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Abstract

The moduli space $\overline{M}_{g,n}$ of n-pointed stable curves of genus g is stratified by the topological type of the curves being parameterized: the closure of the locus of curves with k nodes has codimension k. The one-dimensional components of this stratification are smooth rational curves called F-curves. These are believed to determine all ample divisors.

F-conjecture. *A divisor on $\overline{M}_{g,n}$ is ample if and only if it positively intersects the F-curves.*

In this paper, proving the F-conjecture on $\overline{M}_{g,n}$ is reduced to showing that certain divisors on $\overline{M}_{0,N}$ for $N \leqslant g+n$ are equivalent to the sum of the canonical divisor plus an effective divisor supported on the boundary. Numerical criteria and an algorithm are given to check whether a divisor is ample. By using a computer program called the Nef Wizard, written by Daniel Krashen, one can verify the conjecture for low genus. This is done on $\overline{M}_g$ for $g \leqslant 24$, more than doubling the number of cases for which the conjecture is known to hold and showing that it is true for the first genera such that $\overline{M}_g$ is known to be of general type.

1. Introduction

The moduli space $M_{g,n}$ of smooth n-pointed curves of genus g and its projective closure, the Deligne–Mumford compactification $\overline{M}_{g,n}$, have been studied in many areas of mathematics. This is because properties of families of curves can often be translated into facts about the birational geometry of the moduli space. For example, asking whether almost any curve of genus g occurs as a member of a family given by *free* parameters (i.e. parameterized by an open subset of affine space) is the same as asking whether $\overline{M}_{g,0}$ (written simply as $\overline{M}_g$) is unirational.

To learn about the birational geometry of a projective variety such as $\overline{M}_{g,n}$, it is useful to study its nef and effective divisors. A nef divisor D on a projective variety X is a divisor that nonnegatively intersects every effective curve on X. The nef divisors on X parameterize morphisms from X to any projective variety, since to every regular map $f : X \longrightarrow Y$ from X to a projective variety Y there corresponds a nef divisor, f^*A, where A is ample on Y. The nef and effective divisors of a variety X form cones inside the Néron–Severi space of X. Interior to the nef cone is the cone of ample divisors. By studying these cones, one can say a lot about the space X. For example, one of the strongest results about the birational geometry of $\overline{M}_g$ is that for $g = 22$ and for $g \geqslant 24$, the moduli space is of general type. The result for $g \geqslant 24$ was proved by Harris

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and Mumford (and later Eisenbud) while that for $g = 22$ was proved by Farkas, who also showed that $\overline{M}_{23}$ has positive Kodaira dimension [Far00, Far06, Har77]. After learning enough about the cone of effective divisors, these authors were able to show that for $g = 23$ and for $g \geqslant 24$, the canonical divisor of $\overline{M}_g$ is interior to it and, furthermore, does not touch the sides. In particular, in this range $\overline{M}_g$ is not unirational, and so the general curve of genus g does not appear as a member of a family of curves parameterized by an open subset of affine space.

The cone of nef divisors of a projective variety X is always contained inside the effective cone of divisors of X. For $\overline{M}_g$, the nef cone is strictly interior to the effective cone in the sense that they intersect only at the origin (see [Gib00]). As a result of this fact, there is no projective morphism with connected fiber from $\overline{M}_g$ to any lower-dimensional variety other than a point. This is another example illustrating that the cones of nef and effective divisors are extremely important tools for understanding the birational geometry of a projective variety X. Much more information would be gained if one could further clarify the relationship between the nef and effective cones. Ideally, one would like to describe the nef cone explicitly.

One might hope to specify which divisors on a projective variety X are nef by finding a collection of curves $\{C_i\}_{i \in I}$ that determine all effective curves, i.e. which span the extremal rays of the Mori cone of curves. If such a collection of curves exists, then one could say that a divisor D on X is nef if and only if it intersects these curves. Finding such curves for a given variety X is a very difficult, and often impossible, task. However, for $\overline{M}_{g,n}$, there are smooth rational curves called F-curves that seem to be the right candidates to consider.

In order to describe the F-curves, a few facts about the structure of $\overline{M}_{g,n}$ will be given. Points in $\overline{M}_{g,n}$ correspond to stable n-pointed curves of genus g. A stable curve has at worst nodal singularities. The locus of curves with k nodes has codimension k in $\overline{M}_{g,n}$. Since the dimension of $\overline{M}_{g,n}$ is $3g - 3 + n$, the (closure of the) locus of curves with $3g - 4 + n$ nodes is one-dimensional. Any curve that is numerically equivalent to a component of this one-dimensional locus is called an F-curve. An F-divisor is any divisor that nonnegatively intersects all the F-curves. The F-conjecture asserts that the F-cone of divisors is the same as the nef cone of divisors of $\overline{M}_{g,n}$.

F-CONJECTURE. *A divisor on $\overline{M}_{g,n}$ is nef if and only if it nonnegatively intersects a class of curves called the F-curves.*

In this paper, proving the F-conjecture on $\overline{M}_{g,n}$ is reduced to showing that certain divisors in $\overline{M}_{0,N}$ for $N \leqslant g + n$ are equivalent to the sum of the canonical divisor plus an effective divisor supported on the boundary (Theorem 3.1). As an application of the reduction, numerical criteria are given which, if satisfied by a divisor D on $\overline{M}_g$, guarantee that D is nef (see Corollaries 5.1–5.5). An algorithm is described for using the reduction to check that a given F-divisor is nef (Theorem/Algorithm 4.8). Using a computer program called the Nef Wizard, one can show that the criteria and the algorithm completely determine all nef divisors on $\overline{M}_g$ for $g \leqslant 24$. This computer package, written by Daniel Krashen, can be found at http://www.math.uga.edu/∼dkrashen/nefwiz/index.html.

Most of the criteria are phrased in such a way that they can be applied to showing that F-divisors on $\overline{M}_g$ are nef. However, since by [GKM01, Theorem 0.7] any F-divisor in $\widetilde{M}_{0,g} = \overline{M}_{0,g}/S_g$ is the pullback of an F-divisor on $\overline{M}_g$, the criteria can also be used to prove that F-divisors on this space are nef.

It is worth noting that because, as it turns out, there are a finite number F-curves to begin with, if the F-conjecture were true, it would mean that there are finitely many extremal rays of the cone of curves. This is surprising, since the most general thing one can say about the shape of the cone of curves for an arbitrary variety X is that the part of the cone corresponding to curves which negatively intersect the canonical divisor is polyhedral; on this part of the cone there are countably many extremal rays and they are spanned by irreducible, rational curves. The cone of curves for $\overline{M}_{g,n}$ is not K-negative; in fact, since for $n=0$, $g=23$ and $g \geqslant 24$ the space $\overline{M}_g$ is of general type, very much the opposite is true. It is for this reason that in this work the F-conjecture is checked for genus up to 24; for higher genera there does not seem to be any feature of the spaces that might prevent the conjecture from being true. Also, the list of generators of the cone of F-divisors grows extremely fast, so it takes the computer a long time to run through the list of divisors to check that the criteria are met and the divisors are nef.

Previous results. Prior to this work, the F-conjecture was known to be true on $\overline{M}_g$ for $g \leqslant 11$ and for $g=13$. The first cases, of $g=3$ and 4, were proved by Carel Faber, after whom F-curves and F-divisors are named. In [GKM01], it was shown that the problem of describing the nef divisors on $\overline{M}_{g,n}$ can be reduced to solving the F-conjecture on $\overline{M}_{0,g+n}/S_g$. Results of Keel and McKernan [KM96], when combined with [GKM01], establish the truth of the conjecture for $g \leqslant 11$. Farkas and I were able to extend these results to $g=13$.

2. Definitions and notation

Standard definitions are used for cones of divisors and curves as well as for the basic divisor classes on $\overline{M}_{g,n}$ (see, e.g., [FG03, GKM01, Kol91]). Since numerical details are referred to specifically, the F-curves and F-divisors will now be defined. Following that, formulas for the pullback of a divisor along certain morphisms will be derived. Since the formulas in §§ 2.2, 2.3 and 2.4 are very combinatorially involved, the reader may wish to skip ahead to § 3 and refer back as necessary.

2.1 Faber curves and divisors

An F-curve on $\overline{M}_{g,n}$ is any curve that is numerically equivalent to a component of the locus of points in $\overline{M}_{g,n}$ having $3g-4+n$ nodes. A subset of the boundary classes $\delta_{i,I}$, taken together with the tautological classes $\psi_i = -\delta_{0,i}$ and the Hodge class λ, forms a basis for the Picard group of $\overline{M}_{g,n}$. By writing a divisor in terms of these classes and intersecting it with the various F-curves, one can see that if the divisor is an F-divisor, then its coefficients satisfy certain inequalities. These inequalities, which can be taken to define an F-divisor, are listed below.

Definition/Theorem 2.1 (cf. [GKM01, Theorems 2.1 and 2.2]). For $N=\{1\dots n\}$, consider the divisor

$$D = a\lambda - b_0\delta_0 - \sum_{\substack{0\leqslant i\leqslant \lfloor g/2\rfloor, I\subseteq N\\ |I|\geqslant 1 \text{ if } i=0}} b_{i,I}\delta_{i,I}$$

on $\overline{M}_{g,n}$ (with the convention that for given g and n, we omit any terms for which the corresponding boundary divisor does not exist). Consider the following inequalities:

(i) $a - 12b_0 + b_{1,\emptyset} \geqslant 0$;

(ii) $b_0 \geqslant 0$;

(iii) $b_{i,I} \geqslant 0$ for $g-2 \geqslant i \geqslant 0$;

(iv) $2b_0 - b_{i,I} \geqslant 0$ for $g-1 \geqslant i \geqslant 1$;

(v) $b_{i,I} + b_{j,J} \geqslant b_{i+j,I\cup J}$ for $i, j \geqslant 0$, $i+j \leqslant g-1$ and $I \cap J = \emptyset$;

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(vi) $b_{i,I} + b_{j,J} + b_{k,K} + b_{l,L} - (b_{i+j,I\cup J} + b_{i+k,I\cup L} + b_{i+l,I\cup L}) \geqslant 0$ for $i, j, k, l \geqslant 0$, $i+j+k+l = g$ and I, J, K, L a partition of N;

where $b_{i,I}$ is defined to be b_{g-i,I^c} for $i > \lfloor g/2 \rfloor$.

For $g \geqslant 3$, D is an F-divisor if and only if each of the above inequalities holds.

For $g = 2$, D is an F-divisor if and only if (i) and (iii)–(vi) hold.

For $g = 1$, D is an F-divisor if and only if (i), (v) and (vi) hold.

For $g = 0$, D is an F-divisor if and only if (vi) holds.

2.2 Boundary restrictions

Let $f : \overline{M}_{0,g+n} \longrightarrow \overline{M}_{g,n}$ be the morphism obtained by attaching a pointed curve of genus one to each of the first g marked points. The pullback f^*D will often be referred to as the restriction of a divisor D to the flag locus. Divisors in $\overline{M}_{0,g}$ pulled back along certain so-called boundary restriction morphisms will also be considered.

Definition 2.2. For $z \geqslant 2$, $a \geqslant 1$ and disjoint subsets $N_j \subset N = \{1 \ldots n\}$ of order $n_j \geqslant 2$, let $[N_1 : N_2 : \ldots : N_a]$ be the boundary restriction morphism, which we denote by $v_{a,z} : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,n}$ where $n = \sum_{j=1}^{a} n_j + z$, given by attaching an (n_j+1)-pointed genus-zero curve (whose marked points consist of an attaching point and the N_j) to each of the first a marked points and renumbering the remaining z marked points. We say that a is the order of the boundary restriction morphism.

Note that if D is an F-divisor in $\overline{M}_g$, then f^*D is an F-divisor in $\overline{M}_{0,g}$. This derives from the fact that 1-strata on $\overline{M}_{0,g}$ go to 1-strata on $\overline{M}_g$ via f. Likewise, if D is an F-divisor in $\overline{M}_{0,n}$ and $v : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,n}$ is a boundary restriction, then v^*D is an F-divisor in $\overline{M}_{0,a+z}$. As will be shown in Lemma 2.4, for any boundary restriction morphism $v = [N_1 \ldots N_a]$, the pullback v^*f^*D is determined by the orders of the sets N_j. Hence one may denote the boundary restriction morphism $[N_1 \ldots N_a]$ by the a-tuple $[n_1 \ldots n_a]$.

Note 2.3. For $B \subset A$, we will often use the notation

$$\Delta_{A,B}^{Z,y} = \sum_{Y \subset Z, |Y| = y} \delta_{Y \cup B}.$$

Lemma 2.4.

(i) *Let $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ in $\overline{M}_g$ be a divisor. Then*

$$f^*D = b_1 \sum_{i=1}^{g} \psi_i - \sum_{i=2}^{\lfloor g/2 \rfloor} b_i B_i \quad \text{where } B_i = \sum_{\substack{I \subset \{1\ldots g\} \\ |I| = i}} \delta_I.$$

(ii) *Let $v = v_{a,z} = [n_1 \ldots n_a]$ be a boundary restriction of $\overline{M}_{0,g}$. Then*

$$v^*f^*D = \sum_{i \in A = \{1,\ldots,a\}} b_{n_i} \psi_i + b_1 \sum_{i \in Z = A^c} \psi_i - \sum_{\substack{B \subset A, 0 \leqslant y \leqslant |Z| \\ 2 \leqslant y + |B| \leqslant \lfloor (a+z)/2 \rfloor}} b_{y + \sum_{i \in B} n_i} \Delta_{A,B}^{Z,y}.$$

Proof. For both formulas, apply [AC98, p. 106, Lemma 3.3]. □

As one can see from Lemma 2.4, the pullback f^*D is invariant under the natural action of S_g on $\overline{M}_{0,g}$. We shall often be referring to (the image of) f^*D in $\widetilde{M}_{0,g} = \overline{M}_{0,g}/S_g$. In [KM96], Keel

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and McKernan studied $\widetilde{M}_{0,g}$, proving, among other things, that its cone of effective divisors is simplicial and generated by the $\widetilde{B}_i$, the images of B_i in $\widetilde{M}_{0,g}$. This fact will be used in Proposition 4.5, which says that any nontrivial F-divisor on $\widetilde{M}_{0,g}$ is big. Keel and McKernan had previously shown that every nef divisor on $\widetilde{M}_{0,g}$ is big. They also gave an expression for the canonical divisor $K_{\widetilde{M}_{0,n}}$ which is used in the proof of Theorem 4.1.

2.3 Description of the A-averages

The equivalence classes of boundary divisors span $\mathrm{Pic}(\overline{M}_{0,n})$ but are not independent. Consequently, any divisor class in $\overline{M}_{0,n}$, such as the ψ_i, can be expressed in terms of the boundary classes and, moreover, there are different ways of doing so. Given $i, j, k \in \{1 \ldots n\}$, one has that

$$\psi_i = \sum_{\substack{I \subseteq \{1 \ldots n\} \\ i \in I; j,k \notin I}} \delta_I.$$

In particular, there are $\binom{n-1}{2}$ ways of expressing a divisor class ψ_i as a sum of boundary divisors in this manner. By combining these in various ways, one can produce different manifestations of the ψ_i as sums of boundary classes. Suppose that $A \subseteq \{1 \ldots n\}$ and $i \in A$. In this section, four ways of writing ψ_i as a sum of boundary divisors with respect to A will be given. These are used to express a general divisor D on $\overline{M}_{0,n}$ in terms of boundary classes and will enable one to locate where the divisor sits in the Néron–Severi space of $\overline{M}_{0,n}$ with respect to its effective cone of divisors.

The first way to write ψ_i for $i \in A$ as a sum of boundary classes comes from combining all the expressions for ψ_i given above such that $j, k \in A \backslash \{i\}$.

DEFINITION/LEMMA 2.5. Let $A \subseteq \{1 \ldots n\}$ with $a = |A| \geqslant 3$ and let $Z = A^c$ with $z = |Z|$. The *first A-average of ψ_i with $i \in A$* is

$$\psi_i = \sum_{\substack{i \in B \subset A \\ |B| = b}} \sum_{y} \frac{(a-b)(a-b-1)}{(a-1)(a-2)} \Delta_{A,B}^{Z,y}.$$

The second A-average is derived by writing down all such expressions for ψ_i, taking pairs $j \in A \backslash \{i\}$ and $k \in A^c$.

DEFINITION/LEMMA 2.6. Let $A \subset \{1 \ldots n\}$ with $a = |A| \geqslant 2$ and let $Z = A^c$ with $z = |Z| \geqslant 1$. The *second A-average of ψ_i with $i \in A$* is

$$\psi_i = \sum_{\substack{i \in B \subset A \\ |B| = b}} \sum_{y} \frac{(a-b)(z-y)}{(a-1)z} \Delta_{A,B}^{Z,y}.$$

The third A-average of ψ_i is generated by taking all expressions such that $j, k \in A^c$.

DEFINITION/LEMMA 2.7. Let $A \subset \{1 \ldots n\}$ with $a = |A| \geqslant 1$ and $Z = A^c$ with $z = |Z| \geqslant 1$. The *third A-average of ψ_i with $i \in A$* is

$$\psi_i = \sum_{\substack{i \in B \subset A \\ |B| = b}} \sum_{y} \frac{(z-y)(z-y-1)}{z(z-1)} \Delta_{A,B}^{Z,y}.$$

Finally, by taking all possible pairs $j, k \in \{1 \ldots n\} \backslash \{i\}$, one obtains the fourth expression for ψ_i in terms of the boundary classes. This A-average comes from taking the greatest number

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of ways of expressing the ψ_i as a sum of boundary divisors in this manner and is referred to as the *big average of* ψ_i; it can be found in [FG03, Lemma 1].

Definition/Lemma 2.8. For $i \subset \{1 \dots n\}$, the *fourth* or *big* A*-average of* ψ_i is

$$\psi_i = \sum_{Y \subset \{1 \dots n\} \setminus \{i\}} \frac{(n-1-y)(n-2-y)}{(n-1)(n-2)} \delta_{Y \cup \{i\}},$$

where y is the number of elements in the set Y.

Notice that the big A-average of ψ_i is the same as the third A-average when one takes $|Z| = 1$.

2.4 The c-averages of a divisor on $\overline{M}_{0,n}$

The main technique in this work is to use different ways to write certain divisors on $\overline{M}_{0,n}$ as

$$cK_{\overline{M}_{0,n}} + E,$$

where E is an effective sum of boundary classes. These expressions are called c-averages of a divisor.

For $D = \alpha\lambda - \sum_{2 \leqslant i \leqslant \lfloor g/2 \rfloor} b_i \delta_i$ on $\overline{M}_g$, the pullback $v^* f^* D$ on $\overline{M}_{0,n}$ of D along the so-called boundary restriction morphisms can be expressed as

$$v^* f^* D = b_1 \sum_{i \in Z = \{i | n_i = 1\}} \psi_i + \sum_{i \in A = \{1, \dots, a\}} b_{n_i} \psi_i - \sum_{\substack{B \subset A, 0 \leqslant y \leqslant |Z| \\ 2 \leqslant y + |B| \leqslant \lfloor (a+z)/2 \rfloor}} b_{y + \sum_{i \in B} n_i} \Delta_{A,B}^{Z,y}.$$

Upon replacing, in the expression above, the ψ_i for $i \in A$ or $i \in Z$ with combinations of the various A- or Z-averages, respectively, one obtains up to 12 different c-averages of the divisor $v^* f^* D$. When $a = 0$, there is just the big average. To give a flavor of what the expressions look like, three examples are given below.

Upon replacing the ψ_i by their big averages, one obtains the *big average of* D. We will consider the big averages of $f^* D$ and $v^* f^* D$ for D a divisor in $\overline{M}_g$. These big averages are given in the following definition/lemmas.

Definition/Lemma 2.9. Suppose that D is a divisor in $\overline{M}_g$. Let $f : \overline{M}_{0,g} \longrightarrow \overline{M}_g$ be the morphism given by attaching elliptic tails, and let $v : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,g}$ be a boundary restriction morphism. The *big average of* $f^* D$ is

$$f^* D = \sum_{i=2}^{\lfloor g/2 \rfloor} \left(\frac{i(g-i)}{g-1} b_1 - b_i \right) B_i,$$

and the *big average of* $v^* f^* D$ is

$$\begin{aligned} v^* f^* D &= \sum_{\substack{0 \leqslant y \leqslant z \\ B \subset A, |B| = b \\ 2 \leqslant y + b \leqslant \lfloor (a+z)/2 \rfloor}} \left(\frac{b_1 (y(a+z-y-b)(a+z-y-b-1) + (z-y)(y+b)(y+b-1))}{(a+z-1)(a+z-2)} \right. \\ &\quad + \frac{(a+z-y-b)(a+z-y-b-1) \sum_{i \in B} b_{n_i} + (y+b)(y+b-1) \sum_{i \in A \setminus B} b_{n_i}}{(a+z-1)(a+z-2)} \\ &\quad \left. - b_{y + \sum_{i \in B} n_i} \right) \Delta_{A,B}^{Z,y}. \end{aligned}$$

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Proof. Replace the ψ_i in the expressions for f^*D and v^*D in Lemma 2.4 by their big averages. □

In the next definition/lemma, the c-averages of f^*D and v^*f^*D are given.

DEFINITION/LEMMA 2.10. Suppose that D is a divisor in $\overline{M}_g$ and $c \geqslant 0$. Let $f : \overline{M}_{0,g} \longrightarrow \overline{M}_g$ be the morphism given by attaching elliptic tails, and let $v : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,g}$ be a boundary restriction morphism. The *big c-average of* v^*f^*D is

$$v^*f^*D = cK_{\overline{M}_{0,a+z}} + \sum_{\substack{B\subseteq A,|B|=b\\ 0\leqslant y\leqslant z, 2\leqslant y+b\leqslant\lfloor (a+z)/2\rfloor}} \bigg(f_{y,b}(b_1 - c) + g_{y,b}\sum_{i\in B}(b_{n_i} - c) + h_{y,b}\sum_{i\in B^c}(b_{n_i} - c) + 2c - b_{y+\sum_{i\in B} b_{n_i}}\bigg)\Delta^{Z,y}_{A,B},$$

where

$$g_{y,b} = \frac{(a+z-y-b)(a+z-y-b-1)}{(a+z-1)(a+z-2)},\quad h_{y,b} = \frac{(y+b)(y+b-1)}{(a+z-1)(a+z-2)},$$
$$f_{y,b} = y g_{y,b} + (z-y)h_{y,b},\quad \Delta^{Z,y}_{A,B} = \sum_{Y\subset Z,|Y|=y} \delta_{Y\cup B}.$$

Proof. Recall that from Lemma 2.4, if $v = v_{a,z} = [n_1 \dots n_a]$ is a boundary restriction of $\overline{M}_{0,g}$, then

$$v^*f^*D = b_1 \sum_{i\in Z=\{i|n_i=1\}} \psi_i + \sum_{i\in A=\{1,\dots,a\}} b_{n_i}\psi_i - \sum_{\substack{B\subset A, 0\leqslant y\leqslant |Z|\\ 2\leqslant y+|B|\leqslant\lfloor (a+z)/2\rfloor}} b_{y+\sum_{i\in B} n_i}\Delta^{Z,y}_{A,B}.$$

Using the relation $K_{\overline{M}_{0,g}} = \sum_{1\leqslant i\leqslant g}\psi_i - 2\Delta$, we rewrite the expression as

$$\begin{aligned} v^*f^*D &= \sum_{j\in A}(b_{n_j} - c)\psi_j + (b_1 - c)\sum_{j\in Z}\psi_j + c\sum_{j\in A\cup Z}\psi_j - \sum_{\substack{0\leqslant y\leqslant z, B\subseteq A\\ 2\leqslant y+|B|\leqslant\lfloor n/2\rfloor}} b_{y+\sum_{k\in B} n_k}\Delta^{Z,y}_{A,B}\\ &= \sum_{j\in A}(b_{n_j} - c)\psi_j + (b_1 - c)\sum_{j\in Z}\psi_j + cK_{\overline{M}_{0,g}} + \sum_{\substack{0\leqslant y\leqslant z, B\subseteq A\\ 2\leqslant y+|B|\leqslant\lfloor n/2\rfloor}} (2c - b_{y+\sum_{k\in B} n_k})\Delta^{Z,y}_{A,B}. \end{aligned}$$

By big-averaging the ψ_i and distributing the coefficients through the sum, one obtains the expression given in the theorem. □

Note that if $a = 0$, then $z = g$ and $v^*f^*D = f^*D$. Therefore the *big c-average of* f^*D is just

$$f^*D = cK_{\overline{M}_{0,g}} + \sum_{2\leqslant i\leqslant\lfloor g/2\rfloor}\bigg((b_1 - c)\frac{i(g-i)}{(g-1)} + 2c - b_i\bigg)B_i.$$

Moreover, as long as $c > 0$, one can write this as

$$f^*D = c\bigg(K_{\overline{M}_{0,g}} + \sum_{2\leqslant i\leqslant\lfloor g/2\rfloor}\bigg(\frac{2g-2-i(g-i)}{g-1} + \frac{b_1 i(g-i) - b_i(g-1)}{(g-1)c}\bigg)B_i\bigg).$$

DEFINITION/LEMMA 2.11. Suppose that D is a divisor in $\overline{M}_g$ and $c \geqslant 0$. Let $f : \overline{M}_{0,g} \longrightarrow \overline{M}_g$ be the morphism given by attaching elliptic tails, and let $v = [n_1 \dots n_a] : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,g}$ be a boundary restriction morphism such that $a \geqslant 2$ and $z \geqslant 2$. The *second c-average of* v^*f^*D is

$$v^*f^*D = cK_{\overline{M}_{0,a+z}} + \sum_{\substack{B\subseteq A, |B|=b \\ 0\leqslant y\leqslant z, 2\leqslant y+b\leqslant \lfloor (a+z)/2 \rfloor}} C_{y,B}\Delta^{Z,y}_{A,B},$$

where

$$C_{y,B} = \frac{(a-b)(z-y)\sum_{i\in B}(b_{n_i}-c) + by\sum_{i\in B^c}(b_{n_i}-c)}{(a-1)z} + \frac{(b_1-c)((a-b)(z-y)+by)}{a(z-1)} + 2c - b_{y+\sum_{i\in B} n_i}.$$

Proof. Recall that from Lemma 2.4, if $v = v_{a,z} = [n_1 \dots n_a]$ is a boundary restriction of $\overline{M}_{0,g}$, then

$$v^*f^*D = b_1 \sum_{i\in Z=\{i|n_i=1\}} \psi_i + \sum_{i\in A=\{1,\dots,a\}} b_{n_i}\psi_i - \sum_{\substack{B\subset A, 0\leqslant y\leqslant |Z| \\ 2\leqslant y+|B|\leqslant \lfloor (a+z)/2 \rfloor}} b_{y+\sum_{i\in B} n_i}\Delta^{Z,y}_{A,B},$$

where $\Delta^{Z,y}_{A,B} = \sum_{Y\subset Z, |Y|=y} \delta_{Y\cup B}$.

Using the relation $K_{\overline{M}_{0,g}} = \sum_{1\leqslant i\leqslant g} \psi_i - 2\Delta$, the expression can be rewritten as

$$\begin{aligned} v^*f^*D &= \sum_{j\in A}(b_{n_j}-c)\psi_j + (b_1-c)\sum_{j\in Z}\psi_j + c\sum_{j\in A\cup Z}\psi_j - \sum_{\substack{0\leqslant y\leqslant z, B\subseteq A \\ 2\leqslant y+|B|\leqslant \lfloor n/2 \rfloor}} b_{y+\sum_{k\in B} n_k}\Delta^{Z,y}_{A,B} \\ &= \sum_{j\in A}(b_{n_j}-c)\psi_j + (b_1-c)\sum_{j\in Z}\psi_j + cK_{\overline{M}_{0,g}} + \sum_{\substack{0\leqslant y\leqslant z, B\subseteq A \\ 2\leqslant y+|B|\leqslant \lfloor n/2 \rfloor}} (2c - b_{y+\sum_{k\in B} n_k})\Delta^{Z,y}_{A,B}. \end{aligned}$$

Now substitute the second A-average of the ψ_i for $i \in A$, with $Z = A^c$, and distribute the coefficients through the sum; also substitute the second Z-average of the ψ_i for $i \in Z$, with $A = Z^c$, and distribute the coefficients through the sum. This yields the expression given in the theorem. □

One could also combine different averages of the ψ_i. For example, by substituting the second A-average of the ψ_i for $i \in A$ and substituting the big Z-average of the ψ_i for $i \in Z$, with $A = Z^c$, and distributing the coefficients through the sum, one obtains the expression

$$v^*f^*D = c\bigg(K_{\overline{M}_{0,a+z}} + \sum_{\substack{B\subseteq A, |B|=b \\ 0\leqslant y\leqslant z, 2\leqslant y+b\leqslant \lfloor (a+z)/2 \rfloor}} \frac{n(\alpha)}{d(\alpha)} + \frac{n(\beta)}{d(\beta)}\Big/c\ \Delta^{Z,y}_{A,B}\bigg),$$

where

$$\begin{aligned} n(\alpha) &= (a+z-1)(a+z-2)(2(a-1)-b(a-b)) \\ &\quad - (a-1)((a+z-y-b)(a+z-y-b-1)y + (y+b)(y+b-1)(z-y)), \\ d(\alpha) &= (a-1)(a+z-1)(a+z-2), \\ n(\beta) &= (a+z-1)(a+z-2)((z-y)(a-b)\sum_{i\in B} b_{n_i} + yb\sum_{i\in B^c} b_{n_i}) \\ &\quad + b_1(a-1)z((a+z-y-b)(a+z-y-b-1)y + (y+b)(y+b-1)(z-y)), \\ d(\beta) &= (a+z-1)(a+z-2)(a-1)z. \end{aligned}$$

3. Reduction of the F-conjecture

In this section, proving the F-conjecture is reduced to showing that every F-divisor can be expressed as a sum of a nonnegative multiple of the canonical divisor and an effective sum of boundary divisors.

CONJECTURE 1. *Every F-divisor on $\overline{M}_{0,N}$ is of the form $cK_{\overline{M}_{0,N}} + E$ where $c \geqslant 0$ and E is an effective sum of boundary classes.*

It will be shown in Theorem 3.1 that if Conjecture 1 holds, then the F-conjecture holds on $\overline{M}_{g,n}$ for all pairs g and n such that $g + n \leqslant N$. The numerical criteria and algorithm in the next section for showing that a divisor is nef rest on this reduction of the F-conjecture to Conjecture 1.

THEOREM 3.1. *If Conjecture 1 is true on $\overline{M}_{0,N}$ for $N \leqslant g + n$, then the F-conjecture is true on $\overline{M}_{g,n}$. In particular, if Conjecture 1 is true, then the F-conjecture is true.*

Two facts are needed to explain how Conjecture 1 implies the F-conjecture. The first is that if the F-conjecture is true in $\overline{M}_{0,g+n}$, then it is true in $\overline{M}_{g,n}$. More precisely, let $f : \overline{M}_{0,g+n} \longrightarrow \overline{M}_{g,n}$ be the morphism associated to the map given by attaching pointed elliptic tails at each of the first g marked points.

THE BRIDGE THEOREM [GKM01, Theorem 0.3]. *A divisor D on $\overline{M}_{g,n}$ is nef if and only if:*

(i) *D is an F-divisor; and*

(ii) *f^*D is a nef divisor on $\overline{M}_{0,g+n}$.*

The following result is the second important fact needed to prove Theorem 3.1.

THE RAY THEOREM ([FG03, Theorem 4] and [KM96, Theorem 1.2]). *If R is an extremal ray of the cone of curves of $\overline{M}_{0,N}$ and if $R \cdot (K_{\overline{M}_{0,N}} + G) < 0$ where G is any effective sum of boundary components for which $\Delta \setminus G$ is nonnegative, then R is spanned by an F-curve.*

The symbol Δ denotes the sum of boundary classes. So the condition in the Ray Theorem is that $G = \sum_S a_S \delta_S$ such that $0 \leqslant a_S \leqslant 1$ for all S. The Ray Theorem is an extension of a result due to Keel and McKernan which states that if R is an extremal ray of $\overline{NE}(\overline{M}_{0,N})$ and $R \cdot (K_{\overline{M}_{0,N}} + G) \leqslant 0$ for $G = \sum_S a_S \delta_S$ such that $0 \leqslant a_S < 1$, then R is spanned by an F-curve.

Proof of Theorem 3.1. Suppose that whenever one has an F-divisor D on $\overline{M}_{0,N}$, there exists a constant $c \geqslant 0$ for which

$$D = cK_{\overline{M}_{0,N}} + E,$$

where E is an effective sum of boundary classes. We will show that this assumption implies that the F-conjecture is true on $\overline{M}_{g,n}$. By the Bridge Theorem, in order to prove the F-conjecture on $\overline{M}_{g,n}$, it is enough to show that any F-divisor on $\overline{M}_{0,g+n}$ is nef. Hence if can we show that our assumption implies that D is nef, then the theorem is proved.

By definition, if D nonnegatively intersects all the extremal rays of the cone of curves, then D is nef. Suppose that R is an extremal ray of the cone of curves. The first thing to note is that since D is an F-divisor, if R is spanned by an F-curve, then D nonnegatively intersects R. We will prove that there are no other kinds of extremal rays. We do this by induction on the number of marked points. As the base case we take $N = 7$, since the F-conjecture is true for $N \leqslant 7$ (see [KM96]).

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The cone of curves is the closure of $NE(\overline{M}_{0,N})$ in the real vector space $N_1(\overline{M}_{0,N})$. So every extremal ray R either is spanned by an irreducible curve or is the limit of rays spanned by irreducible curves.

Suppose that R is a D-negative extremal ray of the cone of curves of $\overline{M}_{0,N}$ for $N > 7$ which is not spanned by an F-curve. In other words, suppose that

$$R \cdot D = R \cdot (cK_{\overline{M}_{0,N}} + E) < 0.$$

In particular, by the Ray Theorem, $R \cdot E < 0$.

If R is spanned by a curve, then since E is an effective sum of boundary classes, to get a contradiction it is enough to show that D is nef when restricted to the components in the support of E. This results in pulling D back to a space $\overline{M}_{0,n}$ for $n < N$ along a boundary restriction morphism (defined in § 2). Since the pullback of an F-divisor along a boundary restriction morphism is an F-divisor, we can repeat this argument until we end up in $\overline{M}_{0,n}$ for $n \leqslant 7$.

If the extremal ray R is a limit of curves, then one can find a ray R' spanned by a curve which is close enough so that R' intersects both E and D negatively. In this case, one reaches a contradiction as above. □

As will be shown in Theorem 3.2 below, Conjecture 1 is true on $\overline{M}_{0,N}$ for $N \leqslant 6$. It was already known to be true with $c = 0$ (see [FG03]). However, the proof for the case with $c = 0$ is much harder, since showing that a divisor class is in the convex hull of boundary classes is more difficult than showing it is in the convex hull of boundary classes and the canonical divisor. For larger values of N, it seems unlikely that Conjecture 1 would be true with $c = 0$, even when $N = 7$.

Theorem 3.2. *If D is any divisor on $\overline{M}_{0,n}$ for $n = 5$ or 6, then there exists a constant $k > 0$ such that $D = cK_{\overline{M}_{0,n}} + E$ for all $c \geqslant k$, where E is an effective sum of boundary divisors. In particular, Conjecture 1 is true on $\overline{M}_{0,n}$ for $n \leqslant 6$.*

Proof of Theorem 3.2. First, suppose that $n = 5$. By substituting the big averages (see § 2 for definitions) of the divisors ψ_i, one can express the divisor D as

$$\begin{aligned} D &= \sum_{1 \leqslant i \leqslant 5} c_i \psi_i = c\bigg(\sum_{1 \leqslant i \leqslant 5} \psi_i - 2 \sum_{ij \in \{1 \ldots 5\}} \delta_{ij}\bigg) + \sum_{1 \leqslant i \leqslant 5} (c_i - c)\psi_i + 2c \sum_{ij \in \{1 \ldots 5\}} \delta_{ij} \\ &= cK_{\overline{M}_{0,5}} + \sum_{ij \in \{1 \ldots 5\}} \bigg(\frac{1}{2} \sum_{k \in \{i,j\}} (c_k - c) + \frac{1}{6} \sum_{k \in \{i,j\}^c} (c_k - c) + 2c\bigg)\delta_{ij} \\ &= cK_{\overline{M}_{0,5}} + \sum_{ij \in \{1 \ldots 5\}} \bigg(\frac{1}{2} \sum_{k \in \{i,j\}} c_k + \frac{1}{6} \sum_{k \in \{i,j\}^c} c_k + \frac{1}{2} c\bigg)\delta_{ij}. \end{aligned}$$

Similarly, when $n = 6$ one can write D as follows:

$$\begin{aligned} D &= \sum_{1 \leqslant i \leqslant 6} c_i \psi_i - \sum_{ij \in \{2 \ldots 6\}} b_{1ij} \delta_{1ij} \\ &= cK_{\overline{M}_{0,6}} + \sum_{ij \in \{1 \ldots 6\}} \bigg(\frac{1}{2} \sum_{k \in \{i,j\}} (c_k - c) + \frac{1}{10} \sum_{1 \leqslant k \leqslant 6} (c_k - c) + 2c\bigg)\delta_{ij} \end{aligned}$$

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$$+\sum_{ij\in\{2\dots6\}}\bigg(+\frac{3}{10}\sum_{1\leqslant k\leqslant 6}(c_k-c)+2c-b_{1ij}\bigg)\delta_{1ij}$$

$$=cK_{\overline{M}_{0,6}}+\sum_{ij\in\{1\dots6\}}\bigg(\frac{1}{2}(c_i+c_j)+\frac{1}{10}\sum_{1\leqslant k\leqslant 6}c_k+\frac{2c}{5}\bigg)\delta_{ij}$$

$$+\sum_{ij\in\{2\dots6\}}\bigg(\frac{3}{10}\sum_{1\leqslant k\leqslant 6}c_k+\frac{c}{5}-b_{1ij}\bigg)\delta_{ij}.$$

In either case, if c is taken to be big enough, then the assertion is true. □

4. Iterative procedures to show that a divisor on $\overline{M}_g$ is nef

By proving particular cases of Conjecture 1, one can use Theorem 3.1 to define an algorithm for proving that a divisor D in $\overline{M}_g$ is nef (see Theorem 4.7). The first step is the following result.

THEOREM 4.1. *If $D=b_1\sum_{1\leqslant i\leqslant g}\psi_i-\sum_{2\leqslant i\leqslant\lfloor g/2\rfloor}b_i\widetilde{B}_i$ is any F-divisor on $\widetilde{M}_{0,g}=\overline{M}_{0,g}/S_g$, then there exists a constant $c>0$ for which $D=cK_{\widetilde{M}_{0,g}}+E$, where E is an effective sum of boundary classes.*

To prove this, we will begin by showing that any nontrivial F-divisor on $\widetilde{M}_{0,g}$ is, in fact, big.

Notation 4.2. For positive integers i,j,k and $g-(i+j+k)$, denote by $F_{i,j,k,g-(i+j+k)}$ any F-curve determined by a partition $I\cup J\cup K\cup(N\setminus I\cup J\cup K)$ of $N=\{1,\dots,n\}$ with $|I|=i$, $|J|=j$ and $|K|=k$.

DEFINITION 4.3. Let $g=2k-1$ be a nonnegative odd integer; for a nonnegative integer l such that $1\leqslant 2l+1\leqslant g-3$, put

$$S^{2k-1}_{2l+1}=\sum_{i=1}^{k-l-2}F_{1,2l+1,i,2k-(i+2l+3)};$$

for a positive integer l such that $1\leqslant 2l\leqslant g-3$, put

$$S^{2k-1}_{2l}=\frac{1}{2}F_{1,2l,k-l-1,k-l-1}+\sum_{i=1}^{k-l-2}F_{1,2l,i,2k-(i+2l+2)}.$$

Let $g=2k$ be a nonnegative even integer; for $j=2l$ with $1\leqslant j\leqslant g-3$, put

$$S^{2k}_{2l}=\sum_{i=1}^{k-2l-1}F_{1,2l,i,2k-(i+2l+1)};$$

for $j=2l+1$ with $1\leqslant j\leqslant g-3$, put

$$S^{2k}_{2l+1}=\frac{1}{2}F_{1,2l+1,k-l-1,k-l-1}+\sum_{i=1}^{k-2l-2}F_{1,2l,i,2k-(i+2l+1)}.$$

LEMMA 4.4. *If $D=b_1\sum_{i=1}^{g}\psi_i-\sum_{i=2}^{\lfloor g/2\rfloor}b_iB_i$ is an F-divisor on $\widetilde{M}_{0,g}$, then:*

(i) $D\cdot S^{2k-1}_{2l+1}=(k-l-1)b_1+(k-l-2)b_{2l+1}-(k-l-1)b_{2l+2}$;

(ii) $D\cdot S^{2k-1}_{2l}=((2k-2l-1)/2)b_1+((2k-2l-3)/2)b_{2l}-((2k-2l-1)/2)b_{2l+1}$;

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(iii) $D \cdot S_{2l}^{2k} = (k-l)b_1 + (k-l-1)b_{2l} - (k-l)b_{2l+1}$;

(iv) $D \cdot S_{2l+1}^{2k} = ((2k-2l-1)/2)b_1 + ((2k-2l-3)/2)b_{2l+1} - ((2k-2l-1)/2)b_{2l+2}$.

Proof. To show (i),

$$\begin{aligned} S_{2l+1}^{2k-1} &= \sum_{i=1}^{k-l-2} D \cdot F_{1,2l+1,i,2k-(i+2l+2)} \\ &= \sum_{i=1}^{k-l-2} (b_1 + b_{2l+1} + b_i + b_{i+2l+2} - b_{2l+2} - b_{1+i} - b_{i+2l+1}) \\ &= (k-l-2)(b_1 + b_{2l+1} - b_{2l+2}) + \sum_{i=1}^{k-l-2} (b_i + b_{i+2l+2} - b_{1+i} - b_{i+2l+1}) \\ &= (k-l-2)(b_1 + b_{2l+1} - b_{2l+2}) + b_1 + b_{k+l} - b_{k-l-1} - b_{2l+2} \\ &= (k-l-1)b_1 + (k-l-2)b_{2l+1} - (k-l-1)b_{2l+2}. \end{aligned}$$

Since $g = 2k-1$ in this case, one has that $b_{k+l} = b_{2k-1-(k+l)} = b_{k-l-1}$. The computations for (ii)–(iv) are analogous. □

Proposition 4.5. If $D = b_1 \sum_{i=1}^{g} \psi_i - \sum_{i=2}^{\lfloor g/2 \rfloor} b_i \widetilde{B}_i$ is a nontrivial F-divisor on $\widetilde{M}_{0,g} \cong \overline{M}_{0,g}/S_g$, then for $i \in \{2, \ldots, \lfloor g/2 \rfloor\}$ one has that $(i(g-i)/(g-1))b_1 - b_i > 0$. In particular, D is big.

Proof. One can average the ψ_i to express D as $D = \sum_{i=2}^{\lfloor g/2 \rfloor} \big((i(g-i)/(g-1))b_1 - b_i\big)\widetilde{B}_i$; so if the first assertion of the proposition is true, then D is big. Let us first show that each of the coefficients is nonnegative; we then show that if any of the coefficients is zero, D is trivial.

Notice that for $g = 2k-1$,

$$\frac{2(g-2)}{(g-1)} b_1 - b_2 = \frac{2k-3}{(k-1)} b_1 - b_2 = \frac{1}{(k-1)} D \cdot S_1^{2k-1},$$

while for $g = 2k$,

$$\frac{2(g-2)}{(g-1)} b_1 - b_2 = \frac{4(k-1)}{(2k-1)} b_1 - b_2 = \frac{2}{(2k-1)} D \cdot S_1^{2k}.$$

The next computation will show that for $j \geqslant 2$, upon putting $D \cdot S_j^g = c_1 b_1 + c_j b_j - c_{j+1} b_{j+1}$ one has that

$$\frac{(j+1)(g-j-1)}{(g-1)} b_1 - b_{j+1} = \frac{1}{c_{j+1}} \left(S_j^g + c_j \left(\frac{j(g-j)}{(g-1)} b_1 - b_j \right) \right).$$

In the following, the details will be given for the cases where $g = 2k-1$ and $j = 2l+1$ are both odd and where $g = 2k$ and $j = 2l$ are both even. The other two cases are completely analogous.

When g is odd,

$$\begin{aligned} &\frac{1}{(k-l-1)} \left(D \cdot S_j^{2k-1} + (k-1-2) \left(\frac{j(g-j)}{(g-1)} b_1 - b_j \right) \right) \\ &\quad = \frac{1}{(k-l-1)} \big((k-l-1)b_1 + (k-l-2)b_j - (k-l-1)b_{j+1} \big) \\ &\qquad + \frac{(k-1-2)}{(k-l-1)} \left(\frac{(2l+1)(2k-2l)}{2(k-1)} b_1 - b_j \right) \end{aligned}$$

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$$= \frac{1}{(k-l-1)}\left((k-l-1)b_1 + \frac{(2l+1)(k-l-2)(k-l-1)}{(k-1)}b_1 - (k-l-1)b_{j+1}\right)$$
$$= \frac{1}{(k-l-1)}\left(\frac{(k-l-1)((k-l)+(2l+1)(k-l-2))}{(k-1)}b_1 - (k-l-1)b_{j+1}\right)$$
$$= \frac{(k-l)+(2l+1)(k-l-2)}{(k-1)}b_1 - b_{j+1} = \frac{(l+1)(2k-2l-3)}{(k-1)}b_1 - b_{j+1}$$
$$= \frac{(j+1)(g-(j+1))}{(g-1)}b_1 - b_{j+1}.$$

When g is even,

$$\frac{1}{(k-l)}\left(D\cdot S_j^{2k} + (k-l-1)\left(\frac{j(2k-j)}{(2k-1)}b_1 - b_j\right)\right)$$
$$= b_1 + \frac{(k-l-1)j(2k-j)}{(k-l)(2k-1)}b_1 - b_{j+1} = \frac{(k-l)(2l+1)(2k-2l-1)}{(k-l)(2k-1)}b_1 - b_{j+1}$$
$$= \frac{(j+1)(2k-j-1)}{(2k-1)}b_1 - b_{j+1}.$$

When $j = 2l+1$ is odd,

$$\frac{2}{2k-2l-1}\left(S_{2l+1}^{2k} + \frac{(2k-2l-3)}{2}\left(\frac{(2l+1)(2k-2l-1)}{(2k-1)}b_1 - b_{2l+1}\right)\right)$$
$$= \frac{(2l+2)(2k-2l-2)}{(2k-1)}b_1 - b_{2l+2}.$$

Next, we show that if $D\cdot S_1 = 0$, then D is trivial. Suppose that $D\cdot S_1 = 0$; then, in particular, $D\cdot F_{1,1,i,g-i-2} = 0$ for all $1 \leqslant i \leqslant g-3$ and $(2(g-2)/(g-1))b_1 - b_2 = 0$. Note that $D\cdot F_{1,1,1,g-3} = 3b_1 + b_3 - 3b_2 = 0$ and $(2(g-2)/(g-1))b_1 - b_2 = 0$ imply that

$$0 = 3b_1 + b_3 - 3b_2 = 3b_1 + b_3 - 3\frac{2(g-2)}{(g-1)}b_1 = b_3 - \frac{3(g-3)}{(g-1)}b_1,$$

and so $(3(g-3)/(g-1))b_1 - b_3 = 0$. Now, if

$$\frac{i(g-i)}{(g-1)}b_1 - b_i = 0 \quad \text{and} \quad \frac{(i+1)(g-(i+1))}{(g-1)}b_1 - b_{i+1} = 0,$$

then since

$$D\cdot F_{1,1,i,g-(i+2)} = 2b_1 + b_i + b_{i+2} - b_2 - 2b_{i+1} = 0,$$

we have

$$b_{i+2} = \left(\frac{2(g-2)}{(g-1)} + \frac{2(i+1)(g-(i+1))}{(g-1)} - \frac{2(g-1)}{(g-1)} - \frac{i(g-i)}{(g-1)}\right)b_1 = \frac{(i+2)(g-(i+2))}{(g-1)}b_1.$$

Finally, if any of the coefficients is zero, then $D\cdot S_1 = 0$, which implies that D is trivial. □

Proof of Theorem 4.1. Assume that $D = b_1\sum_{1\leqslant i\leqslant g}\psi_i - \sum_{2\leqslant i\leqslant\lfloor g/2\rfloor} b_i\widetilde{B}_i$ is any F-divisor on $\widetilde{M}_{0,g} = \overline{M}_{0,g}/S_g$. Using the fact (from [KM96, Lemma 4.5]) that

$$K_{\widetilde{M}_{0,g}} = -\frac{(g+1)}{2(g-1)}\widetilde{B}_2 + \sum_{3\leqslant i\leqslant\lfloor g/2\rfloor}\frac{2(g-1)-i(g-i)}{(g-1)}\widetilde{B}_i,$$

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we consider the c-average of D:

$$D = c\bigg(K_{\widetilde{M}_{0,g}} + \frac{c(g+1)+2(2(g-2)b_1-(g-1)b_2)}{2c(g-1)}\widetilde{B}_2 + \sum_{3\leqslant i\leqslant \lfloor g/2\rfloor}\bigg(\frac{2g-2-i(g-i)}{g-1}+\frac{b_1 i(g-i)-b_i(g-1)}{(g-1)c}\bigg)\widetilde{B}_i\bigg).$$

Assume first that $g \geqslant 8$, and for $i \geqslant 3$ let $c_i = \alpha_i + (\beta_i/c)$, where $\alpha_i = (2g-2-i(g-i))/(g-1)$ and $\beta_i = (i(g-i)b_1 - (g-1)b_i)/(g-1)$. Note that by Proposition 4.5, since D is a nontrivial F-divisor, we have that $i(g-i)b_1 > (g-1)b_i$. In particular, for $c \geqslant 0$, the coefficient of $\widetilde{B}_2$ is positive. Moreover, $\beta_i > 0$ for all $i \geqslant 3$. As will be shown, $\alpha_i < 0$ for all $i \geqslant 3$. Indeed, the numerator $n(i) = 2g-2-i(g-i)$ is negative: $n(i)$ is decreasing with respect to i since $\partial n/\partial i = i-g$, and $n(3) = 7-g < 0$ for $g \geqslant 8$.

We shall argue that there is a positive c for which all the coefficients c_i, for $i \geqslant 3$, are positive. For all i, the functions $\alpha_i + \beta_i/c$ have vertical asymptotes at $c = 0$.

For $i \geqslant 3$, the function $\alpha_i + \beta_i/c$ is concave up and decreasing, crossing the c axis when $c = \beta_i/\alpha_i > 0$. Hence we can take

$$c = \min\bigg\{\frac{\beta_i}{\alpha_i}\ \bigg|\ 3 \leqslant i \leqslant \bigg\lfloor\frac{g}{2}\bigg\rfloor\bigg\}.$$

In the $g = 7$ case, one has the relation $2b_1 \geqslant b_3$ from intersecting D with the F-curve given by the 4-tuple $[1:1:2:3]$. In this case,

$$D = c\bigg(K_{\widetilde{M}_{0,7}} + \bigg(\frac{2c+5b_1-3b_2}{3c}\bigg)\widetilde{B}_2 + \bigg(\frac{2b_1-b_3}{c}\bigg)\widetilde{B}_3\bigg).$$

In particular, one must take c so that $c + 5b_1 - 3b_2 \geqslant 0$. But, by Proposition 4.5, $5b_1 - 3b_2 \geqslant 0$, so this just requires that $c \geqslant 0$. Since, by Theorem 3.2, the result holds more generally for $g \leqslant 6$, Theorem 4.1 is proved. $\square$

This result was known to be true for $c = 0$ (see [FG03]). As was pointed out in [FG03], the problem of showing that a particular F-divisor D on $\overline{M}_g$ is nef can therefore be reduced to showing that $f^*D = E$ is nef when restricted to all of the boundary divisors in the support of E. However, as will be shown in Theorem 4.7, that it works for $c > 0$ is a drastic improvement, since one can then immediately reduce the problem of showing that a particular F-divisor is nef to showing that it is nef when restricted to the boundary divisors in the support of E having coefficient larger than c.

To prove the next theorem, it will be necessary to refer to the boundary restriction morphisms and c-averages defined in § 2.4. In particular, so-called 'necessary' boundary restriction morphisms will be considered.

Definition 4.6. Let D be a divisor on $\overline{M}_{0,g}$, and suppose that the c-average of v^*D is of the form $cK_{\overline{M}_{0,a+z}} + E$ where $c \geqslant 0$, E is an effective sum of distinct boundary classes, and $v_{a,z} = [n_1 \ldots n_{a-1}] : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,g}$ is any boundary restriction morphism. We define *necessary boundary restrictions* to be the boundary restrictions v_S and v_{S^c} such that the coefficient of δ_S in this expression is greater than c. Here, for $S \subset \{p_1 \ldots p_{a+z}\}$, one defines v_S to be the boundary restriction morphism

$$v_S = \bigg[\sum_{i\in S\cap A} n_i + |S \cap Z|, \{n_i\}_{i\in S^c\cap A}\bigg].$$

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Recall that $A = \{p_i \in \{p_1 \dots p_{a+z}\} \mid n_i \geqslant 2\}$ is the set of attaching points of the boundary restriction morphism and that $Z = \{p_i \in \{p_1 \dots p_{a+z}\} \mid n_i = 1\}$ is the set of points to which nothing is attached.

THEOREM 4.7. *Consider an F-divisor of the form $D = b_1 \sum_{i=1}^{g} \psi_i - \sum_{i=2}^{\lfloor g/2 \rfloor} b_i \widetilde{B}_i$ on $\widetilde{M}_{0,g} = \overline{M}_{0,g}/S_g$. If for each composition of necessary boundary restrictions v there exists a constant $c_v \geqslant 0$ such that*

$$v^* D = c_v K + E,$$

where E is an effective sum of boundary classes, then D is nef.

Proof. By Theorem 4.1, the divisor D is of the form $cK_{\widetilde{M}_{0,g}} + E$ where $c \geqslant 0$ and E is an effective sum of distinct boundary classes. Exactly as in the proof of Theorem 3.1, the Ray Theorem gives that D is nef as long as it nonnegatively intersects all curves in the support of any component of E with coefficient larger than c. In other words, supposing that the coefficient of δ_S is larger c, it is then enough to show that D is nef when restricted to Δ_S; that is, it is enough to show that $v^* D$ is nef for $v_S = [S]$ and $v_{S^c} = [S^c]$. By hypothesis,

$$v_S^* D = c_v K_{\overline{M}_{0,1+g-|S|}} + E,$$

where E is an effective sum of distinct boundary classes. Repeating this argument, it suffices to show that for each composition of necessary boundary restrictions v, there exists a constant $c_v \geqslant 0$ such that

$$v^* D = c_v K + E,$$

where E is an effective sum of boundary classes. Eventually, the process must stop, since the F-conjecture is known to be true on $\overline{M}_{0,N}$ for $N \leqslant 7$ (see [KM96]). □

To have a computer check that any composition of necessary boundary restrictions of an F-divisor on $\widetilde{M}_{0,g} = \overline{M}_{0,g}/S_g$ always restricts to a divisor on $\overline{M}_{0,a+z}$ of the form $cK_{\overline{M}_{0,a+z}} + E$, one can use any of the c-averages defined in § 2.4.

THEOREM/ALGORITHM 4.8. *Let D be an F-divisor of the form $a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ on $\overline{M}_g$. If the c-average $v^* f^* D = cK + E$ of any necessary boundary restriction v of $f^* D$ is effective, then D is nef.*

Proof. This follows from Theorem 4.7 and the Bridge Theorem. □

5. Numerical criteria

In this section, as an application of Theorem 3.1 and the iterative procedures given in § 4, we give numerical criteria which guarantee that divisors on $\overline{M}_g$ are nef. These criteria can be viewed as a way of carving the cone of F-divisors on $\overline{M}_g$ into nef sub-cones. As is explained in the next section, these sub-cones cover the entire F-cone for $g \leqslant 24$.

COROLLARY 5.1. *Let $D = a\lambda - \sum_{0 \leqslant i \leqslant \lfloor g/2 \rfloor} b_i \delta_i$ be an F-divisor on $\overline{M}_g$. If for $i \in \{2, \dots, \lfloor g/2 \rfloor\}$,*

$$-b_0(g-1) \leqslant i(g-i)(b_1 - b_0) + (g-1)(b_0 - b_i) \leqslant 0,$$

then D is nef.

Proof. First, using Mumford's identity on $\overline{M}_g$, we have

$$-\delta_0 = -12\lambda + \kappa_1 + \sum_{1\leqslant i\leqslant \lfloor g/2\rfloor} \delta_i.$$

Write

$$D = (a - 12b_0)\lambda + b_0\kappa_1 + \sum_{1\leqslant i\leqslant \lfloor g/2\rfloor} (b_0 - b_i)\delta_i;$$

then, by Lemma 2.4, one has

$$f^*D = b_0\kappa_1 + (b_1 - b_0)\sum_{1\leqslant i\leqslant g} \psi_i + \sum_{2\leqslant i\leqslant \lfloor g/2\rfloor} (b_0 - b_i)B_i.$$

Substituting the relation $\kappa_1 = K_{\overline{M}_{0,g}} + \sum B_i$ and then big-averaging the ψ_i, one obtains

$$\begin{aligned} f^*D &= b_0K_{\overline{M}_{0,g}} + (b_1 - b_0)\sum_{1\leqslant i\leqslant g} \psi_i + \sum_{2\leqslant i\leqslant \lfloor g/2\rfloor} (2b_0 - b_i)B_i \\ &= b_0K_{\overline{M}_{0,g}} + \sum_{2\leqslant i\leqslant \lfloor g/2\rfloor} \left(\frac{i(g-i)}{(g-1)}(b_1 - b_0) + 2b_0 - b_i\right)B_i. \end{aligned}$$

It is enough to show that under the given hypothesis, the coefficients of the B_i above are nonnegative and no greater than b_0, so that by the Ray Theorem f^*D, and hence D, is nef; that is, for $2 \leqslant i \leqslant \lfloor g/2\rfloor$,

$$\begin{aligned} -b_0(g-1) &\leqslant i(g-i)b_1 + (i^2 - ig + g - 1)b_0 - (g-1)b_i \\ &= i(g-i)(b_1 - b_0) + (g-1)(b_0 - b_i) \leqslant 0. \end{aligned}$$

But this is true by assumption. □

Corollary 5.2. *Let $D = a\lambda - \sum_{0\leqslant i\leqslant \lfloor \frac{g}{2}\rfloor} b_i\delta_i$ be an F-divisor on $\overline{M}_g$. If there exists a constant $c \geqslant 0$ such that*

$$\frac{2g - 2 - i(g-i)}{g-1} + \frac{b_1 i(g-i) - b_i(g-1)}{(g-1)c} \leqslant c$$

for all $i \in \{2 \ldots \lfloor g/2\rfloor\}$, then D is nef.

Proof. By the Bridge Theorem, D is nef as long as f^*D is nef. To show that the assumptions in the theorem guarantee that f^*D is nef, use the proof of Theorem 4.1 in conjunction with the Ray Theorem. □

It appears that Corollary 5.2 cannot be improved using Mumford's criteria.

A divisor that does not meet the conditions above can, of course, still be nef. For example, in Corollary 5.2, if no matter what constant c is tried there is a boundary class in the support of D with a coefficient larger than c, then more needs to be done to show that D is nef. In particular, one can still prove D is nef by showing that the divisor is nef when restricted to the boundary component whose class has coefficient bigger than c. By assuming more about the divisor (for instance, that every boundary restriction has to be nef), one obtains the criteria stated in the next two corollaries. The first, Corollary 5.3, comes from applying Theorem 4.7 with $c = 0$. The remaining criteria of the section are all consequences of this fact; each provides an easy-to-check condition which guarantees that a divisor on $\overline{M}_g$ is nef.

Corollary 5.3. *An F-divisor $D = a\lambda - \sum_{i=0}^{\lfloor g/2\rfloor} b_i\delta_i$ on $\overline{M}_g$ is nef provided that $b_i \leqslant b_1$ for all $i \geqslant 2$.*

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Proof. Let D be as described in the hypothesis. It will be shown that any boundary restriction of an F-divisor f^*D is equivalent to an effective sum of boundary classes. For simplicity of notation, put $D = f^*D$.

Let $v = v_{a,z} : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,g}$ be a boundary restriction where we attach an (n_i+1)-pointed curve (note that $n_i + 1 \geqslant 3$ here) to each point $p_i \in A$ with $|A| = a$. Then, as we have seen in Lemma 2.4,

$$v^*D = b_1 \sum_{i \in Z=\{i|n_i=2\}} \psi_i + \sum_{i\in A=\{1,\ldots,a\}} b_{n_i}\psi_i - \sum_{\substack{B\subset A, y\leqslant |Z|\\ 2\leqslant y+|B|\leqslant \lfloor g/2\rfloor}} b_{y+\sum_{i\in B} n_i}\Delta^{Z,y}_{A,B},$$

where $\Delta^{Z,y}_{A,B} = \sum_{Y\subset Z,|Y|=y} \delta_{Y\cup B}$.

The proof is divided into three cases: $z \geqslant 4$, $z = 3$ and $z = 2$. First, suppose that $z \geqslant 4$. Let $Z = \{1, \ldots, z\}$. By averaging the ψ_i, the divisor D_z satisfies

$$D_z = \sum_{i\in Z} \psi_i - \sum_{\substack{S\subset Z\\ 2\leqslant s=|S|\leqslant \lfloor z/2\rfloor}} \delta_S = \sum_{\substack{S\subset Z\\ 2\leqslant s=|S|\leqslant \lfloor z/2\rfloor}} \left(\frac{(s-1)z - s^2 + 1}{z-1}\right)\delta_S$$

in $\overline{M}_{0,z}$, with each coefficient positive as long as $z \geqslant 4$. To see this, put $f(s) = (s-1)z - s^2 + 1$. Then $f'(s) = z - 2s \geqslant 0$ since $s \leqslant z/2$. So the function $f(s)$ is increasing in the range that we are interested in. Now, as $f(2) \geqslant 1$, f is always positive. In particular, $\pi_a^*(D_z)$ is an effective sum of boundary classes in $\overline{M}_{0,a+z}$. Let $\pi_a : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,z}$ be the morphism which drops the attaching points $p_i \in A$. Then

$$v^*D - b_1\pi_a^*(D_z) = \sum_{i\in A} b_{n_i}\psi_i - \sum_{I\subset A} b_{\sum_{i\in I} n_i}\Delta^{Z,0}_{A,I} + \sum_{\substack{y>0, I\subset A\\ 0\leqslant |I|\leqslant a}} (b_1 - b_{y+\sum_{i\in I} n_i})\Delta^{Z,y}_{A,I}.$$

For $y > 0$, the coefficients of the classes $\Delta^{Z,y}_{A,I}$ are nonnegative since, by hypothesis, $b_1 \geqslant b_i$ for all i. Fix two elements $p, q \in Z$. Then, for $i \in A$, we have $\psi_i = \sum_{I\subset\{p,q\}^c} \delta_{I\cup i}$ and so

$$\sum_{i\in A} b_{n_i}\psi_i - \sum_{I\subset A} b_{\sum_{i\in I} n_i}\Delta^{Z,0}_{A,I} = \sum_{I\subset A}\left(\sum_{i\in I} b_{n_i} - b_{\sum_{i\in I} n_i}\right)\Delta^{Z,0}_{A,I} + E,$$

where E is an effective sum of boundary classes. That the coefficients $(\sum_{i\in I} b_{n_i} - b_{\sum_{i\in I} n_i})$ are nonnegative is a consequence of the assumption of D being an F-divisor, so that its coefficients satisfy property (v) of Definition/Theorem 2.1.

Now suppose that $z = 3$. Upon replacing the ψ_i for $i \in Z$ by their averages and using the same partial average for the ψ_i for $i \in A$, as was done in the previous case, we get that

$$\begin{aligned} v^*D = &\sum_{\substack{B\subset A,|B|=b\\ 2\leqslant b\leqslant a}} \left(\frac{b_1 3b(b-1)}{(a+1)(a+2)} + \left(\sum_{i\in B} b_{n_i} - b_{\sum_{i\in B} n_i}\right)\right)\Delta^{Z,0}_{A,B} \\ &+ \sum_{\substack{B\subset A,|B|=b\\ 1\leqslant b\leqslant a-1}} \left(\frac{b_1 2(2+b)(1+b)}{(a+2)(a+1)} + (b_1 - b_{1+\sum_{i\in B} n_i})\right)\Delta^{Z,y}_{A,B}. \end{aligned}$$

These coefficients are nonnegative, since by hypothesis $b_1 \geqslant b_i \geqslant 0$ for all i and by assumption D is an F-divisor so that, owing to Definition/Theorem 2.1, $(\sum_{i\in I} b_{n_i} - b_{\sum_{i\in I} n_i}) \geqslant 0$.

Consider the case $z = 2$. For $p \in Z$, we form a partial average of ψ_p by taking $q = Z \setminus p$ and fixing any $i \in A$ such that $\psi_p = \sum_{I\subset A\setminus\{i\}} \delta_{I\cup p}$. There are a ways of fixing such a point $i \in A$.

So $a\psi_p = \sum_{I\subset A,|I|=i}(a-i)\delta_{I\cup p}$ and hence, for $Z=\{p,q\}$,

$$(\psi_p+\psi_q) = \sum_{\substack{I\subset A,|I|=i\\ 1\leqslant i\leqslant a-1}} \left(\frac{a-i}{a} + \frac{a-(a-i)}{a}\right)\Delta^{Z,1}_{A,I} = \sum_{\substack{I\subset A\\ 1\leqslant |I|\leqslant a-1}} \Delta^{Z,1}_{A,I}.$$

Once again, by replacing the ψ_i for $i\in A$ as was done in the two previous cases, we get that

$$v^*D = \sum_{\substack{B\subset A\\ 2\leqslant |B|\leqslant a}} \left(\sum_{i\in B} b_{n_i} - b_{\sum_{i\in B} n_i}\right)\Delta^{Z,0}_{A,B} + \sum_{\substack{B\subset A\\ 1\leqslant |B|\leqslant a-1}} (b_1 - b_{1+\sum_{i\in B} n_i})\Delta^{Z,y}_{A,B}.$$

These coefficients are nonnegative by assumption. Therefore, any F-divisor $D = b_1\sum_{i=1}^{g}\psi_i - \sum_{i=2}^{\lfloor g/2\rfloor} b_i B_i$ in $\overline{M}_{0,g}$ such that $b_i \leqslant b_1$ for all i is nef. □

COROLLARY 5.4. *An F-divisor $D = a\lambda - \sum_{i=0}^{\lfloor g/2\rfloor} b_i\delta_i$ on $\overline{M}_g$ is nef provided that*

$$2\min\{b_i \mid i\geqslant 1\} \geqslant \max\{b_i \mid i \geqslant 1\}.$$

Proof. Let D be as described in the hypothesis. It will be shown that any boundary restriction of an F-divisor f^*D is equivalent to $cK+E$, where E is an effective sum of boundary classes for some $c\geqslant 0$. For simplicity of notation, put $D = f^*D$.

Let $v = v_{a,z} : \overline{M}_{0,a+z} \longrightarrow \overline{M}_{0,g}$ be a boundary restriction where we attach an (n_i+1)-pointed curve to each point $p_i\in A$, with $|A| = a$, and renumber the z points $q_i \in Z$. Then, as we have seen in Lemma 2.4, for $Z = \{i \mid n_i = 1\}$ and $A = \{i \mid n_i > 1\}$,

$$\begin{aligned}
v^*D &= b_1\sum_{i\in Z}\psi_i + \sum_{i\in A} b_{n_i}\psi_i - \sum_{\substack{B\subset A, y\leqslant |Z|\\ 2\leqslant y+|B|\leqslant \lfloor g/2\rfloor}} b_{y+\sum_{i\in B} n_i}\Delta^{Z,y}_{A,B}\\
&= (b_1 - c)\sum_{i\in Z}\psi_i + \sum_{i\in A}(b_{n_i}-c)\psi_i + c\sum_{i\in A\cup Z}\psi_i - \sum_{\substack{B\subset A, y\leqslant |Z|\\ 2\leqslant y+|B|\leqslant \lfloor g/2\rfloor}} b_{y+\sum_{i\in B} n_i}\Delta^{Z,y}_{A,B}\\
&= (b_1 - c)\sum_{i\in Z}\psi_i + \sum_{i\in A}(b_{n_i}-c)\psi_i + cK_{\overline{M}_{0,a+z}} + \sum_{\substack{B\subset A, y\leqslant |Z|\\ 2\leqslant y+|B|\leqslant \lfloor g/2\rfloor}} (2c - b_{y+\sum_{i\in B} n_i})\Delta^{Z,y}_{A,B},
\end{aligned}$$

where $\Delta^{Z,y}_{A,B} = \sum_{Y\subset Z,|Y|=y}\delta_{Y\cup B}$. Recall, as explained in § 2, that each class ψ_i is equivalent to an effective sum of boundary classes. So, as long as $c \leqslant b_i \leqslant 2c$ for all i, we have $v^*D = cK_{\overline{M}_{0,a+z}} + E$ as required. Just take $c\in[\max\{b_i \mid i\geqslant 1\}/2, \min\{b_i \mid i \geqslant 1\}]$, which, by hypothesis, is a nonempty interval. □

COROLLARY 5.5. *Let $D = a\lambda - \sum_{i=0}^{\lfloor g/2\rfloor} b_i\delta_i$ be an F-divisor on $\overline{M}_g$. If g is odd and $b_j = 0$ or if g is even and $b_j = 0$ for $j < g/2$, then D is nef.*

To prove Corollary 5.5, the following result will be used.

LEMMA 5.6. *If $D = a\lambda - \sum_{i=0}^{\lfloor g/2\rfloor} b_i\delta_i$ is an F-divisor in $\overline{M}_g$ such that $b_i = 0$, then:*

(i) *$b_j = b_k$ for all j and k such that $j + k = i$;*

(ii) *$b_j = b_{i+j}$ for all $j\geqslant 1$ such that $i + j \leqslant g-1$.*

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Proof. Since D is an F-divisor, $b_{g-(j+k)} + b_j - b_k \geqslant 0$ and $b_{g-(j+k)} + b_k - b_j \geqslant 0$. But $b_{g-(j+k)} = b_i = 0$, so $b_j = b_k$. The second assertion follows from the fourth type of inequality $b_i + b_j \geqslant b_{i+j}$, which, since $b_i = 0$, gives that $b_j \geqslant b_{i+j}$. By substituting $b_j = b_{g-j}$ and $b_{i+j} = b_{g-(i+j)}$, one obtains $b_{i+j} \geqslant b_j$. □

Proof of Corollary 5.5. Let $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ in $\overline{M}_g$ be an F-divisor such that $b_j = 0$ for some j. The result will be proved by induction on j. Of course, if $b_1 = 0$, the divisor is trivial and there is nothing to prove. If $b_2 = 0$, then by Lemma 5.6, $b_2 = b_{2x} = 0$ for all x such that $2x \leqslant g - 1$ and $b_1 = b_{1+2x}$ for all x such that $1 + 2x \leqslant g - 1$. Therefore, $b_i \leqslant b_1$ for all i so that, by Corollary 5.3, D is nef.

Suppose that $b_k = 0$ for some $3 \leqslant k < \lfloor g/2 \rfloor$ and that the statement is true when $b_i = 0$ for all $i < k$. Consider m such that $mk \leqslant g - 1$ but $(m + 1)k > g - 1$. By Lemma 5.6, $0 = b_k = b_{mk} = b_{g-mk}$. Then $g - mk < k$, and so $b_{g-mk} = 0$ means that, by induction, the statement is true.

Now suppose that $g = 2n - 1$ is odd and $b_{\lfloor g/2 \rfloor} = b_n = 0$. Then, by Lemma 5.6, $b_n = b_{2n} = b_1 = 0$. Hence $b_i = 0$ for all $i \geqslant 1$ and D satisfies Corollary 5.3. □

6. Using the Nef Wizard to show that the criteria prove Conjecture 1 for low values of *g*

One can show via a computer check that all the F-divisors in $\overline{M}_g$, for at least $g \leqslant 24$, are nef.

THEOREM 6.1. *The F-conjecture is true on $\overline{M}_{0,g}/S_g$ for $g \leqslant 24$.*

COROLLARY 6.2. *The F-conjecture is true on $\overline{M}_g$ for $g \leqslant 24$.*

Proof of Corollary 6.2. Apply [GKM01, Theorem 0.7]. □

The procedure for doing so is explained in this section. The starting point is that, by [GKM01], the conjecture on $\overline{M}_g$ is equivalent to the conjecture on $\widetilde{M}_{0,g} = \overline{M}_{0,g}/S_g$. In particular, if one can prove that the extremal F-divisors on $\widetilde{M}_{0,g}$ are nef, then the F-conjecture is true on $\overline{M}_g$. The computer program Nef Wizard generates the extremal F-divisors on $\widetilde{M}_{0,g}$ in terms of the sums of boundary classes $\widetilde{B}_i$. Nef Wizard finds F-divisors on $\overline{M}_g$ that pull back to the extremal divisors via f so that the criteria may be applied.

To prove Theorem 6.1, the following result will be used.

LEMMA 6.3. *Let $E = \sum_{2 \leqslant i \leqslant \lfloor g/2 \rfloor} e_i\widetilde{B}_i$ be a divisor on $\widetilde{M}_{0,g}$, and consider*

$$D_E = a\lambda - b_0\delta_0 - b_1\delta_1 - \sum_{2 \leqslant i \leqslant \lfloor g/2 \rfloor} \left(\frac{i(g-i)}{(g-1)}b_1 - e_i\right)\delta_i$$

where:

(i) $b_1 = \max\{0, ((g-1)/(i(g-i)))e_i, ((g-1)/(2ij))(e_i + e_j - e_{i+j}) \mid 1 \leqslant i, j;\ I + j \leqslant g - 1\}$;

(ii) $b_0 = \frac{1}{2}\max\{b_i \mid i \geqslant 1\}$; *and*

(iii) $a = 12b_0 - b_1$.

Then $f^*D_E = E$ and if E is an F-divisor, so is D_E.

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Proof. To see that $f^*D_E = E$, use Lemmas 2.4 and 2.9:

$$\begin{aligned} f^*D_E &= b_1 \sum_{1\leqslant i\leqslant g} \psi_i - \sum_{2\leqslant i\leqslant \lfloor g/2\rfloor} \left(\frac{i(g-i)}{(g-1)}b_1 - e_i\right)\widetilde{B}_i \\ &= \sum_{2\leqslant i\leqslant \lfloor g/2\rfloor} \left(\frac{i(g-i)}{(g-1)}b_1 - \left(\frac{i(g-i)}{(g-1)}b_1 - e_i\right)\right)\widetilde{B}_i. \end{aligned}$$

Now suppose that E is an F-divisor. To show that D_E is also an F-divisor one just has to check that it satisfies the five inequalities of Theorem 2.1. The first four are true by definition of D_E. For example, to see that $b_i + b_j - b_{i+j} \geqslant 0$, we calculate that

$$\begin{aligned} &\left(\frac{i(g-i)}{(g-1)}b_1 - e_i\right) + \left(\frac{j(g-j)}{(g-1)}b_1 - e_j\right) - \left(\frac{(i+j)(g-(i+j))}{(g-1)}b_1 - e_{i+j}\right) \\ &= \frac{2ijb_1}{(g-1)} - (e_i + e_j - e_{i+j}), \end{aligned}$$

which is nonnegative as long as

$$b_1 \geqslant \frac{(g-1)}{2ij}(e_i + e_j - e_{i+j}).$$

The fifth inequality holds because $f^*D_E = E$. □

Proof of Theorem 6.1. By using a computer program such as LRS [AF01], one can generate a list of extremal divisors E for the F-cone of $\overline{M}_{0,g}/S_g$. This computation is convenient to perform by considering divisors expressed in the basis for $\mathrm{Pic}(\overline{M}_{0,g}/S_g)$ given by $\{\widetilde{B}_i\}_{2\leqslant i\leqslant \lfloor g/2\rfloor}$. To change these extremal divisors into the form necessary to apply the theorems, one can solve for D_E as in Lemma 6.3 and then pull back. Finally, to check that the divisors are all nef, we ran them through the program Nef Wizard. □

7. Relevance of the *F*-conjecture

If the F-conjecture holds, it would imply that the extremal rays of the cone of curves $\overline{NE}(\overline{M}_g)$ are spanned by the F-curves. This would be very good information to have since, as was illustrated in the introduction, $\overline{NE}(\overline{M}_g)$ reveals information about the birational geometry of $\overline{M}_g$. Moreover, it would mean that $\overline{NE}(\overline{M}_g)$ is an interesting example of a cone of curves. To explain why, I shall say a little bit about the minimal model program (MMP).

The MMP generalizes the birational classification of smooth surfaces using certain kinds of projective morphisms called contractions. Contractions are morphisms $f : X \longrightarrow Y$ between projective varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$; they are determined by the faces of the cone of curves. Unlike the situation for surfaces, for higher-dimensional projective varieties contractions are not so resolutely understood, nor is their existence guaranteed.

In order to classify X using contractions $X \longrightarrow Y$, one studies the image variety Y and the fibers of the contraction morphism. There are a couple of possibilities depending on whether or not the image Y has the same dimension as X. If $\dim X > \dim Y$, this is a fibral type contraction. As was mentioned above, by [Gib00] there are no fibral type contractions of $\overline{M}_g$. The other possibility is that $\dim X = \dim Y$. For $\overline{M}_g$ and other higher-dimensional varieties X, two things can happen. The first is that the morphism $X \longrightarrow Y$ is a so-called divisorial contraction: this is

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the analog of the surface case wherein X is the blowup of Y. By [GKM01, Proposition 6.4], for $g \geqslant 5$ the only divisorial contraction of $\overline{M}_g$ is a blowdown of elliptic tails. When $g=3$ there is another divisorial contraction [Rul], and the problem is open when $g=4$. The remaining kind of contraction does not have an analog in the classification of surfaces. It is called a small contraction and is essentially the case where the image variety Y has bad singularities so that one has to surgically repair it (i.e. do flips or flops) in order to proceed with the program. As stated in the introduction, since there are a finite number of F-curves to begin with, if the F-conjecture is true, then the cone of curves is polyhedral, like the cone of curves for a Fano variety. This is counter-intuitive, since for $g=22$ and $g \geqslant 24$ the moduli space is of general type.

Finally, when one considers $\overline{M}_g$ to be defined over a field of positive characteristic, then every extremal face of $\overline{NE}(\overline{M}_g)$ gets contracted. This is also surprising since contractions of a variety X are only guaranteed for K_X-negative extremal rays, and only one of the F-curves is $K_{\overline{M}_g}$-negative.

In any case, for low genus when the nef cones and the F-cones of $\overline{M}_g$ are the same, one has a series of explicit examples of cones of curves that have finitely many extremal rays, each spanned by a smooth, irreducible and rational curve. Moreover, when the characteristic of the field is positive, every face of the cones gets contracted, none of the contractions is fibral and, in fact, all but one are small contractions. Hence one has a rich collection of examples which help to deepen our understanding of the birational geometry of the spaces $\overline{M}_g$. Furthermore, although admittedly not the simplest of examples, these cones broaden our understanding of cones of curves in general.

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