Mon. Not. R. astr. Soc. (1969) **146**, 161–169.

NUMERICAL EXPERIMENTAL CHECK OF LYNDEN-BELL STATISTICS—II

THE CORE–HALO STRUCTURE AND THE ROLE OF THE VIOLENT RELAXATION

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(Received 1969 June 23)

SUMMARY

In a previous paper, Lynden-Bell's statistical mechanics of collisionless stellar systems was checked by numerical experiments on four initial configurations of a one-dimensional self-gravitating system. Three of the configurations reached a final state which differed from the predicted one: a relatively small fraction of the particles formed a high energy halo (which contained most of the energy of the system) around a strongly degenerate, low energy core (which contained most of the particles).

In the present experiment the core-halo structure appearing in all four configurations considered by us was investigated; in addition, the fourth configuration was continued to a final state, and it was the state predicted by the statistics. A detailed analysis of the particle trajectories showed, that the configuration that reached the predicted final state, and one of those that did not, relaxed equally as violently. The other two configurations, by comparison, relaxed weakly.

This limited experimental evidence does not support a strong correlation between the violence of the relaxation and the satisfaction of Lynden-Bell's statistics.

In our previous paper (1), we investigated four different initial phase space configurations, all having the same (constant) fine grained phase space density (η), total energy (E) and total mass (M). Lynden-Bell's (2) statistics predicts for the final state of the coarse grained phase space density ($\bar{f}$)

$$\bar{f} = \frac{\eta}{e^{\beta(\epsilon-\mu)}+1}$$

where β and μ are determined from E and M. The constants β and μ determine the shape of the distribution function in the same way as in Fermi–Dirac statistics. Our values of E and M predicted a weakly degenerate final state. But, in common with previous numerical experimental results (Lecar (3); Lecar & Cohen (4); Cohen & Lecar (5); Henon (6), (7); Hohl & Campbell (8), in the three configurations that reached a steady state, the bulk of the particles formed a strongly degenerate core (which could be fitted nicely by a Lynden-Bell distribution but with different β and μ), while a smaller fraction of the particles formed a high energy halo, which however, contained the bulk of the energy of the system. This high energy tail of the distribution was not in accordance with the statistics. In fact, it looked as if the high energy particles were just left behind from the initial configuration to free stream and phase mix in the gravitational field of the core. In order to check this conjecture, in the present investigation, we analysed the trajectories of the individual particles, and attempted some quantitative estimates of the strength of the relaxation.

In addition, one of the configurations (Case A) which did not reach a steady state in the earlier experiment, was continued to a final state, and it did reach the state predicted by the statistics (as did, one of the three configurations investigated by Hohl & Campbell (**8**)).

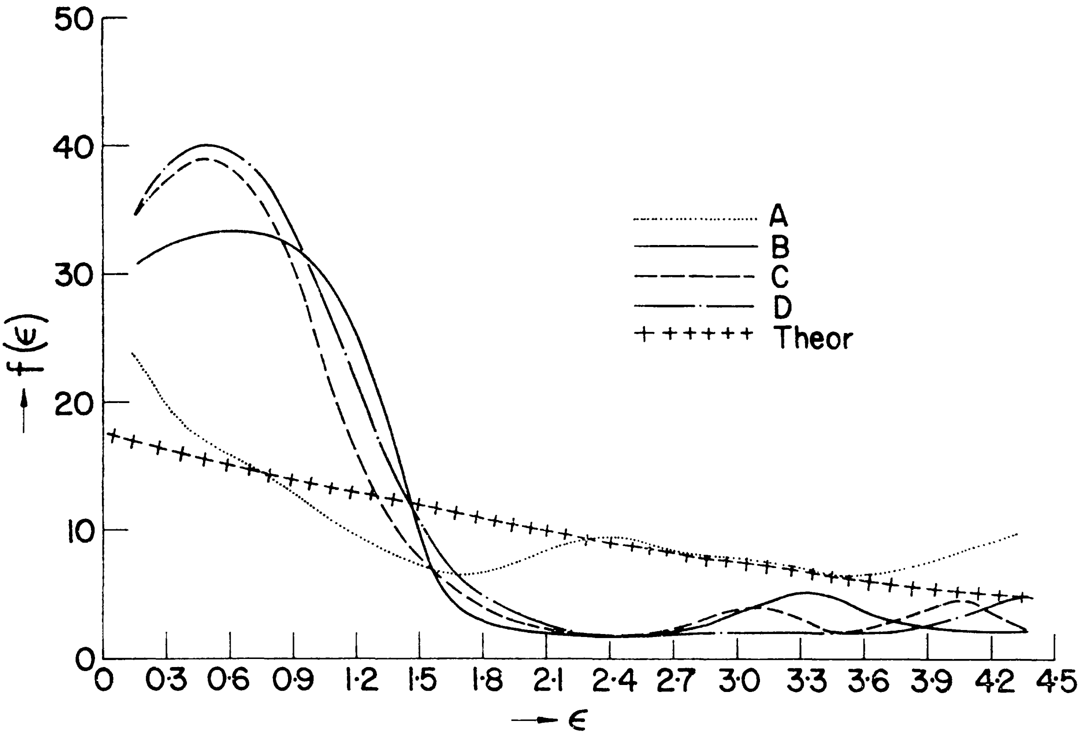

FIG. 1. *The final, time-averaged coarse-grained phase space densities.*

The final, time-averaged, coarse-grained phase space densities are shown in Fig. 1. The two extremes are represented by Case A, which agrees well with the prediction, and Cases C and D which do not. Cases B, C, D define a core of low energy particles and a halo of high energy particles, the distribution function breaking rather sharply at $\epsilon \simeq 2{\cdot}1$. In Table I, we summarize for the four initial configurations, the fraction of particles in the core ($\epsilon < 2{\cdot}1$), N_C, and the halo ($\epsilon > 2{\cdot}1$), N_H, at the beginning ($t = 0$) and end (t_f) of the runs, and the fraction of particles that originated *and* ended in the core or halo, and those that switched. If we take the fraction of particles that switched as a rough measure of the violence of the relaxation, we can dismiss Cases C and D as comparatively unrelaxed, while Cases A and B seem to have relaxed equally as violently. However, Case A evolved toward the prediction, while Case B evolved away from it.

In what follows, we discuss other measures of the strength of the relaxation, which just support the rough estimate given above.

A standard measure of relaxation is the relative change in energy of particles during one period. We define the reciprocal of the relaxation time (T_R) per collective period (τ) as:

$$\frac{\tau}{T_R} = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{V_i^2(o, t) - V_i^2(o, t+\tau_i)}{V_i^2(o, t)} \right|$$

TABLE I

Evolution of the core and halo populations

	Case A	Case B	Case C	Case D
Duration of experiment (in number of oscillation periods of the total kinetic energy)	49	30	30	37
$N_C(0)$	0·38	0·42	0·47	0·60
$N_C(t_f)$	0·44	0·61	0·59	0·64
$N_H(0)$	0·62	0·58	0·53	0·40
$N_H(t_f)$	0·56	0·39	0·41	0·36
Fraction that remained:				
in core	0·14	0·25	0·39	0·56
in halo	0·33	0·22	0·32	0·32
	0·47	0·47	0·71	0·88
Fraction that switched:				
from core to halo	0·24	0·18	0·08	0·04
from halo to core	0·29	0·35	0·21	0·08
	0·53	0·53	0·29	0·12

where $V_i(0, t)$ is the velocity of the ith particle as it crosses $x_i = 0$ in the direction of positive $x(V_i > 0)$, and $V_i(0, t+\tau_i)$ is the same quantity for the next positive crossing (one period later). In effect, we are measuring the work done on the particle around a complete cycle. The relaxation times at the beginning (t_0) and the end (t_f) of the runs are displayed in Table II.

TABLE II

Relaxation times

	A	B	C	D
All particles				
t_0	0·8	4·2	1·7	0·5
t_f	0·4	0·4	0·3	0·2
Only core particles				
t_0	1·1	9·4	3·1	0·7
t_f	0·6	0·5	0·5	0·2
Only halo particles				
t_0	0·7	0·5	0·5	0·1
t_f	0·2	0·06	0·04	0·04

The value τ/T_R can be dominated by a small number of particles whose initial energies are close to zero, and this happened in Case B. A more uniform measure of relaxation is the quantity

$$\alpha(t) = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{V_i^2(0, t) - V_i^2(0, 0)}{\frac{1}{2}[V_i^2(0, t) + V_i^2(0, 0)]}\right|$$

$$\equiv \frac{1}{N}\sum_{i=1}^{N}|\alpha_i(t)|$$

where $V_i(0, t)$ is the velocity of the ith particle the last time it crossed $x = 0$.

Fig. 2 shows $\alpha(t)$ near the beginning of the runs, and Fig. 3 shows $\alpha_i(t_f)$ for the individual particles at the end of the runs.

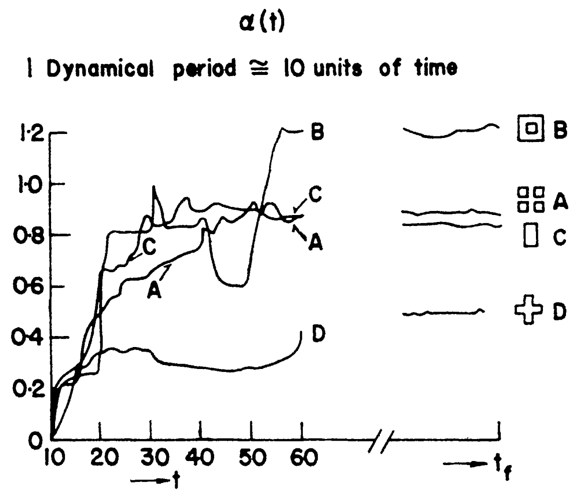

FIG. 2. *The evolution in time of the average change in particle energy as defined by the quantity* $\alpha(t)$.

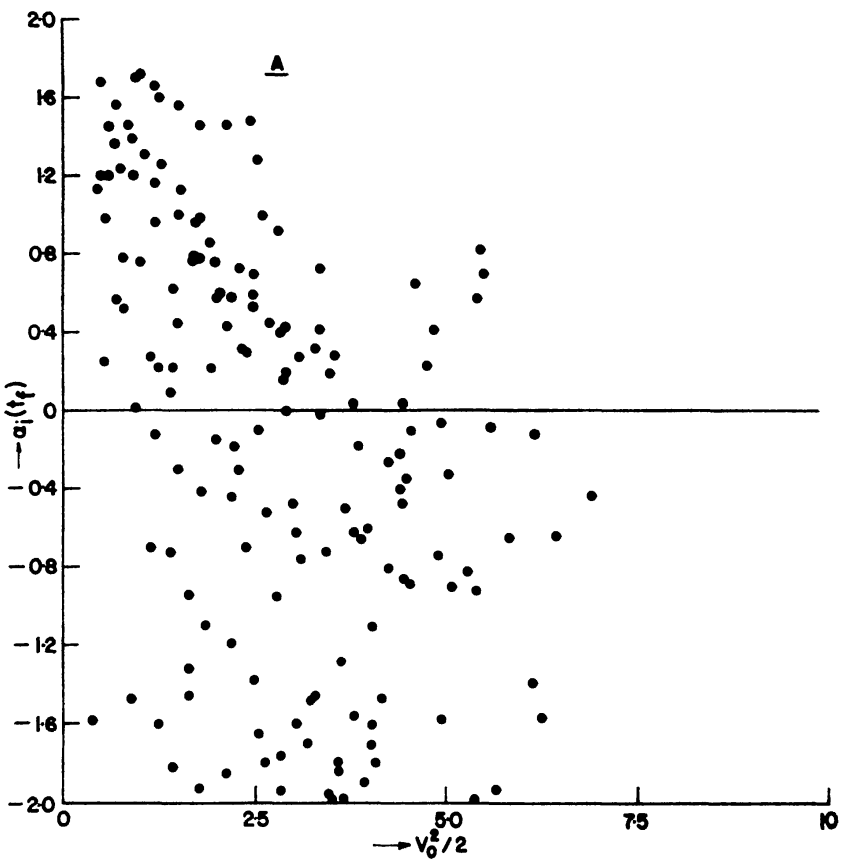

FIG. 3. *The change in the energy of individual particles as defined by the quantity* $\alpha_i(t_f)$.
(*a*): *Case A.*

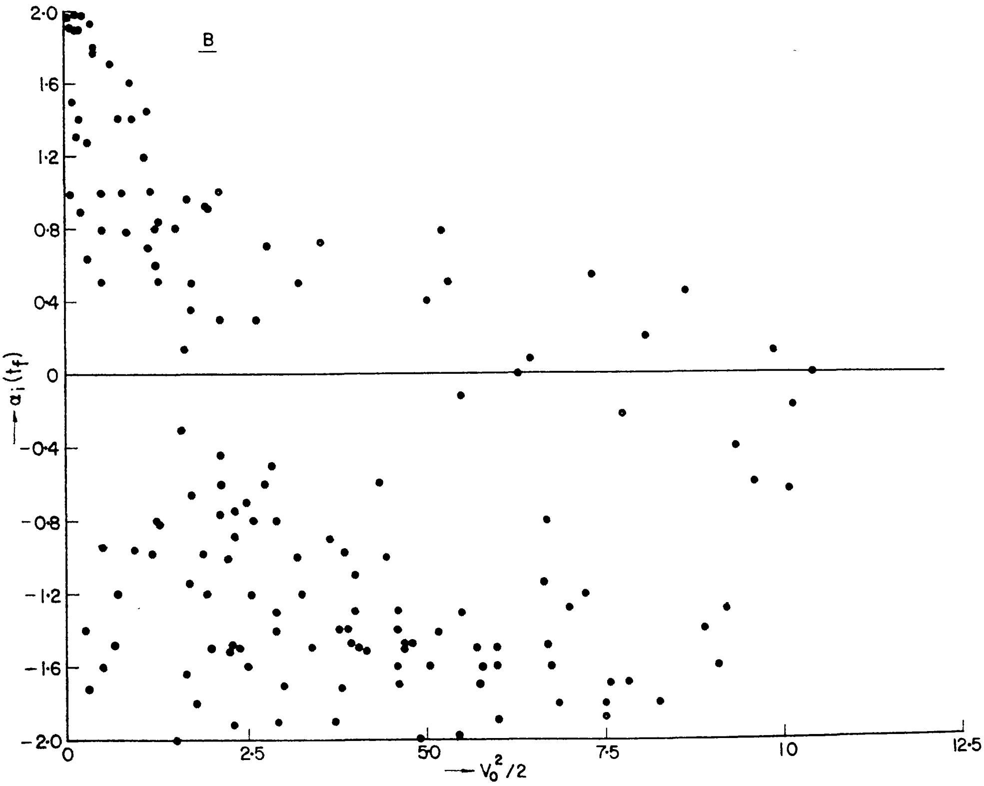

FIG. 3(*b*): *Case B.*

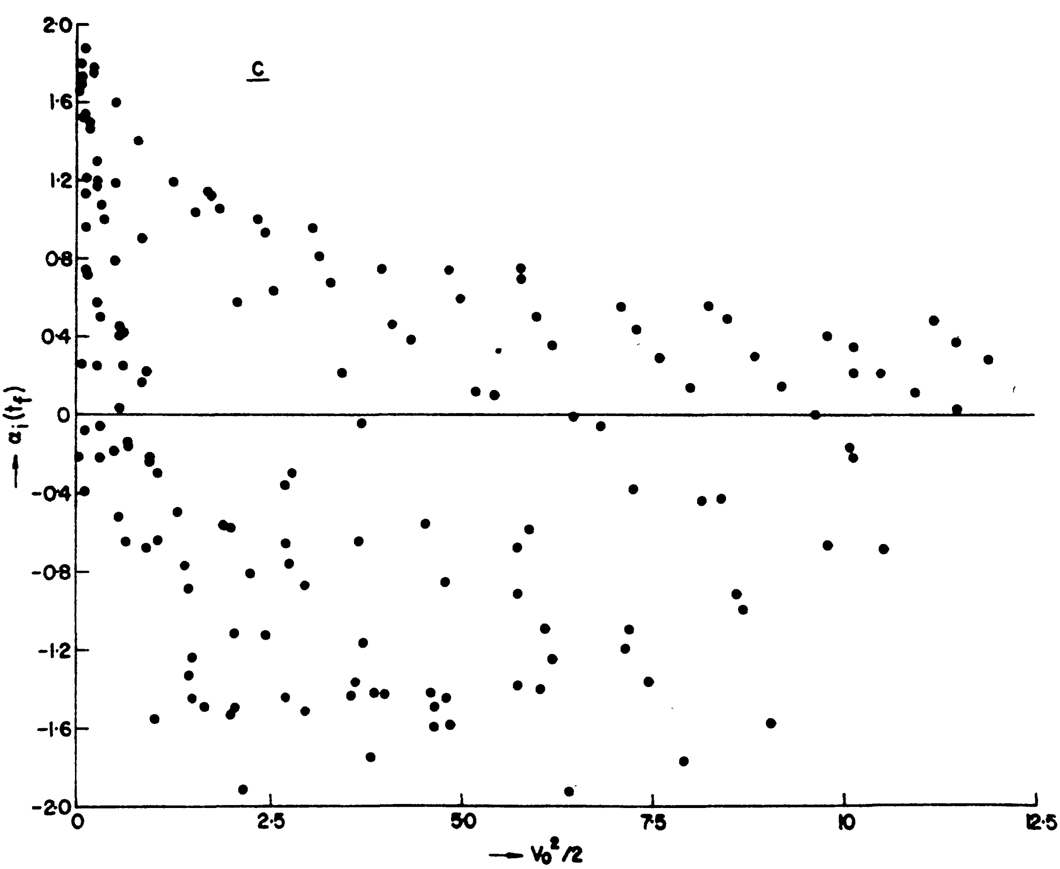

FIG. 3(*c*): *Case C.*

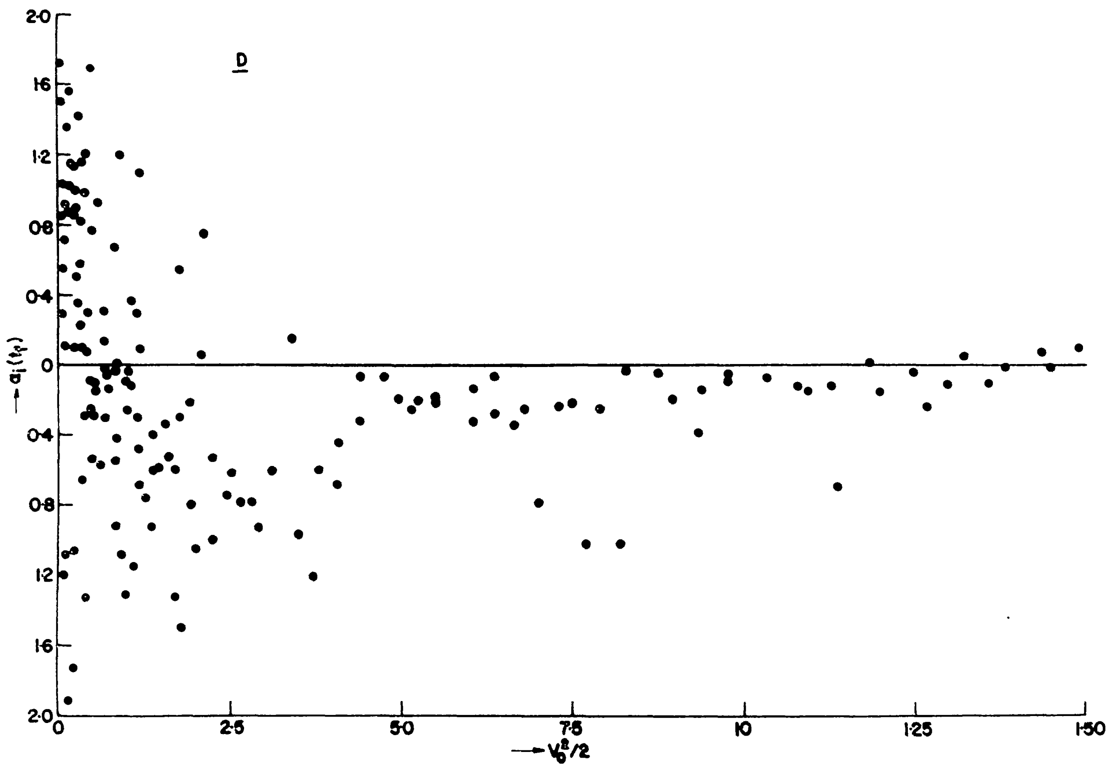

FIG. 3(*d*): *Case D.*

Finally, the sum of the individual particle energies is not a conserved quantity (because the potential energies are not additive), and in fact changes (in Case A) by 15 per cent. A measure of relaxation which is not affected by a shift in the sum of the individual energies of the whole system, is the following: Order the V's in increasing magnitude and assign an index γ for the ith particle such that $V_i^{\gamma} > V_i^{\gamma-1} (V > 0)$.

Let

$$\Gamma_i(t) = \gamma_i(t) - \gamma_i(0).$$

Then

$$\beta(t) = \frac{2}{N^2} \sum_{i=1}^{N} |\Gamma_i(t)|$$

measures the changes in ordering by energy of the particles. Fig. 4 shows $\beta(t)$'s near the beginning of the runs, as well as their asymptotic values.*

In our analysis, we felt the lack of a quantitative definition of collective relaxation. We recognize that changes in the energies of individual particles do not necessarily imply relaxation of the distribution function. For example, an exchange of velocities in a binary collision, would affect the individual particle energies but not the distribution function (which does not take account of particle labels).

In fact, in our model, particles do not exchange velocities during a collision. When two particles (which represent infinite sheets in the one dimensional model) pass through each other, their velocities vary continuously, but their accelerations jump by an amount proportional to $1/N$. The contribution of this effect to T_R has been estimated (Lecar & Cohen (4)) to go as $1/N^3$ and is negligible in this experiment.

* We remind the reader that V_i as used in the definitions of T_R, α and β refers to the velocity of the ith particle as it crosses $x = 0$. As we zero the potential energy at $x = 0$, $\frac{1}{2}V_i^2$ is the total energy of the particle.

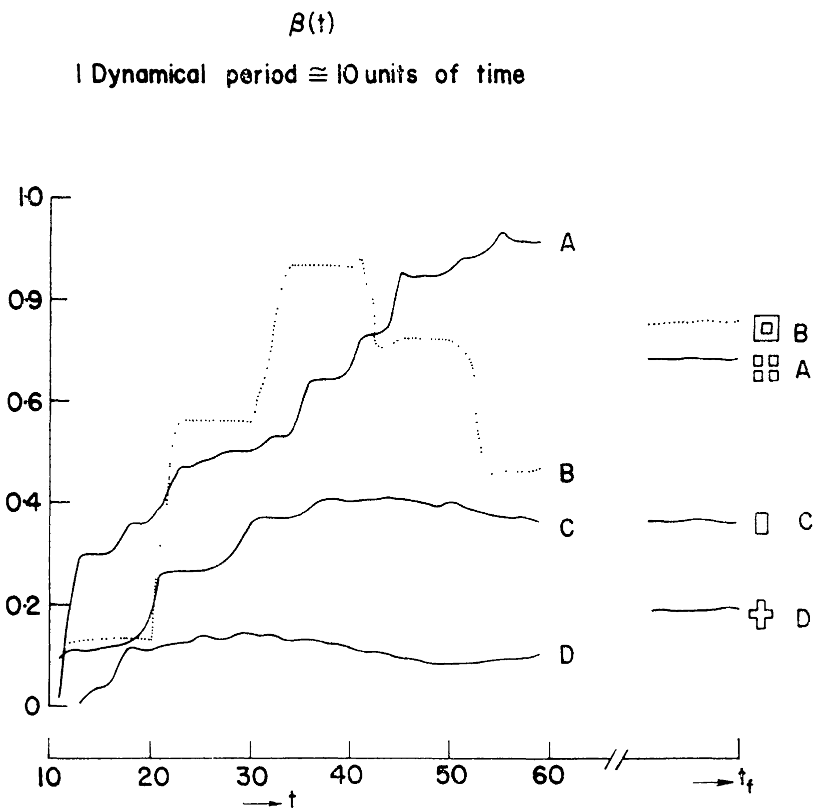

FIG. 4. *The evolution in time of the degree of mixing of particles in energy space as defined by the quantity* $\beta(t)$.

There is also a contribution to T_R which invalidates the Vlasov approximation and consequently destroys degeneracy. This effect has been estimated by Eldridge & Feix (**9**) and measured in numerical experiments by Dawson (**10**), Lecar (**3**) and Hohl & Broadus (**11**) to go as $1/N^2$. Finally, statistical (square root of N) fluctuations are estimated (in the appendix) to contribute to T_R as $1/N$. This latter estimate seems to account for the observed values of T_R of the core particles after the system reached a steady state, so we believe that the larger values (by factors of two to twenty) observed at the start of the runs were due to collective interactions.

It is clear that collective relaxation has none of the inevitabilities of collisional relaxation; collective relaxation does not invariably lead to the same final state. While we have a rough rule of thumb for collisional relaxation that a mono-energetic directed beam will thermalize in ~ 6 relaxation times, we have no such guide for collective relaxation.

Further, we are not confident that we have identified all the important mechanisms of collective relaxation. For example, the core of configuration D continued to oscillate, sinusoidally, for more than 25 periods. As the core has almost constant density, the particles behave as harmonic oscillators with the same frequency, which explains the absence of phase mixing. But, it is not clear why the energy of the oscillations is not transferred to particle kinetic energies, (i.e., Landau Damping).

In conclusion, while we admit only a primitive understanding of collective relaxation, the experimental results do not support a one-to-one correlation between the violence of the relaxation and the satisfaction of the Lynden-Bell statistics.

ACKNOWLEDGMENT

This work was supported by Grant SFC-8-7006 from the Smithsonian Institution, Washington D.C.

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REFERENCES

(1) Goldstein, S., Cuperman, S. & Lecar, M., 1969. *Mon. Not. R. astr. Soc.*, **143**, 209.
(2) Lynden-Bell, D., 1967. *Mon. Not. R. astr. Soc.*, **136**, 101.
(3) Lecar, M., 1967. *Mém. Soc. R. Sci. Liège*, Cinquieme Serie, Tome XI, 227.
(4) Lecar, M. & Cohen, L., 1967. *Symposium on Computer Simulation of Plasma and Many Body Problems*, NASA SP-153, p. 299.
(5) Cohen, L. & Lecar, M., 1968. *Bull. Astronomique*, Tome III, Fascicule I, 213.
(6) Henon, M., 1968. *Bull. Astronomique*, Tome III, Fascicule I.
(7) Henon, M., 1967. *Symposium on Computer Simulation of Plasma and Many Body Problems*, NASA SP-153, p. 349.
(8) Hohl, F. & Campbell, J., 1968. *Astr. J.*, **73**, 611.
(9) Eldridge, D. C. & Feix, M., 1963. *Phys. Fluids*, **6**, 398.
(10) Dawson, J., 1964. *Phys. Fluids*, **7**, 419.
(11) Hohl, F. & Broadus, D. T., 1967. *Phys. Letters*, **25A**, 713.

APPENDIX

Statistical fluctuations

For a system symmetrical about $x = 0$, the acceleration of the ith particle from the centre is

$$\ddot{x}_i \equiv a_i = -4\pi GmN_i$$

where $x_i > 0$, m is the surface density of a sheet, and N_i is the number of sheets with $0 \leqslant x \leqslant x_i$. If we assume a fluctuation $\sqrt{N_i}$ in N_i, then the corresponding fluctuation in the acceleration is

$$\delta a_i \sim 4\pi GmN_i^{1/2}$$

and the resulting change in velocity is

$$\delta V_i \sim 4\pi GmN_i^{1/2}t_i$$

where t_i is the duration of the fluctuation.

Thus

$$\left(\frac{\delta V_i}{V_i}\right)^2 = \frac{(4\pi Gm)^2 N_i t_i^2}{V_i^2}$$

and

$$\frac{1}{T_R} = \frac{d}{dt}\left(\frac{\delta V_i}{V_i}\right)^2 \simeq \frac{1}{t_i}\left(\frac{\delta V_i}{V_i}\right)^2 \simeq \frac{(4\pi Gm)^2 N_i t_i}{V_i^2}.$$

If the density of sheets is constant in the interval $0 \leqslant x \leqslant X_i$, then

$$N_i = \frac{N}{2L} x_i, \quad \omega^2 = 4\pi G m \frac{N}{2L} \quad \text{and} \quad V_i^2 = \omega^2 x_i^2.$$

Taking $t_i = \frac{1}{2}\tau = \frac{1}{2}(2\pi/\omega)$,

$$\frac{\tau}{T_R} = \frac{4\pi^2}{N}\frac{L}{x}.$$

Taking $L/2$ as a typical value for x, and $N = 160$

$$\tau/T_R = \frac{8\pi^2}{N} \simeq 0.5.$$

The observed values for the core particles at the end of runs range from 0·2 to 0·6.