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Numerical Experiments on Eigenvalues of Weakly Singular Integral Equations Using Product Simpson's Rule

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Abstract This paper discusses the use of Product Simpson's rule to solve the integral equation eigenvalue problem $\lambda f(x) = \int_{-1}^{1} k(|x-y|)f(y)dy$ where $k(t) = \ln|t|$ or $k(t) = t^{-\alpha}$, $0 < \alpha < 1$, λ, f and are unknowns which we wish to obtain. The function $f(y)$ in the integral above is replaced by an interpolating function $L_n^f(y) = \sum_{i=0}^{n} f(x_i)\phi_i(y)$, where $\phi_i(y)$ are Simpson interpolating elements and $x_0, x_1, \ldots, x_n$ are the interpolating points and they are chosen to be the appropriate non-uniform mesh points in $[-1, 1]$. The product integration formula $\int_{-1}^{1} k(y)f(y)dy \approx \sum_{i=0}^{n} w_i f(x_i)$ is used, where the weights w_i are chosen such that the formula is exact when $f(y)$ is replaced by $L_n^f(y)$ and $k(y)$ as given above. The five eigenvalues with largest moduli of the two kernels $K(x,y) = \ln|x-y|$ and $K(x,y) = |x-y|^{-\alpha}$, $0 < \alpha < 1$ are given.

Keywords eigenvalue, product integration, singular kernel, integral equation.

Abstrak *Kertas kerja ini membincangkan penggunaan aturan Simpson darab untuk menyelesaikan masalah nilai eigen persamaan kamiran* $\lambda f(x) = \int_{-1}^{1} k(|x-y|)f(y)dy$ *dengan* $k(t) = \ln|t|$ *atau* $k(t) = t^{-\alpha}$, $0 < \alpha < 1$, λ *dan* f *adalah anu yang hendak didapatkan. Fungsi interpolasi* $L_n^f(y) = \sum_{i=0}^{n} f(x_i)\phi_i(y)$, *dengan* $\phi_i(y)$ *unsur interpolasi Simpson dan* $x_0, x_1, \ldots, x_n$ *adalah titik-titik dan ianya dipilih supaya menjadi titik-titik interpolasi tak seragam yang tertentu dalam* $[-1, 1]$. *Rumus pengamiran darab* $\int_{-1}^{1} k(y)f(y)dy \approx \sum_{i=0}^{n} w_i f(x_i)$ *digunakan dengan pemberat* w_i *dipilih supaya rumus adalah tepat apabila* $f(y)$ *diganti oleh* $L_n^f(y)$ *dan* $k(y)$ *seperti di atas. Lima nilai eigen dengan modulus terbesar bagi kedua-dua inti* $K(x,y) = \ln|x-y|$ *dan* $K(x,y) = |x-y|^{-\alpha}$, $0 < \alpha < 1$ *diberikan.*

Katakunci *nilai eigen, pengamiran darab, inti singular, persamaan kamiran.*

1 Introduction

The numerical solution of the integral equation eigenvalue problem

$$\lambda f(x) = \int_{-1}^{1} K(x,y)f(y)dy \tag{1}$$

for an eigenvalue λ and a corresponding eigenfunction $f(x)$ is considered in this paper. If λ is an eigenvalue of the kernel $K(x,y)$, then there is at least one non-null function $f(x)$ satisfying (1). The function $f(x)$ is called a left-eigenfunction or only eigenfunction corresponding to the eigenvalue λ. If there exists a function $g(x)$ such that $\lambda g(x) = \int_{-1}^{1} \overline{K(x,y)}dy$ then $g(x)$ is called the right-eigenfunction corresponding to λ.

In this paper, we shall discuss only when $K(x,y)$ is a weakly singular kernel and has the form

$$K(x,y) = |x-y|^{-\alpha}, 0 < \alpha < 1 \text{ and } K(x,y) = \ln|x-y| \tag{2}$$

These two kernels are Hermitian and compact in $C[-1,1]$, and hence have countable infinite numbers of eigenvalues with zero the only possible limit point (Atkinson [2]).

Solution of (1) is closely related to the solution of a $n \times n$ algebraic eigenvalue problem. Indeed, the main goal of the numerical methods to solve (1) is to reduce it approximately to an algebraic form. Then, the algebraic eigenvalue problem is solved and the solution is taken to be the approximate solution of (1). The numerical treatment of an integral equation involving weakly singular kernel should take into account the nature of this singularity. The available numerical methods are modified quadratures, product integration, collocation and Galerkin method and smoothing the kernels. Razali [10] used product integration methods with piecewise polynomials (Midpoint, Trapezoidal and Simpson rules) with uniform mesh points $x_i = \frac{i}{n}, i = 0,1,\ldots,n$ to find the eigenvalue with largest modulus and its corresponding eigenfunction of the kernel in (2) with $0 \leq x, y \leq 1$.

If the function $f(x)$ is smooth and $f \in C^{m+1}[-1,1]$, de Hoog and Weiss [8] showed that, if the product integration methods with piecewise interpolating polynomial of degree m, is used with uniform mesh points $x_i = -1 + \frac{2}{n}i, i = 0,1,\ldots,n$ to solve the Fredholm second kind integral equation with the kernel given in (2), then the method is of order $O(n^{-m-2+\alpha})$ for the first kernel and $O(n^{-m-2}\ln n)$ for the second kernel in (2). Weakly singular integral equations have solutions containing mild singularities at the end points $\{-1,1\}$ introduced by the kernel. The best result for a uniform mesh, as shown in Chandler [5], is $O(n^{-2+2\alpha})$ for the first kernel and $O(n^{-2}\ln n)$ for the second kernel in (2). Schneider [11] showed that the order of convergence is $O(n^{-m-2+\alpha})$ for the first kernel and $O(n^{-m-2}\ln n)$ for the second kernel in (2) if appropriate non-uniform mesh points are used.

Baratella [4] proved that if the product integration with piecewise polynomial of degree m is used in solving a Fredholm second kind integral equation with the kernel $K(x,y) = |x-y|^{-\alpha}, 0 < \alpha < 1$, then the convergence of the method is optimal if the method is used with $(Nm+1)$ non-uniform mesh points $x_{mi+j} = \tau_i + s_j(\tau_{i+1} - \tau_i)$ where $i = 0,1,\ldots,N-1, j = 0,1,\ldots,m-1, s_j \in [0,1]$ and

$$\tau_i = \begin{cases} -1 + (\frac{2i}{N})^q, & 0 \leq i \leq \frac{N}{2} \\ -\tau_{N-i}, & \frac{N}{2} < i \leq N \end{cases} \tag{3}$$

where $q = \frac{m-1}{1-\alpha}$ and N is an even integer. It is a common belief that this method is the most efficient in the solution of (1). By taking the local polynomial degree m to arbitrarily

Table 1: Estimated condition numbers

N	m=2 q=2.6	m=3 q=3.1	m=4 q=3.6	m=5 q=4.1	m=6 q=4.6
4	3.6E+00	4.4 E+00	4.9 E+00	5.3 E+00	4.6 E+12
8	4.0E+00	5.0 E+00	5.7 E+00	6.4 E+00	5.2E+31
16	4.6 E+00	5.3 E+00	5.9 E+00	7.0 E+04	2.4 E+65
32	4.8 E+00	5.3 E+00	2.7 E+04	3.1 E+12	
64	5.0 E+00	5.4 E+00	1.5 E+09	2.6 E+19	

large, we can obtain an order of convergence as high as we want. In practice, however, using computer arithmetic, this last statement does not appear to be true. Indeed, as the local degree increases, or when q is large, the concentration of the knots near the end points of the interval of integration is so high, which increases as n becomes large, resulting in the final linear system becoming more rapidly ill-conditioned. Moreover, this implementation becomes more expensive. For example, Table 1, as given in Monegato and Scuderi [9], reported some values of the condition numbers estimated, when the product integration with piecewise polynomials of degree m was applied to the equation (1) with $K(x, y) = \ln|x - y|$.

In this paper, the product integration method with piecewise polynomial of degree $m = 2$ (Simpson's rule) is used to solve the equation (1) with the non-uniform mesh points (3) when $q = 2$. Here, the kernels being considered are weakly singular kernels of the types in (2). In Section 2, the product Simpson integration rule is obtained. To reduce an integral equation eigenvalue problem into an algebraic eigenvalue problem, we need to calculate the necessary matrix K_n. This is discussed in Section 3. The numerical results of this work are displayed in Section 4.

2 Product integration methods

Product integration is a simple technique for handling integrals of the form

$$I(f) = \int_{-1}^{1} k(x)f(x)dx \tag{4}$$

where $k(x)$ is a real-valued absolutely integrable function, which needs not be continuous or of one sign, and $f(x)$ is any continuous function on $[-1, 1]$. Integrals with finite end points other than -1 and 1 can be transformed to the form (4) by a simple linear transformation.

A product integration rule for is an expression of the form

$$I_n(f) = \sum_{j=0}^{n} w_j f(x_j) \tag{5}$$

where $x_j, j = 0, 1, \ldots, n$ are a set of distinct points in $[-1, 1]$, and $w_j, j = 0, 1, \ldots, n$ are suitable weights. To obtain (5), the function $f(x)$ in (4) is replaced by an interpolating

function $f_n(x)$, where $f_n(x_j) = f(x_j), j = 0, 1, \ldots, n$ such that the integral $\int_{-1}^{1} k(x)f_n(x)dx$ can be integrated exactly or, at least, very accurately and the weights are determined by requiring the rule (5) is exact when f is replaced by f_n , i.e. $I(f_n) = I_n(f_n)$

The interpolating function $f_n(x)$ can be written as

$$f_n(x) = \sum_{j=0}^{n} \phi_j(x) f(x_j) \tag{6}$$

where $\phi_j(x), j = 0, 1, \ldots, n$ are suitable interpolating elements. Therefore,

$$\begin{aligned} I_n(f) &= \int_{-1}^{1} k(x) f_n(x) dx \\ &= \int_{-1}^{1} k(x) (\sum_{j=0}^{n} \phi_j(x) f(x_j)) dx \\ &= \sum_{j=0}^{n} (\int_{-1}^{1} k(x)\phi_j(x)dx) f(x_j) \\ &= \sum_{j=0}^{n} w_j f(x_j) \end{aligned} \tag{7}$$

where

$$w_j = \int_{-1}^{1} k(x)\phi_j(x)dx, j = 0, 1, \ldots, n \tag{8}$$

can be computed exactly or very accurately.

In this paper, the function $f_n(x)$ is chosen to be a piecewise interpolating polynomial of degree two (Product Simpson's rule method) with (n+1) non-uniform mesh points $x_{2i} = \tau_i$, $i = 0, 1, \ldots, N$ and $x_{2i+1} = \frac{1}{2}(\tau_i + \tau_{i+1}), i = 0, 1, \ldots, N-1$ where $N = \frac{n}{2}$, N is an even integer, and

$$\tau_i = \begin{cases} -1 + (\frac{2i}{n})^2, & 0 \le i \le \frac{N}{2} \\ -\tau_{N-i}, & \frac{N}{2} < i \le N \end{cases} \tag{9}$$

The function $f(x)$ is approximated at each sub interval $[x_{2i}, x_{2i+2}]$, $i = 0, 1, \ldots, \frac{N}{2} - 1$ by a quadratic polynomial interpolating $f(x)$ at the points x_{2i}, x_{2i+1} and x_{2i+2}.

By defining, $\Delta_{i,j} \equiv x_j - x_i$, then the interpolating elements $\{\phi_j(x)\}_{j=0}^{n}$ in 6 are:

$$\begin{aligned} \phi_{2j}(x) &= \begin{cases} \dfrac{(x - x_{2j-2})(x - x_{2j-1})}{\Delta_{2j-2,2j}\Delta_{2j-1,2j}}, & x_{2j-2} < x \le x_{2j} \\ \dfrac{(x - x_{2j+1})(x - x_{2j+2})}{\Delta_{2j+1,2j}\Delta_{2j+2,2j}}, & x_{2j} < x < x_{2j+2} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } j = 1, 2 \ldots, \frac{N}{2} - 1 \\ \phi_0(x) &= \frac{(x - x_1)(x - x_2)}{\Delta_{1,0}\Delta_{2,0}}, \quad x_0 \le x \le x_2, \\ \phi_n(x) &= \frac{(x - x_{n-2})(x - x_{n-1})}{\Delta_{n-2,n}\Delta_{n-1,n}}, \quad x_{n-2} \le x \le x_n, \end{aligned} \tag{10}$$

and

$$\phi_{2j+1}(x) = \begin{cases} \dfrac{(x-x_{2j})(x-x_{2j+2})}{\Delta_{2j,2j+1}\Delta_{2j+2,2j+1}}, & x_{2j} < x < x_{2j+2} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } j = 1, 2 \ldots, \frac{N}{2} - 1$$

The error in the approximation above is given by

$$\begin{aligned} |I(f) - I_n(f)| &\leq \int_{-1}^{1} |k(x)|\,|f(x) - L_n^f(x)|\,dx \\ &\leq \|k\|_1 \|f - L_n^f\| \end{aligned} \tag{11}$$

provided that

$$\|k\|_1 = \int_{-1}^{1} |k(x)|dx \quad \text{exists and bounded}$$

and

$$\|f - L_n^f\| = \max_{x\in[-1,1]} |f(x) - L_n^f(x)|.$$

It is clear that the interpolating piecewise polynomial L_n^f converges uniformly to $f(x)$ for all $f(x) \in C[-1,1]$, provided that $\lim_{n\to\infty} \max_{1\leq i\leq n} |x_i - x_{i-1}| = 0$. Since, $\lim_{n\to\infty} \max_{1\leq i\leq n} |x_i - x_{i-1}| = 0$ when x_i, $i = 0, 1, \ldots, n$, are as given in 9, then $I_n(f) \to I(f)$ as $n \to \infty$ for all $f \in C[-1,1]$ provided only $\|k\|_1$ exists and bounded.

3 The matrix elements

To solve (1) numerically, we reduce it to the algebraic eigenvalue problem

$$\lambda^{(n)}\mathbf{f} = K_n\mathbf{f} \tag{12}$$

where $\lambda^{(n)}$ is the approximate value to λ and K_n is the $n\times n$ matrix which we obtain from the kernel $K(x,y)$. Under suitable conditions, the eigenvalues of (12) will approximate those of (1). Equation (12) will always have eigenvalues, in general distinct, while equation (1), in general, may have none, a finite number, or a denumerable infinite number of eigenvalues. So we can not claim to have solved (1) completely using numerical methods.

The eigenvalues of (1), in general, are complex numbers, but if the kernel is Hermitian then all eigenvalues are real and the right eigenfunction is equal to the left eigenfunction corresponding to the same eigenvalue. In this paper, the kernel $K(x,y)$ is assumed to be weakly singular as in (2) which is symmetric, so all its eigenvalues are real. Indeed, it has an infinite number of real eigenvalues with zero as its limit point (Baker [3]).

The integral equation eigenvalue problem (1) is reduced to a problem finding the eigenvalue of the matrix K_n in (12) as follows. The function $f(y)$ in (1) is replaced by

$$f_n(y) = \sum_{j=0}^{n} \phi_j(y) f(x_j) \tag{13}$$

where $\phi_j(y)$, $j = 0, 1, \ldots, n$ are as given in (10), and x_j,$j = 0, 1, \ldots, n$ are as in (9). Then

$$\lambda f(x) = \int_{-1}^{1} K(x,y) f_n(y) dy + R_n(x) \tag{14}$$

where the error, $R_n(x)$, is given by

$$R_n(x) = \int_{-1}^{1} K(x,y)(f(x) - f(y))dx \tag{15}$$

Using (5), (8) in (14) we obtain

$$\lambda f(x) = \sum_{j=0}^{n} w_j(x) f(x_j) + R_n(x) \tag{16}$$

where

$$w_j(x) = \int_{-1}^{1} K(x,y)\phi_j(y)dx, \quad j = 0, 1, \ldots, n \tag{17}$$

In (16), on ignoring $R_n(x)$, by successively setting $x = x_i$, $i = 0, 1, \ldots, n$ and replacing $f(x)$ by $f_n(x)$, λ by $\lambda^{(n)}$, the resulting equation is

$$\lambda^{(n)} f(x_i) = \sum_{j=0}^{n} w_j(x_i) f(x_j), \quad i = 0, 1, \ldots, n \tag{18}$$

where $\lambda^{(n)}$ is an approximate value for λ.

System (18) represents an algebraic eigenvalue problem

$$(K_n - \lambda^{(n)} I)\mathbf{f}_n = \mathbf{0} \tag{19}$$

where

$$(K_n)_{ij} = w_j(x_i), \quad i, j = 0, 1, \ldots, n \tag{20}$$

and

$$\mathbf{f}_n = (f(x_0) \quad f(x_1) \quad \ldots \quad f(x_n))^T \tag{21}$$

Then $\lambda^{(n)}$ is an eigenvalue of K_n and $\mathbf{f}_n$ its corresponding eigenvector.

Suppose that λ is an eigenvalue of the symmetric kernel $K(x,y)$ and $f(x)$ is the corresponding eigenfunction with $\|f(x)\|_2$, where $K(x,y)$ is given in (2). Since $K(x,y)$ is symmetric, $f(x)$ is the right and left eigenfunction corresponding to λ, i.e.

$$\lambda f(x) = \int_{-1}^{1} K(x,y) f(y)dy = \int_{-1}^{1} f(y) K(y,x)dy, \quad -1 \leq x \leq 1.$$

Suppose also that the approximate eigenvalue is $\lambda^{(n)}$ and the approximate eigenfunction $f_n(x)$ gives rise to a function

$$\eta(x) = \int_{-1}^{1} K(x,y) f_n(y)dy - \lambda^{(n)} f_n(x).$$

The function $\eta(x)$ can be computed since $\lambda^{(n)}$ and $f_n(x)$ are known. Then

$$\int_{-1}^{1} \eta(x) f(x)dx = \int_{-1}^{1} \int_{-1}^{1} K(x,y) f_n(y) f(x)dydx - \lambda^{(n)} \int_{-1}^{1} f_n(x) f(x)dx$$

that is,

$$(\eta, f) = (Kf_n, f) - \lambda^{(n)}(f_n, f) = (f_n, Kf) - \lambda^{(n)}(f_n, f) = (\lambda - \lambda^{(n)})(f_n, f) \tag{22}$$

Thus, to estimate $\lambda - \lambda^{(n)}$ we need to compute $\eta(x)$. Suppose that $f_n(x)$ is scaled so that $\|f(x) - f_n(x)\|_\infty \to 0$, where $f(x)$ is the fixed normalized eigenfunction corresponding to λ. Then it has been shown in Baker [3] that

$$\|f_n(x)\|_2 \to 1, \quad \text{and} \quad (f_n, f) \to (f, f) = 1.$$

Therefore, for n sufficiently large, $(f_n, f) \neq 0$. Thus, from (22) and since $\|f(x)\|_2 = 1$,

$$|\lambda - \lambda^{(n)}| \leq \frac{|(\eta, f)|}{|(f_n, f)|} \leq \frac{\|\eta(x)\|_2}{|(f_n, f)|} \leq \frac{\|\eta(x)\|_2}{|(f, f)|}\{1 + O(1)\}.$$

Therefore

$$|\lambda - \lambda^{(n)}| \leq \|\eta(x)\|_2\{1 + O(1)\} \leq \sqrt{2}\|\eta(x)\|_\infty\{1 + O(1)\},$$

and we have an asymptotic bound for $|\lambda - \lambda^{(n)}|$ in terms of $\eta(x)$.

4 The numerical results

Case 1: $K(x, y) = |x - y|^{-1/2}$
From (20), the matrix elements are given by

$$(K_n)_{ij} = w_j(x_i), \quad i, j = 0, 1, \ldots, n$$

where x_i, $i = 0, 1, \ldots, n$ are given in (9), $\Delta_{i,j} = x_j - x_i$, n is an even integer. Then from (17)

$$w_j(x_i) = \int_{-1}^{1} |x_i - y|^{-1/2}\phi_j(y)dx, \quad i, j = 0, 1, \ldots, n \tag{23}$$

The matrix elements are given in Appendix A. Table 2 shows the five eigenvalues of the largest moduli for the kernel $K(x, y) = |x - y|^{-1/2}$ obtained with varying orders of the matrix K_n using inverse iteration method.

Case 2: $K(x, y) = \ln|x - y|$
From (20), the matrix elements are given by

$$(K_n)_{ij} = w_j(x_i), \quad i, j = 0, 1, \ldots, n$$

where x_i, $i = 0, 1, \ldots, n$ are given in (9), $\Delta_{i,j} = x_j - x_i$, n is an even integer. Then from (17)

$$w_j(x_i) = \int_{-1}^{1} \ln|x_i - y|\phi_j(y)dx, \quad i, j = 0, 1, \ldots, n \tag{24}$$

The matrix elements are given in Appendix B. Table 3 shows the five eigenvalues of the largest moduli for the kernel $K(x, y) = \ln|x - y|$ obtained with varying orders of the matrix K_n using inverse iteration method.

Table 2: Five eigenvalues of the largest moduli of $|x-y|^{-1/2}$

	$n=128$	$n=256$	$n=512$	$n=1024$
λ_1	3.794218403203	3.794219417592	3.794219544603	3.794219560631
λ_2	1.692241206139	1.692242499027	1.692242644505	1.692242661217
λ_3	1.304165823783	1.304167694375	1.304167877801	1.304167897931
λ_4	1.087026120637	1.087030930371	1.087031312905	1.087031344743
λ_5	0.957630643222	0.957639508606	0.957640168011	0.957640219525

Table 3: Five eigenvalues of the largest moduli of $\ln|x-y|$

	$n=128$	$n=256$	$n=512$	$n=1024$
λ_1	-1.76423854863	-1.76423854617	-1.764238546033	-1.764238546026
λ_2	-1.56600479947	-1.56600508801	-1.566005107041	-1.566005108285
λ_3	-0.78833193093	-0.78833270962	-0.7883327596266	-0.788332762815
λ_4	-0.61295592575	-0.61295819569	-0.6129583418883	-0.612958351179
λ_5	-0.45609158915	-0.45609603958	-0.4560963270175	-0.456096345285

5 Conclusion

In this paper, we have used Product Simpson's rule with non-uniform mesh points to find the eigenvalues of a weakly singular integral equation. The order of the convergence is optimal. The five eigenvalues with largest moduli are found. The results obtained are found to be of great accuracy.

Appendix A

To evaluate the integral (23) when $\phi_j(y)$ is as in (11), define

$$P(i,k)\equiv\int_{\Delta_{i,2k}}^{\Delta_{i,2k+2}}|u|^{-\alpha}du, P(i,k)\equiv\int_{\Delta_{i,2k}}^{\Delta_{i,2k+2}}u|u|^{-\alpha}du \text{ and } P(i,k)\equiv\int_{\Delta_{i,2k}}^{\Delta_{i,2k+2}}u^2|u|^{-\alpha}du.$$

Then

$$P(i,k) = \begin{cases} \frac{1}{1-\alpha}(\Delta_{i,j+2})^{1-\alpha} & , \quad i=2k \\ \frac{1}{1-\alpha}((\Delta_{i-1,j})^{1-\alpha}+(\Delta_{i,j+1})^{1-\alpha}) & , \quad i=2k+1 \\ \frac{1}{1-\alpha}((\Delta_{i,2k+2})^{1-\alpha}-(\Delta_{i,2k})^{1-\alpha}) & , \quad i<2k \\ \frac{1}{1-\alpha}((\Delta_{2k,i})^{1-\alpha}-(\Delta_{2k+2,i})^{1-\alpha}) & , \quad i>2k+1 \end{cases}$$

$$Q(i,k) = \begin{cases} \frac{1}{2-\alpha}(\Delta_{i,j+2})^{2-\alpha} & , \quad i=2k \\ \frac{1}{2-\alpha}((\Delta_{2k+1,2k+2})^{2-\alpha}-(\Delta_{2k,2k+1})^{2-\alpha}) & , \quad i=2k+1 \\ \frac{1}{2-\alpha}((\Delta_{i,2k+2})^{2-\alpha}-(\Delta_{i,2k})^{2-\alpha}) & , \quad i<2k \\ \frac{1}{2-\alpha}((\Delta_{2k+2,i})^{2-\alpha}-(\Delta_{2k,i})^{2-\alpha}) & , \quad i>2k+1 \end{cases}$$

and

$$R(i,k) = \begin{cases} \frac{1}{3-\alpha}(\Delta_{i,i+2})^{3-\alpha} & , \quad i = 2k \\ \frac{1}{3-\alpha}((\Delta_{2k,2k+1})^{3-\alpha} + (\Delta_{2k+1,2k+2})^{3-\alpha}) & , \quad i = 2k+1 \\ \frac{1}{3-\alpha}((\Delta_{i,2k+2})^{3-\alpha} - (\Delta_{i,2k})^{3-\alpha}) & , \quad i < 2k \\ \frac{1}{3-\alpha}((\Delta_{2k,i})^{3-\alpha} - (\Delta_{2k+2,i})^{3-\alpha}) & , \quad i > 2k+1 \end{cases}$$

Hence, using integration by substitution with $h = (y - x_i)/u$ in (23), we get

$$w_0(x_i) = \frac{1}{\Delta_{1,0}\Delta_{2,0}}(R(i,0) + (\Delta_{1,i} + \Delta_{2,i})Q(i,0) + \Delta_{1,i}\Delta_{2,i}P(i,0)),$$
$$\text{for} \quad i = 0, 1, \ldots, n$$

$$w_{2j}(x_i) = \frac{R(i,j-1) + (\Delta_{2j-2,i} + \Delta_{2j-1,i})Q(i,j-1) + \Delta_{2j-2,i}\Delta_{2j-1,i}P(i,j-1)}{\Delta_{2j-2,2j}\Delta_{2j-1,2j}}$$
$$+ \frac{R(i,j) + (\Delta_{2j+1,i} + \Delta_{2j+2,i})Q(i,j) + \Delta_{2j+1,i}\Delta_{2j+2,i}P(i,j)}{\Delta_{2j+1,2j}\Delta_{2j+2,2j}},$$
$$\text{for} \quad j = 0, 1, \ldots, \frac{n}{2} - 1 \quad \text{and} \quad i = 0, 1, \ldots, n$$

$$w_n(x_i) = \frac{R(i, \frac{n}{2} - 1) + (\Delta_{n-2,i} + \Delta_{n-1,i})Q(i, \frac{n}{2} - 1) + \Delta_{n-2,i}\Delta_{n-1,i}P(i, \frac{n}{2} - 1)}{\Delta_{n-2,n}\Delta_{n-1,n}},$$
$$\text{for} \quad i = 0, 1, \ldots, n$$

and

$$w_{2j+1}(x_i) = \frac{R(i,j) + (\Delta_{2j,i} + \Delta_{2j+2,i})Q(i,j) + \Delta_{2j,i}\Delta_{2j+2,i}P(i,j)}{\Delta_{2j,2j+1}\Delta_{2j+2,2j+1}},$$
$$\text{for} \quad j = 0, 1, \ldots, \frac{n}{2} - 1 \quad \text{and} \quad i = 0, 1, \ldots, n$$

Appendix B

To evaluate the integral (24) when $\phi_j(y)$ as in (10), define

$$P(i,k) \equiv \int_{\Delta_{i,2k}}^{\Delta_{i,2k+2}} \ln|u|du, \quad Q(i,k) \equiv \int_{\Delta_{i,2k}}^{\Delta_{i,2k+2}} u\ln|u|du \quad \text{and} \quad R(i,k) \equiv \int_{\Delta_{i,2k}}^{\Delta_{i,2k+2}} u^2\ln|u|du.$$

Then

$$P(i,k) = \begin{cases} \Delta_{i,i+2}\ln\Delta_{i,i+2} - \Delta_{i,i+2} & , \quad i = 2k \\ \Delta_{2k,2k+1}\ln\Delta_{2k,2k+1} - \Delta_{2k,2k+1} & \\ \qquad +\Delta_{2k+1,2k+2}\ln\Delta_{2k+1,2k+2} - \Delta_{2k+1,2k+2} & , \quad i = 2k+1 \\ \Delta_{i,2k+2}\ln\Delta_{i,2k+2} - \Delta_{i,2k+2} - \Delta_{i,2k}\ln\Delta_{i,2k} + \Delta_{i,2k} & , \quad i < 2k \\ \Delta_{2k,i}\ln\Delta_{2k,i} - \Delta_{2k,i} - \Delta_{2k+2,i}\ln\Delta_{2k+2,i} + \Delta_{2k+2,i} & , \quad i > 2k+1 \end{cases}$$

$$Q(i,k) = \begin{cases} \frac{1}{2}(\Delta_{i,i+2})^2\ln\Delta_{i,j+2} - \frac{1}{4}(\Delta_{i,i+2})^2 & , \quad i = 2k \\ \frac{1}{2}(\Delta_{2k+1,2k+2})^2\ln\Delta_{2k+1,2k+2} - \frac{1}{4}(\Delta_{2k+1,2k+2})^2 & \\ \qquad -\frac{1}{2}(\Delta_{2k,2k+1})^2\ln\Delta_{2k,2k+1} + \frac{1}{4}(\Delta_{2k,2k+1})^2 & , \quad i = 2k+1 \\ \frac{1}{2}(\Delta_{i,2k+2})^2\ln\Delta_{i,2k+2} - \frac{1}{4}(\Delta_{i,2k+2})^2 & \\ \qquad -\frac{1}{2}(\Delta_{i,2k})^2\ln\Delta_{i,2k} + \frac{1}{4}(\Delta_{i,2k})^2 & , \quad i < 2k \\ \frac{1}{2}(\Delta_{2k+2,i})^2\ln\Delta_{2k+2,i} - \frac{1}{4}(\Delta_{2k+2,i})^2 & \\ \qquad -\frac{1}{2}(\Delta_{2k,i})^2\ln\Delta_{2k,i} + \frac{1}{4}(\Delta_{2k,i})^2 & , \quad i > 2k+1 \end{cases}$$

and

$$R(i,k) = \begin{cases} \frac{1}{3}(\Delta_{i,i+2})^3\ln\Delta_{i,i+2} - \frac{1}{9}(\Delta_{i,i+2})^3 & , \quad i = 2k \\ \frac{1}{3}(\Delta_{2k,2k+1})^3\ln\Delta_{2k,2k+1} - \frac{1}{9}(\Delta_{2k,2k+1})^3 & \\ \qquad +\frac{1}{3}(\Delta_{2k+1,2k+2})^3\ln\Delta_{2k+1,2k+2} - \frac{1}{9}(\Delta_{2k+1,2k+2})^3 & , \quad i = 2k+1 \\ \frac{1}{3}(\Delta_{i,2k+2})^3\ln\Delta_{i,2k+2} - \frac{1}{9}(\Delta_{i,2k+2})^3 & \\ \qquad -\frac{1}{3}(\Delta_{i,2k})^3\ln\Delta_{i,2k} + \frac{1}{9}(\Delta_{i,2k})^3 & , \quad i < 2k \\ \frac{1}{3}(\Delta_{2k,i})^3\ln\Delta_{2k,i} - \frac{1}{9}(\Delta_{2k,i})^3 & \\ \qquad -\frac{1}{3}(\Delta_{2k+2,i})^3\ln\Delta_{2k+2,i} + \frac{1}{9}(\Delta_{2k+2,i})^3 & , \quad i > 2k+1 \end{cases}$$

Hence, using integration by substitution with $h = (y - x_i)/u$ in (24), we get

$$w_0(x_i) = \frac{1}{\Delta_{1,0}\Delta_{2,0}}(R(i,0) + (\Delta_{1,i} + \Delta_{2,i})Q(i,0) + \Delta_{1,i}\Delta_{2,i}P(i,0)),$$
$$\text{for } \; i = 0, 1, \ldots, n$$

$$w_{2j}(x_i) = \frac{R(i,j-1) + (\Delta_{2j-2,i} + \Delta_{2j-1,i})Q(i,j-1) + \Delta_{2j-2,i}\Delta_{2j-1,i}P(i,j-1)}{\Delta_{2j-2,2j}\Delta_{2j-1,2j}}$$
$$+\frac{R(i,j) + (\Delta_{2j+1,i} + \Delta_{2j+2,i})Q(i,j) + \Delta_{2j+1,i}\Delta_{2j+2,i}P(i,j)}{\Delta_{2j+1,2j}\Delta_{2j+2,2j}},$$
$$\text{for } \; j = 1, \ldots, \frac{n}{2} - 1 \;\text{ and }\; i = 0, 1, \ldots, n$$

$$w_n(x_i) = \frac{R(i,\frac{n}{2}-1) + (\Delta_{n-2,i} + \Delta_{n-1,i})Q(i,\frac{n}{2}-1) + \Delta_{n-2,i}\Delta_{n-1,i}P(i,\frac{n}{2}-1)}{\Delta_{n-2,n}\Delta_{n-1,n}},$$
$$\text{for } \; i = 0, 1, \ldots, n$$

and

$$w_{2j+1}(x_i) = \frac{R(i,j) + (\Delta_{2j,i} + \Delta_{2j+2,i})Q(i,j) + \Delta_{2j,i}\Delta_{2j+2,i}P(i,j)}{\Delta_{2j,2j+1}\Delta_{2j+2,2j+1}},$$
$$\text{for } \; j = 0, 1, \ldots, \frac{n}{2} - 1 \;\text{ and }\; i = 0, 1, \ldots, n$$

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