be* path-wise unique *if $X_1 = \{X_1(t) : t \geq 0\}$ and $X_2 = \{X_2(t) : t \geq 0\}$ are solutions to (2.40), then $\mathbb{P}(X_1(t) = X_2(t)$ for all $t \geq 0) = 1$.*

The following conditions are imposed on the coefficient functions b, σ and F in order to guarantee the existence of a unique solution to (2.40). We call them the "Growth" and "Lipschitz" conditions respectively.

- **Growth condition:**

$$\|\sigma(x)\|^2 + \|b(x)\|^2 + \int_{\|z\| \leq c} \|F(x,z)\|^2 \nu(dz) \leq K(1 + \|x\|^2), \; x \in \mathbb{R}^n, \tag{2.41}$$

  for some positive constant K.

- **Lipschitz condition:**

$$\begin{aligned} &\|\sigma(x) - \sigma(y)\|^2 + \|b(x) - b(y)\|^2 + \int_{\|z\| \leq c} \|F(x,z) - F(y,z)\|^2 \nu(dz) \\ &\leq K\|x - y\|^2, \quad x, y \in \mathbb{R}^n, \end{aligned} \tag{2.42}$$

  for some positive constant K.

Theorem 2.6 ([15], Theorem 9.1, pp. 230-231)**.** *If b, σ and F, satisfy conditions (2.41) and (2.42), then there exists a unique n-dimensional càdlàg process $X = \{X(t) : t \geq 0\}$ that satisfies equation (2.40).*

Example 2.3.1. *Let Z be a d-dimensional Lévy process with Lévy triplet $(a, 0, \nu)$. Then Z has the following Lévy-Itô decomposition*

$$Z(t) = at + \int_0^t \int_{\|z\|\leq 1} z\tilde{N}(dz, ds) + \int_0^t \int_{\|z\|>1} zN(dz, ds). \tag{2.43}$$

Let $x \in \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^n$ a vector-valued Lipschitz continuous function, and $h : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ a matrix-valued Lipschitz continuous function. Consider the SDE

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t h(X(s-))dZ(s),\ \ t \in [0, T]. \tag{2.44}$$

Equation (2.44) can formally be written in integral form as

$$\begin{aligned} X(t) = x + \int_0^t b(X(s))ds + \int_0^t h(X(s))ads + \int_0^t \int_{\|z\|\leq 1} h(X(s-))z\widetilde{N}(dz, ds) \quad (2.45)\\ + \int_0^t \int_{\|z\|>1} h(X(-s))zN(dz, ds). \end{aligned}$$

Let $b_1(x) = b(x) + h(x)a$. Then (2.45) can be rewritten as follows,

$$\begin{aligned} X(t) = x + \int_0^t b_1(X(s))ds + \int_0^t \int_{\|z\|\leq 1} h(X(s-))z\widetilde{N}(dz, ds)\\ + \int_0^t \int_{\|z\|>1} h(X(-s))zN(dz, ds). \end{aligned} \tag{2.46}$$

Remark 2.3.1. *Observe that (2.46) is a particular case of (2.40) with $c = 1$, $\sigma \equiv 0$, $F(x, z) = h(x)z$, $\|z\| \leq 1$ and $G(x, z) = h(x)z$, $\|z\| > 1$. Also, since b and h are Lipschitz continuous, it follows that b_1 is Lipschitz continuous.*

The following proposition guarantees the existence of a unique solution to equation (2.46).

Proposition 2.3.1. *Let b and h be as in Example 2.3.1, then (2.46) has unique solution.*

Proof. We only need to verify condition (2.42). Indeed, let $x, y \in \mathbb{R}^n$. Then, since b and h are Lipschitz continuous and ν is a Lévy measure, it follows that

$$\begin{aligned}
&\|b_1(x) - b_1(y)\|^2 + \int_{\|z\|\leq 1} \|h(x)z - h(y)z\|^2 \nu(dz) \\
&\qquad \leq K_1\|x-y\|^2 + K_2\|x-y\|^2 \int_{\|z\|\leq 1} \|z\|^2 \nu(dz) \\
&\qquad = K\|x-y\|^2,
\end{aligned} \tag{2.47}$$

where $K = K_1 + K_2 \int_{\|z\|\leq 1} \|z\|^2 \nu(dz)$. Thus, condition (2.42) is verified. Condition (2.41) follows from (2.47), since, if we let $x \in \mathbb{R}^d$, then we obtain

$$\begin{aligned}
&\|b_1(x)\|^2 + \int_{\|z\|\leq 1} \|h(x)z\|^2 \nu(dz) \\
&\qquad \leq \|b_1(x) - b_1(0)\|^2 + \int_{\|z\|\leq 1} \|(h(x) - h(0))z\|^2 \nu(dz) \\
&\qquad\quad + \|b_1(0)\|^2 + \|h(0)\|^2 \int_{\|z\|\leq 1} \|z\|^2 \nu(dz) \\
&\qquad \leq K\|x\|^2 + \|b_1(0)\|^2 + \|h(0)\|^2 \int_{\|z\|\leq 1} \|z\|^2 \nu(dz) \\
&\qquad \leq C\left(1 + \|x\|^2\right)
\end{aligned} \tag{2.48}$$

where $C = \max\left\{K, \|b_1(0)\|^2 + \|h(0)\|^2 \int_{\|z\|\leq 1} \|z\|^2 \nu(dz)\right\}$. □

Remark 2.3.2. *(i) The equation (2.43) rarely admits a closed form solution. Thus, the need for numerical simulations.*

(ii) When $b \equiv 0$, we obtain

$$X(t) = x + \int_0^t h(X(s-))dZ(s), \ t \in [0, T]. \tag{2.49}$$

We observe here that if Z has a jump size of z, then $X(t)$ will have a jump of size $h(X(t-))z$. In literature, when one talks of an SDE driven by a Lévy process, generally, one is referring to (2.49).

(iii) Note, we write $X(t-)$ instead of $X(t)$ in order that the integrand be predictable, so the integral is well defined as an Itô integral.

(iv) The solution X to equation (2.43), is a homogeneous Markov process. See Theorem 6.4.6 of [1], pp. 388.

We now give an example of (2.40) with an exact solution.

Proposition 2.3.2 ([10], Proposition 8.21, pp. 284-285)**.** *Let $Z = \{Z(t) : t \geq 0\}$ be a Lévy process in $\mathbb{R}$ with Lévy triplet (a, σ^2, ν). Then there exists a unique càdlàg process $(X(t) : t \geq 0)$ in $\mathbb{R}$ such that*

$$X(t) = 1 + \int_0^t X(s-)dZ(s), \tag{2.50}$$

given by

$$X(t) = \exp\left(Z(t) - \frac{\sigma^2}{2}t\right) \prod_{0 \leq s \leq t} (1 + \Delta Z(s))\, e^{-\Delta Z(s)}. \tag{2.51}$$

Chapter 3

Regular and Jump-Adapted Approximation Schemes for Jump-Diffusion SDEs–An Overview

The deterministic Taylor expansion has proven to be an indispensable tool in both theoretical and practical investigations. It allows one to approximate a sufficiently smooth function in a neighborhood of a given point to any desired order of accuracy. A similar expansion can be obtained in the stochastic case. The stochastic Taylor expansion is a generalization of the deterministic Taylor formula. It permits one to expand the increments of smooth functions of Itô processes in terms of *multiple stochastic integrals*. It is the main tool for the construction of stochastic numerical methods. For a detail treatment of this topic, see [26] and [24]. This chapter is divide into two sections. In Section 3.1, we give an overview of the stochastic Taylor expansion, and in Section 3.2, we review how the stochastic Taylor expansion is used to construct numerical schemes for (1.6).

3.1 Stochastic Taylor Expansion

It often imperative to be able to expand the increments of smooth functions of solutions of SDEs. Therefore, a stochastic expansion with similar properties to the deterministic Taylor formula can be extremely useful. This expansion is basically obtain by a repeated application of Itô's formula. Here, we consider the following SDE

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) + \int_0^t \int_E G(X(s-), z)N(ds, dz) \quad (3.1)$$

where, x, b, G, B, N are as in (2.40) and E is a subset of $\mathbb{R}_0^d$, such that $\nu(E) < \infty$. That is (3.1) is an SDE driven by a Brownian motion and a compound Poisson process which we call a *jump-diffusion* SDE.

Remark 3.1.1. *Observe that (3.1) is a special case of (2.40) with $F \equiv 0$ and a finite Lévy measure ν.*

We denote by $p(t) = N(E \times [0, t]),\ t \in [0, T]$, the Poisson process which counts the number jumps occurring in the time interval $[0, t]$, for all $t \geq 0$. Let

$$\{\tau_i, i = 1, 2, \cdots, p(t)\}, \quad (3.2)$$

denote the jump times generated by N with corresponding jump sizes (or marks)

$$\{V_i, i = 1, 2, \cdots, p(t)\}. \quad (3.3)$$

With this notation, (3.1) can be written as follows

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) + \sum_{i=1}^{p(t)} G(X_{\tau_i}, V_i). \quad (3.4)$$

Discrete time approximations of (3.1) are constructed using the stochastic Taylor expansion of its solution. Smooth functions of (3.1) can be expanded in terms of *multiple stochastic integrals*. In order to express this expansion, we introduce below certain notations.

3.1.1 Multiple Stochastic Integrals

In order to describe the multiple stochastic integrals, we start by defining the notion of a *multi-index* α.

Definition 3.1.1. *A row vector $\alpha = (j_1, j_2, \cdots, j_l)$, where $j_i \in \{-1, 0, 1, \cdots, d\}$ for $i \in \{1, 2, \cdots, l\}$, is called a multi-index of length $l = l(\alpha) \in \mathbb{N}$. Here, d represents the number of components of B in (3.1).*

For $d \in \mathbb{N}$, the set of multi-indices α will be denoted by

$$\mathcal{M} = \{(j_1, \cdots, j_l) : j_i \in \{-1, 0, 1, 2, \cdots, d\}, \{i \in \{1, 2, \cdots, l\}, l \in \mathbb{N}\} \cup \{v\} \qquad (3.5)$$

where v denotes the void multi-index, i.e., v has zero length. A component $j \in \{1, 2, \cdots, d\}$ of the multi-index will denote a multiple stochastic integral with respect to the j^{th} Brownian motion, a component $j = 0$ will denote integration with respect to time and finally, a component $j = -1$ will denote integration with respect to the PRM N. In addition, $n(\alpha)$ will denote the number of components of α that are equal to zero and $s(\alpha)$ the number of components of α that are equal to -1. We denote by $\alpha-$ the multi-index obtained by deleting the last component of α and $-\alpha$ the multi-index obtained by deleting the first component of α.

Example 3.1.1. *If we let $d = 1$, then we get the following*

$$\begin{aligned} l((0,1,-1)) &= 3, \;\; n((0,1,-1)) = 1 \\ s((0,-1)) &= 1, \;\; (0,1,-1)- = (0,1) \\ -(0,1,-1) &= (1,-1). \end{aligned} \tag{3.6}$$

In order to define multiple stochastic integrals, one needs to define sets of adapted stochastic processes $g = \{g(t) : t \in [0,T]\}$ that will qualify as integrands for theses multiple stochastic integrals in the stochastic Taylor expansion. These sets are defined as follows:

$$\begin{aligned} \mathcal{S}_v &= \left\{ g : \sup_{0 \le t \le T} \mathbb{E}\left[\|g(t)\| \right] < \infty \right\}, \\ \mathcal{S}_{(0)} &= \left\{ g : \mathbb{E}\left[\int_0^T \|g(s)\| ds \right] < \infty \right\}, \\ \mathcal{S}_{(-1)} &= \left\{ g : \mathbb{E}\left[\int_0^T \int_E \|g(s,z)\|^2 \nu(dz) ds \right] < \infty \right\}, \\ \mathcal{S}_{(j)} &= \left\{ g : \mathbb{E}\left[\int_0^T \|g(s)\|^2 ds \right] < \infty \right\}, \;\; j = 1, 2, \cdots, d. \end{aligned} \tag{3.7}$$

The set $\mathcal{S}_\alpha$ for an arbitrary multi-index $\alpha \in \mathcal{M}$ with $l(\alpha) > 1$, is defined below.

Definition 3.1.2. *Let ρ and τ be two stooping times such that $0 \le \rho \le \tau \le T$ almost surely. For a multi-index $\alpha \in \mathcal{M}$ and an adapted process $g(.) \in \mathcal{S}_\alpha$, the multiple stochastic integral $I_\alpha[g(.)]_{\rho,\tau}$ is defined recursively as*

$$I_\alpha[g(.)]_{\rho,\alpha} = \begin{cases} g(\tau), \;\; \text{if } \alpha = v, \\ \int_\rho^\tau I_{\alpha-}[g(.)]_{\rho,u} du, \;\; \text{if } l \ge 1 \text{ and } j_l = 0, \\ \int_\rho^\tau I_{\alpha-}[g(.)]_{\rho,u} dW_u^{j_l}, \;\; \text{if } l \ge 1 \text{ and } j_l = 1, 2, \cdots, d, \\ \int_\rho^\tau \int_E I_{\alpha-}[g(.)]_{\rho,u-} N(dz_{s(\alpha)}, du) \;\; \text{if } l \ge 1 \text{ and } j_l = -1, \end{cases} \tag{3.8}$$

where $g(.) = g\left(., z_1, \cdots, z_{s(\alpha)}\right)$ and $u-$ denotes the left hand limit of u.

In order to simplify notation, we will omit dependence of g on one or more of the variables $z_1, \cdots, z_{s(\alpha)}$ of the vector z expressing the jumps of the PRM.

Definition 3.1.3. *For every multi-index $\alpha \in \mathcal{M}$ with $l(\alpha) > 1$, the sets $\mathcal{S}_\alpha$ are recursively defined as sets of adapted stochastic processes $g = \{g(t) : t \geq 0\}$ such that the integral process $\{I_{\alpha-}[g(.)]_{\rho,t}, t \in [0,T]\}$ satisfies*

$$I_{\alpha-}[g(.)]_{\rho,.} \in \mathcal{S}_{j_l}. \tag{3.9}$$

Example 3.1.2. *Assume $d = 2$.*

1. *Let $\alpha = (1, 0, -1)$, then*

$$\begin{aligned} I_{(1,0,-1)}[g(.)]_{\rho,\tau} &= \int_\rho^\tau \int_E I_{(1,0)-}[g(s,z)]_{\rho,u-} N(dz,ds) \\ &= \int_\rho^\tau \int_E \int_\rho^{s-} I_{(1)-}[g(s_1,z)]_{\rho,s} ds_1 N(dz,ds) \\ &= \int_\rho^\tau \int_E \int_\rho^{s} \int_\rho^{s_1} I_v[g(s_2,z)]_{\rho,s_2-} dW^1_{s_2} ds_1 N(dz,ds) \end{aligned} \tag{3.10}$$

2. *Let $\alpha = (k, 0)$ $k = 1, 2, \cdots, d$, then*

$$\begin{aligned} I_{(2,0)}[g(.)]_{\rho,\tau} &= \int_\rho^\tau I_{(2)-}[g(s)] ds \\ &= \int_\rho^\tau \int_\rho^s I_{(v)}[g(s_1)] dW^k_{s_1} ds \\ &= \int_\rho^\tau \int_\rho^s g(s_1) dW^k_{s_1} ds \end{aligned} \tag{3.11}$$

3.1.2 Coefficient Function

We now conider some sets of sufficiently smooth and integrable functions which will qualify as coefficient functions in the stochastic expansion. Denote by $\mathcal{L}^0$ the set of functions $f:[0,T]\times\mathbb{R}^n\times E^{s(\alpha)}\to\mathbb{R}^n$ for which

$$f(t,x+G(x,z),u)-f(t,x,u)-\sum_{i=1}^{n}G^i(x,z)\frac{\partial}{\partial x^i}f(t,x,u) \tag{3.12}$$

is ν–integrable for all $t\in[0.T],x\in\mathbb{R}^n,u\in E^{s(\alpha)}$ and $f(.,.,u)\in C^{1,2}$. Next, we denote by $\mathcal{L}^k,k=\{1,2,\cdots,d\}$ the set of functions $f:[0,T]\times\mathbb{R}^n\times E^{s(\alpha)}\to\mathbb{R}^n$ whose partial derivative $\partial/\partial x^i f(t,x,u),i=1,2,\cdots,d$ exists. Finally, let $\mathcal{L}^{-1}$ be the set of functions $f:[0,T]\times\mathbb{R}^n\times E^{s(\alpha)}\to\mathbb{R}^n$ for which

$$\|f(t,x+G(x,z),u)-f(t,x,u)\|^2 \tag{3.13}$$

is ν–integrable for all $t\in[0,T],x\in\mathbb{R}^n$ and $u\in E^{s(\alpha)}$. We now define the following operators for functions $f(t,x,u)\in\mathcal{L}^k$

$$\begin{aligned} L^0f(t,x,u) &= \frac{\partial}{\partial t}f(t,x,u)+\sum_{i=1}^{n}b^i(x)\frac{\partial}{\partial x^i}f(t,x,u) \\ &\quad +\frac{1}{2}\sum_{i,l=1}^{n}\sum_{j=1}^{m}\sigma^{i,j}(x)\sigma^{l,j}(x)\frac{\partial}{\partial x^i\partial x^l}f(t,x,u) \end{aligned} \tag{3.14}$$

$$L^kf(t,x,u)=\sum_{i=1}^{n}\sigma^{i,k}(x)\frac{\partial}{\partial x^i}f(t,x,u), \tag{3.15}$$

for $k\in\{1,2,\cdots,d\}$, and finally

$$L_v^{-1}f(t,x,u)=f(t,x+G(x,z),u)-f(t,x,u), \tag{3.16}$$

for all $t\in[0,T],x\in\mathbb{R}^n$ and $u\in E^{s(\alpha)}$. We employ the notation above to define the coefficient functions f_α, where $\alpha\in\mathcal{M}$.

Definition 3.1.4. *For all* $\alpha \in (j_1, \cdots, j_{l(\alpha)}) \in \mathcal{M}$ *and a function* $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$*, the Itô coefficient functions are recursively defined as follows*

$$f_\alpha(t,x,u) = \begin{cases} f(t,x) \text{ for } l(\alpha) = 0, \\ L^{j_1} f_{-\alpha}(t,x,u_1,\cdots,u_{s(-\alpha)}), \text{ for } l(\alpha) \geq 1, j_1 \in \{0,1,\cdots,d\}, \\ L^{-1}_{u_{s(\alpha)}} f_{-(\alpha)}(u_1,\cdots,u_{s(-\alpha)}), \text{ for } l(\alpha) \geq 1, j_1 = -1. \end{cases} \tag{3.17}$$

Remark 3.1.2. *In Definition 3.1.4, it is assumed that the coefficients of the SDE (3.1) and the function* f *satisfy the smoothness and integrality conditions needed for the operators (3.14)–(3.16) to be well defined.*

Example 3.1.3. *Let* $f(t,x) = x$*, and let* $n = d = 1$*. Then we get the following*

$$\begin{aligned} f_{(-1,0)}(t,x,u) &= L^{-1}_{u_{s(-(1,0))}} f_{-(-1,0)}(t,x,u_1,\cdots,u_{s(-(-1,0))}) \\ &= L^{-1}_u f_{(0)}(t,x,u) = L^{-1}_u L^0 f_v(t,x,u) \\ &= L^{-1}_u L^0 f(t,x) = L^{-1}_u L^0 x \\ &= L^{-1}_u L^0 b(x) = b(G(x,z)) - b(x). \end{aligned} \tag{3.18}$$

The stochastic Taylor expansion is defined on a special kind of set known as a *hierarchical set*. This set plays an important role in the construction of numerical schemes for (3.1), and also determines the order of convergence of the constructed numerical scheme.

Definition 3.1.5. *(i) A subset* $\mathcal{H} \subset \mathcal{M}$ *is called a hierarchical set if* $\mathcal{H}$ *is non-empty, the multi-indices in* $\mathcal{H}$ *are uniformly bounded in length, i.e.,* $\sup_{\alpha \in \mathcal{H}} l(\alpha) < \infty$*, and* $-\alpha \in \mathcal{H}$ *for each* $\alpha \in \mathcal{H} \setminus \{v\}$*.*

(ii) The remainder set $\mathcal{R}(\mathcal{H})$ *of a hierarchical set* $\mathcal{H}$ *is defined as*

$$\mathcal{R}(\mathcal{H}) = \{\alpha \in \mathcal{M} \setminus \mathcal{H} : -\alpha \in \mathcal{H}\}. \tag{3.19}$$

The remainder set consist of all next following multi-indices with respect to the given hierarchical set.

Example 3.1.4. *Let $d = 1$ and consider the set $\mathcal{H}_{0.5} = \{v, (-1), (0), (1)\}$. This set corresponds to the Euler approximation of (3.1) as we will see later. Its corresponding remainder set is*

$$\mathcal{R}(\mathcal{H}_{0.5}) = \{(-1,-1), (-1,0), (1,-1), (1,0), (-1,0), (0,0), (-1,1), (0,1), (1,1)\}.$$

Theorem 3.1 ([26], Theorem 4.4.1, pp. 206)**.** *For two stopping times ρ and τ with $0 \le \rho \le \tau \le T$ almost surely, a hierarchical set $\mathcal{H} \subset \mathcal{M}$, and a function $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, we have the corresponding stochastic expansion*

$$f(\tau, X(\tau)) = \sum_{\alpha \in \mathcal{H}} I_\alpha \left[f_\alpha(\rho, X(\rho)) \right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{R}(\mathcal{H})} I_\alpha \left[f_\alpha(., X(.)) \right]_{\rho,\tau}, \tag{3.20}$$

assuming that the function f and the coefficients of the SDE (3.1) are sufficiently smooth and integrable such that the arising coefficient functions f_α are well defined and the corresponding multiple stochastic integrals exists.

Remark 3.1.3. *Let $f(t,x) = x$. Then the solution X of (3.1) can be represented as follows*

$$X(\tau) = \sum_{\alpha \in \mathcal{H}} I_\alpha \left[f_\alpha(\rho, X(\rho)) \right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{R}(\mathcal{H})} I_\alpha \left[f_\alpha(., X(.)) \right]_{\rho,\tau} \tag{3.21}$$

where ρ and τ are stopping times such that $0 \le \rho \le \tau \le T$.

Example 3.1.5. *Consider the SDE (3.1) with $d = 1$ and $G \equiv 0$. Let $\rho = 0$ and $\tau = t$, and consider the hierarchical set $\mathcal{H} = \{v\}$. Then the remainder set $\mathcal{R}(\mathcal{H}) = \{(0), (1)\}$.*

Then by (3.20) in Theorem 3.1 with $f(t,x) = f(x)$ we obtain the following expansion

$$\begin{aligned} f(X(t)) &= f(X(0)) + I_{(0)}[f_{(0)}(s, X(s)]_{0,t} + I_{(1)}[f_{(1)}(s, X(s)]_{0,t} \\ &= f(X(0)) + I_{(0)}[L^0 f(X(s))]_{0,t} + I_{(1)}[L^1 f(X(s))]_{0,t} \\ &= f(X(0)) + \int_0^t L^0 f(X(s))ds + \int_0^t L^1 f(X(s))dB(s) \\ &= f(X(0)) + \int_0^t \left(b(X(s))f'(X(s)) + \frac{1}{2}\sigma(X(s))f''(X(s)) \right) ds \\ &\qquad + \int_0^t \sigma(X(s))f'(X(s))dB(s). \end{aligned} \tag{3.22}$$

Remark 3.1.4. *We observe from Example 3.1.5 that (3.20) is a generalization of the well known Itô's formula.*

Example 3.1.6. *Let $\mathcal{H}_{0.5}$ be the hierarchical set in Example 3.1.4 with $d = 1$. By Theorem 3.1 and Remark 3.1.3 we obtain*

$$\begin{aligned} X(\tau) &= I_v[f_v(\rho, X(\rho)]_{\rho,\tau} + I_{(0)}[f_{(0)}(\rho, X(\rho)]_{\rho,\tau} + I_{(1)}[f_{(1)}(\rho, X(\rho)]_{\rho,\tau} \\ &\quad + I_{(-1)}[f_{(-1)}(\rho, X(\rho)]_{\rho,\tau} + R \\ &\quad = X(\rho) + b(X(\rho))(\tau - \rho) + \sigma(X(\rho))(B(\tau) - B(\rho)) + \int_\rho^\tau \int_E G(X(\rho), z)N(dz, ds) + R, \end{aligned} \tag{3.23}$$

where R is the remainder given by

$$\begin{aligned} R &= \int_\rho^\tau \int_\rho^s L^0 b(X(u))duds + \int_\rho^\tau \int_\rho^s L^1 b(X(u))dB(u)ds + \int_\rho^\tau \int_\rho^s L^0 b(X(u))dudB(s) \\ &\quad + \int_\rho^\tau \int_\rho^s \int_E L_v^{-1} b(X(u))N(du, dz)ds + \int_\rho^\tau \int_\rho^s L^1 b(X(u))dB(u)dB(s) \\ &\quad + \int_\rho^\tau \int_\rho^s \int_E L_v^{-1} b(X(u))N(du, dz)dB(s) + \int_\rho^\tau \int_E \int_\rho^s L^0 G(X(u), z)duN(dz, ds) \\ &\quad + \int_\rho^\tau \int_E \int_\rho^s L^1 G(X(u), z)dB(u)N(dz, ds) \\ &\quad + \int_\rho^\tau \int_E \int_\rho^s \int_E L_v^{-1} G(X(u), z)N(dz, du)N(du, ds) \end{aligned} \tag{3.24}$$

Remark 3.1.5. *The most important property of the stochastic expansion is that, it permits a function of a process to be expanded as finite sum of multiple stochastic integrals with constant integrands. Just like the deterministic Taylor formula, it can be conveniently used for approximating the increments of solution of SDEs on small time intervals.*

3.2 Stochastic Taylor Approximation

In this section, we give an overview of numerical approximation of the solution of (3.4), which are constructed by truncating the stochastic Taylor expansion (3.21) to include as many terms are desired. As mentioned in the introduction, there are two kinds of approximations–*regular approximations* and *jump-adapted approximations*. The names regular and jump-adapted arise from the method of discretization of the time interval $[0, T]$ is discretized.

3.2.1 Regular Strong and Weak Taylor Approximations

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0, T]$ where $N \in \mathbb{N}$. For a given maximum step size $\Delta \in (0, 1)$ time discretization

$$t_\Delta = \{0 = t_0 < t_1 < \cdots < t_N = T\}, \tag{3.25}$$

is required to satisfy the following conditions:

$$\mathbb{P}(t_{n+1} - t_n \leq \Delta) = 1, \tag{3.26}$$

where

$$t_{n+1} \text{ is } \mathcal{F}_{t_n} - \text{measurable}, \tag{3.27}$$

for $n \in \{0, 1, \cdots, N-1\}$ and,

$$n_t = \max\{n \in \mathbb{N} : t_n \leq t\}, \tag{3.28}$$

denotes the largest integer n such that $t_n \leq t$, for all $t \in [0, T]$. This kind of discretization is called a *regular time discretization*. An example, is an equidistant time discretization where the n^{th} discretization time $t_n = n\Delta$, $n = 0, 1, \cdots, N$, and the time step size is $\Delta = T/N$. Discretization times can also be random as is needed if there is the desires to control the step size. Conditions (3.26), (3.27) and (3.28) imposes some restrictions on the choice of the random discretization times. Condition (3.26) requires that the maximum step size in the discretization cannot be larger than $\Delta \in (0, 1)$, (3.27) ensures the length $\Delta_n = t_{n+1} - t_n$ of the next time step depends only the information available at time t_n, and finally (3.28) guarantees a finite number of discretization points in any bounded interval $[0, t]$.

For any approximation scheme, it is important to specify the mode of convergence. We will consider two types of convergence, namely, "strong" and "weak" convergence.

Definition 3.2.1. *A numerical approximation $\{Y(t) : t \in [0, T]\}$ on a time discretization $(t)_\Delta$ of a stochastic process $\{X(t) : t \in [0, T]\}$ is said to converge strongly to X with strong order of convergence $\gamma > 0$ if*

$$\mathbb{E}\left[\|Y(T) - X(T)\|^2\right] \leq C\Delta^{2\gamma} \tag{3.29}$$

for some $C > 0$ which does not depend on Δ.

Definition 3.2.2. *A numerical approximation $\{Y(t) : t \in [0, T]\}$ on a time discretization $(t)_\Delta$ of a stochastic process $\{X(t) : t \in [0, T]\}$ is said to converge weakly to X with weak order of convergence $\beta > 0$ if for some smooth enough function g, we* *have that*

$$|\mathbb{E}\left[g(Y(T)) - g(X(T))\right]| \leq C\Delta^{\beta} \tag{3.30}$$

for some $C > 0$ which does not depend on Δ.

We now give a brief summary of strong and weak Taylor approximations of (3.1) for a given strong order $\gamma \in \{0.5, 1, 1.5, \cdots\}$ and a given weak order $\beta \in \{1, 2, \cdots\}$ respectively. The construction of these approximations is based on the stochastic Taylor expansion (3.20) and the choice of the hierarchical set. For a given hierarchical set, one obtains an approximation with a specific strong or weak order of convergence. We begin with the strong approximation. The following hierarchical set is

$$\mathcal{H}_{\gamma} = \left\{\alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\right\}. \tag{3.31}$$

use to construct strong regular approximation. For a regular time discretization $(t)_{\Delta}$ with a maximum step size $\Delta \in (0, 1)$, the *strong order γ approximation Y* is defined as follows

$$Y_{n+1} = Y_n + \sum_{\alpha \in \mathcal{H}_{\gamma}} I_{\alpha}\left[f_{\alpha}\left(t_n, Y_n\right)\right]_{t_n, t_{n+1}}, \tag{3.32}$$

for $n = 0, 1, \cdots, n_T - 1$, with $f(t, x) = x$.

The *weak regular β approximation* is constructed in a similarly using the hierarchical set

$$\mathcal{H}_{\beta} = \left\{\alpha \in \mathcal{M} : l(\alpha) \leq \beta\right\}. \tag{3.33}$$

In this case,

$$Y_{n+1} = Y_n + \sum_{\alpha \in \mathcal{H}_\beta} I_\alpha \left[f_\alpha \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}}, \tag{3.34}$$

The orders of convergence of (3.32) and (3.34), are accessed through specific interpolation $\{Y(t) : t \in [0, T]\}$ given by

$$Y(t) = \sum_{\alpha \in \mathcal{H}_\gamma} I_\alpha \left[f_\alpha \left(t_{n_t}, Y(t_{n_t}) \right) \right]_{t_{n_t}, t}, \tag{3.35}$$

and

$$Y(t) = \sum_{\alpha \in \mathcal{H}_\beta} I_\alpha \left[f_\alpha \left(t_{n_t}, Y(t_{n_t}) \right) \right]_{t_{n_t}, t}, \tag{3.36}$$

respectively, starting from $x \in \mathbb{R}^n$. Note here that $Y_n \stackrel{def}{=} Y(t_n)$. Before stating the main convergence theorems which enables one to construct strong and weak approximation schemes to the solution X of the SDE (3.4), we give a an example of the construction of a strong approximation using a particular hierarchical set.

Example 3.2.1 (The strong Taylor 1 approximation)**.** *Let n=d=1, and consider the hierarchical set*

$$\mathcal{H}_1 = \{\alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\} = \{v, (0), (1), (-1), (1,1), (1,-1), (-1,1), (-1,-1)\}$$

The strong order 1 Taylor approximation is given by

$$\begin{aligned}
Y_{n+1} &= Y_n + \sum_{\alpha \in \mathcal{H}_1} I_\alpha \left[f_\alpha \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} \\
&= Y_n + I_{(0)} \left[f_{(0)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} + I_{(1)} \left[f_{(1)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} \\
&\quad + I_{(-1)} \left[f_{(-1)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} + I_{(1,1)} \left[f_{(1,1)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} \\
&\quad + I_{(1,-1)} \left[f_{(1,-1)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} + I_{(-1,1)} \left[f_{(-1,1)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}} \\
&\quad + I_{(-1,-1)} \left[f_{(-1,-1)} \left(t_n, Y_n \right) \right]_{t_n, t_{n+1}},
\end{aligned} \tag{3.37}$$

which simplifies to

$$\begin{aligned}
Y_{n+1} = Y_n &+ b(Y_n)(t_{n+1} - t_n) + \sigma(Y_n)(B(t_{n+1}) - B(t_n)) \\
&+ \int_{t_n}^{t_{n+1}} \int_E G(Y_n, z) N(dz, ds) + \sigma(Y_n)\sigma'(Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dB(s_1) dB(s) \\
&+ \sigma(Y_n) \int_{t_n}^{t_{n+1}} \int_E \int_{t_n}^{s} G(Y_n, z) dB(s_1) N(dz, ds) \\
&+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_E \left\{ \sigma(Y_n + G(Y_n, z)) - \sigma(Y_n) \right\} N(dz, ds_1) dB(s) \\
&+ \int_{t_n}^{t_{n+1}} \int_E \int_{t_n}^{s} \int_E \left\{ G(Y_n + G(Y_n, z_1), z) - G(Y_n, z) \right\} N(dz_1, ds_1) N(dz, ds)
\end{aligned} \tag{3.38}$$

Remark 3.2.1. *In order to implement the scheme in Example 3.2.1, one needs a method on of approximating multiple stochastic integrals in (3.38), which is usually not an easy task.*

Theorem 3.2 ([26], Theorem 6.4.2, pp. 291)**.** *For given $\gamma \in \{0.5, 1, 1.5, \cdots\}$, let $Y = \{Y(t) : t \in [0, T]\}$ be the strong order γ Taylor approximation defined in (3.32), corresponding to a regular time discretization $(t)_\Delta$ with a maximum step size $\Delta \in (0, 1)$. Suppose that the coefficient functions f_α satisfy the following conditions:*

(i) *For* $\alpha \in \mathcal{H}, t \in [0,T], u \in E^{s(\alpha)}$ *and* $x, y \in \mathbb{R}^n$ *the coefficient function* f_α *satisfies the Lipschitz condition*

$$\|f_\alpha(t,x,u) - f_\alpha(t,y,u)\| \le K_1(u)\|x-y\|, \tag{3.39}$$

where $K_1(u)^2$ *is* $\nu(du^1) \times \nu(du^{s(\alpha)})$*-integrable.*

(ii) *For all* $\alpha \in \mathcal{H}_\gamma \cup \mathcal{R}(\mathcal{H}_\gamma)$*, we assume*

$$f_{-\alpha} \in C^{1,2} \text{ and } f_\alpha \in \mathcal{S}_\alpha, \tag{3.40}$$

there exists a set G *with* $\mathcal{H}_{\gamma-1} \subset G \subset \mathcal{H}_\gamma$*, where for all* $\alpha \in (G \cup \mathcal{R}(G)) \setminus \{v\}$*:* $f_{-\alpha} \in \cap_{k=-1}^{d} \mathcal{L}^k$*, for all* $\alpha \in G : f_\alpha(\tau_n, x_{\tau_n}) \in \mathcal{M}_{s(\alpha)}, n = 0, 1, \cdots n_T$*,*

(iii) *and for all* $\alpha \in \mathcal{R}(G) : f_\alpha(., x_.) \in \mathcal{M}_{s(\alpha)}$*, and for all* $\alpha \in \mathcal{H}_\gamma \cup \mathcal{R}(\mathcal{H}_\gamma), t \in [0,T], u \in E^{s(\alpha)}$ *and* $x \in \mathbb{R}^n$*, we require*

$$\|f(t,x,u\|^2 \le K_2(u)\left(1 + \|x\|^2\right), \tag{3.41}$$

where $K_1(u)^2$ *is* $\nu(du^1) \times \nu(du^{s(\alpha)})$*-integrable. Then the estimate*

$$\mathbb{E}\left[\sup_{0\le s\le T} \|X(s) - Y(s)\|^2\right] \le K_3 h^{2\gamma} \tag{3.42}$$

holds, where the constant K_3 *does not depend on* h*.*

Remark 3.2.2 ([26], Remark 6.4.2, pp. 292)**.** *The conditions on the coefficients* b, σ *and* G *of the SDE (3.4) which imply conditions (3.39)–(3.41) on the coefficient functions* f_α *is that* $b^k, \sigma^{k,j}, G^k$ *for* $k = 1, 2, \cdots, n$ *and* $j = 1, 2, \cdots, d$ *should be* $2(\gamma+1)$ *times continuously differentiable, uniformly bounded, with uniformly bounded derivatives.*

Theorem 3.3 ([26], Theorem 12.3.4, pp. 519)**.** *For a given* $\beta \in \{1, 2, \cdots\}$*, let* $Y = \{Y(t) : t \in [0,T]\}$ *be the weak order* β *Taylor approximation defined in*

(3.34) corresponding to a regular time discretization $(t)_h$ with maximum step size $h \in (0,1)$. We suppose that b, σ, G are Lipschitz continuous with components $b^k, \sigma^{k,j}, G^k \in C_P^{2(\beta+1)}$ for all $k \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots m\}$ and that the coefficient functions f_α with $f(t,x) = x$, satisfy the linear growth condition

$$\|f_\alpha(t,x)\| \leq K(1 + \|x\|), \tag{3.43}$$

with $K < \infty$, for all $t \in [0,T], x \in \mathbb{R}^n$ and $\alpha \in \Gamma_\beta \cup \mathcal{R}(\Gamma_\beta)$. Then for any $g \in C_P^{2(\beta+1)}$, there exists a positive constant C, independent of h, such that

$$\left|\mathbb{E}\left[g(X(T)) - g(Y^h(T))\right]\right| \leq Ch^\beta \tag{3.44}$$

Remark 3.2.3 ([26], Remark 12.3.5, pp. 520)**.** *The linear growth condition (3.63) on f_α is satisfied when b, σ and G are uniformly bounded.*

3.2.2 Jump-Adapted Strong and Weak Approximations

From Remark 3.2.1, we observe that the higher order schemes can become more complex since they involve the approximation of mixed multiple stochastic integrals involving a PRM. In order to avoid carrying out this tidiuos process, one can employ *jump-adapted approximations* that significantly reduces the complexity of the scheme. Here, we consider a *jump-adapted time discretization* $0 = t_0 < t_1 < \cdots < t_{n_T}$ of the interval $[0,T]$ on which a jump-adapted approximation $Y = \{Y(t) : t \in [0,T]\}$ of the solution of (3.4) is constructed. As before, let

$$n_t = \max\{n \in \mathbb{N} \cup \{0\} : t_n \leq t\} < \infty \tag{3.45}$$

The jump-adapted time discretization includes the jump times $\{\tau_1, \cdots \tau_{p(t)}\}$. Moreover, for any maximum step size $\Delta \in (0,1)$, we require the time jump-adapted time discretization $t_\Delta = \{0 = t_0 < t_1 < \cdots < t_{n_T} = T\}$, to satisfy condition

(3.26), where t_{n+1} is $\mathcal{F}_{t_n}$ − measurable, for $n \in \{0, 1, \cdots, n_T - 1\}$ if it is not a jump time. For example, we could construct a jump-adapted discretization $(t)_\Delta$ by a superposition of the jump times $\{\tau_1, \cdots \tau_{p(t)}\}$ and a deterministic equidistant discretization $0 = T_1 < \cdots < T_N = T$, with step size $\Delta = T/N$ of $[0, T]$. In this case, we obtain the discretization

$$\{t_1, \cdots, t_{n_T}\} = \{\tau_1, \cdots \tau_{p(t)}\} \cup \{T_1, \cdots T_N\}$$

where $n_T = \text{Card}\left(\{\tau_1, \cdots \tau_{p(t)}\} \cup \{T_1, \cdots T_N\}\right)$. Since the jumps can only arise at discretization times, the diffusive part of the dynamics of (3.1) is separated from the jumps within this time grid. Thus, between the jump times, the diffusive part can be approximated with a pure strong or weak scheme for the diffusion process. The effect of the jump is then added when a jump time is encountered. In order to construct the jump-adapted approximations, we consider the following hierarchical set

$$\widehat{\mathcal{M}} = \{(j_1, \cdots, j_l) : j_i \in \{1, 2, \cdots, m\}, i \in \{1, 2, \cdots, l\} \text{ for } l \in \mathbb{N}\} \cup \{v\}. \tag{3.46}$$

Definition 3.2.3. *For all $\alpha = (j_1, \cdots, j_{l(\alpha)}) \in \widehat{\mathcal{M}}$ and a function $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, the Itô coefficient functions f_α is defined as follows*

$$f_\alpha(t, x) = \begin{cases} f(t, x) \text{ for } l(\alpha) = 0, \\ L^{j_1} f_{-\alpha}(t, x) \text{ for } l(\alpha) \geq 1, \end{cases} \tag{3.47}$$

assuming the coefficients of the SDE (3.4) are sufficiently smooth for the operator $L^{j_1}, j = 1, 2, \cdots$ in (3.47) to be well defined.

In order to construct the jump-adapted approximations we start by denoting the almost sure left-hand limit of Y at time t_{n+1} by

$$Y(t_{n+1}-) = \lim_{s \uparrow t_{n+1}} Y(s).$$

For the strong jump-adapted approximation with strong order of convergence γ, we take $f(t,x) = x$, and consider the hierarchical set

$$\widehat{\mathcal{H}_\gamma} = \left\{ \alpha \in \widehat{\mathcal{M}} : l(\alpha) + n(\alpha) \leq \gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2} \right\}. \tag{3.48}$$

With this hierarchical set, one obtains the following scheme

$$Y(t_{n+1}-) = Y(t_n) + \sum_{\alpha \in \hat{\mathcal{H}} \setminus \{v\}} I_\alpha \left[f_\alpha(t_n, Y(t_n)) \right]_{t_n, t_{n+1}}, \tag{3.49}$$

and

$$Y(t_{n+1}) = Y(t_{n+1}-) + \int_E G(Y(t_{n+1}-), z) N(dz, \{t_{n+1}\}), \tag{3.50}$$

for $n \in \{0, 1, \cdots, n_T - 1\}$. The order of strong convergence is accessed through the interpolation

$$Y(t_{n+1}-) = \sum_{\alpha \in \hat{\mathcal{H}}_\gamma \setminus \{v\}} I_\alpha \left[f_\alpha(t_n, Y(t_n)) \right]_{t_{n_t}, t} \tag{3.51}$$

as there are no jumps between the jump times

We now state the convergence theorem for jump-adapted strong Taylor approximations.

Theorem 3.4 ([26], Theorem 8.7.1, pp. 364)**.** *For a given $\gamma \in \{0.5, 1, 1.5, \cdots,\}$, let $Y = \{Y(t) : t \in [0,T]\}$ be the jump-adapted strong order γ Taylor approximation corresponding to a jump-adapted time discretization with maximum step size $\Delta \in (0,1)$. Suppose that the coefficient functions f_α satisfy the following conditions*

(i) For $\alpha \in \mathcal{H}, t \in [0,T]$ and $x, y \in \mathbb{R}^n$ the coefficient function f_α satisfies the Lipschitz condition

$$\|f_\alpha(t,x) - f_\alpha(t,y)\| \leq K_1 \|x - y\|, \tag{3.52}$$

(ii) For all $\alpha \in \mathcal{H}_\gamma \cup \mathcal{R}(\mathcal{H}_\gamma)$, we assume

$$f_{-\alpha} \in C^{1,2} \text{ and } f_\alpha \in \mathcal{S}_\alpha, \tag{3.53}$$

(iii) and for all $\alpha \in \widehat{\mathcal{H}_\gamma} \cup \mathcal{R}(\widehat{\mathcal{H}_\gamma}), t \in [0,T]$ and $x \in \mathbb{R}^n$, we require

$$\|f(t,x)\|^2 \leq K_2 \left(1 + \|x\|^2\right). \tag{3.54}$$

Then the estimate

$$\mathbb{E}\left[\sup_{0\leq s\leq T} \|X(s) - Y(s)\|^2\right] \leq K_3 h^{2\gamma} \tag{3.55}$$

holds, where the constant K_3 does not depend on Δ.

The jump-adapted weak order β Taylor approximation is constructed similarly. Only that in this case we consider the following hierarchical set

$$\hat{\Gamma}_\beta = \left\{\alpha \in \widehat{\mathcal{M}} : l(\alpha) \leq \beta\right\}. \tag{3.56}$$

The jump-adapted weak order β Taylor approximation is given by

$$Y(t_{n+1}-) = Y(t_n) + \sum_{\alpha \in \widehat{\Gamma}_\beta \setminus \{v\}} I_\alpha \left[f_\alpha(t_n, Y(t_n))\right]_{t_n, t_{n+1}} \tag{3.57}$$

and

$$\begin{aligned} Y(t_{n+1}) &= Y(t_{n+1}-) + \int_E G(Y(t_{n+1}-), z) N(dz, \{t_{n+1}\}) \\ &= \begin{cases} Y(t_{n+1}-) + G(Y(t_{n+1}-), V_{t_{n+1}}) \text{ if } t_{n+1} \text{ if a jump time} \\ Y^(t_{n+1}-) \text{ otherwise,} \end{cases} \end{aligned} \tag{3.58}$$

for $n \in \{0, 1, \cdots, n_T - 1\}$. The order of weak convergence is accessed through the interpolation

$$Y(t_{n+1}-) = \sum_{\alpha \in \hat{\Gamma}_\beta \setminus \{v\}} I_\alpha \left[f_\alpha(t_n, Y(t_n)) \right]_{t_{n_t}, t} \tag{3.59}$$

Before stating the result on weak convergence, we give an example of a weak order 1 jump-adapted approximation of the solution X of the SDE (3.1).

Example 3.2.2. *Consider the hierarchical set*

$$\hat{\Gamma}_1 = \{\alpha \in \widehat{\mathcal{M}} : l(\alpha) \leq 1\} = \{v, (0), (1)\}. \tag{3.60}$$

By (3.57), we get

$$\begin{aligned} Y(t_{n+1}-) &= Y(t_n) + I_{(0)}[f_{(0)}(t_n, Y(t_n)]_{t_n, t_{n+1}} + I_{(1)}[f_{(1)}(t_n, Y(t_n)]_{t_n, t_{n+1}} \\ &= Y(t_n) + b(Y(t_n))(t_{n+1} - t_n) + \sigma(Y(t_n)) \left(B(t_{n+1}) - B(t_n) \right) \end{aligned} \tag{3.61}$$

The jump-correction is given by (3.58). Therefore,

$$Y(t_{n+1}) = \begin{cases} Y(t_{n+1}-) + G(Y(t_{n+1}-), V_{t_{n+1}}) \text{ if } t_{n+1} \text{ if a jump time,} \\ Y(t_{n+1}-) \text{ otherwise.} \end{cases} \tag{3.62}$$

Notice that (3.61) is the Euler scheme for (3.1) with $G \equiv 0$, which has a weak order of convergence $\beta = 1$.

Theorem 3.5 ([26], Theorem 12.3.4, pp. 536)**.** *For a given $\beta \in \{1, 2, \cdots\}$, let $Y = \{Y(t) : t \in [0, T]\}$ be the weak order β Taylor approximation defined in (3.58)–(3.59) corresponding to a jump-adapted time discretization $(t)_\Delta$ with maximum step size $\Delta \in (0, 1)$. We suppose that b, σ, G are Lipschitz continuous with components $b^k, \sigma^{k,j}, G^k \in C_P^{2(\beta+1)}$ for all $k \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots m\}$ and that the* *coefficient functions* f_α *with* $f(t,x) = x$*, satisfy the linear growth condition*

$$\|f_\alpha(t,x)\| \leq K(1 + \|x\|), \tag{3.63}$$

with $K < \infty$*, for all* $t \in [0,T], x \in \mathbb{R}^n$ *and* $\alpha \in \hat{\Gamma}_\beta \cup \mathcal{R}\left(\hat{\Gamma}_\beta\right)$*. Then for any* $g \in C_P^{2(\beta+1)}$*, there exists a positive constant* C*, independent of* Δ*, such that*

$$\left|\mathbb{E}\left[g(X(T)) - g(Y(T))\right]\right| \leq C\Delta^\beta. \tag{3.64}$$

Chapter 4

Numerical Approximation of SDEs Driven by Lévy Processes with Infinite Jump Activity

This chapter and the next constitute the main work in this dissertation. The goal here, is to give numerical approximations of the solution of (1.1), both strong and weak of any desired order as oppose to just the Euler scheme. To this effect, we construct a *jump-diffusion* SDE, which will serve as a good approximation to (1.1) when Z has infinitely many jumps. This will be accomplished piecemeal. We combine the ideas of Assmusen and Rosiński [2] (see also [9]) and in the spirit of [21], with the numerical schemes developed by Bruti and Platen in [26] to construct numerical approximations to the solution of (1.1). We will begin with a "simple" model and then move to a more general model. This chapter consists of two sections. In Section 4.1 we formulate our model, i.e., the construction of the jump-diffusion SDE (4.27). In Section 4.2, we state and proof the main results in this dissertation. Theorem 4.4 and Corollary 4.9.2, gives an L^p, $p \geq 2$, error estimates for approximating (1.1) with (4.27). In Theorem 4.7 and Corollary 4.9.2, we give a weak error estimate, for the same jump-diffusion approximation of (1.1). The proof technique here follows from the ideas in [21]. Finally, in Theorem 4.8 and Theorem 4.9, we give error estimates resulting from the numerical approximation of (1.1) using the schemes in [26].

4.1 Model Formulation

4.1.1 The Driving Lévy Process

Let $Z = \{Z(t) : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, be a d-dimensional pure jump Lévy process, i.e., Z has the following Lévy-Itô decomposition

$$Z(t) = at + \int_0^t \int_{\|z\| \leq 1} z\widetilde{N}(dz, ds) + \int_0^t \int_{\|z\|>1} zN(dz, ds), \tag{4.1}$$

where $a \in \mathbb{R}^d, N(dz, ds)$ is a Poisson random measure on $\mathbb{R}^d \times [0, \infty)$ with intensity measure $\nu(dz)ds$ where ν is a Lévy measure, i.e.,

$$\int_{\mathbb{R}^d} \min\left(1, \|z\|^2\right) \nu(dz) < \infty, \tag{4.2}$$

and $\widetilde{N}(dz, ds)$ is the compensated version of $N(dz, ds)$. We further assume that $\nu(\mathbb{R}^d) = \infty$, i.e., Z has an infinite number of jumps on any interval of nonzero length almost surely. Without loss of generality, we assume that

$$\int_{\mathbb{R}^d} \|z\|^p \nu(dz) \leq k^p, \text{ where, } p \geq 2, \text{ and } \|a\| \leq k, \tag{4.3}$$

for some positive constant k.

4.1.2 The SDE under Consideration

We consider the n-dimensional stochastic process X, the solution of the SDE

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t h(X(s-))dZ(s), \ t \in [0, T], \tag{4.4}$$

where, $b : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function, $\sigma, h : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are matrix-valued Lipschitz functions, Z is as in (4.1), W is a Brownian motion which is independent of Z, $X(0) = x \in \mathbb{R}^n$ is the initial value of X, and $T < \infty$. We first consider the case when $\sigma = 0$, i.e., X is the solution to the following SDE

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t h(X(s-))dZ(s), \ t \in [0, T] \tag{4.5}$$

with b and h as in (4.4). Note here that the second integral (Gaussian part) in (4.4) is absent. Without loss of generality we assume that the coefficients b and h satisfy the following inequalities

$$\|b(x_1) - b(x_2)\| \le k_1\|x_1 - x_2\|, \ \|h(x_1) - h(x_2)\| \le k_1\|x_1 - x_2\|, \ x_1, x_2 \in \mathbb{R}^n, \tag{4.6}$$

for some positive constant k_1. Condition (4.6) together with (4.3) clearly imply the Growth and Lipschitz conditions (2.41) and (2.42) respectively. Therefore, it follows from Theorem 2.6 and Proposition 2.3.1 that (4.4) has a unique solution.

Remark 4.1.1. *When Z is a Brownian motion W, we obtain the classical SDE with respect to W which is given by*

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t h(X(s-))dW(s), \ t \in [0, T]. \tag{4.7}$$

As mentioned earlier in the introduction, there is an extensive amount of literature on the simulation of this equation. See for example, the monographs [20] and [24]. But *when Z is a pure jump Lévy process, for example an α-stable process or a tempered α-stable process, the literature on the numerical approximation, in particular the strong approximation of the corresponding SDE is still vey scarce.*

4.1.3 Gaussian Approximation of Small Jumps

As noted in Remark 2.2.1 that the main difficulty in the simulation of a Lévy process stems from the fact that there is no general algorithm for simulating its increments. In Figure 2.3, we observed that for large values of α (the index of stability), the sample path of an α-stable process resembles a Brownian motion. The graphs in Figures 2.1, 2.2 and 2.3 leads one to think that an α-stable process can be obtained from a combination of a compound Poisson process and Brownian motion. This indeed is the case for stable processes and other Lévy processes. Asmussen and Rosiński in [3], established necessary and sufficient conditions under which the small jumps of a one-dimensional Lévy process can be approximated by a Brownian motion. Their result, which was later extended to the multidimensional case by Cohen and Rosiński in [31], will be crucial to us in the construction of our jump-diffusion SDE. Let Z be the Lévy process given in (4.1), and suppose that for every $\epsilon \in (0, 1]$, we have the decomposition

$$\nu = \nu_\epsilon + \nu^\epsilon, \tag{4.8}$$

where

$$\int_{\mathbb{R}^d} \|z\|^2 \nu_\epsilon(dz) < \infty \quad \text{and} \quad \nu^\epsilon\left(\mathbb{R}^d\right) < \infty. \tag{4.9}$$

An example of such a decomposition is when $\nu^\epsilon(dz) = \mathbb{1}_{\{z:\|z\|>\epsilon\}}\nu(dz)$, i.e., the truncation of jumps smaller than ϵ in magnitude. We shall denote by N_ϵ and N^ϵ the PRMs with corresponding intensities measures ν_ϵ and ν^ϵ respectively, and, by $\widetilde{N}_\epsilon$

and $\widetilde{N}^\epsilon$ the compensated versions of N_ϵ and N^ϵ respectively. Let

$$Z = R_\epsilon + P^\epsilon + a_\epsilon, \tag{4.10}$$

be the corresponding decomposition of the Lévy process Z into a sum of independent terms R_ϵ, P^ϵ and a_ϵ. The process $R_\epsilon = \{R_\epsilon(t), t \geq 0\}$ is a Lévy process with characteristic function

$$\mathbb{E}\left[e^{\langle z, R_\epsilon(t)\rangle}\right] = \exp\left\{t \int_{\mathbb{R}^d} \left[e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right] \nu_\epsilon(dx)\right\}. \tag{4.11}$$

Therefore R_ϵ has mean zero Lévy process and covariance matrix

$$\mathbb{E}\left[R_\epsilon(t) R_\epsilon(t)^T\right] = t\Sigma_\epsilon, \tag{4.12}$$

where,

$$\Sigma_\epsilon = \int_{\mathbb{R}^d} zz^T \nu_\epsilon(dz) \tag{4.13}$$

is a non-singular symmetric positive definite matrix. In one dimension, with $\nu^\epsilon(dz) = \mathbb{1}_{\{z:|z|>\epsilon\}}\nu(dz)$, we get

$$\sigma(\epsilon)^2 = \sigma_\epsilon^2 = \int_{|z|\leq\epsilon} |z|^2 \nu(dz). \tag{4.14}$$

We refer to R_ϵ as the 'small-jump' part of Z, $P^\epsilon = \{P^\epsilon(t), t \geq 0\}$ is a compound Poisson process with Lévy measure ν^ϵ, and $a_\epsilon = \{a_\epsilon t, t \geq 0\}$ is a drift given by

$$a_\epsilon = a + \int_{\|z\|>1} z\nu_\epsilon(dz) - \int_{\|z\|\leq 1} z\nu^\epsilon(dz). \tag{4.15}$$

Denote by $B = \{B(t), t \geq 0\}$ a standard Brownian motion in $\mathbb{R}^d$, and by " $\xrightarrow{d}$ " weak convergence in the space $D([0,\infty))$ of càdlàg functions from $[0,\infty]$ into $\mathbb{R}^d$ equipped with the Skorohod topology (see Section 1 of [17], pp. 324–329, for details on the Skorohod topology).

Theorem 4.1 (Theorem 2.2 of [31]). *With the above notation, suppose Σ_ϵ is non-singular for every $\epsilon \in (0,1]$. Then as $\epsilon \to 0$,*

$$\Sigma_\epsilon^{-1/2} R_\epsilon \xrightarrow{d} B \tag{4.16}$$

if and only if for every $\kappa > 0$

$$\int_{\left\langle \Sigma_\epsilon^{-1} z, z \right\rangle > \kappa} \left\langle \Sigma_\epsilon^{-1} z, z \right\rangle \nu_\epsilon(dz) \to 0. \tag{4.17}$$

Theorem 4.1 is a generalization of the following result by Asmussen and Rosinski [3] in the one dimensional setting.

Theorem 4.2 (Theorem 2.1 of [3]). $\sigma(\epsilon)^{-1} R_\epsilon \xrightarrow{d} B$ *as $\epsilon \to 0$ if and only if for each $\kappa > 0$*

$$\sigma(\kappa\sigma(\epsilon) \wedge \epsilon) \sim \sigma(\epsilon), \ \text{as } \epsilon \to 0. \tag{4.18}$$

In practice, it is difficult to verify the conditions (4.17) and (4.18). For sufficient conditions for (4.17) which are easier to verify, see Theorem 2.4 and Theorem 2.5 of [31]. For the one dimensional case, we have the following sufficient condition for (4.18).

Proposition 4.1.1 (Proposition 2.1 of [3]). *Condition (4.6) is implied by*

$$\lim_{\epsilon \to 0} \frac{\sigma(\epsilon)}{\epsilon} = \infty \tag{4.19}$$

Example 4.1.1. *Let Z be a symmetric α-stable process with Lévy measure*

$$\nu(dz) = \left(\frac{c_1}{|z|^{1+\alpha}} \mathbb{1}_{\{z<0\}} + \frac{c_2}{z^{1+\alpha}} \mathbb{1}_{\{z>0\}} \right) dz, \tag{4.20}$$

where, $c_1, c_2 \geq 0, \alpha \in (0,2)$, and $c_1 + c_2 > 0$. Then

$$\sigma(\epsilon)^2 = \sigma_\epsilon^2 = \int_{|z|\leq\epsilon} z^2 \nu(dz) = \frac{c_1 + c_2}{2-\alpha}\epsilon^{2-\alpha}, \tag{4.21}$$

from which we obtain

$$\frac{\sigma(\epsilon)}{\epsilon} = \left(\frac{c_1 + c_2}{2-\alpha}\right)^{1/2} \frac{1}{\epsilon^{\alpha/2}} \to \infty, \text{ as } \epsilon \to 0. \tag{4.22}$$

Thus, condition (4.19) is satisfied and the normal approximation holds.

Remark 4.1.2. *If Theorem 4.1 applies, then R_ϵ can be approximated by $\Sigma_\epsilon^{1/2} B$, where the Brownian motion B is independent of P^ϵ. Consequently we get*

$$Z \stackrel{d}{=} a_\epsilon + \Sigma_\epsilon^{1/2} B + P^\epsilon \stackrel{def}{=} Z_\epsilon. \tag{4.23}$$

$\Sigma_\epsilon^{1/2}$ denotes the square root of Σ_ϵ, i.e., there exists a unique positive definite matrix Σ_ϵ^1 such that $\Sigma_\epsilon = (\Sigma_\epsilon^1)^2$.

Proposition 4.1.2. *Let $\epsilon \in (0,1]$, and let ν_ϵ and Σ_ϵ be given by (4.8) and (4.13) respectively. Then,*

$$\left\|\Sigma_\epsilon^{1/2}\right\|^2 = \int_{\mathbb{R}^d} \|z\|^2 \nu_\epsilon d(z) = \operatorname{trace}(\Sigma_\epsilon). \tag{4.24}$$

Proof. Let $\{e_1, \cdots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Then by the Pythagorean theorem and the fact that Σ_ϵ is symmetric, it follows that

$$\begin{aligned} \|\Sigma_\epsilon^{1/2}\|^2 &= \sum_{i=1}^d \left\|\left(\Sigma_\epsilon^{1/2}\right)_i\right\|^2 = \sum_{i=1}^d \left\langle e_i, \left(\Sigma_\epsilon^{1/2}\right)^T \Sigma_\epsilon^{1/2} e_i \right\rangle = \sum_{i=1}^d (\Sigma_\epsilon)_{ii} \\ &= \operatorname{trace}(\Sigma_\epsilon) = \sum_{i=1}^d \int_{\mathbb{R}^d} z_i z_i \nu_\epsilon(dz) = \int_{\mathbb{R}^d} \|z\|^2 \nu_\epsilon(dz). \end{aligned} \tag{4.25}$$

$\square$

Remark 4.1.3. *From Proposition 4.1.2 and (4.9), it follows that*

$$\sup_{\epsilon\in(0,1]} \|\Sigma_\epsilon^{1/2}\|^2 < \infty. \tag{4.26}$$

4.2 The Jump-Diffusion Approximation of an SDE Driven by a Lévy Process with an Infinite Lévy Measure

4.2.1 Construction of Jump-Diffusion SDE

In this section, we consider the solution X^ϵ of a jump-diffusion SDE, and show that it approximates the solution X of the SDE (4.4) in $L^p, p \geq 2$ and in a weak sense. Let $\epsilon \in (0,1]$ be arbitrary, and consider the following stochastic differential equation

$$X^\epsilon(t) = x + \int_0^t b(X^\epsilon(s))ds + \int_0^t h(X^\epsilon(s))dZ^\epsilon(s), \quad t \in [0,T], \tag{4.27}$$

where b and h are as in (4.4) and Z^ϵ is as in Remark 4.1.2. We further assume that $\mathbb{E}[\|Z\|^p] < \infty$, for $p \geq 2$. Then, we can rewrite X and X^ϵ the solutions of (4.2) and (4.27) respectively as follows

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t h(X(s))\tilde{a}ds + \int_0^t \int_{\mathbb{R}^d} h(X(s-))z\widetilde{N}(dz,ds), \tag{4.28}$$

and

$$\begin{aligned} X^\epsilon(t) = x &+ \int_0^t b(X^\epsilon(s))ds + \int_0^t h(X^\epsilon(s))\tilde{a}ds + \int_0^t h(X^\epsilon(s))\Sigma_\epsilon^{1/2}dB(s) \\ &+ \int_0^t \int_{\mathbb{R}^d} h(X^\epsilon(s-))z\widetilde{N^\epsilon}(dz,ds), \end{aligned} \tag{4.29}$$

where

$$\tilde{a} = a + \int_{\|z\|>1} z\nu(dz). \tag{4.30}$$

Without loss of generality, we assume that

$$\|\tilde{a}\| \leq M. \tag{4.31}$$

If we let $b_1(y) = b(y) + h(y)\tilde{a}$, then it follows from(4.6) and (4.31) that

$$\|b_1(y_1) - b_1(y_2)\| \leq K_1 \|y_1 - y_2\| \tag{4.32}$$

for some positive constant K_1. We show that there exists a unique solution to (4.27) and then give moment estimates to the solutions of (4.4) and (4.27). In order to accomplish this task, we will need an extension of the Burkholder inequality. We assume that we are given a Brownian motion $B = \{B(t) : t \geq 0\}$ and a PRM N on a measurable space $\mathcal{Z}$. Consider a d-dimensional semimaritngale $Y = \{Y(t) : t \geq 0\}$ represented by

$$Y(t) = x + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathcal{Z}} G(z,s)\widetilde{N}(dz,ds) \tag{4.33}$$

Theorem 4.3 ([29], Theorem 2.11, pp. 332)**.** *For any $p \geq 2$, there exists a positive constant C_p such that*

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq t} \|Y(s)\|^p\right] \leq C_p \Bigg\{ \|x\|^p + \mathbb{E}\left[\int_0^t \|b(s)\|^p ds\right] + \mathbb{E}\left[\left(\int_0^t \|\sigma(s)\|^2 ds\right)^{p/2}\right] \\ + \mathbb{E}\left[\left(\int_0^t \int_{\mathcal{Z}} \|G(z,s)\|^2 \nu(dz)ds\right)^{p/2}\right] + \mathbb{E}\left[\int_0^t \int_{\mathcal{Z}} \|G(z,s)\|^p \nu(dz)ds\right]\Bigg\} \end{aligned} \tag{4.34}$$

holds for any semimartingale X represented by (4.33).

A more general result can be found in [12] which involves an arbitrary martingale M with an arbitrary stopping time T. See also [23] for an extension of this result to Hilbert spaces. The main advantage of this inequality over the maximal inequalities of Burkholder, Davis and Gundy is that the right-hand side is expressed in terms of predictable "ingredients", rather than in terms of the quadratic variation. We are now in a position to establish the existence of a unique solution to (4.27)

Lemma 4.3.1. *1) Under the above setting, the SDE (4.27) has a unique solution. Moreover, for each fixed $\epsilon \in (0,1]$, it holds that*

$$\mathbb{E}\left[\sup_{0\le t\le T}\|X^\epsilon(t)\|^p\right] \le C_1(1+\|x\|^p),\ \ p\ge 2 \tag{4.35}$$

for some positive constant C_1 which does not depend on ϵ.

2) The following estimate holds for the solution X of the SDE (4.6)

$$\mathbb{E}\left[\sup_{0\le t\le T}\|X(t)\|^p\right] \le C_2(1+\|x\|^p),\ \ p\ge 2 \tag{4.36}$$

for some positive constant C_2.

Proof. The existence of a unique solution to the SDE (4.27) follows from Theorem 2.6 due to the condition (4.6) imposed on the coefficient functions b and h. By Theorem 4.3, there exists a positive constant C (which changes from line to line) such that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\|X^\epsilon(s)\|^p\right] \le C\mathbb{E}\Bigg[&\|x\|^p + \int_0^t \|b_1(X^\epsilon(s))\|^p\,ds \\ &+ \left(\int_0^t \|h(X^\epsilon(s))\|^2\left\|\Sigma_\epsilon^{1/2}\right\|^2 ds\right)^{p/2} \\ &+ \left(\int_0^t\int_{\mathbb{R}^d}\|h(X^\epsilon(s-))\|^2\,\|z\|^2\nu^\epsilon(dz)ds\right)^{\frac{p}{2}} \\ &+ \int_0^t\int_{\mathbb{R}^d}\|h(X^\epsilon(s-))\|^p\,\|z\|^p\nu^\epsilon(dz)ds\Bigg].\end{aligned} \tag{4.37}$$

But

$$\sup_{\epsilon\in(0,1]} \left\|\Sigma_\epsilon^{1/2}\right\|^2 < \infty, \quad \text{and} \quad \int_{\mathbb{R}^d} \|z\|^p \nu^\epsilon(dz) < \infty. \tag{4.38}$$

Therefore,

$$\mathbb{E}\left[\sup_{0\le s\le t} \|X^\epsilon(s)\|^p\right] \le C\mathbb{E}\left[\|x\|^p + \int_0^t \|b_1(X^\epsilon(s))\|^p \, ds + \left(\int_0^t \|h(X^\epsilon(s))\|^2 \, ds\right)^{\frac{p}{2}}\right]. \tag{4.39}$$

By Jensen's inequality and the fact that b and h are Lipschitz continuous, we obtain

$$\mathbb{E}\left[\sup_{0\le s\le t} \|X^\epsilon(s)\|^p\right] \le C\mathbb{E}\left[1 + \|x\|^p + \int_0^t \|X^\epsilon(s))\|^p \, ds\right]. \tag{4.40}$$

Finally, by Gronwall's inequality we get

$$\mathbb{E}\left[\sup_{0\le s\le t} \|X^\epsilon(s)\|^p\right] \le C\mathbb{E}\left[1 + \|x\|^p\right] \tag{4.41}$$

Similarly, we obtain the estimate (4.36). □

4.2.2 L^2 Error Estimate

We start by giving an L^2-error estimate, which we will then generalized to an L^p–error estimates, with $p \ge 2$. We assume here that $\mathbb{E}[\|Z\|^2] < \infty$, and then give an upper bound for

$$\mathbb{E}[\sup_{0\le t\le T} \|X(s) - X^\epsilon(s)\|^2]. \tag{4.42}$$

From (4.28) and (4.29), we have that

$$
\begin{aligned}
X(t) - X^{\epsilon}(t) = \int_0^t & \left[b_1(X(s)) - b_1(X^{\epsilon}(s)) \right] ds \\
& + \int_0^t \int_{\mathbb{R}^d} \left[h(X(s-)) - h(X^{\epsilon}(s-)) \right] z \tilde{N}(dz, ds) \\
& + \int_0^t \int_{\mathbb{R}^d} h(X^{\epsilon}(s-)) z \tilde{N}_{\epsilon}(dz, ds) - \int_0^t h(X^{\epsilon}(s)) \Sigma_{\epsilon}^{1/2} dB(s).
\end{aligned}
\tag{4.43}
$$

Let

$$
e(t) = \mathbb{E}\left[\sup_{0 \le s \le t} \| X(s) - X^{\epsilon}(s) \|^2 \right]. \tag{4.44}
$$

It follows from the Cauchy-Schwartz inequality that

$$
e(t) \le 4 \left(e_1(t) + e_3(t) + e_4(t) + e_5(t) \right), \tag{4.45}
$$

where

$$
\begin{aligned}
e_1(t) &= \mathbb{E}\left[\sup_{0 \le s \le t} \left\| \int_0^s \left[b_1(X(u)) - b_1(X^{\epsilon}(u)) \right] du \right\|^2 \right] \\
e_2(t) &= \mathbb{E}\left[\sup_{0 \le s \le t} \left\| \int_0^s \int_{\mathbb{R}^d} \left[h(X(u-)) - h(X^{\epsilon}(u-)) \right] z \widetilde{N}(dz, du) \right\|^2 \right] \\
e_3(t) &= \mathbb{E}\left[\sup_{0 \le s \le t} \left\| \int_0^s \int_{\mathbb{R}^d} h(X^{\epsilon}(u-)) z \widetilde{N_{\epsilon}}(dz, du) \right\|^2 \right] \\
e_4(t) &= \mathbb{E}\left[\sup_{0 \le s \le t} \left\| \int_0^s h(X^{\epsilon}(u)) \Sigma_{\epsilon}^{1/2} dB(u) \right\|^2 \right].
\end{aligned}
\tag{4.46}
$$

Lemma 4.3.2. *Let $e_j(t),\ j = 1, 2, 3, 4$ be given by (4.46). Then the following estimates hold*

$$
\begin{aligned}
e_j(t) &\le C_j \int_0^t e(s) ds \ \textit{for},\ j = 1, 2, \\
e_j(t) &\le C_j \operatorname{trace}(\Sigma_{\epsilon}),\ \textit{for}\ j = 3, 4,
\end{aligned}
\tag{4.47}
$$

for some positive constants $C_j, j = 1, 2, 3, 4,$ where $C_j, j = 3, 4$ does not depend on ϵ.

Proof. By Cauchy-Schwarz inequality, (4.32), and Lemma 4.3.1, it follows that

$$e_1(t) \leq K_1 t \int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s} \|X(u) - X^\epsilon(u)\|^2\right] ds \leq C_1 \int_0^t e(s)ds. \tag{4.48}$$

Now by Doob's inequality, (4.3), (4.6) and by Lemma 4.3.1, it follows that

$$e_2(t) \leq C_2 \mathbb{E}\left[\int_0^t \|X(s) - X^\epsilon(s)\|^2 ds\right]. \tag{4.49}$$

for some positive constant C_2. Therefore

$$e_2(t) \leq C_2 \int_0^t e(s)ds. \tag{4.50}$$

Again by Doob's inequality and Lemma 4.3.1, we get that

$$\begin{aligned} e_3(t) &\leq k_1 \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \|X^\epsilon(s)\|^2 \|z\|^2 \nu_\epsilon(dz)ds\right] \\ &\leq k_1 \int_{\mathbb{R}^d} \|z\|^2 \nu_\epsilon(dz) \int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s} \|X^\epsilon(u)\|^2\right] ds \\ &\leq C k_1 \text{trace}\left(\Sigma_\epsilon\right) = C_3 \text{trace}\left(\Sigma_\epsilon\right) \end{aligned} \tag{4.51}$$

Similarly,

$$e_4(t) \leq C_4 \text{trace}\left(\Sigma_\epsilon\right). \tag{4.52}$$

Therefore

$$e_j(t) \leq C_j \int_0^t e(s)ds, \text{ for } j = 1, 2, \tag{4.53}$$

and

$$e_j(t) \leq C_j \text{trace}\left(\Sigma_\epsilon\right), \text{ for } j = 3, 4. \tag{4.54}$$

$\square$

Proposition 4.2.1. *Let X and X^ϵ be the unique solutions of (4.4) and (4.27) respectively. Then the following estimate holds*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|X(t) - X^\epsilon(t)\|^2\right] \leq C\text{trace}(\Sigma_\epsilon) \tag{4.55}$$

for some positive constants C which does not depend on ϵ.

Proof. By (4.45), Lemma 4.3.2 and Proposition 4.1.2, it follows that

$$e(t) \leq C_4 \text{trace}\left(\Sigma_\epsilon\right) + C_3 \int_0^t e(s)ds. \tag{4.56}$$

Therefore by Gronwall's inequality, we obtain $e(T) \leq C_4 e^{C_3 T}\text{trace}(\Sigma_\epsilon) = C\text{trace}(\Sigma_\epsilon)$.

$\square$

Corollary 4.3.1. *For each $T > 0$, and $A > 0$,*

$$\lim_{\epsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,T]} \|X(t) - X^\epsilon(t)\| > A\right) = 0 \tag{4.57}$$

Proof. Indeed, let $A > 0$. Then

$$\begin{aligned} \mathbb{P}\left(\sup_{t\in[0,T]} \|X(t) - X^\epsilon(t)\| > A\right) &\leq \frac{1}{A^2}\mathbb{E}\left[\sup_{0\leq t\leq T} \|X(t) - X^\epsilon(t)\|^2\right] \\ &\leq \frac{C}{A^2}\text{trace}(\Sigma_\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{4.58}$$

$\square$

4.2.3 L^p Error Estimate

We will now generalize Proposition 4.2.1. To this effect, we assume that the driving process Z has moments of order $p \geq 2$, then give an upper bound for

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|X(t) - X^\epsilon(t)\|^p\right]. \tag{4.59}$$

Proposition 4.2.1 will then become a special case of Theorem 4.4 with $p = 2$.

Theorem 4.4. *Let X and X^ϵ be the unique solutions of (4.4) and (4.27) respectively. Then for p ≥ 2 the following estimate holds*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|X(t) - X^\epsilon(t)\|^p\right] \leq C \int_{\mathbb{R}^d} \|z\|^p \nu_\epsilon(dz), \tag{4.60}$$

for some positive constants C which does not depend on ϵ.

Proof. From (4.43) and Theorem 4.3 we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T} \|X(t) - X^\epsilon(t)\|^p\right] &\leq \mathbb{E}\left[\int_0^t \|b_1(X(s)) - b_1(X^\epsilon(s))\|^p\, ds\right] \\ &+ \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \|h(X(s)) - h(X^\epsilon(s))\|^2 \|z\|^2 \nu(dz) ds\right)^{p/2}\right] \\ &+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \|h(X(s)) - h(X^\epsilon(s))\|^p \|z\|^p \nu(dz) ds\right] \\ &+ \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \|h(X^\epsilon(s))\|^2 \|z\|^2 \nu_\epsilon(dz) ds\right)^{p/2}\right] \\ &+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \|h(X^\epsilon(s))\|^p \|z\|^p \nu_\epsilon(dz) ds\right] \\ &+ \mathbb{E}\left[\left(\int_0^t \|h(X^\epsilon(s))\|^2 \|\Sigma_\epsilon^{1/2}\|^2 ds\right)^{p/2}\right].\end{aligned} \tag{4.61}$$

By Jensen's inequality, conditions (4.3) and (4.6), it follows that

$$\mathbb{E}\left[\sup_{0\le t\le T}\|X(t)-X^{\epsilon}(t)\|^{p}\right]\le C\Bigg(\mathbb{E}\left[\int_{0}^{t}\|X(s)-X^{\epsilon}(s)\|^{p}\right] + \int_{\mathbb{R}^d}\|z\|^{p}\nu_{\epsilon}(dz)\mathbb{E}\left[\int_{0}^{t}(1+\|X^{\epsilon}(s)\|^{p}\,ds\right]\Bigg), \quad (4.62)$$

for some positive constant C. By Lemma 4.3.1, we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\|X(t)-X^{\epsilon}(t)\|^{p}\right]\le C_1\Bigg(\mathbb{E}\left[\int_{0}^{t}\|X(s)-X^{\epsilon}(s)\|^{p}\right]ds + C_1(1+\|x\|^{p})\int_{\mathbb{R}^d}\|z\|^{p}\nu_{\epsilon}(dz)\Bigg) \quad (4.63)$$

for some positive constant C_1. Finally, by Gronwall's inequality we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\|X(t)-X^{\epsilon}(t)\|^{p}\right]\le C\int_{\mathbb{R}^d}\|z\|^{p}\nu_{\epsilon}(dz) \quad (4.64)$$

□

Corollary 4.4.1. *Let f defined on $\mathbb{R}^n$ be Lipschitz continuous with Lipschitz constant L_f. Then for some positive constant C*

$$\left|\mathbb{E}\left[f(X(s))-f(X^{\epsilon}(s))\right]\right|\le C\left(\int_{\mathbb{R}^d}\|z\|^{p}\nu_{\epsilon}(dz)\right)^{1/p} \quad (4.65)$$

Proof. From the Lipschitz continuity of f, Holder's inequality and Theorem 4.4, it follows that

$$\begin{aligned}\left|\mathbb{E}\left[f(X(T))-f(X^{\epsilon}(T))\right]\right| &\le L_f\mathbb{E}\left[\|X(s)-X^{\epsilon}(s)\|^{p}\right]^{1/p}\mathbb{E}[1^{q}]^{1/q} \\ &\le C\left(\int_{\mathbb{R}^d}\|z\|^{p}\nu_{\epsilon}(dz)\right)^{1/p}\end{aligned} \quad (4.66)$$

where q is such that $1/p+1/q=1$, with $p,q\ge 2$. □

4.2.4 Weak Approximation

Here, we show that X^ϵ approximates X in a weak sense, i.e., for a sufficiently smooth function g, $\mathbb{E}\left[g(X(T)) - g(X^\epsilon(T))\right]$ is "small". We will assume that $\mathbb{E}[\|Z\|^3] < \infty$. Let $X(t) = X^{(s,x)}(t)$ be the solution of (4.4) starting from x at time s, let u be given by

$$u(t,x) \overset{def}{=} \mathbb{E}\left[g(X(T)) \mid X(t) = x\right] = \mathbb{E}\left[g(X^{(t,x)}(T))\right], \tag{4.67}$$

where X is the solution of (4.4) with initial value $x \in \mathbb{R}^n$. Since X is a Markov process (see Remark 2.3.2(iv)), it follows that X has an infinitesimal generator $\mathcal{L}_1$ given by

$$\begin{aligned}(\mathcal{L}_1 f)(t,x) &= \frac{\partial}{\partial t} f(t,x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(t,x) \left(b^i(x) + \sum_{j=1}^{d} h_{ij}(x) a^j \right) \\ &\quad + \int_{\|z\|>1} [f(t,x+h(x)z) - f(t,x)]\nu(dz) \\ &\quad + \int_{\|z\|\le 1} \left[f(t,x+h(x)z) - f(t,x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(t,x) \left(\sum_{j=1}^{d} h_{ij}(x) z^j \right) \right] \nu(dz),\end{aligned} \tag{4.68}$$

where $f \in C_0^2(\mathbb{R}^d)$ (see [1], Theorem 6.7.3, pp. 407). In Lemma 4.6.2, we show that u satisfies the following backwards Kolmogorov equation

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) + \mathcal{L}_1 u(t,x) &= 0 \\ u(T,x) &= g(x).\end{aligned} \tag{4.69}$$

We now state a version of Ito's formula that will be useful to us in this sequel. Let the n-dimensional stochastic process $Y = \{Y(t) : t \geq 0\}$ be a Lévy-type stochastic integral of the form

$$Y(t) = Y(0) + \int_0^t G(s)ds + \int_0^t F(s)dB(s) + \int_0^t \int_{\|z\|\le 1} H(z,s)\tilde{N}(dz,ds) + \int_0^t \int_{\|z\|>1} K(z,s)N(dz,ds), \tag{4.70}$$

where G, F, H and K are such that the integrals are well defined. Here B is a d-dimensional standard Brownian motion which is independent of the PRM N on $\mathbb{R}_0^d \times \mathbb{R}^+$ with compensated version $\widetilde{N}$ and intensity measure ν, which is a Lévy measure. See Chapter 4 of [1], for a detailed treatment of integrals of the form (4.70).

Theorem 4.5 ([1], Theorem 4.4.7, pp. 226). *If Y is as in (4.70), then for each $f \in C^2(\mathbb{R}^d), t \ge 0$, with probability one, we have*

$$\begin{aligned}
f(Y(t)) - f(Y(0)) &= \sum_{i=1}^n \int_0^t \frac{\partial}{\partial y_j} f(Y(s-))dY_c^j(s) \\
&+ \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \int_0^t \frac{\partial}{\partial y_i \partial y_j} f(Y(s-))d[Y_c^j, Y_c^i](s) \\
&+ \int_0^t \int_{\|z\|>1} \left[f(Y(s-)+K(z,s)) - f(Y(s-))\right] N(dz,ds) \\
&+ \int_0^t \int_{\|z\|\le 1} \left[f(Y(s-)+H(z,s)) - f(Y(s-))\right] \widetilde{N}(dz,ds) \\
&+ \int_0^t \int_{\|z\|\le 1} \left[f(Y(s-)+H(z,s)) - f(Y(s-)) + \sum_{i=1}^n H^i(z,s)\frac{\partial}{\partial y_i} f(Y(s-))\right] \nu(dz)ds,
\end{aligned} \tag{4.71}$$

where $Y_c^j(s)$ and $[Y_c^i, Y_c^j](s)$ denotes the continuous parts of Y^j and the quadratic variation process $[Y^i, Y^j](s)$ respectively.

Theorem 4.6 ([22], Theorem 2.9). *Let $f \in C^{1,2}(\mathbb{R}^d)$ and let $\mathcal{L}_1$ be the generator of X the solution to (4.4). Then the process $\{M(t), t \ge 0\}$ given by*

$$M(t) = f(t, X(t)) - f(0, X(0)) - \int_0^t \left\{\frac{\partial f}{\partial s}(s, X(s)) + \mathcal{L}_1 f(s, X(s))\right\} ds \tag{4.72}$$

is a martingale.

The following result gives us bounds on the flow of the solution of the SDE (4.4).

Lemma 4.6.1. *Assume that b, σ and h are in C_b^n, and that Z has moments of order p, with $p \geq 2$. Then*

$$\mathbb{E}\left[\sup_{[0,T]}\left|\frac{\partial^\alpha}{\partial x^\alpha}X^{(t,x)}(T)\right|^p\right] < \infty, \tag{4.73}$$

for any multi-index α such that $0 < |\alpha| \leq n$.

Proof. The proof follows from [29], Theorem 3.3, pp. 342. Here, we need to verify that the coefficients b, σ and h and their derivatives are bounded and Hölder continuous, which clearly follows from the assumptions imposed on b, σ and h. $\square$

Lemma 4.6.2. *Assume that $h, g \in C_b^n([0,T] \times \mathbb{R}^d)$, and that $\mathbb{E}\left[\|Z\|^3\right] < \infty$. Then u satisfies (4.69), $u \in C^{1,n}([0,T] \times \mathbb{R}^d)$, and $\partial^\alpha u/\partial x_\alpha(t,x)$ are uniformly bounded for any multi-index α such that $1 \leq |\alpha| \leq n$.*

Proof. By the stationarity property of Lévy processes, it follow that, for any fixed time T, $X^{(s,x)}(t) = X^{(0,x)}(t-s)$ for $0 \leq s \leq t \leq T$. By Lemma 4.6.1, we can interchange of the derivative and the expected value to obtain

$$\frac{\partial u}{\partial x_k}(t,x) = \mathbb{E}\left[\frac{\partial g}{\partial x_j}(X^{(t,x)}(T))\frac{\partial}{\partial x_k}X_j^{(t,x)}(T)\right], \tag{4.74}$$

for $k = 1, 2, \cdots, n$. The other derivatives with respect to x are obtained by successive differentiation under the expected value. The uniform boundedness follows from the boundedness of g and Lemma 4.6.1. Next, we apply Itô's formula to $g(X^{(t,x)}(T))$ in order to verify differentiation of u with respect to the time variable t. We will proof this only for the one dimensional case, as the generalization to multi-dimensional case is straight forward. By Itô's formula (Theorem 4.5), we get that

$$\begin{aligned} g(X(t)) = g(x) &+ \int_0^t g'(X(s))dX_c(s) \\ &+ \int_0^t \int_{|z|>1} \left[g(X(s-) + h(X(s)z) - g(X(s-)) \right] N(dz, ds) \\ &+ \int_0^t \int_{|z|\le 1} \left[g(X(s-) + h(X(s)z) - g(X(s-)) \right] \widetilde{N}(dz, ds) \\ &+ \int_0^t \int_{|z|\le 1} \left[g(X(s-) + h(X(s)z) - g(X(s-)) - g'(X(s-))h(X(s))z \right] \nu(dz)ds. \end{aligned} \tag{4.75}$$

Replacing $dX_c(s)$ by its value and taking expectation we obtain

$$\begin{aligned} \mathbb{E}\left[g(X(t))\right] &= g(x) + \int_0^t \mathbb{E}\left[g'(X(s))b_1(X(s))\right] ds \\ &+ \int_0^t \mathbb{E}\left[\int_{|z|>1} \left[g(X(s-) + h(X(s)z) - g(X(s-))\right] \nu(dz)\right] ds \\ &+ \int_0^t \mathbb{E}\left[\int_{|z|\le 1} \left[g(X(s-) + h(X(s)z) - g(X(s-)) - g'(X(s-))h(X(s))z\right] \nu(dz)\right] ds. \end{aligned} \tag{4.76}$$

Taking derivatives with respect to t obtain

$$\begin{aligned} &\frac{\partial \mathbb{E}\left[g(X(t))\right]}{\partial t} \\ &\quad = \mathbb{E}\left[g'(X(t))b_1(X(t))\right] + \mathbb{E}\left[\int_{|z|>1} \left[g(X(t-) + h(X(t)z) - g(X(t-))\right] \nu(dz)\right] \\ &\qquad + \mathbb{E}\left[\int_{|z|\le 1} \left[g(X(t-) + h(X(t)z) - g(X(t-)) - g'(X(t-))h(X(t))z\right] \nu(dz)\right] \\ &\quad = \mathbb{E}\left[g'(X(t))b_1(X(t)) + \int_{|z|>1} g'(X(t-))h(X(t))z\nu(dz)\right] \\ &\qquad + \mathbb{E}\left[\int_{\mathbb{R}} \left[g(X(t-) + h(X(t)z) - g(X(t-)) - g'(X(t-))h(X(t))z\right] \nu(dz)\right] \\ &\quad = \mathbb{E}\left[g'(X(t))b_1(X(t)) + \nu\{|z|>1\}g'(X(t-))h(X(t))\right] \\ &\qquad + \mathbb{E}\left[\int_{\mathbb{R}} \left[g(X(t-) + h(X(t)z) - g(X(t-)) - g'(X(t-))h(X(t))z\right] \nu(dz)\right] \end{aligned} \tag{4.77}$$

By Taylor's expansion, it follows that

$$
\begin{aligned}
&\frac{\partial \mathbb{E}\left[g(X(t))\right]}{\partial t} \\
&\quad = \mathbb{E}\left[g'(X(t))b_1(X(t)) + \nu\{|z|>1\}g'(X(t-))h(X(t))\right] \\
&\qquad + \mathbb{E}\left[\int_{\mathbb{R}}\left[\int_0^1 (1-u)(h(X(t))z)^2 g''(X(t-)+uh(X(t)))du\right]\nu(dz)\right] \qquad (4.78)
\end{aligned}
$$

Since b_1 and h are Lipchitz g'' is bounded, it follows that

$$
\left|\frac{\partial \mathbb{E}\left[g(X(t))\right]}{\partial t}\right| \leq C\left(\mathbb{E}\left[1+|X(t)|\right] + \mathbb{E}\left[1+|X(t)|^2\right]\right) \qquad (4.79)
$$

for some positive constant C. The boundedness of $\partial \mathbb{E}\left[g(X(t))\right]/\partial t$ follows from Lemma 4.3.1. Next, we verify that (4.69) holds. This follows from an application of Theorem 4.5 on the function $u(t, X(t))$. □

Corollary 4.6.1. *Let g and u be as in (4.67). Then,*

$$
\mathbb{E}\left[u(T, X(T)) - u(0, X(0))\right] = 0. \qquad (4.80)
$$

Proof. The proof follows from Theorem 4.6 and Lemma 4.6.2. Indeed, let $u(t,x) = f(t,x)$ in Theorem 4.6. Then we have that

$$
M(T) = u(T, X(T)) - u(0,x) - \int_0^T \left\{\frac{\partial f}{\partial s}(s, X(s)) + \mathcal{L}_1 f(s, X(s))\right\} ds. \qquad (4.81)
$$

Taking the expected value of both sides and applying Lemma 4.6.2 we obtain

$$
\begin{aligned}
\mathbb{E}\left[M(T)\right] &= \mathbb{E}\left[u(T,X(T)) - u(0,X(0))\right] - \mathbb{E}\left[\int_0^T \frac{\partial u}{\partial s}(s,X(s)) + \mathcal{L}_1 u(s, X(s)ds\right] \\
&= \mathbb{E}\left[u(T,X(T)) - u(0,X(0))\right] - \int_0^T \mathbb{E}\left[\frac{\partial u}{\partial s}(s,X(s)) + \mathcal{L}_1 u(s, X(s)ds\right] \\
&= \mathbb{E}\left[u(T,X(T)) - u(0,X(0))\right]
\end{aligned}
$$

But $\mathbb{E}\left[M(T)\right] = \mathbb{E}\left[M(0)\right] = 0$ since M is a martingale, and the result follows. $\square$

Lemma 4.6.3. *Let $g, h \in C_b^3(\mathbb{R}^n)$. and assume that $\mathbb{E}\left[\|Z\|^3\right] < \infty$. Let u be defined as in (4.67). Then the following estimate holds,*

$$|\mathbb{E}\left[u(t, X^\epsilon(t)) - u(0, X(0))\right]| \leq C \int_{\mathbb{R}^d} \|z\|^3 \nu_\epsilon(dz) \tag{4.82}$$

for some positive constant C which does not depend on ϵ.

Proof. In the proof, we will omit the summation sign to simplify notation. We are going to divide the proof into seven parts.

Step 1: By Theorem 4.5, Lemmas 4.3.1 and Lemma 4.6.2, we have that

$$\begin{aligned}
u(t, X^\epsilon(t)) - u(0, X^\epsilon(0)) &= \int_0^t \frac{\partial u}{\partial s}(s, X^\epsilon(s))ds + \int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))b^i(X^\epsilon(s))ds \\
&+ \int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon(s))a_\epsilon^j ds \\
&+ \int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ik}(X^\epsilon(s)) \left(\Sigma_\epsilon^{\frac{1}{2}}\right)_{kj} dB_j(s) \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x_i x_j}(s, X^\epsilon(s))h_{ik}(X^\epsilon(s)) \left(\Sigma_\epsilon\right)_{kl} h_{jl}(X^\epsilon(s))ds \\
&+ \int_0^t \int_{\mathbb{R}^d} u(s, X^\epsilon(s-) + h(X^\epsilon(s))z) - u(s, X^\epsilon(s-))N^\epsilon(dz, ds)
\end{aligned} \tag{4.83}$$

Step 2: By Lemma 4.6.2 and the martingale property of Brownian integrals, we take the expected value of both sides in **Step 1** to obtain

$$\begin{aligned}
&\mathbb{E}\left[u(t, X^\epsilon(t)) - u(0, X^\epsilon(t))\right] \\
&\quad = \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial s}(s, X^\epsilon(s))ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon(s))a_\epsilon^j ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))b^i(X^\epsilon(s))ds\right] \\
&\qquad + \mathbb{E}\left[\frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x_i x_j}(s, X^\epsilon(s))h_{ik}(X^\epsilon(s))\left(\Sigma_\epsilon\right)_{kl} h_{jl}(X^\epsilon(s))ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} u(s, X^\epsilon(s-) + h(X^\epsilon(s))z) - u(s, X^\epsilon(s-))\nu^\epsilon(dz)ds\right]
\end{aligned} \tag{4.84}$$

Step 3: Recall, the expression for a_ϵ in (4.15):

$$a_\epsilon^j = a^j + \int_{\|z\|>1} z^j \nu_\epsilon(dz) - \int_{\|z\|\leq 1} z^j \nu^\epsilon(dz). \tag{4.85}$$

We can then rewrite

$$\mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon(s))a_\epsilon^j ds\right] \tag{4.86}$$

the second expectation after the equal sign in (4.84) as follows

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon(s))a_\epsilon^j ds\right] \\
&\quad = \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon(s))a^j ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^t \int_{\|z\|>1} \frac{\partial u}{\partial x_i}(s, X^\epsilon(s)h_{ij}(X^\epsilon(s-))z^j \nu_\epsilon(dz)ds\right] \\
&\qquad - \mathbb{E}\left[\int_0^t \int_{|z|\leq 1} \frac{\partial u}{\partial x_i}(s, X^\epsilon(s)h_{ij}(X^\epsilon(s-))z^j \nu^\epsilon(dz)ds\right],
\end{aligned} \tag{4.87}$$

so that

$$\mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon(s))a^j ds\right] + \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))b^i(X^\epsilon(s))ds\right] \\ = \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))b_1(X^\epsilon(s))ds\right], \tag{4.88}$$

where $b_1(x) = b(x) + h(x)a$. Thus, (4.84) becomes

$$\begin{aligned}\mathbb{E}\left[u(t, X^\epsilon(t)) - u(0, X^\epsilon(t))\right] &= \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial s}(s, X^\epsilon(s))ds\right] \\ &+ \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))b_1^i(X^\epsilon(s))ds\right] \\ &+ \mathbb{E}\left[\int_0^t \int_{\|z\|>1} \frac{\partial u}{\partial x_i}(s, X^\epsilon(s)h_{ij}(X^\epsilon(s-))z^j \nu_\epsilon(dz)ds\right] \\ &- \mathbb{E}\left[\int_0^t \int_{|z|\leq 1} \frac{\partial u}{\partial x_i}(s, X^\epsilon(s)h_{ij}(X^\epsilon(s-))z^j \nu^\epsilon(dz)ds\right] \\ &+ \mathbb{E}\left[\frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x_i x_j}(s, X^\epsilon(s))h_{ik}(X^\epsilon(s))\left(\Sigma_\epsilon\right)_{kl} h_{jl}(X^\epsilon(s))ds\right] \\ &+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} u(s, X^\epsilon(s-) + h(X^\epsilon(s))z) - u(s, X^\epsilon(s-))\nu^\epsilon(dz)ds\right]\end{aligned} \tag{4.89}$$

Step 4: We now in introduce the generator $\mathcal{L}_1$ in to (4.89) to obtain

$$
\begin{aligned}
&\mathbb{E}\left[u(t,X^\epsilon(t)) - u(0,x)\right] \\
&\qquad = \mathbb{E}\left[\int_0^t \frac{\partial u}{\partial s}(s,X^\epsilon(s)) + \mathcal{L}_1 u(s,X^\epsilon(s))ds\right] \\
&\quad + \mathbb{E}\left[\frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x_i x_j}(s,X^\epsilon(s))h_{ik}(X^\epsilon(s))\left(\Sigma_\epsilon\right)_{kl} h_{jl}(X^\epsilon(s))ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\|z\|>1} \frac{\partial u}{\partial x_i}(s,X^\epsilon(s)h_{ij}(X^\epsilon(s-))z^j \nu_\epsilon(dz)ds\right] \\
&\quad - \mathbb{E}\left[\int_0^t \int_{|z|\le 1} \frac{\partial u}{\partial x_i}(s,X^\epsilon(s)h_{ij}(X^\epsilon(s-))z^j \nu^\epsilon(dz)ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} u(s,X^\epsilon(s-)+h(X^\epsilon(s))z) - u(s,X^\epsilon(s-))\nu^\epsilon(dz)ds\right] \\
&\quad - \mathbb{E}\left[\int_0^t \int_{\|z\|>1} u(s,X^\epsilon(s-)+h(X^\epsilon(s))z) - u(s,X^\epsilon(s-))\nu(dz)ds\right] \\
&\quad - \mathbb{E}\left[\int_0^t \int_{\|z\|\le 1} u(s,X^\epsilon(s-)+h(X^\epsilon(s))z) - u(s,X^\epsilon(s-))\right. \\
&\qquad\qquad \left. - \frac{\partial u}{\partial x_i}(s,X^\epsilon(s))h_{ij}(X^\epsilon(s-))z^j \nu(dz)ds\right]
\end{aligned} \tag{4.90}
$$

Step 5: Here, we replace Σ_ϵ by its value (4.13), to obtain

$$
\begin{aligned}
&\mathbb{E}\left[u(t,X^\epsilon(t)) - u(0,X^\epsilon(0))\right] \\
&\quad = \mathbb{E}\left[\frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x_i x_j}(s,X^\epsilon(s))h_{ik}(X^\epsilon(s))(\Sigma_\epsilon)_{kl} h_{jl}(X^\epsilon(s))ds\right] \\
&\qquad - \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} u(s,X^\epsilon(s-)+h(X^\epsilon(s))z) - u(s,X^\epsilon(s-))\right. \\
&\qquad\qquad\qquad \left. - \frac{\partial u}{\partial x_i}(s,X^\epsilon(s))h_{ij}(X^\epsilon)z^j \nu_\epsilon(dz)ds\right] \\
&\quad = \mathbb{E}\left[\frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \frac{\partial^2 u}{\partial x_i x_j}(s,X^\epsilon(s))h_{ik}(X^\epsilon(s))z_k z_l h_{jl}(X^\epsilon(s))\nu_\epsilon(dz)ds\right] \\
&\qquad - \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} u(s,X^\epsilon(s-)+h(X^\epsilon(s))z) - u(s,X^\epsilon(s-))\right. \\
&\qquad\qquad\qquad \left. - \frac{\partial u}{\partial x_i}(s,X^\epsilon(s))h_{ij}(X^\epsilon(s))z^j \nu_\epsilon(dz)ds\right]
\end{aligned} \tag{4.91}
$$

Step 6: Next, we let

$$\begin{aligned} M &= u(s, X^\epsilon(s-) + h(X^\epsilon(s))z) - u(s, X^\epsilon(s-)) - \frac{\partial u}{\partial x_i}(s, X^\epsilon(s))h_{ij}(X^\epsilon)z^j \\ &\quad - \frac{1}{2}\frac{\partial^2 u}{\partial x_i x_j}(s, X^\epsilon(s))h_{ik}(X^\epsilon(s))z^k z^l h_{jl}(X^\epsilon(s)). \end{aligned} \tag{4.92}$$

By Taylor's expansion, we obtain

$$M = \frac{1}{6}\frac{\partial^3 u}{\partial x_i x_j \partial x_k}(s, X^\epsilon(s-) + \theta h(X^\epsilon(s)z))h_{il}(X^\epsilon(s))h_{jm}(X^\epsilon(s))h_{kn}(X^\epsilon(s))z^l z^m z^n \tag{4.93}$$

with $\theta \in (0,1)$. It follows from Lemma 4.6.2 and the Lipschitz continuity of h that there exists a positive constant C such that

$$\begin{aligned} \mathbb{E}\left[|M|\right] &\leq C\left(\mathbb{E}\left[\|h(X^\epsilon(s))\|^3\right]\right)\|z\|^3 \\ &\leq C\left(\mathbb{E}\left[1 + \|X^\epsilon(s-)\|^3\right]\right)\|z\|^3 \\ &\leq C\left(1 + \|x\|^3\right)\|z\|^3. \end{aligned}$$

Thus

$$\begin{aligned} &\left|\mathbb{E}\left[-\int_0^t \int_{\mathbb{R}^d} M\nu_\epsilon(dz)ds\right]\right| \\ &\quad \leq C\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(1 + \|x\|^3\right)\|z\|^3 \nu_\epsilon(dz)ds\right] \end{aligned}$$

Step 7: Combining **Step 1**, **Step 2**, **Step 3**, **Step 4**, **Step 5** and **Step 6**, and invoking Lemma 4.3.1, Corollary 4.6.1, and Fubbini's theorem we obtain

$$|\mathbb{E}\left[g(X(T)) - g(X^\epsilon(T))\right]| \leq C\int_{\mathbb{R}^d}\|z\|^3\nu_\epsilon(dz),$$

and the result follows. □

We now state and proof the main result for this section.

Theorem 4.7. *Let $g, h \in C_b^3(\mathbb{R}^d)$, $k = 1, 2, 3$ and assume that $\mathbb{E}\left[\|Z\|^3\right] < \infty$. Then the following estimate holds,*

$$|\mathbb{E}\left[g(X(T)) - g(X^\epsilon(T))\right]| \leq C \int_{\mathbb{R}^d} \|z\|^3 \nu_\epsilon(dz) \tag{4.94}$$

for some positive constant C which does not depend on ϵ.

Proof. By Corollary 4.6.1 and Lemma 4.6.3 it follows that

$$\begin{aligned} |\mathbb{E}\left[g(X(T)) - g(X^\epsilon(T))\right]| &= |\mathbb{E}\left[u(0, X(0)) - g(X^\epsilon(T))\right]| \\ &= |\mathbb{E}\left[u(0, X(0)) - u(T, X^\epsilon(T))\right]| \\ &\leq C \int_{\mathbb{R}^d} \|z\|^3 \nu_\epsilon(dz). \end{aligned} \tag{4.95}$$

□

4.2.5 Convergence Results for Strong and Weak Numerical Schemes

Here, we give error bounds for numerical approximations of the solution X of (4.4). Before establishing these error estimates, we have the following remark.

Remark 4.2.1. *For each fixed $\epsilon \in (0, 1)$, the SDE*

$$\begin{aligned} X^\epsilon(t) = x &+ \int_0^t b_1(X^\epsilon(s))ds + \int_0^t h(X^\epsilon(s))\Sigma_\epsilon^{1/2} dB(s) \\ &+ \int_0^t \int_{\mathbb{R}^d} h(X^\epsilon(s-))z\tilde{N}^\epsilon(dz, ds), \end{aligned} \tag{4.96}$$

is a jump-diffusion SDE which is a special case of (3.4), with $b_1(x) = b(x) + h(x)\tilde{a}$, $\sigma(x) = h(x)\Sigma_\epsilon^{1/2}$, $G(x, z) = h(x)z, E = \mathbb{R}^d$, and the Lévy measure is ν^ϵ.

Theorem 4.8. *For a given $\gamma \in \{0.5, 1, 1.5, \cdots\}$ and a fixed $\epsilon \in (0,1)$, let $Y = \{Y(t); t \in [0,T]\}$ be the strong order γ Taylor approximation defined in (3.32) of the solution X^ϵ of equation (4.96), corresponding to a time discretization $(t)_\Delta$ with a maximum step size $\Delta \in (0,1)$. Further, assume that the coefficient functions b and h are $2(\gamma+1)$ times continuously differentiable, uniformly bounded, with uniformly bounded derivatives. Then the following estimate holds*

$$\mathbb{E}\left[\|X(T) - Y(T)\|^2\right] \leq C\left(\mathit{trace}(\Sigma_\epsilon) + \Delta^{2\gamma}\right) \tag{4.97}$$

for some positive constant C.

Proof. By the triangle inequality and Theorem 4.2.1., we have that

$$\begin{aligned}\mathbb{E}\left[\|X(T) - Y(T)\|^2\right] &\leq 2\left(\mathbb{E}\left[\|X(T) - X^\epsilon(T)\|^2\right] + \mathbb{E}\left[\|X^\epsilon(T) - Y(T)\|^2\right]\right) \\ &\leq C_1 \mathrm{trace}(\Sigma_\epsilon) + \mathbb{E}\left[\|X^\epsilon(T) - Y(T)\|^2\right]\end{aligned} \tag{4.98}$$

But by Remark 4.2.1 and Theorem 3.2, we have that

$$\mathbb{E}\left[\|X^\epsilon(T) - Y(T)\|^2\right] \leq C_2 \Delta^{2\gamma}. \tag{4.99}$$

for some positive constant C_2. Combining (4.98) and (4.99), we obtain

$$\mathbb{E}\left[\|X(T) - Y(T)\|^2\right] \leq C\left(\mathrm{trace}(\Sigma_\epsilon) + \Delta^{2\gamma}\right) \tag{4.100}$$

for some positive constant C. □

Theorem 4.9. *For a given $\beta \in \{1, 2, \cdots\}$ and a fixed $\epsilon \in (0,1)$, let $Y = \{Y(t) : t \in [0,T]\}$ be the weak order β Taylor approximation defined in (3.34) of the solution X^ϵ of equation (4.96), corresponding to the regular time discretization $(t)_\Delta$ with maximum step size $\Delta \in (0,1)$. Further, we suppose that $b, \sigma, G \in C^{2(\beta+1)}$,* *Lipschitz continuous and uniformly bounded. Then the following estimate holds*

$$|\mathbb{E}\left[g(X(T)) - g(Y(T))\right]| \leq C\left(\int_{\mathbb{R}^d} \|z\|^3 \nu_\epsilon(dz) + \Delta^\beta\right) \tag{4.101}$$

for some positive constant C which does not depend on ϵ.

Proof. By the Triangle inequality, Theorem 3.3 and Theorem 4.9 we get

$$\begin{aligned} &|\mathbb{E}\left[g(X(T)) - g(Y(T))\right]| \\ &\quad\leq |\mathbb{E}\left[g(X(T)) - g(X^\epsilon(T))\right]| + |\mathbb{E}\left[g(X^\epsilon(T)) - g(Y(T))\right]| \\ &\quad\leq C\left(\int_{\mathbb{R}^d} \|z\|^3 \nu_\epsilon(dz) + \Delta^\beta\right) \end{aligned} \tag{4.102}$$

for some positive constant C. □

4.3 A More General Model

4.3.1 Formulation

Here, we consider the following stochastic process $Y = \{Y(t) : t \geq 0\}$, the solution to the SDE

$$Y(t) = x + \int_0^t b(Y(s))ds + \int_0^t \sigma(Y(s))dW(s) + \int_0^t h(Y(s-))dZ(s), \quad t \in [0, T] \tag{4.103}$$

where Z is as in (4.2) and $W = \{W(t) : t \geq 0\}$ is a d-dimensional Brownian motion which is independent of Z. Further, we assume that $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is a Lipschitz continuous function, with b and h are as in (4.4). By the Lévy-Ito decomposition, we can rewrite (4.103) as follows

$$\begin{aligned} Y(t) = x &+ \int_0^t b(Y(s))ds + \int_0^t h(Y(s))ads + \int_0^t \sigma(Y(s))dW(s) \\ &+ \int_0^t \int_{\|z\|\leq 1} h(Y(s-))z\widetilde{N}(dz,ds) + \int_0^t \int_{\|z\|>1} h(Y(s-))zN(dz,ds) \end{aligned} \tag{4.104}$$

Let Z_ϵ be the Lévy process given by (4.23), and consider the following SDE

$$\begin{aligned} Y^\epsilon(t) = x &+ \int_0^t b(Y^\epsilon(s))ds + \int_0^t h(Y^\epsilon(s))a_\epsilon ds + \int_0^t \sigma(Y^\epsilon(s))dW(s) \\ &+ \int_0^t h(Y^\epsilon(s))\Sigma_\epsilon^{1/2}dB(s) + \int_0^t \int_{\mathbb{R}^d} h(Y^\epsilon(s-))zN^\epsilon(dz,ds) \end{aligned} \tag{4.105}$$

By Theorem 2.6, equations (4.104) and (4.105) have uniques solutions due to the assumptions of Lipschitz continuity on the coefficient functions.

4.3.2 L^p Error Estimate

We assume here that $\mathbb{E}[\|Z\|^p] < \infty,\ p \geq 2.$ Then (4.104) and (4.105) can be rewritten as

$$Y(t) = x + \int_0^t b_1(Y(s))ds + \int_0^t \sigma(Y(s))dW(s) + \int_0^t \int_{\mathbb{R}^d} h(Y(s-))z\widetilde{N}(dz,ds), \tag{4.106}$$

and

$$Y^\epsilon(t) = x + \int_0^t b_1(Y^\epsilon(s))ds + \int_0^t \sigma(Y^\epsilon(s))dW(s) + \int_0^t h(Y^\epsilon(s))\Sigma_\epsilon^{1/2}dB(s) \tag{4.107}$$

$$+ \int_0^t \int_{\mathbb{R}^d} h(Y^\epsilon(s-))z\widetilde{N^\epsilon}(dz,ds), \tag{4.108}$$

where $b_1(x) = b(x) + h(x)\tilde{a}$. Observe that

$$M_2(t) = \int_0^t \sigma(Y^\epsilon(s))dW(s) + \int_0^t h(Y^\epsilon(s))\Sigma_\epsilon^{1/2}dB(s) \tag{4.109}$$

is a martingale, so we can apply Burkholder's inequality to the expectation $\mathbb{E}\left[\sup_{0\leq s\leq t} \|M(s)\|^p\right]$, for $p \geq 1$.

Corollary 4.9.1. *In the above setting, it holds that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|Y(s)\|^p\right] \leq C_1\left(1 + \|x\|^p\right), \quad p \geq 2, \tag{4.110}$$

and

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|Y^\epsilon(s)\|^p\right] \leq C_2\left(1 + \|x\|^p\right), \quad p \geq 2, \tag{4.111}$$

for some positive constants C_1, and C_2 which do not depend on ϵ.

The proof technique is in the same lines as the proof of Corollary 4.9.2, so we omit it. From (4.106) and (4.108) we obtain

$$\begin{aligned}
&Y(t) - Y^\epsilon(t)\\
&\quad= \int_0^t \left[b_1(Y(s)) - b_1(Y^\epsilon(s))\right] ds + \int_0^t \left[\sigma(Y(s)) - \sigma(Y^\epsilon(s))\right] dW(s)\\
&\qquad+ \int_0^t \int_{\mathbb{R}^d} \left[h(Y(s-)) - h(Y^\epsilon(s-))\right] z\tilde{N}(dz, ds)\\
&\qquad+ \int_0^t \int_{\mathbb{R}^d} h(Y^\epsilon(s-))z\tilde{N}_\epsilon(dz, ds) - \int_0^t h(Y^\epsilon(s))\Sigma_\epsilon^{1/2}dB(s).
\end{aligned} \tag{4.112}$$

Corollary 4.9.2. *Let Y and Y^ϵ be the unique solutions of (4.104) and (4.105) respectively. Then for $p \geq 2$ the following estimate holds*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|Y(t) - Y^\epsilon(t)\|^p\right] \leq C \int_{\mathbb{R}^d} \|z\|^p \nu_\epsilon(dz) \tag{4.113}$$

for some positive constants C which does not depend on ϵ.

Proof. Let

$$
\begin{aligned}
Y_1(t) - Y_1^\epsilon(t) = &\int_0^t \left[b_1(Y(s)) - b_1(Y^\epsilon(s))\right] ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \left[h(Y(s-)) - h(Y^\epsilon(s-))\right] z\tilde{N}(dz, ds) \\
&+ \int_0^t \int_{\mathbb{R}^d} h(Y^\epsilon(s-))z\tilde{N}_\epsilon(dz, ds) - \int_0^t h(Y^\epsilon(s))\Sigma_\epsilon^{1/2} dB(s).
\end{aligned} \tag{4.114}
$$

Then $Y(t) - Y^\epsilon(t)$ can be written as

$$
Y(t) - Y^\epsilon(t) = Y_1(t) - Y_1^\epsilon(t) + \int_0^t \left[\sigma(Y(s)) - \sigma(Y^\epsilon(s))\right] dW(s). \tag{4.115}
$$

But,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} \|Y(t) - Y^\epsilon(t)\|^p\right] \le C \Bigg(&\mathbb{E}\left[\sup_{0\le t\le T} \|Y_1(t) - Y_1^\epsilon(t)\|^p\right] \\
&\mathbb{E}\left[\sup_{0\le t\le T} \left\| \int_0^t \left[\sigma(Y(s)) - \sigma(Y^\epsilon(s))\right] dW(s) \right\|^p\right] \Bigg).
\end{aligned} \tag{4.116}
$$

for some positive constant C. By Theorem 4.4, we have that that

$$
\mathbb{E}\left[\sup_{0\le t\le T} \|Y_1(t) - Y_1^\epsilon(t)\|^p\right] \le C_3 \int_{\mathbb{R}^d} \|z\|^p \nu_\epsilon(dz), \tag{4.117}
$$

for some positive constant C_3. By Burkholder's inequality, Jensen's inequality and the Lipschitz continuity of σ, it follows that

$$
\mathbb{E}\left[\sup_{0\le s\le t} \left\| \int_0^t \left[\sigma(Y(r)) - \sigma(Y^\epsilon(r))\right] dW(r) \right\|^p\right] \le C_4 \mathbb{E}\left[\int_0^t \|Y(s) - Y^\epsilon(s)\|^p \, ds\right] \tag{4.118}
$$

for some positive constant C_4. Combining (4.114), (4.115), (4.116), (4.117), (4.118) and applying Gronwall's inequality we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\|Y(t)-Y^{\epsilon}(t)\|^p\right]\le C\int_{\mathbb{R}^d}\|z\|^p\nu_{\epsilon}(dz). \tag{4.119}$$

for some positive constant C which does not depend on ϵ. □

4.3.3 Weak Estimate

Next, we consider the weak error $\mathbb{E}\left[g(Y(T))-g(Y^{\epsilon}(T)\right]$ for some smooth function g. Define the function u_Y as follows

$$u_Y(t,x)\stackrel{def}{=}\mathbb{E}\left[g(Y(T))\mid Y(t)=x\right], \tag{4.120}$$

where Y is the solution of (4.104) with initial value $x\in\mathbb{R}^n$. Observe again that Y is a Markov process (see Remark 2.3.2(iv)) with infinitesimal generator $\mathcal{L}_2$. In this case, $\mathcal{L}_2$ is given by

$$\begin{aligned}(\mathcal{L}_2 f)(t,x)&=\frac{\partial}{\partial t}f(t,x)+\sum_{i=1}^n\frac{\partial}{\partial x_i}f(t,x)b_1^i(x)+\left(\mathcal{L}_{\Delta}f\right)(t,x)\\&\quad+\int_{\|z\|>1}[f(t,x+h(x)z)-f(t,x)]\nu(dz)\\&\quad+\int_{\|z\|\le 1}\left[f(t,x+h(x)z)-f(t,x)-\sum_{i=1}^n\frac{\partial}{\partial x_i}f(t,x)\left(\sum_{j=1}^d h_{ij}(x)z^j\right)\right]\nu(dz),\\&=(\mathcal{L}_1 f)(t,x)+(\mathcal{L}_{\Delta}f)(t,x)\end{aligned} \tag{4.121}$$

where

$$\left(\mathcal{L}_{\Delta}f\right)(t,x)=\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}f(t,x)\sigma_{ij}(x) \tag{4.122}$$

and $f \in C_0^2(\mathbb{R}^d)$ (see [1], Theorem 6.7.3, pp. 407). The only difference between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the inclusion of the Laplace operator $\mathcal{L}_\Delta$. u_Y satisfies the following backwards Kolmogorov equation

$$\frac{\partial u_Y}{\partial t}(t,x) + \mathcal{L}_1 u_Y(t,x) = 0. \tag{4.123}$$
$$u_Y(T,x) = g(x),$$

We have the following result which is a corollary to Theorem 4.7.

Corollary 4.9.3. *Let $g, h \in C_b^3(\mathbb{R}^d)$, and assume that $\mathbb{E}\left[\|Z\|^3\right] < \infty$. Then the following estimate holds,*

$$\left|\mathbb{E}\left[g(Y(T)) - g(Y^\epsilon(T))\right]\right| \leq C \int_{\mathbb{R}^d} \|z\|^3 \nu_\epsilon(dz) \tag{4.124}$$

for some positive constant C which does not depend on ϵ.

Proof. With an application of Itô's formula we get

$$\begin{aligned}
&u(t, Y^\epsilon(t)) - u(0, Y^\epsilon(0)) \\
&\quad = \int_0^t \frac{\partial u_Y}{\partial s}(s, Y^\epsilon(s))ds + \int_0^t \frac{\partial u}{\partial y_i}(s, Y^\epsilon(s)) b^i(X^\epsilon(s))ds \\
&\qquad + \int_0^t \frac{\partial u_Y}{\partial y_i}(s, Y^\epsilon(s)) h_{ij}(Y^\epsilon(s)) a_\epsilon^j ds \\
&\qquad + \int_0^t \frac{\partial u_Y}{\partial y_i}(s, Y^\epsilon(s)) h_{ik}(Y^\epsilon(s)) \left(\Sigma_\epsilon^{\frac{1}{2}}\right)_{kj} dB_j(s) \\
&\qquad + \int_0^t \frac{\partial u_Y}{\partial y_i}(s, Y^\epsilon(s)) \sigma_{ij}(Y^\epsilon(s)) dW_j(s) \\
&\qquad + \frac{1}{2}\int_0^t \frac{\partial^2 u_Y}{\partial y_i y_j}(s, Y^\epsilon(s)) h_{ik}(Y^\epsilon(s)) \left(\Sigma_\epsilon\right)_{kl} h_{jl}(Y^\epsilon(s))ds \\
&\qquad + \frac{1}{2}\int_0^t \frac{\partial^2 u_Y}{\partial y_i y_j}(s, Y^\epsilon(s)) \sigma_{ij}(Y^\epsilon(s))ds \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} u(s, Y^\epsilon(s-) + h(Y^\epsilon(s))z) - u(s, Y^\epsilon(s-)) N^\epsilon(dz, ds)
\end{aligned} \tag{4.125}$$

The difference here with **Step 1** in the proof of Lemma 4.6.3 is the inclusion of the terms

$$\int_0^t \frac{\partial u_Y}{\partial y_i}(s, Y^\epsilon(s))\sigma_{ij}(Y^\epsilon(s))dW_j(s) \tag{4.126}$$

and

$$\frac{1}{2}\int_0^t \frac{\partial^2 u_Y}{\partial y_i y_j}(s, Y^\epsilon(s))\sigma_{ij}(Y^\epsilon(s))ds \tag{4.127}$$

with X^ϵ replaced by Y^ϵ. Observe that if we take the expected value of both sides in (4.125) then (4.126) vanishes. (4.127) will be absorbed into the generator $\mathcal{L}_1$ in **Step 4** in the proof Lemma 4.6.3 to give us $\mathcal{L}_2$. The rest of the proof (**Step 5**–**Step 7**) is the same as the proof of Lemma 4.3.1. □

Remark 4.3.1. *For each fixed $\epsilon \in (0,1)$ we define the new Brownian motion B_1 by $B_1(t) = W(t) + \Sigma_\epsilon^{1/2}B(t)$, and $\sigma_1(x) = (\sigma(x), h(x))$ then (4.108) becomes*

$$Y^\epsilon(t) = x + \int_0^t b_1(Y^\epsilon(s))ds + \int_0^t \sigma_1(Y^\epsilon(s))dB_1(s) + \int_0^t\int_{\mathbb{R}^d} h(Y^\epsilon(s-))z\widetilde{N}^\epsilon(dz, ds) \tag{4.128}$$

which is a jump-diffusion SDE of the form (4.96). Thus we have strong and weak numerical schemes for (4.128) with corresponding convergence results as in Theorem 4.8 and Theorem 4.9.

Chapter 5

Numerical Experiment

5.1 The SDE Considered for the Numerical Experiment

For our numerical experiment, we will consider the Doléans-Dade exponential. Here, we simulate the sample paths of the "exact" solution using series representation (see Theorem 2.5) of the driving process. Next, we simulate the sample path of the jump-diffusion using regular Euler scheme, jump-adapted Euler scheme and the strong order–1 schemes respectively. Lastly, we give some numerical error estimates.

5.1.1 Set–Up of Simulation Example

We consider the Doléans-Dade exponential given in Example 2.3.2. Let X be the unique solution to the SDE

$$dX(t) = X(t-)dZ(t),\ X(0) = x,\ t \in [0, T], \tag{5.1}$$

where Z is an infinite activity Lévy process with Lévy triplet $(a, 0, \nu)$. Then the exact solution is given by

$$X(t) = e^{Z(t)} \prod_{0 \leq s \leq t} (1 + \Delta Z(s)) \, e^{-\Delta Z(s)}. \tag{5.2}$$

Let $\epsilon \in (0, 1]$ be arbitrary. The corresponding jump-diffusion equation X^ϵ is given by

$$X^\epsilon(t) = x + \int_0^t X^\epsilon(s) a_\epsilon ds + \int_0^t X^\epsilon(s) \sigma_\epsilon dB(s) + \int_0^t \int_{|z|>\epsilon} X^\epsilon(s-) z N(dz, ds), \tag{5.3}$$

where,

$$a_\epsilon = a - \int_{\epsilon < |z| \leq 1} z \nu(dz). \tag{5.4}$$

We denote by $\tau_1^\epsilon, \tau_2^\epsilon, \cdots, \tau_{N^\epsilon(T)}^\epsilon$, the jump times generated by the PRM $N^\epsilon(dz, ds)$ with corresponding jumps (or marks) $V_1, V_2, \cdots, V_{N^\epsilon(T)}$, where,

$$N^\epsilon(t) = N(\{z : |z| > \epsilon\} \times [0, t]), \tag{5.5}$$

for $t \in [0, 1]$, is a Poisson process with parameter $\lambda_\epsilon t$. We note that $N^\epsilon(t)$ counts the number of jumps in $[0, t]$ that are greater than ϵ in magnitude. Also note that, by Theorem 2.3

$$\lambda_\epsilon = \nu\{z : |z| > \epsilon\} < \infty. \tag{5.6}$$

With this set-up, the exact solution of (5.3) is given by

$$X^\epsilon(t) = e^{\left((a_\epsilon - \frac{\sigma_\epsilon^2}{2})t + \sigma_\epsilon B(t)\right)} \prod_{k=1}^{N^\epsilon(t)} (1 + V_k) \tag{5.7}$$

Next, we let the driving process Z be an exponentially tempered α-stable ($T\alpha S$) process with Lévy triplet $(0, \nu, 0)$ where ν is given by

$$\nu(dz) = \left(\frac{e^{-|z|}}{|z|^{1+\alpha}} \mathbb{1}_{(-\infty,0)} + \frac{e^{-z}}{z^{1+\alpha}} \mathbb{1}_{(0,\infty)} \right) dz, \quad \alpha \in (0,2). \tag{5.8}$$

In this case, the drift a_ϵ is equal to 0. Indeed,

$$\begin{aligned} a_\epsilon &= 0 - \int_{\epsilon<|z|\leq 1} z\nu(dz) \\ &= -\left(\int_{-1}^{-\epsilon} z \frac{e^z}{(-z)^{1+\alpha}} dz + \int_\epsilon^1 z \frac{e^{-z}}{z^{1+\alpha}} dz \right) \\ &= -\left(\int_\epsilon^1 -z \frac{e^{-z}}{z^{1+\alpha}} dz + \int_\epsilon^1 z \frac{e^{-z}}{z^{1+\alpha}} dz \right) \\ &= 0. \end{aligned} \tag{5.9}$$

The variance $\sigma(\epsilon)^2 = \sigma_\epsilon^2$ of the small jumps is

$$\begin{aligned} \sigma_\epsilon^2 &= \int_{|z|\leq\epsilon} z^2 \nu(dz), \\ &= \int_{-\epsilon}^0 z^2 \frac{e^z}{(-z)^{1+\alpha}} dz + \int_0^\epsilon z^2 \frac{e^{-z}}{z^{1+\alpha}} dz, \\ &= 2 \int_0^\epsilon z^{1-\alpha} e^{-z} dz. \end{aligned} \tag{5.10}$$

By Theorem A.3, we obtain

$$\sigma_\epsilon^2 \sim \frac{2}{2-\alpha} \epsilon^{2-\alpha}, \tag{5.11}$$

from which it follows that

$$\sigma_\epsilon \sim \left(\frac{2}{2-\alpha} \right)^{\frac{1}{2}} \epsilon^{1-\frac{\alpha}{2}}. \tag{5.12}$$

Therefore the jump-diffusion approximation X^ϵ is given by

$$X^\epsilon(t) = 1 + \int_0^t X^\epsilon(s)\sigma_\epsilon dB(s) + \int_0^t \int_{|z|>\epsilon} X^\epsilon(s) z N(dz, ds), \tag{5.13}$$

with exact solution

$$X^\epsilon(t) = e^{\left(-\frac{\sigma_\epsilon^2}{2}t + \sigma_\epsilon B(t)\right)} \prod_{k=1}^{N^\epsilon(t)} (1 + V_k), \tag{5.14}$$

with σ_ϵ is given by (5.12).

Remark 5.1.1.

The sample paths of the driving process Z and Z^ϵ are simulated using series representation (see Algorithm 2, in Example 2.2.4), which is then used to construct the sample paths of X and X^ϵ respectively.

5.1.2 Euler Scheme

We now construct the Euler scheme for (5.13). Let $T = 1$, and $0 = t_0 < t_1 < \cdots, t_n = 1$, be an equidistant discretization of the interval $[0, 1]$. That is $t_n = n\Delta$, for $n \in \left\{0, 1, \cdots, \frac{1}{\Delta}\right\}$, where $\Delta \in (0, 1)$ is the time step. The Euler scheme is given by

$$\begin{aligned} Y(t_{n+1}) &= Y(t_n) + \sigma_\epsilon Y(t_n)\Delta B_n + \int_{t_n}^{t_{n+1}} \int_{|z|>\epsilon} Y(t_n) z N(dz, ds) \\ &= Y(t_n)\left(1 + \sigma_\epsilon \Delta B_n + \sum_{k=N^\epsilon(t_n)+1}^{N^\epsilon(t_{n+1})} V_k\right) \end{aligned} \tag{5.15}$$

for $n \in \{0, 1, \cdots, n_1 - 1\}$ with initial value $Y_0 = 1$ with n_1 given by (3.45).

5.1.3 Jump-Adapted Euler Scheme

Let $0 = \tau_0 < \tau_1 < \cdots, \tau_{n_1} = 1$ be an equidistant discretization of $[0,1]$, and let $\tau_1^\epsilon, \tau_2^\epsilon, \cdots, \tau^\epsilon_{N^\epsilon(T)}$, be the jump times of Z^ϵ. Consider the jump-adapted time discretization

$$\{t_0, t_1, \cdots, t_{N_1}\} = \{\tau_0, \tau_1, \cdots, \tau_{n_1}\} \cup \left\{\tau_1^\epsilon, \tau_2^\epsilon, \cdots, \tau^\epsilon_{N^\epsilon(1)}\right\}. \tag{5.16}$$

where $N_1 = n_1 + N^\epsilon(1)$. On this time discretization the jump-adapted Euler scheme is given by

$$Y(t_{n+1}-) = Y(t_n)(1 + \sigma_\epsilon \Delta B_n), \tag{5.17}$$

and

$$Y(t_{n+1}) = \begin{cases} Y(t_n)(1 + \sigma_\epsilon \Delta B_n)\left(1 + V_{N^\epsilon(t_{n+1})}\right), \text{ if } t_{n+1} \text{ is a jump time} \\ Y(t_n)(1 + \sigma_\epsilon \Delta B_n), \text{ if } t_{n+1} \text{ is not a jump time.} \end{cases} \tag{5.18}$$

5.1.4 Strong Jump-Adapted Order-One Scheme

A strong jump-adapted order-one approximation is given by

$$Y(t_{n+1}) = \begin{cases} Y(t_n)(1 + \sigma_\epsilon \Delta B_n + \frac{\sigma_\epsilon^2}{2}\left(\Delta B_n^2 - \Delta_n\right))\left(1 + V_{N^\epsilon(t_{n+1})}\right), \text{ if } t_{n+1} \text{ is a jump time} \\ Y(t_n)(1 + \sigma_\epsilon \Delta B_n + \frac{\sigma_\epsilon^2}{2}\left(\Delta B_n^2 - \Delta_n\right)), \text{ if } t_{n+1} \text{ is not a jump time,} \end{cases} \tag{5.19}$$

where

$$Y(t_{n+1}-) = Y(t_n)\left(1 + \sigma_\epsilon \Delta B_n + \frac{\sigma_\epsilon^2}{2}\left((\Delta B_n)^2 - \Delta_n\right)\right). \tag{5.20}$$

Here, Δ denotes the step size, $\Delta B_n = B(t_{n+1}) - B(t_n)$, $\Delta_n = t_{n+1} - t_n$ and $Y(t_{n+1}-)$ is the left limit of the process Y at t_{n+1}. For other higher order schemes, see [26]. Before giving some error estimates, we display typical sample paths using the three approximation schemes given above for $\alpha = 0.8$ and $\alpha = 1.4$.

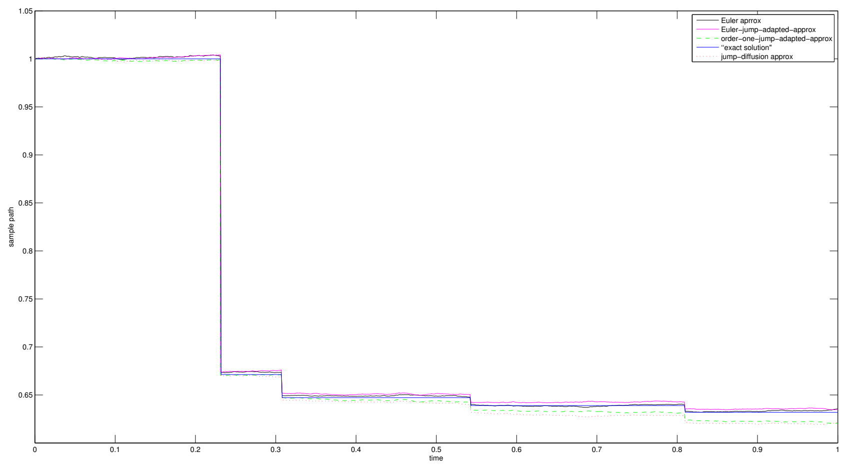

Figure 5.1: Simulated sample paths of an SDE driven by an exponentially tempered α–stable Lévy process with index of stability $\alpha = 0.8$.

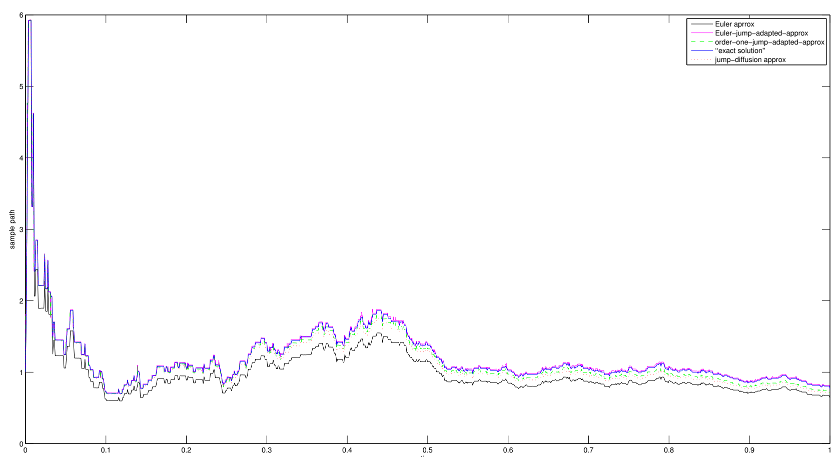

Figure 5.2: Simulated sample paths of an SDE driven by an exponentially tempered–α-stable Lévy process with index of stability $\alpha = 1.4$.

5.1.5 Error Estimates

Denote by M, the number of realizations, and by Y the numerical approximation of X^ϵ. We begin by estimating the L^2 and the weak errors due to the jump-diffusion approximation, which we shall denote by

$$err_{s\epsilon} = \mathbb{E}\left[|X(t) - X^\epsilon(t)|^2\right], \tag{5.21}$$

and

$$err_{w\epsilon} = |\mathbb{E}\left[(X(t) - X^\epsilon(t))\right]|, \tag{5.22}$$

respectively. We examine $err_{s\epsilon}$ and $err_{w\epsilon}$ as $\epsilon \to 0$. These errors are estimated as follows:

$$err_{s\epsilon} = \mathbb{E}\left[|X(T) - X^\epsilon(T)|^2\right] \approx \frac{1}{M}\sum_{k=1}^{M} |X(T,\omega_k) - X^\epsilon(T,\omega_k)|^2, \tag{5.23}$$

and

$$err_{w\epsilon} = |\mathbb{E}\left[X(T) - X^\epsilon(T)\right]| \approx \frac{1}{M}\left|\sum_{k=1}^{M} (X(T,\omega_k) - X^\epsilon(T,\omega_k))\right|, \tag{5.24}$$

where, ω_k is the k^{th} realization. We let $\alpha = 0.95$ and $M = 5000$ realizations. The estimates for (5.23) and (5.24) are given in Table 5.1.

Table 5.1: Numerical errors due to the jump-diffuison approximation.

ϵ	0.1	0.05	0.01	0.005
$\mathrm{err}_{s\epsilon}$	5.09×10^{-4}	1.53×10^{-4}	6.12×10^{-5}	1.55×10^{-9}
$\mathrm{err}_{w\epsilon}$	2.26×10^{-2}	1.24×10^{-2}	7.83×10^{-3}	3.94×10^{-5}
$\mathrm{Time}_{\mathrm{conv}}$/ secs	251.995	253.791	259.923	276.831

Rather the study the error $\mathbb{E}\left[|X^\epsilon(T) - Y(T)|^2\right]$ and $|\mathbb{E}\left[(X^\epsilon(T) - Y(T))\right]|$, we will instead estimate $\mathbb{E}\left[|X(T) - Y(T)|^2\right]$ and $|\mathbb{E}\left[(X(T) - Y(T))\right]|$. That is, for different step sizes Δ, we estimate the strong errors $\mathrm{err}_{\mathrm{s(eu)}}, \mathrm{err}_{\mathrm{s(jadp)}}$ and $\mathrm{err}_{\mathrm{s(jdp1)}}$ due to the numerical approximations, i.e., regular Euler, jump-adapted Euler and the strong–one order schemes. These errors are estimated as follows

$$\mathbb{E}\left[|X(T) - Y(T)|^2\right] \approx \frac{1}{M}\sum_{k=1}^{M} |X(T,\omega_k) - Y(T,\omega_k)|^2 . \tag{5.25}$$

Table 5.2: Strong error estimates.

Δ	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\mathrm{err}_{\mathrm{s(eu)}}$	6.0×10^{-5}	2.7×10^{-4}	3.2×10^{-4}	2.4×10^{-4}	1.2×10^{-6}
$\mathrm{t}_{\mathrm{eu}}/\mathrm{sec}$	35.42	70.15	134.28	257.52	634.63
$\mathrm{err}_{\mathrm{s(jdp)}}$	3.5×10^{-3}	1.9×10^{-3}	2.4×10^{-4}	1.3×10^{-4}	9.0×10^{-5}
$\mathrm{t}_{\mathrm{jdp}}/\mathrm{sec}$	37.28	69.00	129.478	252.33	566.57
$\mathrm{err}_{\mathrm{s(jdp1)}}$	6.0×10^{-3}	1.7×10^{-4}	7.4×10^{-4}	2.1×10^{-4}	5.9×10^{-8}
$\mathrm{t}_{\mathrm{jdp1}}/\mathrm{sec}$	32.11	65.24	128.85	299.33	531.07

We examine the numerical error due to the numerical approximation of the X^ϵ by Y using the the Euler, jump-adapted and and the order one strong jump-adapted scheme. The log base 2 plots are given in Figure 5.3.

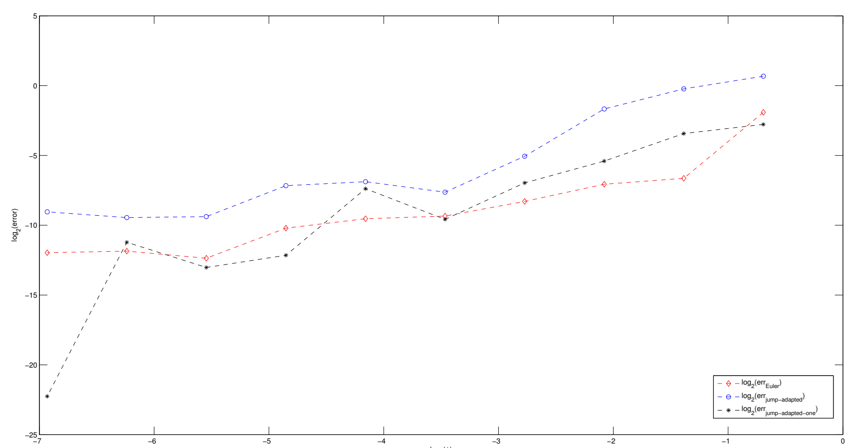

Figure 5.3: Log-log base 2 plot of strong error from the numerical approximation Y, of the jump diffusion X^ϵ versus the time step Δ.

We obtain the following polynomial fits on the errors from the three different schemes. For the Euler scheme, we obtain $y = -1.8 + 0.7x$, for the the jump-adapted scheme, we obtain $y = 0.5 + 0.8x$, and finally, for the order one strong jump-adapted scheme, we get, $y = -0.02 + 1.3x$. The slope of each of these lines, indicates the order of convergence of the given scheme.

Similarly, we denote by $\mathrm{err}_{\mathrm{weu}}$, $\mathrm{err}_{\mathrm{wjadp}}$ and $\mathrm{err}_{\mathrm{wjadp1}}$ the weak errors due to the regular Euler, jump-adapted schemes and weak order one jump–adapted approximations respectively, with the weak error estimate denoted by

$$err_w = |\mathbb{E}\left[X(T) - Y(T)\right]| \tag{5.26}$$

Here, we have taken g to be the identity function. In this case, we have the following estimates

Table 5.3: Weak error estimates

Δ	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\mathrm{err}_{\mathrm{w(eu)}}$	5.3×10^{-3}	9.4×10^{-3}	1.5×10^{-4}	8.6×10^{-6}	5.3×10^{-8}
$\mathrm{t}_{\mathrm{eu}}/\mathrm{sec}$	35.42	70.15	134.28	257.52	634.63
$\mathrm{err}_{\mathrm{w(jdp)}}$	5.9×10^{-2}	9.0×10^{-3}	1.5×10^{-3}	2.2×10^{-3}	1.1×10^{-3}
$\mathrm{t}_{\mathrm{jdp}}/\mathrm{sec}$	37.28	69.00	129.478	252.33	566.57
$\mathrm{err}_{\mathrm{w(jdp1)}}$	3.0×10^{-2}	2.0×10^{-4}	5.4×10^{-5}	1.8×10^{-5}	1.4×10^{-5}
$\mathrm{t}_{\mathrm{jdp1}}/\mathrm{sec}$	32.11	65.24	128.85	299.33	531.07

5.2 Conclusion and Further Directions

In this dissertation, I combine the ideas of Assmusen and Rosiński [3] (see also [31]) and in the spirit of the authors in [21], with the numerical schemes developed by Brutti and Platen in [7] to construct numerical approximations to the solution of a class of stochastic differential equations driven by a Lévy process with infinitely many jumps. My theoretical results are complemented by good error estimates. This extends the work of authors in [7] and [8], to a larger of class of stochastic differential equations, i.e., stochastic differential equations driven by Lévy processes with infinitely many jumps.

Most of the numerical schemes developed so far, including the method in this dissertation, assume that the driving process Z has at least the second moment. This assumption excludes the class of SDEs driven by α-stable processes and other classes with heavy-tailed distributions. I am interested in extending the methods in my dissertation to include such SDEs. It is also worth mentioning that, it will be interesting to relax some of the smoothness conditions on the coefficients of the SDE considered in this dissertation, and still give upper bounds for the error estimates. I believe can be accomplished via the powerful tool of Malliavin calculus.

Bibliography

[1] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, 2004. 8, 14, 28, 66, 67, 83

[2] S. Asmussen and P. W. Glynn. *Stochastic Simulation: Algorithms and Analysis*. Springer, 2007. 17, 50

[3] S. Asmussen and J. Rosiński. Approximation of small jumps of Lévy processes with view towards simulation. *J. Appl. Probab.*, 38:482–483, 2001. 5, 6, 53, 55, 93

[4] O. E. Barndoff-Nielsen, T. Mikosch, and S. I. Resnik. *Lévy Processes: Theory and Application*. Birkhäuser, 2001. 8

[5] J. Bertoin. *Lévy Processes*. Cambridge University Press, 1996. 8

[6] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*. Cambridge University Press, 1989. 102

[7] N. Bruti-Liberati and E. Platen. Strong approximations of stochastic differential equations with jumps. *J. Comput. Appl. Math*, 205:982–1001, 2007. 6, 93, 94

[8] E. Buckwar and M. G. Riedler. Runge-Kutta-methods for jump-diffusion deferential equations. *J. of Comp. and Appl. Math*, 236:1155–1182, 2011. 5, 94

[9] S. Cohen and J. Rosiński. Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes. *Bernoulli*, 13:195–210, 2007. 5, 6, 50

[10] R. Cont and P. Tankov. *Financial Modeling With Jump Processes*. Chapman & Hall/CRC, 2004. 1, 17, 18, 19, 28

[11] S. Dereich. Multilevel Monte Carlo Algorithms for Lévy-Driven SDEs with Gaussian Correction. *The Annals of Applied Probability*, 21:283–311, 2011. 5

[12] K. Dzhaparidze and E. Valkeila. On the Hellinger Type Distance for Filtered Experiments. *Probab. Th. Rel Fields*, 85:105–117, 1990. 59

[13] W. Feller. Diffusion processes in genetics. *In J. Neyman, editor, Proceedings of Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 227–246, 1951. 1

[14] N. Fournier. Simulation and Approximation of Lévy-Driven Stochastic Differential Equations. *ESAIM P&S*, 15:249–269, 2011. 5

[15] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North Holland/Kodnsha, 1989. 26

[16] J. Jacod, T.G. Hurtz, S. Méléard, and P. Protter. The approximate Euler method for Lévy driven stochastic differential equations. *Séminaire de Probabilité XXV Poincaré.*, 1485:138–139, 1991. 4

[17] J. Jacod and A. N. Shiryaev. *Limit Theorems For Stochastic Processes*. Springer, 2003. 55

[18] A. Janicki and A. Weron. *Simulations And Chaotic Behavior Of α-Stable Stochastic Processes*. Dekker, 1994. 16

[19] H. Janssen, K. Oerding, F. van Wijland, and H. Hilhorst. Lévy-flight spreading of epidemic processes leading to percolation cluster. *Eur. Phy. J. B.*, 7:137–145, 1999. 1

[20] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag, 1992. 4, 52

[21] A. Kohatsu-Higa and P. Tankov. Jump-adapted discretization schemes for Lévy-driven SDEs. *Stoch. Proc. and Appl.*, 120:2258–2285, 2010. 5, 6, 50, 51, 93

[22] H. Kunita. Itô stochastic calculus: Its surprising power for applications. *Stochastic process and Appl.*, 120:622–652, 2010. 67

[23] C. Marinelli and M. Rockner. On maximal inequalities for purely discontinuous martingales in infinite dimensions. *arXiv:1308.2648v1 [math PR]*, 2013. 59

[24] G. N. Milstein and M. V. Tretyakov. *Stochastic Numerics for Mathematical Physics*. Springer, 2004. 29, 52

[25] E. Mordecki, A. Szepessy, R. Tempone, and G. E. Zouraris. Adaptive weak approximation of diffusions with jumps. *SIAM J. Numer. Annal.*, 46:1732–1768, 2008. 5

[26] E. Platen and N. Bruti-Liberati. *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*. Springer-Verlag, 2010. 4, 6, 29, 36, 42, 43, 44, 46, 48, 50, 51, 90

[27] P. Protter and D. Talay. The Euler scheme for Lévy driven stochastic differential equations. *Ann. Probab*, 25:393–423, 1997. 4

[28] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer, 2004. 101, 102

[29] M. M. Rao. *Real and Stochastic Analysis: New Perspective*. Birkhäuser, 2004. 58, 68

[30] A. M. Reynolds. Free-flight odor tracing in drosophila is consistent with an optimal intermittent scale free search. *PLos One.*, 2(4:e354):137–145, 2007. 1

[31] J. Rosiński. Series representations of Lévy processes from the perspective of point processes. *In Lévy processes: Theory and Applications, O.E. Barndorff–Nielsen and T. Mikosch and S. I. Resnick (eds)*, pages 401–415, 2001. 20, 21, 22, 53, 55, 93

[32] J. Rosiński. Tempering stable processes. *Stoc. Proc. and Apl.*, 117:677–707, 2006. 16

[33] J. Rosiński. Simulation of Lévy processes. *Encyclopedia of Stat. in Quality and Reliability*, 2007. 22

[34] S. Rubenthaler. Numerical simulation of the solution of a stochastic differential equation driven by a Lévy process. *Stochastic process. Appl.*, 103:311–349, 2003. 4, 5

[35] G. Samorodnitsky and M. S. Taqqu. *Satble Non-Gaussian Random Processes*. Chapman & Hall, 1994. 16

[36] Ken-Iti Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 2002. 8, 10, 11, 14

[37] P. Tankov. High order weak approximation schemes for Lévy-driven SDEs. *arXiv*, 2010. 5

[38] W. A. Woyczyński. Lévy processes in the physical sciences. *In Lévy processes: Theory and Applications, O.E. Barndorff–Nielsen and T. Mikosch and S. I. Resnick (eds)*, pages 241–266. Birkhauser, 2001. 1

[39] C. Zhang. Numerical Weak Approximation of SDE driven by Lévy Processes. *PhD. Dissertation, University of Southern California*, 2010. 5

Appendices

Appendix A

Some Useful Results

A.1 Gronwall's Inequality

Theorem A.1. *Let α be function from $\mathbb{R}_+$ to itself, and suppose that*

$$\alpha(s) \leq c + k \int_0^s \alpha(r)dr \tag{A.1}$$

for $0 \leq s \leq t$. Then

$$\alpha(t) \leq ce^{kt}. \tag{A.2}$$

Moreover, if $c = 0$ then α vanishes identically.

Proof. See e.g., [28], Theorem 68, pp. 349. □

A.2 Burkholder's Inequality

For a given stochastic process $\{X(t), t \geq 0\}$, denote by $X^{\star}(t)$ its supremum process, i.e.,

$$X_t^{\star} = \sup_{0 \leq s \leq t} |X(s)|. \tag{A.3}$$

Theorem A.2. *Let M be a local martingale. Then there exists constants c_p, C_p such that for a finite stopping time T*

$$\mathbb{E}\left[(M_T^{\star})^p\right]^{1/p} \leq c_p \mathbb{E}\left[[M,M]_T^{p/2}\right]^{1/p} \leq C_p \mathbb{E}\left[(M_T^{\star})^p\right]^{1/p} \tag{A.4}$$

for $1 \leq p < \infty$.

Proof. See e.g. [28], Theorem 74, pp. 226. □

A.3 Approximation of Integral Involving Slowly Varying Functions

Theorem A.3. *(Bingham et al., [6], Proposition 1.5.8., pp 26) If L is slowly varying and a is so large that L(z) is locally bounded in $[a, \infty)$, and $\alpha > -1$, then*

$$\int_a^z t^{\alpha} L(t)dt \sim \frac{z^{\alpha+1}L(z)}{\alpha+1}. \tag{A.5}$$

Appendix B

MATLAB Codes

B.1 MATLAB Code for Generating the Graphs in Chapter 2

```
              %  Author Ernest Jum
               % Department of Mathematics
               % University of Tennessee
               % Knoxville
               % Simulating alpha-stable and
               % tempered alpha-stable processes
               % August 2013
to=0; tf=1; m=1000;
dt=(tf-to)/(m-1);
t=to:dt:tf;
alpha=1.3; %index of stability
epsilon=0.001; %truncation level (precision)
lambda=1; %tempering parameter
kappa=1; %constant
tau=1/epsilon^(alpha); %truncation level
```

```
k=0;
T(1)=-log(rand);
while sum(T(1:k)) < tau
    k=k+1;
    T(k)=-log(rand);
    U(k)=rand;
    s=rand;
    if s<0.5
        V(k)=1;
    else
        V(k)=-1;
    end
        eta(k)=-log(rand)/lambda;
        xi(k)=rand;
        eta_xi(k)=eta(k)*((xi(k))^(1/alpha));
end
%% Jump sizes process %%%
const=(alpha/(2*kappa*tf))^(-1/alpha);
%%
Z(1)=0; Z1(1)=0;
for n=2:length(t)
    for i=1:length(T)
    gamma(i)=sum(T(1:i));
        if U(i) <=t(n)
         J(i)=V(i)*(gamma(i)^(-1/alpha));
         J1(i)=V(i)*(min(const*((gamma(i))^(-1/alpha)),eta_xi(i)));
        else
         J(i)=0;
         J1(i)=0;
         end
    end
    Z(n)=sum(J(1:length(T)));
    Z1(n)=sum(J1(1:length(T)));
end
```

```
figure(1)
plot(t, Z,'k', t, Z1)
hleg1 = legend('\alpha--stable','tempered-\alpha-stable');
xlabel('time')
ylabel('smaple path')
```

B.2 Code for the Numerical Experiments in Chapter 5

B.2.1 Function for Counting Number of Jumps on an Interval

```
function num_jumps = find_num_jumps(Uvec, tval)
    l = length(Uvec);
    num_jumps = 0;
    for i=1:l
        if Uvec(i) <= tval
            num_jumps = num_jumps+1;
        end
        if Uvec(i) > tval
            break;
        end
    end
end
```

B.2.2 Main Code

```
tic
clc
close all
clear all
format long
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
            % Developed and written by
            % Ernest Jum
            % Department of Mathematics
            % The University of Tennessee Knoxville
            % January 2015
%%  Initial time and Final time %%%%%
k=3;
%m=2^(k);
m=10^(k);
t0=0;tf=1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                    %% Regular time grid %%%%
Dt=(tf-t0)/m;%time step
t=0:Dt:tf;
kappa=1; lambda=1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                    %% Coefficient Function
h=@(x)x;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                    %% Index of stability and epsilon %%%%
alpha=1.3;
epsilon=0.009;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                %% Variance of small jumps %%%%
sigmaeps=(((2*kappa)/(2-alpha))^(1/2))*(epsilon^(1-(alpha/2)));
```

```
ra=1/alpha;
                %% Precision %%%%%
tau=alpha/(2*kappa*tf*(epsilon^(alpha)));%precision
%% number of iterations
N=10000;
%% Initialization
X=zeros(length(t), N);
Z=zeros(length(t), N);
W=zeros(length(t), N);
Xeps=zeros(length(t), N);
Y_eu=zeros(length(t), N);
%%  Simulation of Jumps
S=-log(rand);
k=0;
while S < tau
    k=k+1;
        T(k)=-log(rand);
        gam(k)=S; %store the Poisson arrival times
        S=S +T(k);
        U(k)=rand;
        jump_times(k)=tf*U(k);
        z=rand; %%%%% simulate random signs %%%%%%
        if z <=0.5
        e(k)=1;
        else
        e(k)=-1;
        end
        eta(k)=-log(rand)/lambda;
        xi(k)=rand;
        eta_xi(k)=eta(k)*(xi(k))^(1/alpha);
end
     %% Jump sizes of tempered stable process %%%
g_alpha=gam.^(-ra);
const=(alpha/(2*kappa*tf))^(-ra);
```

```
new_g_alpha=const*g_alpha;
eta_xi1=eta_xi;
jump_size=e.*min(new_g_alpha, eta_xi);
%%%%%% Dolean-Dades Exponential %%%%%%%%
 %% Simulating the almost exact solution %%
U1=sort(jump_times);
t1=sort([U1,t]);
m1=length(t1);
%BM increments for jump-adapted approx
dW1=zeros(length(t1), N);
% BM incrments for Euler
dW=zeros(length(t), N);
 % Initialize jump-adapted approx
Y_jadp=zeros(length(t1), N);
% Initialize jump-adapted stong-order-1 approx
Y_jadp_one=zeros(length(t1), N);

for n=1:N
dW(1,n)=normrnd(0, t(1));
dW1(1,n)=normrnd(0, t(1));
    for k=2:m
    dW(k, n)=normrnd(0, Dt);
    W(k,n)=sum(dW(1:k, n));
    end
    for k=2:m1
    dW1(k, n)=normrnd(0,t1(k)-t1(k-1));
    W1(k,n)=sum(dW1(1:k, n));
    end
Z(1,n)=0;
X(1,n)=exp(Z(1,n));
Xeps(1,n)=X(1,n);
%%Exact Solution
jp=zeros(length(t), 1);
jmp=zeros(length(t), 1);
```

```
Z(1,n)=0; X(1,n)=1; Xeps(1,n)=1;
for j=2:length(t)
    r=find_num_jumps(U1, t(j));
    if r~=0
         jmp(j)=prod((1+jump_size(1:r)));
         jp(j)=prod((1+jump_size(1:r)))*prod(exp(-jump_size(1:r)));
         Z(j, n)=sum(jump_size(1:r));
         X(j,n)=exp(Z(j,n))*jp(j);
         Xeps(j, n)=exp(-((sigmaeps^(2))/2)*t(j)+sigmaeps*W(j,n))*jmp(j);
    else
         Z(j, n)=sum(jump_size(1:r-1));
         X(j, n)=exp(Z(j,n));
        Xeps(j, n)=exp(-((sigmaeps^(2))/2)*t(j)+sigmaeps*W(j,n));
    end
end
%% Euler with constant step size
A=[];B=[];
Y_eu(1, n)=1;
number_jumps = length(U1);
p=number_jumps;
js=zeros(length(t), 1);
 for j=1:length(t)-1
%     fprintf('iteration: %d\n', j);
        l = find_num_jumps(U1, t(j));
        s = find_num_jumps(U1, t(j+1));
 if l == 0 && s == 0
        Y_eu(j+1, n)=h(Y_eu(j, n))*(1 + (sigmaeps* dW(j+1,n)));
 elseif l+1 > number_jumps
        js(j)=0;
        Y_eu(j+1, n)=h(Y_eu(j, n))*(1 + (sigmaeps* dW(j+1,n))+js(j));
 else
        js(j)=sum(jump_size(l+1:s));
        Y_eu(j+1, n)=h(Y_eu(j, n))*(1 + (sigmaeps* dW(j+1,n))+js(j));
 end
```

```
 end
%% Jump-adapted Euler
 jump_flag = 0;
 Y_jadp(1,n)=1;
 for k=2:m1
     for j=1:length(U)
         if t1(k)==U1(j)
             num_jumps = find_num_jumps(U1,t1(k));
             Y_jadp(k,n)=Y_jadp(k-1, n)*(1 +sigmaeps*dW1(k, n))*...
                                 (1+jump_size(num_jumps));
             jump_flag = 1;
            break
         end
     end
     if jump_flag == 0
         Y_jadp(k,n)=Y_jadp(k-1, n)*(1 +sigmaeps*dW1(k, n));
     end
    jump_flag = 0;
 end
 %% jump-adapted-strong-order-one
 jump_flag1 = 0;
 Y_jadp_one(1,n)=1;
 for k=2:m1
     for j=1:length(U)
         if t1(k)==U1(j)
           num_jumps = find_num_jumps(U1,t1(k));
           Y_jadp_one(k,n)=Y_jadp_one(k-1, n)*(1 +sigmaeps*dW1(k, n)...
               +(sigmaeps^(2)/2)*((dW1(k, n)^(2))-(t1(k)-t1(k-1))))...
                                       *(1+jump_size(num_jumps));
             jump_flag1 = 1;
            break
         end
     end
     if jump_flag1 == 0
```

```
        Y_jadp_one(k,n)=Y_jadp_one(k-1, n)*(1 +sigmaeps*dW1(k, n)...
                +(sigmaeps^(2)/2)*((dW1(k, n)^(2))-(t1(k)-t1(k-1))));
     end
    jump_flag1 = 0;
 end
end
%% Error due to jump diffusion approx of orginal SDE
 err_s_epsilon=mean((X(length(t),:)-Xeps(length(t),:)).^(2));
err_w_epsilon=abs(mean(X(length(t),:)-Xeps(length(t),:)))
err_eps=[err_s_epsilon, err_w_epsilon]
 %% strong errors
glob_s_err_eu= mean((X(length(t),:)-Y_eu(length(t),:)).^(2));
glob_s_err_eu1= mean((Xeps(length(t),:)-Y_eu(length(t),:)).^(2));

glob_s_err_jadp= mean((X(length(t),:)-Y_jadp(length(t),:)).^(2));
glob_s_err_jadp1= mean((Xeps(length(t),:)-...
                                     Y_jadp(length(t),:)).^(2));

glob_s_err_jadp11= mean((X(length(t),:)-Y_jadp_one(length(t),:)).^(2));
glob_s_err_jadp111= mean((Xeps(length(t),:)-...
                                    Y_jadp_one(length(t),:)).^(2));
 % weak errors
glob_w_err_jadp1=abs(mean(X(length(t),:)-Y_jadp_one(length(t),:)));
glob_w_err_jadp11=abs(mean(Xeps(length(t),:) ...
                           -Y_jadp_one(length(t),:)));
glob_w_err_jadp2=abs(mean(X(length(t),:)-Y_jadp(length(t),:)))
glob_w_err_jadp22=abs(mean(Xeps(length(t),:)-Y_jadp(length(t),:)))
glob_w_err_eu=abs(mean(X(length(t),:)-Y_eu(length(t),:)));
glob_w_err_eu1=abs(mean(Xeps(length(t),:)-Y_eu(length(t),:)));
                %% Plots of sample paths %%
figure(1)
plot( t, Y_eu(:,N), 'k', t1, Y_jadp(:, N),'m', t1, Y_jadp_one(:, N), ...
                '--g', t, X(:, N), 'b', t, Xeps(:,N),':r');
hleg1 = legend('Euler aprrox','Euler-jump-adapted-approx',...
```

```
                                'order—one—jump—adapted—approx',...
                        '``exact solution"','jump—diffusion approx');
xlabel('time')
ylabel('sample path')
toc
```

Vita

Ernest Jum was born in Akwaya, South West Region, of Cameroon on May 18 1978. After graduating from the Cameroon College of Arts, Science and Technology (CCAST) Bambili in 1997, he attended the University of Buea, Cameroon from 1999 to 2002. He graduated with a Bachelor of Science in Mathematics, and a minor in Computer Science. In 2005, he received received a Master of Science in Mathematics from the University of Buea. From 2005 to 2007, Ernest was an Instructor at the University of Buea where he taught Statistics and Computer Science.

In August 2007, Ernest joined the graduate program in mathematics at East Tennessee State University where he obtained another Master of Science in Mathematics in 2009. He then joined the Department of Mathematics at the University of Tennessee, Knoxville in 2009 and graduated with his Doctorate of Philosophy in Mathematics and a Masters of Science in Statistics in August 2015.