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Abstract

We summarise Runge-Kutta type methods for the solution of ordinary differential equations in models of economic dynamics. In this work we are going to present explicit Runge-Kutta type methods, a family of methods to solve numerically systems of ordinary differential equations, without the need to evaluate high-order derivatives.

We apply this numerical approach to solve a dynamic, general equilibrium growth model of North-South technological-knowledge diffusion by imitation.

Keywords: North-South; Technological Knowledge Diffusion; Convergence; Numerical Computations.
JEL Classification: C63, O31, O33, O47.

1 Introduction

A wide variety of economic problems is modelled through differential equations. Since most differential equations are not soluble analytically, numerical solution of ordinary differential equations is a fundamental technique in continuous time dynamics to obtain information about the model's behaviour.

A differential equation is an equation involving an unknown function, y, and its derivatives. A first order *ordinary differential equation* (ODE) has the form

$$y'(t) = \frac{dy}{dt}(t) = f(t, y(t)) \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and $y(t) : [t_0, t_T] \subset \mathbb{R} \to \mathbb{R}^n$.[1] For $m = 1$ we have a single equation and for $m \geq 2$ we have a system of equations. Using matrix notation we may write

$$y(t) = \begin{bmatrix} y_1(t) \\ y_1(t) \\ \vdots \\ y_{nm}(t) \end{bmatrix}, \quad f(t, y(t)) = \begin{bmatrix} f_1(t, y(t)) \\ f_2(t, y(t)) \\ \vdots \\ f_m(t, y(t)) \end{bmatrix}.$$

Thus, basically, we know the derivative of an unknown function $y(t) \in \mathbb{R}^m$ as a (nonlinear) function f of t (the independent variable) and $y(t)$ (the dependent variable). From the resolution of (1) we obtain a family of solutions determined by a constant. A particular solution is computed by requiring that it goes through a specific point, say (t_0, y_0). In other words,

$$y(t_0) = y_0, \tag{2}$$

$y_0 = (y_{0,1}, y_{0,2}, \cdots, y_{0,m})$. The initial condition can also be a terminal condition if it is imposed at the end of the interval. In practice a time span $[t_0, t_T]$, the interval of time over which the problem is to be solved, is also given. The problem specified by (1) and (2) is called an *initial value problem* (IVP), and usually $t_0 = 0$, $t_T = T$. Solving the IVP is to predict the path that a quantity will take during a certain time interval, given an initial quantity.

Problems involving ODEs of higher order can be reduced to a system of first order ordinary differential equations by introducing new variables. We will not deal with other kind of problems like *boundary value problems* (BVP). The system (1) is called non-autonomous but often economic problems are time-autonomous, that is of the form

[1] The complex space can also be considered.

$$y'(t) = \frac{dy}{dt}(t) = f(y(t)). \tag{3}$$

Sufficient conditions are known under which an IVP has a unique solution. For a unique continuous, differentiable function $y(t)$ to exist, f must be continuous in t and Lipschitz[2] continuous in y ($\|f(t,y) - f(t,x)\| \leq L \|y - x\|$, where $L > 0$ is the Lipschitz constant).

Unfortunately it is seldom that these equations have solutions which can be expressed in closed form. And if they can, the analytical form is often too cumbersome and the solution techniques are generally unable to deal with large and nonlinear systems of equations that arise in real problems. Numerical methods can often produce a solution to any degree of accuracy that the computer can represent.

Our aim is to present (explicit) Runge-Kutta methods for the numerical solution of IVP issued from economic problems. In particular, here we apply this methodology to solve a dynamic, general equilibrium growth model of North-South technological-knowledge diffusion by imitation, drawing heavily on Afonso ([1], ch. 3) . In this model, it is assumed that the North has higher (i) exogenous productivity related with the quality of institutions, (ii) initial level of human capital, (iii) aggregate domestic quality index measuring technological-knowledge. This latter feature occurs because the North innovates – Schumpeterian R&D, aimed at improving the quality of intermediate goods, as in Aghion and Howitt [2] –, whereas the South imitates Northern innovations and imitation is a vehicle for international technological-knowledge transfer, as in Grossman and Helpman ([7], ch. 11-12) and Barro and Sala-i-Martin [4], among others.

2 How to solve IVPs in a computer?

To solve a continuous problem in a computer we need first to discretize it. Initial value problems can be numerically solved using finite difference methods and recursive procedures.

The numerical procedures to be developed are based on approximations $y_0, y_1, \cdots, y_T$ to the exact solution $y(t_0), y(t_1), \cdots, y(t_T)$ at the grid points: $a = t_0 < t_1 < \cdots < t_T = b$. We call step sizes to the distances $h_n = t_n - t_{n-1}$, $n = 1, \cdots, T$ and we are going to consider equal step sizes, that is, uniform grids, where $h = (b - a)/T$. The aim is, starting with the initial value

[2]Rudolph Otto Sigismund Lipschitz (May 14, 1832 - October 7, 1903) was a German mathematician.

$y_0 = y(t_0)$, find y_n, which approximates $y(t_n)$, by recurrence relations in such a way that the value of y_{n+1} could be stated as a function of y_n, using relations of the form

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h). \tag{4}$$

This numerical approach gives rise to the class of *one-step* (or *self-starting*) and *explicit* methods, which use only data gathered in the current step.

Out of the scope of this work are the *multistep* methods where the value of y_{n+1} is stated as a function of y_k, $k = n - r + 1, \cdots, i$ (r-step method) and the *implicit* methods where y_{n+1} depends implicitly from itself through f: for the one-step case

$$y_{n+1} = y_n + h\Phi(t_n, t_{n+1}, y_n, y_{n+1}; h). \tag{5}$$

A method is said to be *convergent* if

$$\lim_{\substack{h\to 0 \\ nh=t-a}} y_n = y(t_n) \tag{6}$$

and is said to be *consistent* if

$$\Phi(t_n, y_n; 0) = f(t_n, y_n). \tag{7}$$

2.1 Taylor series based methods

A first approach to solve equation (1) can be obtained by assuming that f and its derivatives are well-defined over the interval of interest so that we can use the truncated Taylor[3] series expansion for $y(t)$ in t_0,

$$y(t) = y(t_0) + \sum_{k=1}^{p} \frac{1}{k!}(t - t_0)^k y^{(k)}(t_0) + O(h^{p+1}).$$

The derivatives are not known since we don't know the solution function. Using the notation $f^{(j)} = \frac{d^j f}{dt^j}$ and $t_n = t_0 + nh$, $n \in \mathbb{Z}^+$, we can approximate $y(t_1)$ by $y_1, y(t_1) \approx y_1 = y(t_0) + \sum_{k=1}^{p} \frac{1}{k!} h^k f^{(k-1)}(t_0, y_0) + O(h^{p+1})$. We now know the approximate value of y at time t_1 and we can obtain similarly $y_2 \approx y(t_2)$. Now we start over and by this way we can obtain for each p a method (explicit one-step) to approximate the solution $y_n \approx y(t_n)$ of the form (4)

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h) \text{ with } \Phi(t_n, y_n; h) = \sum_{k=1}^{p} \frac{1}{k!} h^k f^{(k-1)}(t_n, y_n). \tag{8}$$

[3]Brook Taylor (August 18, 1685 – December 29, 1731) was an English mathematician.

These are called *Taylor methods*. The simplest Taylor type method is obtained for $p = 1$, $\Phi(t_n, y_n; h) = f(t_n, y_n)$, and it is know as the *explicit Euler*[4] method

$$y_{n+1} = y_n + hf(t_n, y_n); \tag{9}$$

the Euler method is therefor a method of order 2 ($O(h^2)$).

To improve the previous method we can truncate the Taylor series after the first order. This will keep the errors small but has the disadvantage of requiring the evaluation of higher derivatives. The Runge-Kutta approach is to keep the desirable feature of Taylor method but with the replacement of the high order derivatives evaluation requirement with the evaluation of f at some points within the integration step from each t_n to t_{n+1}.

2.2 Runge-Kutta methods

The family of *Runge*[5]*-Kutta*[6] (RK) methods, that we will present next, is then designed to approximate Taylor series but with the advantage of not requiring explicit evaluations of the derivatives of f. The idea is to consider time steps and to assume that the exact value of the slope of the step can be written as a linear combination of the function evaluated at certain points in the step.

Runge-Kutta methods compute approximations y_n with initial values $y_0 = y(t_0)$ using the Taylor series expansion (8). Runge in 1875, based on the knowledge of $y(t_n)$, took $y(t_n + h) \approx y(t_n) + hf\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right)$ and computed $y\left(t_n + \frac{h}{2}\right)$ using the Euler method with step $\frac{h}{2}$:

$$\begin{aligned} y_{n+1} &= y_n + hk_2 \\ k_1 &= f(t_n; y_n) \\ k_2 &= f(t_n + \frac{h}{2}; y_n + \frac{h}{2}k_1). \end{aligned}$$

This method does not need to evaluate the derivatives of f and it is more accurate than the Euler method. The explicit s-stage Runge-Kutta methods are a generalization of this idea.

[4]Leonhard Euler (April 15, 1707 - September 18, 1783) was a Swiss mathematician and physicist.

[5]Carle David Tolmé Runge (August 30, 1856 – January 3, 1927) was a German mathematician, physicist, and spectroscopist.

[6]Martin Wilhelm Kutta (November 3, 1867 - December 25, 1944) was a German mathematician.

3 Explicit s-stage Runge-Kutta methods

A *Runge-Kutta s-stage* (RKs) method is obtained by doing s function evaluations per step, giving rise to

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h), \quad \Phi(t_n, y_n; h) = \sum_{i=1}^{s} w_i k_i, \tag{10}$$

where

$$k_i = f\left(t_n + hc_i, y_n + h\sum_{j=1}^{i-1} a_{i,j}k_j\right), \quad c_1 = 0, \tag{11}$$

for an *explicit method* and

$$k_i = f\left(t_n + hc_i, y_n + h\sum_{j=1}^{s} a_{i,j}k_j\right) \tag{12}$$

for an implicit one. A usual way to present such methods is to use the *Butcher*[7] *tableau*

$$\begin{array}{c|c} c & A \\ \hline & w \end{array}$$

where $w = [w_1, w_2, \cdots, w_s]$, $c = [c_1, c_2, \cdots, c_s]^T$ and $A = [a_{i,j}]$, $i = 2, \cdots, s$, $j = 1, \cdots, s-1$. The components of the vector w are the weights in the combination of the intermediate values k_i, the components of vector c are the increments of t_n and the entries of the matrix A are the multipliers of the approximate slopes. Explicit methods can be viewed as a subset of implicit methods with $a_{i,j} = 0$, $j \geq i$ and $c_1 = 0$. For an *implicit* RKs method the j index goes from 1 to s and the Butcher tableau is represented for $A = [a_{i,j}]$, $i, j = 1, \cdots, s$.

For a method of order p we wish to find values for Butcher's tableau such that equation (10) matches the first $p+1$ terms in equation (8).

3.1 Explicit 2-stage and 2^{nd} order Runge-Kutta methods

So lets consider that we want to build a 2-stage and 2^{nd} order method, $s = p = 2$.

[7] J. C. Butcher is Honorary Research Professor at the Department of Mathematics of the University of Auckland, New Zealand.

From equation (10) we have

$$\begin{aligned} y_{n+1} &= y_n + hw_1k_1 + hw_2k_2 \\ k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + hc_2, y_n + ha_{2,1}k_1) \end{aligned} \tag{13}$$

or using the Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ c_2 & a_{2,1} & \\ \hline & w_1 & w_2 \end{array}.$$

But

$$\begin{aligned} k_2 &= f(t_n, y_n + ha_{2,1}k_1) + hc_2\frac{\partial f}{\partial x}(t_n, y_n + ha_{2,1}k_1) + O(h^2) \\ &= f(t_n, y_n) + ha_{2,1}k_1\frac{\partial}{\partial y}f(t_n, y_n) + \\ &\quad + hc_2\frac{\partial}{\partial x}f(t_n, y_n + ha_{2,1}k_1) + O(h^2) \\ &= f(t_n, y_n) + ha_{2,1}k_1\frac{\partial f}{\partial y}(t_n, y_n) + \\ &\quad + hc_2\frac{\partial}{\partial x}\left[f(t_n, y_n) + ha_{2,1}k_1\frac{\partial f}{\partial y}(t_n, y_n)\right] + O(h^2) \\ &= f(t_n, y_n) + hc_2\frac{\partial f}{\partial x}(t_n, y_n) + ha_{2,1}k_1\frac{\partial f}{\partial y}(t_n, y_n) + O(h^2); \end{aligned}$$

So equation (13) becomes

$$\begin{aligned} y_{n+1} &= y_n + hw_1 f(t_n, y_n) + hw_2. \\ &\quad .\left(f(t_n, y_n) + hc_2\frac{\partial f}{\partial x}(t_n, y_n) + ha_{2,1}\frac{\partial f}{\partial y}(t_n, y_n).f(t_n, y_n)\right) + \\ &\quad + O(h^3) \\ &= y_n + h(w_1 + w_2).f(t_n, y_n) + h^2. \\ &\quad .\left(w_2c_2\frac{\partial f}{\partial x}(t_n, y_n) + w_2a_{2,1}\frac{\partial f}{\partial y}(t_n, y_n).f(t_n, y_n)\right) + O(h^3). \end{aligned} \tag{14}$$

From equation (8) we shall write

$$\begin{aligned} y_{n+1} &= y_n + hf(t_n, y_n) + \frac{1}{2!}h^2 f'(t_n, y_n) + O(h^3) \\ &= y_n + hf(t_n, y_n) + h^2. \\ &\quad .\left(\frac{1}{2}\frac{\partial f}{\partial t}(t_n, y_n) + \frac{1}{2}\frac{\partial f}{\partial y}(t_n, y_n).f(t_n, y_n)\right) + O(h^3). \end{aligned} \tag{15}$$

To achieve a 2-stage 2^{nd} order method we must take into account equations (14) and (15) :

$$\begin{cases} w_1 + w_2 = 1 \\ w_2 c_2 = \frac{1}{2} \\ w_2 a_{2,1} = \frac{1}{2} \end{cases} \iff \begin{cases} w_1 = 1 - \frac{1}{2c_2} \\ w_2 = \frac{1}{2c_2} \\ a_{2,1} = c_2 \end{cases} .$$

Several solutions can be obtained since this is an underdeterminated linear system of equations. Well-know methods can be obtained:

- *modified Euler method*
  - for $c_2 = \frac{1}{2} : w_1 = 0, w_2 = 1, a_{2,1} = \frac{1}{2}$

$$y_{n+1} = y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right],$$

that is $\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$;

- *improved Euler method*
  - for $c_2 = 1 : w_1 = \frac{1}{2}, w_2 = \frac{1}{2}, a_{2,1} = 1$

$$y_{n+1} = y_n + \frac{1}{2}h\left[f\left(t_n, y_n\right) + f\left(t_n + h, y_n + hf\left(t_n, y_n\right)\right)\right],$$

that is $\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$.

3.2 Explicit 4-stage and 4^{th} order Runge-Kutta methods

We can proceed similarly to develop higher order methods, but this requires hard and complicated work. One of the most frequently used methods of the

Runge-Kutta family is the (classical) 4^{th} order method:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\ k_1 &= f\left(t_n, y_n\right) \\ k_2 &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \\ k_3 &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right) \\ k_4 &= f\left(t_n + h, y_n + hk_3\right). \end{aligned} \tag{16}$$

The values of k_2 and k_3 represent approximations to the derivative y' at points intermediate between $(t_n, y(t_n))$ and $(t_{n+1}, y(t_{n+1}))$; the value of $\Phi(t_n, y_n; h)$ is the weighted average of the k_i, $i = 1, \cdots, 4$. Using Butcher tableau we have

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array} .$$

The slope used in this case is the weighted mean of the slope of these 4 points, being the 2 midpoint values those which contribute the most for the slope.

3.3 Accuracy of Runge-Kutta methods

Taking into consideration equations (7) and (10) we obtain

$$\sum_{i=1}^{s} w_i = 1$$

as a condition for RK methods to be consistent. It is easy to verify that all the RK versions derived above fulfill this requisite. All RK methods are convergent since consistency is a necessary and sufficient for convergence [8].

When solving ODE's in a computer using numerical algorithms we must take care of *roundoff* and *truncation errors*. The first are due to finite precision arithmetic and they increase proportionately to the total number of integration steps used, while the later are due to the truncation of the Taylor series. From the first, we shall point out that the use of very small step length can be risky.

We should also distinguish between *local* and *global (truncation) errors*. The local truncation error, T_{n+1}, is introduced in each step,

$$T_{n+1} = \|y(t_{n+1}) - y(t_n) - h\Phi(t_n, y(t_n); h)\| \tag{17}$$

and the global truncation error, e_{n+1}, corresponds to the overall error resulting from all integration steps

$$e_{n+1} = \|y(t_{n+1}) - y_{n+1}\| . \tag{18}$$

A $p-$order method has local error of $O(h^{p+1})$ and global error of $O(h^p)$. To prove the last proposition, we can write from (18)

$$\begin{aligned}
e_{n+1} &= \|y(t_{n+1}) - y_{n+1}\| \\
&= \|y(t_{n+1}) - y_n - h\Phi(t_n, y_n; h)\| \\
&\leq \|[y(t_{n+1}) - y(t_n) - h\Phi(t_n, y(t_n); h)] + \\
&\quad + [y(t_n) - y_n] + [h\Phi(t_n, y(t_n); h) - h\Phi(t_n, y_n; h)]\| \\
&\leq T_{n+1} + e_n + hLe_n = T_{n+1} + (1 + hL)\, e_n \\
&= T_{n+1} + (1 + hL)\, [T_n + (1 + hL)\, e_{n-1}] \\
&= T_{n+1}\, (1 + (1 + hL)) + (1 + hL)\, e_{n-1} \\
&\leq \ldots \\
&\leq T_{n+1} \sum_{i=0}^{n} (1 + hL)^i + (1 + hL)^{n+1}\, e_0 \\
&= \frac{T_{n+1}}{h} \frac{(1 + hL)^{n+1} - 1}{L};
\end{aligned}$$

as $T_{n+1} = O(h^{p+1})$ it follows that $e_{n+1} = O(h^p)$.

3.4 Relation between s-stage and p-order in Runge-Kutta methods and embedded Runge-Kutta methods

From the previous developments it is not difficult to understand that it is not possible to obtain an explicit RK method of order p with less than $s = p$ stages. What it is not so immediate to conclude is that this equality is only true until $p = 4$. Indeed, to obtain a RK method of order greater than p we need to add more than $s = p$ stages. The minimum number of stages necessary to develop a method of order p is still an open problem. For example for $p = 5$ one needs $s = 6$ and for $p = 8$ one needs $s = 11$; for values

of p larger than 8, the minimum number for s is bounded but not determined yet. From this simple facts, it is clear why RK4 methods are so popular.

To implement an efficient code, one needs to have access to error estimates in order to be able to obtain valid numerical results. In the case of Runge-Kutta methods we need to have access to information during the computation to control the step length on a computationally inexpensive way. Several techniques can be used. One of them consists on embedded pairs of Runge-Kutta methods of orders p and $p+1$; the difference between the higher and lower order solutions, $y_{n+1} - \widehat{y}_{n+1}$, is used to control the local error. Maybe the most famous of these methods of order 4 and 5 are the Runge-Kutta-Fehlberg and the explicit Runge-Kutta (4,5) pair of Dormand and Price [6]. The later is drawn in the following Butcher tableau (imagine all the hard work needed to develop such a method !...)

	0							
	$\frac{1}{5}$	$\frac{1}{5}$						
	$\frac{3}{10}$	$\frac{3}{40}$	$\frac{9}{40}$					
	$\frac{4}{5}$	$\frac{44}{45}$	$-\frac{56}{15}$	$\frac{32}{9}$				
	$\frac{8}{9}$	$\frac{19372}{6561}$	$-\frac{25360}{2187}$	$\frac{64448}{6561}$	$-\frac{212}{729}$			
	1	$\frac{9017}{3168}$	$-\frac{355}{33}$	$\frac{46732}{5247}$	$\frac{49}{176}$	$-\frac{5103}{18656}$		
	1	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$-\frac{2187}{6784}$	$\frac{11}{84}$	
$\widehat{y}_{n+1}$		$\frac{5179}{57600}$	0	$\frac{7571}{16695}$	$\frac{393}{640}$	$-\frac{92097}{339200}$	$\frac{187}{2100}$	$\frac{1}{40}$
y_{n+1}		$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$-\frac{2187}{6784}$	$\frac{11}{84}$	0
$y_{n+0.5}$		$\frac{5783653}{57600000}$	0	$\frac{466123}{1192500}$	$-\frac{41347}{1920000}$	$\frac{16122321}{339200000}$	$-\frac{7117}{20000}$	$\frac{183}{10000}$

4 Economic evidence

For the numerical results of the economic problem considered, to be shown in this section, we consider the explicit Runge-Kutta (4,5) pair of Dormand and Price. We require a precision of 10^{-5} for the relative error, re, and of 10^{-7} for the absolute error, a, in order to obtain an approximation that satisfies

$$|y_i(t_n) - y_{n,i}| \leq re\,|y_i(t_n)| + ae_i$$

for each component of the solution.

4.1 Modelisation of economies

Final goods, continuously indexed by $n \in [0,1]$, are produced in perfect competition. Following the Schumpeterian set-up, each final good is produced

by human capital, H, and by a continuum of intermediate goods indexed by $j \in [0, J]$. The output of n, Y_n, at time t is,

$$Y_n(t) = A \left[\int_0^J \left(q^{k(j,t)} \; x_n(k,j,t) \right)^{1-\alpha} dj \right] \; \left[H_{w,n}(t) \right]^{\alpha}. \tag{19}$$

The production function is the same in both countries, except for term A, which is a positive exogenous variable representing the level of productivity, dependent on the country's institutions (where $A_S < A_N$, indexing the South by S and the North by N). The integral sums up the contributions of intermediate goods to production. In the Schumpeterian tradition, the quantity of each j, x, is quality-adjusted. The constant quality upgrade is $q > 1$, and k is the highest quality rung at time t. The expression with exponent $\alpha \in]0, 1[$ represent the role of the H input, and index w in H_n identifies the quantity of H employed in the production of n, *i.e.*, that works and earns a wage (as opposed to H in accumulating human capital).

In particular, by taking into consideration the profit maximisation by producers of final goods and the profit maximising limit pricing by monopolist producers of intermediate goods, the aggregate output (normalising its price at each time t to one), Y, is

$$Y(t) = \int_0^1 p_n(t) \; Y_n(t) \; dn = A^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{q} \right)^{\frac{1-\alpha}{\alpha}} \; Q(t) \; H_w(t), \tag{20}$$

where $Q(t) \equiv \int_0^J q^{k(j,t)\left[\frac{1-\alpha}{\alpha}\right]} \, dj$ is the aggregate domestic quality index, measuring domestic technological knowledge.

With regards to the R&D sector, the value of the leading-edge patent depends on the duration of the monopoly, which, in turn, depends on the probability of successful R&D [2]. Let Z index the country, and $pb_Z(k,j,t)$ denote the instantaneous probability of successful innovation $Z = N$ or imitation $Z = S$ in the next higher quality $[k(j,t)+1]$ of intermediate good j. Formally,

$$pb_Z(k,j,t) = rs_Z(k,j,t) \cdot DC_Z(k,j,t) \cdot \left[CU_i(t) \right]^{\Gamma_Z}, \tag{21}$$

where: (i) $rs_Z(k,j,t)$ is the flow of domestic Y in (20) devoted to R&D in j; (ii) $DC_Z = \beta_Z \; q^{k_Z(j,t)\left(\frac{\alpha-1}{\alpha}\right)} \cdot H_{w,Z}^{-1}$, $\beta_Z > 0$, represents the domestic causes promoting domestic R&D; (iii) $\left[CU_i(t) \right]^{\Gamma_Z}$ (where $\Gamma_N = 0$ and $\Gamma_S = 1$) is a catching-up term, specific to the South, which sums up the positive effects of imitation capacity and backwardness on the probability of successful imitation. By considering $i = 1, 2, 3$, we take the following three different specifications:

$$CU_1(t) = \exp(s_{R\&D}) \cdot \left\{ f\left[\widetilde{H}_w(t)\right] \right\}^{\overline{\sigma}_1} \cdot \left\{ g\left[\widetilde{Q}(t), d\right] \right\}^{-\overline{\sigma}_2 + \widetilde{Q}(t)}; \tag{22a}$$

$$CU_2(t) = \exp(s_{R\&D}) \cdot \exp\left\{ \overline{\sigma}_3 \; h\left[\widetilde{H}_w(t), \widetilde{Q}(t), d\right] \right\}; \tag{22b}$$

$$CU_3(t) = \exp(s_{R\&D}) \cdot \left\{ f\left[\widetilde{H}_w(t)\right] \right\}^{\overline{\sigma}_1} \cdot \exp\left\{ g\left[\widetilde{Q}(t), d\right] \right\}^{\overline{\sigma}_4 + \frac{\overline{\sigma}_5}{\widetilde{Q}(t)}}; \tag{22c}$$

where: $\overline{\sigma}_1,\ \overline{\sigma}_2,\ \overline{\sigma}_4,\ \overline{\sigma}_5,\ s_{R\&D} > 0$; $\overline{\sigma}_2 > \widetilde{Q}$; $\overline{\sigma}_3 > 1$; $0 < \widetilde{H}_w \equiv \frac{H_{w,S}}{H_{w,N}}, \widetilde{Q} \equiv \frac{Q_S}{Q_N} < 1$; $\widetilde{H}_w$ is the South's relative level of employed human capital and $\widetilde{Q}$ is the relative technological-knowledge level of the South's intermediate good; the exponential $\exp(s_{R\&D})$ captures one important determinant of imitation capacity, which are the policies promoting R&D [3]; and functions f, g and h are formally given, at each t, by:

$$f\left(\widetilde{H}_w\right) = 1 + \frac{\exp\left(\widetilde{H}_w\right)}{1 + \exp\left(\widetilde{H}_w\right)}; \tag{23a}$$

$$g\left(\widetilde{Q}, d\right) = \begin{cases} 0 & \text{, if } 0 < \widetilde{Q} \leq d \\ -\widetilde{Q}^2 + (1+d)\widetilde{Q} - d & \text{, if } d < \widetilde{Q} < 1 \end{cases}; \tag{23b}$$

$$h\left(\widetilde{H}_w, \widetilde{Q}, d\right) = g\left(\widetilde{Q}, d\right) \; \widetilde{H}_w. \tag{23c}$$

That is, in the lines of, *e.g.*, Nelson and Phelps [11], from f, human capital at work enhances the imitation capacity and so speeds up convergence with the North, and parameter $\overline{\sigma}_1$ indicates how quickly $pb_S(k, j, t)$ rises as $\widetilde{H}_w$ also rises. Function g attempts to capture the benefits obtained from relative backwardness, *i.e.*, provided that the technological-knowledge gap is not very wide, the South can benefit from an advantage of backwardness.[8] This is because low-income countries, which are far from the technological-knowledge frontier, are stagnant and show no potential for rapid growth [13]. Function h brings together the role of the gap in human-capital employed and the role of the technological-knowledge gap. Equations (23b) and (23c)

[8] However, when the gap is wider such that $\widetilde{Q}$ is below threshold d, backwardness is no longer an advantage. Hence, the rule that the wider the initial technological-knowledge gap, the higher the catching up, does not apply unconditionally.

guarantee that functions g and h are non-negative and so economically feasible. Moreover, they are quadratic over the range of main interest, and, once affected by the exponents, yield an increasing (in the technological-knowledge gap) advantage of backwardness.

Assuming free-entry equilibrium into R&D, which is defined by the equality between expected revenue and resources spent, the equilibrium growth rate of the technological knowledge is given by

$$\widehat{Q}(t) = \left\{ \frac{\beta_Z}{\zeta_Z} \left[CU_i(t) \right]^{\Gamma_Z} \left(\frac{q-1}{q} \right) \left[A_Z \left(1-\alpha \right) \right]^{\frac{1}{\alpha}} - r_Z(t) \right\} \left[q^{\left(\frac{1-\alpha}{\alpha} \right)} - 1 \right]. \tag{24}$$

Concerning the consumption and human-capital accumulation, a time-invariant number of heterogeneous individuals decides on the allocation of time and income. Time is divided between accumulation of human capital and working to earn a share of Y, proportional to the individual's human capital. Income is partly spent directly on the consumption of Y, and partly lent in return for future interest.

The maximization of the lifetime utility (assuming a CIES instantaneous utility function), taking into account the individual budget constraint, which equalizes savings (*i.e.*, the accumulation of financial assets K, with return r) to income earned minus consumption, and the production function of human capital *à la* Lucas [10],[9] yields the solution for the consumption path, which is the standard Euler equation

$$\widehat{c}(t) = \widehat{C}(t) = \frac{1}{\theta} \left[r(t) - \rho \right], \tag{25}$$

where: $\widehat{c}$ is the growth rate of c; c is the individual consumption; C is the aggregate consumption; $\theta > 0$ is the constant elasticity of marginal utility with respect to consumption; and $\rho > 0$ is the homogeneous discount rate. An interior solution to the maximization problem requires positive amounts of both assets, K and H, which is not sustainable unless their returns are equalized at all times, and the following resulting condition ensures this:

$$\widehat{w}(t) = r(t) - \chi_T + \delta, \tag{26}$$

where: $w(t)$ is the wage per unit of human capital; χ_T measures the productivity of formation; and δ is the depreciation rate of human capital.

[9] That is, the productivity of the time spent in human capital accumulation, $u_T(t)$, increases with the amount of human capital at each time t. Thus, $u_w(t) = 1 - u_T(t)$ is the fraction of time t that is spent at work.

4.2 Steady state and Southern transitional dynamics

As Y, X, R and C are all multiples of $Q \cdot H_w$, the constant and unique steady-state endogenous growth rate, which, through the Euler equation (25), also implies a constant steady-state interest rate, r^*, denoted by g^* is

$$g^* = \widehat{Q}^* + \widehat{H}_w^* = \widehat{Y}^* = \widehat{X}^* = \widehat{R}^* = \widehat{C}^* = \frac{r^* - \rho}{\theta}. \tag{27}$$

Thus, r^* is obtained by first plugging (24) into the human capital demand path obtained from (20), resulting in $\widehat{w}(t) = \widehat{Q}(t)$, and then by equating this latter expression to the condition for optimization by individuals (26). In particular, the constant r_N^* (and g_N^*) is a direct result. However, the constant r_S^* (and g_S^*) requires that $\widetilde{Q}^*$ and $\widetilde{H}_w^*$ must be constant as well. Thus, it is due to the North-South technological-knowledge diffusion that both countries grow at the same rate, $g^* = g_N^* = g_S^*$.

Having established that in steady state there is a world growth rate common to both countries. However, it is necessary to find out whether the South converges towards that steady state or not, by assuming that the North is and remains always in steady state. Thus, first, the system of differential equations governing the transitional dynamics needs to be obtained, and then it can be solved through numerical integration. By considering the human capital market equilibrium, the free-entry condition into R&D, individual utility maximization with individual optimal time allocation and that the North is in steady state, we will be able to characterize the Southern transitional dynamics.

Due to the exploitation of technological-knowledge backwardness, the Southern growth rate can be higher than the Northern one during the transitional phase, *i.e.*, from $t = 0$, when imitation starts, until t^*, when the South reaches the Northern steady state. In this phase, the probability of successful imitation changes with variations in both $\widetilde{Q}$ and $\widetilde{H}_w$ towards $\widetilde{Q}^*$ and $\widetilde{H}_w^*$, respectively. The Southern transitional path is fully described by a system of differential equations in $\widetilde{Q}$, $\widetilde{H}_w$, $\vartheta \equiv C/Q_S H_{w,S}$, $u_{T,S}$ and $u_{w,S} \equiv 1 - u_{T,S}$, which must be constants in steady state.

4.3 Numerical computation

As stated, for the numerical solution of the ODEs system describing Southern transitional dynamics, we use the explicit Runge-Kutta (4,5) pair of Dormand and Price ([6]; [14]). We solve the above system for a set of baseline parameter values and initial conditions in the Appendix. In particular, the values for $\overline{\sigma}_3$, $\overline{\sigma}_4$ and $\overline{\sigma}_5$ in the two alternative specifications for the catching-up term

	t^*	$\widetilde{Q}^*$	$\widetilde{H}_w^*$	$u_{w,S}^*$
case 1	56	0.55	0.40	0.72
case 2	200	0.55	0.26	0.72
case 3	360	0.55	0.14	0.72

Table 1: Steady-state values of the relative technological-knowledge level of the South $\widetilde{Q}$, of the South's relative level of employed human capital $\widetilde{H}_w$ and of the Southern fraction of time spent at work $u_{w,S}$ for the three cases considered.

are chosen in order to get the same steady-state technological-knowledge gap between countries under all specifications. In all computations we require a relative error tolerance of 10^{-5} and an absolute error tolerance of 10^{-7}.

Table 1 condenses the main results, by comparing initial and steady-state values of the relevant variables under the three catching-up specifications.

First, from Table 1, we can immediately observe that the adjustment processes are globally stable. Moreover, the speed of convergence towards the steady state is different according to the case. The fastest speed of convergence is observed in case 1, where the steady state is reached after 56 years. In cases 2 and 3, the steady state is reached after, respectively, 200 and 360 years. The prolonged time scale towards the steady state is not however surprising. For example, in the model without human-capital accumulation and with scale effects proposed by Papageorgiou [12], the steady state emerges at the end of 160 years. The long time towards the steady state suggests that transitional dynamics is important and should not be neglected.

Regardless of the case, the South improves its relative technological knowledge (from 0.34 to 0.55) and, in case 1, its relative level of human capital at work (from 0.30 to 0.40). However, under cases 2 and 3, its relative human capital at work decreases (from 0.30 to 0.26 and 0.14, respectively), which partly offsets the benefit obtained in terms of technological knowledge. This question is not analyzed in the standard technological-knowledge literature, since human-capital accumulation is not taken into account.

Table 1 also shows, as expected, that the fraction of time spent working in steady state is the same in all cases, which, in turn, corresponds to the Northern value.

Figure 1 completes the set of results, by showing the path of variables $\widetilde{Q}$, $\widetilde{H}_w$ and $u_{w,S}$ under the catching-up specification $(22a)$. For the other two cases, the behaviour of these variables is similar.

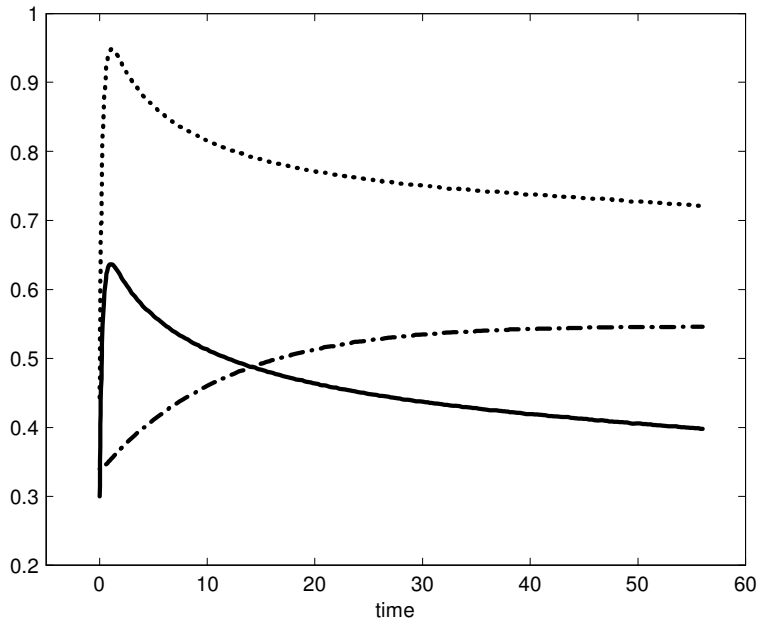

Figure 1: Transitional dynamics towards the steady state of $\widetilde{Q}$ (dashdot line), $\widetilde{H}_w$ (solid line) and $u_{w,S}$ (dotted line).

Thus, $\widetilde{Q}$ increases from its initial value, 0.34, towards its steady-state value, 0.55, because initially the probability of successful imitation rises. However, as the rung of quality left to be copied decreases, the cost of imitation increases and the probability of successful imitation falls towards the steady-state value.

The paths of $\widetilde{H}_w$ and $u_{w,S}$ indicate that there is an initial abrupt surge in the South's relative level of employed human capital owing to the immediate increase in the share of time dedicated at work. After that, $\widetilde{H}_w$ drops due to the greater share of time devoted to work in the South in comparison to the North. Thus, the drop is a result of the (relatively) smaller Southern human-capital accumulation during the transition phase. As shown in Table 1, the biggest drop occurs under case 3, which generates the smallest Southern human-capital accumulation. In light of the initial values, at the end of the adjustment process, a new higher steady-state level of $\widetilde{H}_w$ is only reached in case 1.

The joint behaviour of $\widetilde{Q}$ and $\widetilde{H}_w$ implies that initially the South grows at a higher rate than the North owing to the immediate jump in the Southern share of time devoted to work. Afterwards, it is the magnitude of the

probability of successful imitation when compared with the probability of successful innovation that accounts for the higher Southern growth rate. Finally, both countries grow at the same rate due to the successive increases in the cost of imitation, which represents a form of diminishing returns. In the end, both the technological-knowledge gap and the gap in human capital at work remain constant. That is, the interest rate and the growth rate of the South fall steadily towards their (or Northern) steady-state values.

The initial conditional value of $u_{w,S}$ indicates that the South starts with a relative scarcity of human capital, due to the smaller value of $u_{w,S}$ at $t = 0$. The differential is greater in case 1 (see the Appendix). As reported above, at the beginning of the transitional phase, the growth rate of human capital at work drives economic growth, due to both the previous increase in the amount of human capital and the reallocation of human capital to production. After that, when the economy moves towards the steady state, R&D becomes the main engine of growth.

We also checked the robustness of the results of the transitional dynamics to shocks. The results were obtained from numerical simulations in which one parameter or an initial condition at a certain time is allowed to deviate from its baseline value. The general conclusion is that the model's qualitative behaviour is similar for the ranges of parameter values tested. In fact, similar stable saddle paths to steady state were obtained, differing only slightly in the specific levels of the steady state of the variables which they approach.

We report that the simulation only took 1.0, 0.5 and 24.5 second for cases 1, 2 and 3, respectively, to be performed on a PIV 3.0 GHz Hyper-Threaded machine, in spite of (and due to) the sophisticated numerical method.

5 Concluding remarks

In this paper, we summarise Runge-Kutta type methods for the solution of ordinary differential equations in models of economic dynamics. In particular, we apply this methodology in an endogenous growth model through human-capital accumulation and R&D, which, in turn, is induced by innovations in the North and by imitations in the South.

The argument is based on the premise that the process of Northern technological-knowledge progress can only favour some Southern countries. In this case, technological-knowledge imitation is a window of opportunity for the South, since, during the transition towards the steady state, it achieves higher growth rates. Moreover, it is shown that the Southern convergence depends on the connection between the North-South technological-knowledge gap and the South's relative level of employed human capital.

Appendix: Baseline parameter values and initial conditions

Baseline parameter values

A_N	1.56	$\beta_S = \overline{\sigma}_1 = s_{R\&D}$	0.25	$\overline{\sigma}_5$	0.23
A_S	1.00	d	0.10	θ	1.05
α	0.60	$\overline{\sigma}_2$	0.60	ρ	0.03
q	2.50	$\overline{\sigma}_3$	3.92	δ	0.02
β_N	0.80	$\overline{\sigma}_4$	0.05	χ_T	0.09

Parameters are chosen to calibrate the steady-state world growth rate around 2%, which approximately matches the average per capita growth rate of the U.S. in the post-war period [9]. For some parameters the choice is guided by empirical findings, while other parameter values are based on theoretical specifications. When the range of choice is large we have opted for a value close to some critical value.

Initial values of the variables

$\widetilde{Q}$	0.34	$u_{w,S}\vert_{\text{case 1}}$	0.44
$\widetilde{H}_w$	0.30	$u_{w,S}\vert_{\text{case 2}}$	0.51
ϑ	0.20	$u_{w,S}\vert_{\text{case 3}}$	0.50

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