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Numerical Conformal Mapping via a Fredholm Integral Equation using Fourier Method

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ABSTRACT

An interpolation formula based on Fourier method for the numerical solution of a Fredholm integral equation related to conformal mapping of a simply connected region onto a unit disc is presented. The integral equation involves the Neumann kernel. The numerical results obtained from the interpolation formula based on Fourier method are then compared with the numerical results obtained from the interpolation formula based on Nyström's method. Numerical comparison shows that the interpolation formula based on Nyström's method gives better performance. Numerical implementations on some test regions are presented.

INTRODUCTION

It has been established that Fourier method is equivalent to Nyström's method for the numerical solution of Fredholm integral equation (Berrut and Trummer, 1987). This implies that both methods will produce the same approximation to the solution of a Fredholm integral equation at the collocation points. However, Berrut and Trummer (1987) did not give any numerical examples to support their findings. Furthermore, no numerical comparison has been given for the performance of the interpolation formulas based on Fourier method and Nyström's method. The aim of this paper is to provide relevant numerical examples to fill up this gap through numerical conformal mapping.

Conformal mapping has been a familiar tool of science and engineering for generations. The practical limitation has always been that only for certain special regions are exact conformal mapping known, while for the rest, they must be computed numerically. Thus several methods have been developed for numerical conformal mapping. The mapping of a simply connected region onto a unit disc is known as a Riemann mapping and the mapping function is known as Riemann mapping function. The most commonly used method to compute the Riemann mapping function is derived from integral equations involving the boundary correspondence function that relates the two regions' boundaries pointwise. Typically the

boundaries are discretized at n points, so that the integral equation reduces to an algebraic system.

A method to compute the Riemann mapping function via the Bergman kernel is presented in Razali *et al*. (1997) which expresses the Bergman kernel as the solution of a second kind integral equation involving the Neumann kernel. In Razali *et al.* (1997), the integral equation via the Bergman kernel has been solved using the interpolation formula based on Nyström's method with trapezoidal rule. In this paper, we shall use interpolation formula based on Fourier method to solve the integral equation and compare numerically the performance of these two types of interpolation formulas. The numerical solution of the integral equation will then be used to approximate the boundary correspondence function for several test regions. The exact boundary correspondence functions for several test regions will also be calculated and compared with the approximate results.

The organization of this paper is as follows. Section 2 contains a brief review of the integral equation for the Bergman kernel. In Section 3 we show how to treat the integral equation numerically using Fourier method. In Section 4, we make some numerical comparisons between interpolation formulas based on Nyström's and Fourier methods using some test regions and draw some conclusion.

INTEGRAL EQUATION FOR THE BERGMAN KERNEL

Let Ω be a simply connected region in the complex plane whose boundary Γ is assumed to be analytic C^2 Jordan curve. Suppose $z = z(t)$ is the parametric representation of Γ with

$$T(z) = \frac{z'(t)}{|z'(t)|}$$

denotes the unit tangent in the direction of increasing parameters at the point z.

Let $a \in \Omega$ and let R be the Riemann map of Ω whose boundary is Γ, with the usual normalization

$$R(a) = 0, \quad R'(a) > 0, \quad a \in \Omega.$$

Due to Γ being analytic, R can be extended to a function that is analytic on $\overline{\Omega} = \Omega \cup \Gamma$. A classical relationship between the Riemann map and the Bergman kernel is given by

$$R(z) = \frac{1}{\mathrm{i}} T(z) \frac{B(z,a)}{|B(z,a)|}, \quad z \in \Gamma,$$

where $B(z,a)$ is the Bergman kernel which is analytic on $\overline{\Omega}$ (Razali *et al.*, 1997). In Razali *et al.* (1997), it is shown that the function $\hat{B}(z,a) = T(z) B(z,a)$ is the unique continuous solution to the integral equation

$$\hat{B}(z,a) + \int_{\Gamma} N(z,w) \hat{B}(w,a) |dw| = -\frac{1}{\pi} \frac{\overline{T(z)}}{\left(\overline{z} - \overline{a}\right)^2}, \quad z \in \Gamma, \tag{1}$$

where

$$N(z,w) = \begin{cases} \dfrac{1}{\pi} \mathrm{Im}\left[\dfrac{T(z)}{z-w}\right], & z, w \in \Gamma, w \neq z \\ \dfrac{1}{2\pi} \dfrac{\mathrm{Im}\left[z''(t)\overline{z'(t)}\right]}{|z'(t)|^3}, & z = w \in \Gamma. \end{cases} \tag{2}$$

The real kernel N is the familiar Neumann kernel which arises frequently in the integral equations of potential theory and conformal mapping (Henrici, 1986). Since Γ is analytic, the Neumann kernel N is certainly continuous at all points $(z,w) \in \Gamma \times \Gamma$.

Assume the parametric representation of Γ is $z = z(t)$, $0 \le s,t \le \beta$. Let $w = z(s)$. Consequently, integral equation (1) and Neumann kernel (2) can be expressed as

$$\phi(t) - \lambda \int_0^{\beta} v(t,s) \phi(s) ds = \psi(t), \tag{3}$$

where $\lambda = -1$, and for $0 \le s,t \le \beta$,

$$\phi(t) = |z'(t)| \hat{B}(z(t),a), \tag{4}$$

$$\phi(s) = \left|z'(s)\right| \hat{B}\left(z(s), a\right), \tag{5}$$

$$\psi(t) = -\frac{1}{\pi} \frac{\overline{z'(t)}}{\left(\overline{z(t)} - \overline{a}\right)^2}, \tag{6}$$

$$v(t,s) = \begin{cases} \dfrac{1}{\pi} \operatorname{Im}\left[\dfrac{z'(t)}{z(t) - z(s)}\right], & \text{if } t \neq s, \\ \dfrac{1}{2\pi} \operatorname{Im}\left[\dfrac{z''(t)}{z'(t)}\right], & \text{if } t = s. \end{cases} \tag{7}$$

Note that β is finite, v is a given function of two variables, ψ is a given function, and ϕ is to be determined. An important fact is that $\lambda = -1$ is not an eigenvalue of v. The homogenous equation corresponding to equation (3) thus has the trivial solution. By the Fredholm alternative (Kreyszig, 1978), the non-homogeneous equation has exactly one continuous solution ϕ for any continuous function ψ.

NUMERICAL IMPLEMENTATION USING FOURIER METHOD

If we set $\beta = 2\pi$, equation (3) becomes

$$\phi(t) + \int_0^{2\pi} v(t,s)\phi(s)\,ds = \psi(t) \tag{8}$$

Note that the functions ϕ, ψ and v are 2π-periodic.

To implement Fourier method, we choose N equidistant collocation points $s_j = t_j = 2\pi j/N$, and interpolate the kernel v and the inhomogeneity ψ by trigonometric polynomials, i.e.,

$$\hat{v}(t,s) = \sum_{n=-M'}^{M} \sum_{m=-M}^{M'} a_{nm} e^{\mathrm{i}nt} e^{\mathrm{i}ms} \tag{9}$$

and

$$\hat{\psi}(t) = \sum_{n=-M'}^{M} d_n e^{\mathrm{i}nt} \tag{10}$$

where

$$M' = \begin{cases} M, \text{ if } N \text{ is odd, } N = 2M+1; \\ M-1, \text{ if } N \text{ is even, } N = 2M. \end{cases}$$

The coefficients a_{nm} and d_n are the elements of the discrete Fourier transforms given by (Berrut and Trummer, 1987)

$$a_{nm} = \frac{1}{N^2} \sum_{j,k=0}^{N-1} v(t_j, s_k) w^{-jn} w^{-km} \tag{11}$$

$$d_n = \frac{1}{N} \sum_{j=0}^{N-1} \psi(t_j) w^{-jn} \tag{12}$$

where $w = \exp(2\pi \mathrm{i}/N)$, $v_{jk} = v(t_j, s_k)$ and $\psi_j = \psi(t_j)$ are the interpolated values. We seek a solution at the collocation points of the form

$$\hat{\phi}(t_j) = \sum_{n=-M'}^{M} b_n e^{\mathrm{i}nt_j}(t_j) \tag{13}$$

with unknown coefficient b_n. Replacing the functions ϕ, ψ, and v in equation (8) by their respective approximations $\hat{\phi}$, $\hat{\psi}$, and $\hat{v}$, we obtain

$$\sum_{n=-M'}^{M} b_n e^{\mathrm{i}nt} + \int_0^{2\pi} \left[\sum_{n=-M'}^{M} \sum_{m=-M}^{M'} a_{nm} e^{\mathrm{i}nt} e^{\mathrm{i}ms} \right] \left[\sum_{k=-M'}^{M} b_k e^{\mathrm{i}ks} \right] ds = \sum_{n=-M'}^{M} d_n e^{\mathrm{i}nt}.$$

or

$$\sum_{n=-M'}^{M} b_n e^{\mathrm{i}nt} + \sum_{n=-M'}^{M} \sum_{m=-M}^{M'} \sum_{k=-M'}^{M} a_{nm} b_k e^{\mathrm{i}nt} \int_0^{2\pi} e^{\mathrm{i}ms} e^{\mathrm{i}ks} ds = \sum_{n=-M'}^{M} d_n e^{\mathrm{i}nt}. \tag{14}$$

Observe that equation (14) can be simplified by using the orthogonality relations for the periodic complex exponentials. These orthogonality relations have the following property (Kammler, 2000):

$$\int_0^p e^{2\pi \mathrm{i}kx/p} e^{-2\pi \mathrm{i}lx/p} dx = \begin{cases} p, & \text{if } k = l \\ 0, & \text{otherwise.} \end{cases} \qquad k, l = 0, \pm 1, \pm 2, \ldots.$$

Applying the above orthogonality relations with $p = 2\pi$, equation (14) becomes

$$\sum_{n=-M'}^{M} b_n e^{\mathrm{i}nt} + \sum_{n=-M'}^{M} \sum_{m=-M}^{M'} 2\pi a_{nm} b_{-m} e^{\mathrm{i}nt} = \sum_{n=-M'}^{M} d_n e^{\mathrm{i}nt}.$$

On multiplying both sides by $e^{-\mathrm{i}kt}$ and then integrate each term from 0 to 2π, we get

$$\sum_{n=-M'}^{M} b_n \int_0^{2\pi} e^{\mathrm{i}nt} e^{-\mathrm{i}kt} dt + \sum_{n=-M'}^{M} \sum_{m=-M}^{M'} 2\pi a_{nm} b_{-m} \int_0^{2\pi} e^{\mathrm{i}nt} e^{-\mathrm{i}kt} dt = \sum_{n=-M'}^{M} d_n \int_0^{2\pi} e^{\mathrm{i}nt} e^{-\mathrm{i}kt} dt.$$

Applying the orthogonality properties, we obtain

$$b_k (2\pi) + \sum_{m=-M}^{M'} 2\pi a_{km} b_{-m} (2\pi) = d_k (2\pi), \qquad k = -M', \ldots, M$$

which implies

$$b_n + \sum_{m=-M}^{M'} 2\pi a_{nm} b_{-m} = d_n, \qquad n = -M', \ldots, M. \tag{15}$$

Hence, equation (15) is a system of linear equations for the b_n.

Since v and ψ are known functions, equations (11) and (12) can be used readily to compute the coefficients a_{nm} and d_n. Substitute these values into the system (15), we can then solve the values for b_n. Finally, substitute b_n into equation (13) will give us the approximate solution for equation (8).

Equation (15) can be rewritten as an N by N system

$$\left(\mathbf{I}+\mathbf{A}\right)\mathbf{b}=\mathbf{d}, \tag{16}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{A}$ is a matrix defined by $2\pi a_{nm}$, $\mathbf{b}=\{b_n\}$ and $\mathbf{d}=\{d_n\}$, for $n=-M',...,M$, $m=M,...,M'$. Note that the coefficient matrix $\left(\mathbf{I}+\mathbf{A}\right)$, $\mathbf{b}$ and $\mathbf{d}$ are complex. Once the discretized solutions $\hat{\phi}\left(t_j\right)$ are known at the collocation points, equation (13) provides an interpolation formula for ϕ based on Fourier method:

$$\hat{\phi}\left(t\right)=\sum_{n=-M'}^{M} b_n e^{int}. \tag{17}$$

Another appealing procedure for solving equation (8) numerically is using the Nyström's method with the trapezoidal rule. Choosing N equidistant collocation points $t_j=2\pi j/N$ and the trapezoidal rule for Nyström's method to discretize equation (8), we obtain (Razali *et al*., 1997)

$$\tilde{\phi}\left(t_j\right)+\frac{2\pi}{N}\sum_{k=0}^{N-1} v\left(t_j,t_k\right)\tilde{\phi}\left(t_k\right)=\psi\left(t_j\right), \qquad 0\le j\le N-1. \tag{18}$$

Defining the matrix $\mathbf{Q}$ by $Q_{jk}=2\pi v\left(t_j,t_k\right)\big/N$, and $x_j=\tilde{\phi}\left(t_j\right)$, $y_j=\tilde{\psi}\left(t_j\right)$, equation (18) can be rewritten as an N by N system

$$\left(\mathbf{I}+\mathbf{Q}\right)\mathbf{x}=\mathbf{y}. \tag{19}$$

Since equation (8) has a unique solution, then for a wide class of quadrature formula, the system (19) also has a unique solution, as long as N is sufficiently large (Atkinson, 1986). Similarly, once the discretized solutions $\tilde{\phi}\left(t_j\right)$ are known at the collocation points, the interpolation formula based on Nyström's method is given by

$$\tilde{\phi}\left(t\right)=\psi\left(t\right)-\frac{2\pi}{N}\sum_{j=0}^{N-1} v\left(t,t_j\right)\tilde{\phi}\left(t_j\right). \tag{20}$$

Suppose $\theta(t)$ is the boundary correspondence function to a representation $z = z(t)$, $0 \le t \le 2\pi$, of Γ. Then

$$R\left(z(t)\right) = e^{\mathrm{i}\theta(t)}, \tag{21}$$

where R is the Riemann mapping function. The boundary correspondence function can be computed (without integration) by the formula (Razali *et al.*, 1997)

$$\theta(t) = \arg\left(-\mathrm{i}\phi(t)\right). \tag{22}$$

COMPUTATION ON SEVERAL TEST REGIONS

In this section the numerical scheme discussed in Section 3 is applied to several test regions with the normalization $R(0) = 0$, $R'(0) > 0$. We have used the MATHEMATICA 5.0 to carry out the entire numerical procedure. We list the sup-norm error $\left\|\theta(t) - \theta_P(t)\right\|_\infty$ where $\theta(t)$ is the exact boundary correspondence function for the test regions, and $\theta_P(t)$ is the approximation obtained by means of equation (22) using P equally spaced interpolation points in the interval $[0, 2\pi]$, most of which are not the original collocation points. In all our experiments, we have chosen $P = 36$.

These allow exact comparisons with the numerical results obtained from interpolation formula based on Nyström's method given in Razali *et al.* (1997).

Example 1: Ellipse $\left(0 \le \varepsilon < 1,\ \text{axis ratio} = (1+\varepsilon)/(1-\varepsilon)\right)$.

$$z(t) = e^{\mathrm{i}t} + \varepsilon e^{-\mathrm{i}t},$$

$$\theta(t) = t + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{\varepsilon^k}{1+\varepsilon^{2k}} \sin(2kt).$$

TABLE 1: Error norm $\left\|\theta(t)-\theta_P(t)\right\|_\infty$ for ellipse

N	Axis ratio				
	1.2	1.5	2.0	3.0	5
4	5.3(-01)	-	-	-	-
8	9.9(-02)	-	-	-	-
16	1.6(-03)	5.5(-02)	-	-	-
32	1.7(-07)	1.1(-04)	1.0(-02)	-	-
64	2.7(-15)	7.4(-10)	4.8(-06)	1.4(-02)	-
128	-	1.3(-15)	2.9(-13)	6.4(-07)	-
256	-	-	4.9(-15)	1.2(-13)	3.6(-07)
512	-	-	-	-	5.5(-11)

Example 2: Inverted Ellipse $(0 < p \le 1)$.

$$z(t)=\sqrt{1-\left(1-p^2\right)\cos^2(t)}e^{it},$$

TABLE 2: Error norm $\left\|\theta(t)-\theta_P(t)\right\|_\infty$ for inverted ellipse

N	Values of p		
	0.8	0.5	0.2
4	3.4(-01)	-	-
8	5.0(-02)	6.1(-01)	-
16	8.2(-04)	8.7(-02)	-
32	1.6(-07)	1.2(-03)	9.8(-01)
64	6.2(-15)	2.9(-07)	4.2(-02)
128	-	1.2(-14)	8.0(-05)

Example 3: Oval of Cassini $(|z-\alpha||z+\alpha|=1, 0\le\alpha\le 1)$.

$$z(t)=\left(\alpha^2\cos 2t+\sqrt{1-\alpha^4\sin^2 2t}\right)^{\frac{1}{2}}e^{it},$$

$$\theta(t)=t-\frac{1}{2}\arg\left(w(t)\right),$$

$$w(t)=\sqrt{1-\alpha^4\sin^2 2t}+\mathrm{i}\alpha^2\sin 2t\,.$$

TABLE 3: Error norm $\left\|\theta(t)-\theta_P(t)\right\|_\infty$ for oval of Cassini

N	Values of α					
	0.2	0.5	0.8	0.9	0.99	0.999
4	5.9(-02)	3.7(-01)	-	-	-	-
8	5.1(-04)	2.4(-02)	-	-	-	-
16	1.2(-07)	2.3(-04)	2.5(-02)	1.5(-01)	-	-
32	1.2(-14)	3.0(-08)	1.5(-04)	2.9(-03)	-	-
64	8.9(-16)	1.8(-15)	4.1(-08)	1.3(-05)	-	-
128	-	-	2.7(-15)	2.2(-10)	1.7(-03)	-
256	-	-	1.8(-15)	1.8(-15)	8.6(-07)	-
512	-	-	-	-	2.0(-12)	2.1(-04)
1024	-	-	-	-	4.0(-13)	5.3(-08)

Example 4: Epitrochoid ("Apple") $(0 \le \alpha \le 1)$.

$$z(t) = e^{it} + \frac{\alpha}{2} e^{2it},$$
$$\theta(t) = t.$$

TABLE 4: Error norm $\left\|\theta(t)-\theta_P(t)\right\|_\infty$ for epitrochoid

N	Values of α					
	0.2	0.3	0.4	0.6	0.8	0.9
4	4.5(-05)	3.9(-04)	1.9(-03)	2.4(-02)	6.5(-01)	-
8	1.0(-08)	4.8(-07)	7.8(-06)	4.8(-04)	2.2(-02)	-
16	8.9(-16)	5.7(-13)	1.5(-10)	4.0(-07)	3.1(-04)	1.6(-02)
32	-	8.9(-16)	8.9(-16)	5.8(-13)	5.6(-07)	5.3(-04)
64	-	-	-	8.9(-16)	8.0(-13)	1.9(-07)
128	-	-	-	-	8.9(-16)	1.6(-13)

Even though Berrut and Trummer (1987) has established the equivalence of the methods of Fourier and Nyström for solving Fredholm integral equation of the second kind; our numerical results have shown that the interpolation formula based on Nyström's method in Razali *et al*. (1997) gives better accuracy compared with the interpolation formula based on Fourier method. These numerical experiments suggest that one should prefer the interpolation formula based on Nyström's method in order to obtain approximations of high accuracy for numerical conformal mapping of interpolation points on the boundary.

REFERENCES

Atkinson, K. E. 1996. *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations*. Philadephia: Society for Industrial and Applied Mathematics.

Berrut, J. P. and M. R. Trummer. 1987. Equivalence of Nyström's Method and Fourier Methods for the Numerical Solution of Fredholm Integral Equations. *Mathematics of Computation* **48**(1987): 617 – 623.

Henrici, P. 1986. *Applied and Computational Complex Analysis* Volume 3. New York: John Wiley & Sons.

Kammler, D. W. 2000. *A First Course in Fourier Analysis*. New Jersey: Prentice-Hall.

Kreyszig, E. 1978. *Introductory Functional Analysis with Applications*. New York: John Wiley & Sons.

Razali, M. R. M., M. Z. Nashed and A. H. M. Murid. 1997. Numerical Conformal Mapping via the Bergman Kernel. *Journal of Computational and Applied Mathematics* **82**(1997): 333 – 350.