

Substituting (34) in (32), and equating coefficients of like powers of (x, y) , we obtain the recursion formulae

$$(35) \quad \sum_{j+k=n} B_{ij} B_{0k} [j][2k - j + 1] = \sum_{j+k=n-1} B_{i+1,j} B_{0k} [i + 1][j - k]; \quad i: 0, 1, \dots$$

From (10), it is readily verified that $B_{i0} = 0$ for $i \neq 0$, so that equations (35) give solutions for the B_{ij} in terms of the B_{0k} . These solutions are of interest since they show a one-to-one correspondence between the functions $G(0, y)$ and $G(x, y)$, for $(x, y) \in [R \cap S]$.

NUMERICAL INTEGRATION FOR LINEAR SUMS OF EXPONENTIAL FUNCTIONS

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1. Introduction. The methods of numerical integration going by the names trapezoidal rule, Simpson's rule, Weddle's rule, and the Newton-Cotes formulae are of the type

$$(1) \quad \int_{-1}^1 f(x) dx \simeq \sum_{i=0}^n \lambda_{in} f(x_{in})$$

where the abscissae $\{x_{in}\}$ are uniformly distributed on a finite interval, chosen as $(-1, 1)$ for convenience,

$$(2) \quad x_{in} = -1 + \frac{2i}{n}, \quad i = 0, 1, 2, \dots, n,$$

and where the set of constants $\{\lambda_{in}\}$ depend on the name of the rule and the value of n but not on the function $f(x)$. Throughout this note all abscissae will be assumed to be uniformly distributed on $(-1, 1)$ unless the contrary is explicitly stated.

Since correspondence relation (1) involves $(n + 1)$ constants $\{\lambda_{in}\}$, it might be possible to choose $(n + 1)$ arbitrary functions $g_j(x)$, $j = 0, 1, 2, \dots, n$, and require that the set $\{\lambda_{in}\}$ be the solution, if such exists, of the $(n + 1)$ simultaneous linear equations

$$(3) \quad \int_{-1}^1 g_j(x) dx = \sum_{i=0}^n \lambda_{in} g_j(x_{in}), \quad j = 0, 1, 2, \dots, n.$$

Indeed, the selection

$$(4) \quad g_j(x) = x^j, \quad j = 0, 1, 2, \dots, n,$$

will give a set of $(n + 1)$ simultaneous equations of form (3) and the solution $\{\lambda_{in}\}$ is the set of Newton-Cotes weights for that value of n . The numerical evaluation

¹ This work was performed with the financial support of the Office of Naval Research of the Navy Department.

of $\{\lambda_{in}\}$ is best accomplished by other and more sophisticated methods, however.²

Because of linearity in both the integral and the finite summation, once the constants $\{\lambda_{in}\}$ have been determined for a specific set of functions $\{g_j(x)\}$, correspondence relation (1) is exact for any linear combination of that fundamental set. Thus, for example, for the fundamental set (4), correspondence relation (1) with the appropriate values $\{\lambda_{in}\}$ is exact for all polynomials of degree less than or equal to n .

Although tradition favors the set of functions (4), there is nothing compelling about such a selection. Indeed, two other possible choices might be

$$(5) \quad g_j(x) = e^{jx}, \quad j = 0, 1, 2, \dots, n,$$

and

$$(6) \quad g_j(x) = e^{jx}, \\ j = -m, -m + 1, \dots, 0, 1, \dots, m - 1, m; n = 2m.$$

These choices would seem to be appropriate whenever numerical methods are being applied to exponential growth curves or exponential decay curves.

2. Use of the basic set $g_j(x) = e^{jx}$. If integration relation (1) be made exact for the set $\{e^{jx}\}, j = 0, 1, \dots, n$ with evenly spaced x abscissae, the set (3) of $(n + 1)$ simultaneous linear equations in the unknowns $\{\lambda_{in}\}, i = 0, 1, \dots, n$ is obtained. Call the solution of this system $\{a_{in}\}$, solution values for $n = 1, 2, 3, 4, 5, 6$ are tabulated below.

For the symmetric case where integration relation (1) is made exact for $\{e^{jx}\}, j = -m, -m + 1, \dots, m - 1, m; n = 2m$, a similar but different set of linear equations (3) results for the unknowns $\{\lambda_{in}\}$. Call the solution of this system $\{b_{in}\}$. As implied above, only even values of n are used in order to preserve the symmetry, and values of $\{b_{in}\}$ are tabulated below for $n = 2, 4, 6$.

$n = 1,$	$a_{01} =$	1.31303	5285		
	$a_{11} =$	0.68696	4715		
$n = 2,$	$a_{02} =$	0.21805	032 ⁺	$b_{02} =$	0.32260 623 ⁻
	$a_{12} =$	1.49780	742	$b_{12} =$	1.35478 755
	$a_{22} =$	0.28414	226 ⁻	$b_{22} =$	0.32260 623 ⁻
$n = 3,$	$a_{03} =$	0.51324	284		
	$a_{13} =$	0.22445	055		
	$a_{23} =$	1.08155	527		
	$a_{33} =$	0.18075	134		
$n = 4,$	$a_{04} =$	-0.13716	639 ⁺	$b_{04} =$	0.15048 171
	$a_{14} =$	1.40098	548	$b_{14} =$	0.73243 318

² Whittaker and Robinson, *The Calculus of Observations*, 4th Edition, (1946), London, pp. 152-156.

	$a_{24} = -0.30895$	914	$b_{24} = 0.23417$	022
	$a_{34} = 0.91710$	903	$b_{34} = 0.73243$	318
	$a_{44} = 0.12803$	103 ⁻	$b_{44} = 0.15048$	171
$n = 5,$	$a_{05} = 0.68919$	3		
	$a_{15} = -1.07644$	3		
	$a_{25} = 2.12534$	6		
	$a_{35} = -0.63595$	6		
	$a_{45} = 0.79933$	8		
	$a_{55} = 0.09852$	18		
$n = 6,$	$a_{06} = -0.83607$		$b_{06} = 0.09443$	5
	$a_{16} = 3.54128$		$b_{16} = 0.53464$	7
	$a_{26} = -3.88102$		$b_{26} = 0.01139$	3
	$a_{36} = 3.32254$		$b_{36} = 0.71905$	0
	$a_{46} = -0.94685$		$b_{46} = 0.01139$	3
	$a_{56} = 0.72075$		$b_{56} = 0.53464$	7
	$a_{66} = 0.07937$	5 ⁺	$b_{66} = 0.09443$	5

The computing service of the Institute for Numerical Analysis has supplied the author with most of the coefficients tabulated above.

3. Estimates of the error term. The choices of the coefficients $\{a_{in}\}$ and $\{b_{in}\}$ are such that integration relation (1) is exact whenever

$$(7) \quad f(x) = A_0 + A_1 e^x + \cdots + A_n e^{nx} \quad \text{and} \quad \lambda_{in} = a_{in},$$

and whenever

$$(8) \quad f(x) = B_{-m} e^{-mx} + B_{-m+1} e^{-(m-1)x} + \cdots + B_0 + \cdots + B_m e^{mx} \quad \text{and} \quad \lambda_{in} = b_{in}.$$

When $f(x)$ is not of these prescribed forms, the error in using correspondence (1) may be of some importance. By making the transformation

$$(9) \quad u = e^x, \quad f(x) = f(\log u) = g(u)$$

integration relation (1) becomes

$$(10) \quad \int_{e^{-1}}^e g(u) \frac{du}{u} \simeq \sum_{i=0}^n \lambda_{in} g(u_{in})$$

where the $\{u_{in}\}$ are not evenly distributed. By approximating $g(u)$ by its Taylor's series with a remainder term, the following expressions for the error in using correspondence (1) can be obtained:

Using the coefficients $\{a_{in}\}$,

$$(11) \quad \text{Error} \leq \frac{(e^2 - 1)^{n+1}}{(n+1)!} \left[2 + \sum_{i=0}^n |a_{in}| \right] \left[\max_{-1 \leq x \leq 1} \left(e^{-x} \frac{d}{dx} \right)^{n+1} f(x) \right]$$

and, using the coefficients $\{b_{in}\}$,

$$(12) \quad \text{Error} \leq \frac{(e^2 - 1)^{2m+1}}{(2m + 1)!} \left[\frac{e^m - e^{-m}}{m} + \sum_{i=0}^{2m} \frac{|b_{i,2m}|}{e^{mx_i,2m}} \right] \cdot \left[\max_{-1 \leq x \leq 1} \left(e^{-x} \frac{d}{dx} \right)^{2m+1} e^{mx} f(x) \right].$$

Neither of these error expressions can be said to be very practical in actual computation, and neither appears suitable for establishing convergence properties of the type

$$(13) \quad \lim_{n \rightarrow \infty} \sum_{i=0}^n \lambda_{in} f(x_{in}) = \int_{-1}^1 f(x) dx.$$

However, both (11) and (12) reduce to zero when $f(x)$ is of the form prescribed by (7) or (8) respectively.

4. Numerical examples. As illustrative numerical examples, the case $n = 4$ was selected and several typical functions were integrated approximately by the positive power exponential rule, the symmetrical exponential rule and the Newton-Cotes formula,

$$\int_{-1}^1 f(x) dx = \frac{1}{48} [7f(-1) + 32f(-\frac{1}{2}) + 12f(0) + 32f(\frac{1}{2}) + 7f(1)].$$

Values of $\{a_{ii}\}$ and $\{b_{ii}\}$ are given in the tables in part 2. The typical functions used were x^2 , e^{2x} , $1/(x + 3)$, e^{-x^2} , xe^x , x^6 , and $e^{2.2x}$. The following results were obtained:

Function	Positive Power Exponential		Symmetrical Exponential		Newton-Cotes		8 Decimal Approximation to Exact Value	
x^2	.5703	8827	.6671	8001	.6666	6666	.6666	6667
e^{2x}	3.6268	6044	3.6268	6041	3.6317	3108	3.6268	6041
$1/(x + 3)$.6828	6353	.6931	5792	.6931	7460	.6931	4718
e^{-x^2}	1.4930	1396	1.4857	2754	1.4887	4582	1.4936	4827
xe^x	.7292	4338	.7353	6007	.7361	7480	.7357	5888
x^6	.0270	8487	.3238	5196	.3333	3332	.2857	1429
$e^{2.2x}$	4.0527	7287	4.0530	7585	4.0607	7415	4.0519	1379

From this tabulation, it would appear that the symmetrical exponential method compares favorably with the Newton-Cotes method for such typical functions as $1/(x + 3)$, e^{-x^2} , xe^x , x^6 , and $e^{2.2x}$. Note that the choice of x^2 or e^{2x} is not really a fair choice when comparing these two methods, since Newton-Cotes is derived so as to give exactness for x^2 and the symmetrical exponential so as to give exactness for e^{2x} .