Numerische Mathematik 3, 381–397 (1961)

Numerical integration of ordinary differential equations based on trigonometric polynomials

By

WALTER GAUTSCHI *

There are many numerical methods available for the step-by-step integration of ordinary differential equations. Only few of them, however, take advantage of special properties of the solution that may be known in advance. Examples of such methods are those developed by BROCK and MURRAY [2], and by DENNIS [4], for exponential type solutions, and a method by URABE and MISE [5] designed for solutions in whose Taylor expansion the most significant terms are of relatively high order. The present paper is concerned with the case of periodic or oscillatory solutions where the frequency, or some suitable substitute, can be estimated in advance. Our methods will integrate exactly appropriate trigonometric polynomials of given order, just as classical methods integrate exactly algebraic polynomials of given degree. The resulting methods depend on a parameter, $v=h\omega$, where h is the step length and ω the frequency in question, and they reduce to classical methods if $v\to 0$. Our results have also obvious applications to numerical quadrature. They will, however, not be considered in this paper.

1. Linear functionals of algebraic and trigonometric order

In this section $[a, b]$ is a finite closed interval and $C^s[a, b]$ $(s\geqq 0)$ denotes the linear space of functions $x(t)$ having s continuous derivatives in $[a, b]$. We assume $C^s[a, b]$ normed by

$$\|x\| = \sum_{\sigma=0}^{s} \max_{a\leqq t\leqq b} |x^{(\sigma)}(t)|. \tag{1.1}$$

A linear functional L in $C^s[a, b]$ is said to be of *algebraic order p*, if

$$Lt^r = 0 \qquad (r = 0, 1, \ldots, p); \tag{1.2}$$

it is said to be of *trigonometric order p, relative to period T*, if

$$L1 = L\cos\left(r\frac{2\pi}{T}t\right) = L\sin\left(r\frac{2\pi}{T}t\right) = 0 \qquad (r = 1, 2, \ldots, p). \tag{1.3}$$

Thus, a functional L is of algebraic order p if it annihilates all algebraic polynomials of degree $\leqq p$, and it is of trigonometric order p, relative to period T, if it annihilates all trigonometric polynomials of order $\leqq p$ with period T.

Functionals of trigonometric order p are comparable with those of algebraic order $2p$, in the sense that both involve the same number of conditions. The

* Oak Ridge National Laboratory, operated by Union Carbide Corporation for the U. S. Atomic Energy Commission, Oak Ridge, Tennessee.

relationship turns out to be much closer if we let L depend on the appropriate number of parameters. In fact, consider functionals of the form

$$Lx = \beta_1 L_1 x + \cdots + \beta_{2p} L_{2p} x + L_{2p+1} x, \tag{1.4}$$

where L_λ ($\lambda \geqq 1$) are fixed linear continuous functionals in $C^s[a, b]$ and β_λ real parameters. Then the following theorem holds.

Theorem 1. *Let the functionals L_λ in (1.4) satisfy the following conditions:*

(i) $L_\lambda 1 = 0$ $(\lambda = 1, 2, \ldots, 2p+1)$.

(ii) *There is a unique set of parameters, $\beta_\lambda = \beta_\lambda^0$, such that the functional L in (1.4) is of algebraic order $2p$, that is to say,*

$$\det(L_\lambda t^\varkappa) \neq 0 \quad \begin{pmatrix} \varkappa \text{ row index, } \lambda \text{ column index} \\ \varkappa, \lambda = 1, 2, \ldots, 2p \end{pmatrix}. \tag{1.5}$$

Then, for T sufficiently large, there is also a unique set of parameters, $\beta_\lambda = \beta_\lambda(T)$, such that L is of trigonometric order p relative to period T. Furthermore,

$$\beta_\lambda(T) \to \beta_\lambda^0 \quad \text{as} \quad T \to \infty. \tag{1.6}$$

Proof. The main difficulty in the proof is the fact that in the limit, as $T \to \infty$, equations (1.3) degenerate into one single equation, $L\,1 = 0$. We therefore transform (1.3) into an equivalent set of equations from which the behavior of the solution at $T = \infty$ can be studied more easily.

In this connection the following trigonometric identities are helpful,

$$\sin^{2r}\frac{x}{2} = \sum_{\varrho=1}^{r} \sigma_{r\varrho}(1 - \cos\varrho x) \qquad (r = 1, 2, 3, \ldots), \tag{1.7}$$

where $\sigma_{r\varrho}$ are suitable real numbers and $\sigma_{rr} \neq 0$. The existence of such numbers is obvious, if one observes that $\sin^{2r}\frac{x}{2} = [(1-\cos x)/2]^r$ can be written as a cosine-polynomial of exact order r. Differentiating both sides in (1.7) gives also

$$\sin^{2r-1}\frac{x}{2}\cos\frac{x}{2} = \sum_{\varrho=1}^{r} \tau_{r\varrho}\sin\varrho x \qquad (r = 1, 2, 3, \ldots), \tag{1.8}$$

where $\tau_{r\varrho} = \varrho\sigma_{r\varrho}/r$, and in particular $\tau_{rr} = \sigma_{rr} \neq 0$.

Equations (1.3) are equivalent to

$$L\,1 = 0,$$

$$L\left(1 - \cos r\frac{2\pi}{T}t\right) = L\sin r\frac{2\pi}{T}t = 0 \qquad (r = 1, 2, \ldots, p).$$

Because of assumption (i) the first of these equations is automatically satisfied. The remaining equations are equivalent to

$$\sum_{\varrho=1}^{r} \sigma_{r\varrho} L\left(1 - \cos\varrho\frac{2\pi}{T}t\right) = \sum_{\varrho=1}^{r} \tau_{r\varrho} L\sin\varrho\frac{2\pi}{T}t = 0 \qquad (r = 1, 2, \ldots, p). \tag{1.9}$$

Using (1.7) and the linearity of L we have

$$\sum_{\varrho=1}^{r} \sigma_{r\varrho} L\left(1-\cos\varrho \frac{2\pi}{T} t\right) = L \sum_{\varrho=1}^{r} \sigma_{r\varrho}\left(1-\cos\varrho \frac{2\pi}{T} t\right) = L\left[\sin^{2r}\left(\frac{\pi}{T} t\right)\right].$$

Similarly, using (1.8), we find

$$\sum_{\varrho=1}^{r} \tau_{r\varrho} L \sin\varrho \frac{2\pi}{T} t = L\left[\sin^{2r-1}\left(\frac{\pi}{T} t\right)\cos\frac{\pi}{T} t\right].$$

Therefore, letting

$$u = \frac{\pi}{T}, \tag{1.10}$$

we can write (1.9), after suitable multiplications, as follows:

$$\left.\begin{aligned} L\left[\left(\frac{\sin u t}{u}\right)^{2r-1}\cos u t\right] &= 0 \\ L\left[\left(\frac{\sin u t}{u}\right)^{2r}\right] &= 0 \end{aligned}\right\} \quad (r = 1, 2, \ldots, p). \tag{1.11}$$

This represents a system of $2p$ linear algebraic equations in the same number of unknowns β_λ, the coefficient matrix and known vector of which both depend on the parameter u. We show that in the limit as $u\to 0$ the system (1.11) goes over into the system of equations $Lt^r = 0$ $(r = 1, 2, \ldots, 2p)$.

In fact, it is readily seen, by expansion or otherwise, that for any integers $\sigma \geqq 0$, $r \geqq 1$, as $u\to 0$,

$$\frac{d^\sigma}{dt^\sigma}\left[\left(\frac{\sin u t}{u}\right)^{2r-1}\cos u t\right] \to \frac{d^\sigma}{dt^\sigma} t^{2r-1},$$

$$\frac{d^\sigma}{dt^\sigma}\left(\frac{\sin u t}{u}\right)^{2r} \to \frac{d^\sigma}{dt^\sigma} t^{2r},$$

the convergence being uniform with respect to t in any finite interval. In particular,

$$\left.\begin{aligned} \left\|\left(\frac{\sin u t}{u}\right)^{2r-1}\cos u t - t^{2r-1}\right\| &\to 0 \\ \left\|\left(\frac{\sin u t}{u}\right)^{2r} - t^{2r}\right\| &\to 0 \end{aligned}\right\} \quad (u \to 0),$$

so that, by the continuity of the L_λ, also

$$\left.\begin{aligned} L_\lambda\left[\left(\frac{\sin u t}{u}\right)^{2r-1}\cos u t\right] &\to L_\lambda t^{2r-1} \\ L_\lambda\left(\frac{\sin u t}{u}\right)^{2r} &\to L_\lambda t^{2r} \end{aligned}\right\} \quad (u \to 0).$$

From this our assertion follows immediately.

Since the limiting system, by assumption, has a unique solution, β_λ^0, the matrix of the system (1.11) is nonsingular for $u=0$, and hence remains so for u sufficiently small. It follows that for sufficiently large T there is a unique solution, $\beta_\lambda(T)$, of (1.11), satisfying (1.6). Theorem 1 is proved.

Remark 1. Assumption (i) in Theorem 1 is not essential, but convenient for some of the applications made later. The theorem holds without the assumption (i) if the functional L in (1.4) is made to depend on $2p+1$ parameters,

$$(1.4') \qquad L\,x = \beta_0 L_0\, x + \beta_1 L_1\, x + \cdots + \beta_{2p} L_{2p}\, x + L_{2p+1}\, x,$$

and assumption (1.5) is modified, accordingly, to

$$(1.5') \qquad \det(L_\lambda t^\varkappa) \neq 0 \qquad \begin{pmatrix} \varkappa \text{ row index, } \lambda \text{ column index} \\ \varkappa, \lambda = 0, 1, \ldots, 2p \end{pmatrix}.$$

The proof remains the same.

Remark 2. For particular choices of the L_λ it may happen that the functional L can be made of higher algebraic order than the number of parameters would indicate. Even if the excess in order is a multiple of 2, this does not mean necessarily that a similar increase in trigonometric order is possible. For example,

$$L\,x = \beta\, x(0) + x(1) - \tfrac{1}{2} x'(0) - \tfrac{1}{2} x'(1), \qquad \beta = -1$$

if of algebraic order 2, but in general cannot be made of trigonometric order 1, since

$$L\left[\frac{\sin u\,t}{u} \cos u\,t\right] = \frac{\sin 2u}{2u} - \frac{1}{2}(1 + \cos 2u) > 0 \qquad \left(0 < u < \frac{\pi}{2}\right).$$

2. Linear multi-step methods

Linear functionals in C^1 play an important rôle in the numerical solution of first order differential equations

$$(2.1) \qquad x' = f(t, x), \qquad x(t_0) = x_0,$$

in that they provide the natural mathematical setting for a large class of numerical methods, the so-called linear multi-step methods. These are methods which define approximations x_m to values $x(t_0 + m h)$ of the desired solution by a relation of the following form

$$(2.2) \qquad \begin{gathered} x_{n+1} + \alpha_1 x_n + \cdots + \alpha_k x_{n+1-k} = h(\beta_0 x'_{n+1} + \beta_1 x'_n + \cdots + \beta_k x'_{n+1-k}) \\ (n = k-1, k, k+1, \ldots), \end{gathered}$$

where

$$x'_m = f(t_0 + m\,h,\, x_m).$$

Once k "starting" values $x_0, x_1, \ldots, x_{k-1}$ are known, (2.2) is used to obtain successively all approximations x_m $(m \geqq k)$ desired.

The integer $k > 0$ will be called the *index* of the multi-step method, assuming, of course, that not both α_k and β_k vanish. (2.2) is called an *extrapolation method* if $\beta_0 = 0$, and an *interpolation method* if $\beta_0 \neq 0$. Interpolation methods require the solution of an equation at each stage, because x'_{n+1} in (2.2) is itself a function of the new approximation x_{n+1}.

It is natural to associate with (2.2) the linear functional

$$(2.3) \qquad L\,x = \sum_{\lambda=0}^{k} \left[\alpha_\lambda\, x(t_0 + (n+1-\lambda)\,h) - h\,\beta_\lambda\, x'(t_0 + (n+1-\lambda)\,h)\right] \qquad (\alpha_0 = 1).$$

The multi-step method (2.2) is called of algebraic order p, if its associated linear functional (2.3) is of algebraic order p; similarly one defines trigonometric order of a multi-step method.

Since any linear transformation $t' = at + b$ $(a \neq 0)$ of the independent variable transforms an algebraic polynomial of degree $\leq p$ into one of the same kind, it is clear that (2.2) is of algebraic order p if and only if the functional

$$L^1 x = \sum_{\lambda=0}^{k} [\alpha_\lambda x(k-\lambda) - \beta_\lambda x'(k-\lambda)] \tag{2.4}$$

is of algebraic order p. Here, the parameter h has dropped out, so that the coefficients α_λ, β_λ of a multi-step method of algebraic order do not depend on h. The situation is somewhat different in the trigonometric case, where a linear transformation other than a translation (or reflexion) changes the period of a trigonometric polynomial. By a translation, however, it is seen that (2.2) is of trigonometric order p, relative to period T, if and only if

$$L^h x = \sum_{\lambda=0}^{k} \{\alpha_\lambda x[(k-\lambda)h] - h\beta_\lambda x'[(k-\lambda)h]\} \tag{2.5}$$

is of trigonometric order p relative to period T.

For a multi-step method to be useful it must be numerically stable, which above all imposes certain restrictions on the coefficients α_λ (see, e.g., [*1*, sec. 9]). In view of this we shall consider the α_λ as prescribed numbers satisfying the conditions of stability. Also they shall satisfy

$$\sum_{\lambda=0}^{k} \alpha_\lambda = 0 \qquad (\alpha_0 = 1) \tag{2.6}$$

to insure algebraic and trigonometric order $p=0$.

It is then well known ([*1*, sec. 6]) that to any given set of $k+1$ coefficients α_λ satisfying (2.6) there corresponds a unique extrapolation method with index k and algebraic order k. Letting therefore $k=2p$ we can apply Theorem 1 to $L = L^h$, identifying

$$L_\lambda x = -h x'[(2p-\lambda)h] \quad (1 \leq \lambda \leq 2p), \qquad L_{2p+1} x = \sum_{\lambda=0}^{2p} \alpha_\lambda x[(2p-\lambda)h]. \tag{2.7}$$

It follows that there exists a unique extrapolation method with *even* index $k=2p$ and trigonometric order p relative to any sufficiently large period T. Again, as is well known, given $k+1$ coefficients α_λ, there corresponds a unique interpolation method with index k and algebraic order $k+1$. Letting now $k+1=2p$, a similar application of Theorem 1 shows the existence, for T sufficiently large, of an interpolation method with *odd* index $k=2p-1$ and trigonometric order p relative to period T. Furthermore, in the limit as $T \to \infty$, the resulting methods of trigonometric order p reduce to those of algebraic order $2p$.

The essential parameter is actually not T, but h/T, as is seen if the conditions (1.11) of trigonometric order p are written down for the functional L^h. Since

$$\frac{d}{dt}\left[\left(\frac{\sin ut}{u}\right)^{2r-1} \cos ut\right] = \left(\frac{\sin ut}{u}\right)^{2r-2} (2r\cos^2 ut - 1),$$
$$\frac{d}{dt}\left[\left(\frac{\sin ut}{u}\right)^{2r}\right] = 2r\left(\frac{\sin ut}{u}\right)^{2r-1} \cos ut$$

one finds[1]

$$h\sum_{\lambda=0}^{k}\beta_\lambda\left(\frac{\sin[u(k-\lambda)h]}{u}\right)^{2r-2}\left(2r\cos^2[u(k-\lambda)h]-1\right)$$
$$=\sum_{\lambda=0}^{k}\alpha_\lambda\left(\frac{\sin[u(k-\lambda)h]}{u}\right)^{2r-1}\cos[u(k-\lambda)h],$$
$$2rh\sum_{\lambda=0}^{k}\beta_\lambda\left(\frac{\sin[u(k-\lambda)h]}{u}\right)^{2r-1}\cos[u(k-\lambda)h]=\sum_{\lambda=0}^{k}\alpha_\lambda\left(\frac{\sin[u(k-\lambda)h]}{u}\right)^{2r}.$$

Dividing the first relation by h^{2r-1}, and the second relation by h^{2r}, and letting

$$v=2uh=\frac{2\pi}{T}h,$$

one gets[1]

$$\sum_{\lambda=0}^{k}\beta_\lambda\left(\frac{2\sin[\frac{1}{2}(k-\lambda)v]}{v}\right)^{2r-2}\left(2r\cos^2\left[\frac{1}{2}(k-\lambda)v\right]-1\right)$$
$$=\sum_{\lambda=0}^{k}\alpha_\lambda\left(\frac{2\sin[\frac{1}{2}(k-\lambda)v]}{v}\right)^{2r-1}\cos\left[\frac{1}{2}(k-\lambda)v\right], \tag{2.8}$$
$$2r\sum_{\lambda=0}^{k}\beta_\lambda\left(\frac{2\sin[\frac{1}{2}(k-\lambda)v]}{v}\right)^{2r-1}\cos\left[\frac{1}{2}(k-\lambda)v\right]=\sum_{\lambda=0}^{k}\alpha_\lambda\left(\frac{2\sin[\frac{1}{2}(k-\lambda)v]}{v}\right)^{2r}$$
$$(r=1,2,\ldots,p).$$

We summarize our findings in the following

Theorem 2. *In correspondence to each set of coefficients α_λ with zero sum there exist unique sets of coefficients $\beta_\lambda(v)$, $\beta_\lambda^*(v)$ depending on the parameter*

$$v=2\pi h/T,$$

such that for v sufficiently small,

$$x_{n+1}+\alpha_1x_n+\cdots+\alpha_{2p}x_{n+1-2p}=h\left[\beta_1(v)x_n'+\cdots+\beta_{2p}(v)x_{n+1-2p}'\right] \tag{2.9}$$

is an extrapolation method of trigonometric order p relative to period T, and

$$x_{n+1}+\alpha_1x_n+\cdots+\alpha_{2p-1}x_{n+2-2p}$$
$$=h\left[\beta_0^*(v)x_{n+1}'+\beta_1^*(v)x_n'+\cdots+\beta_{2p-1}^*(v)x_{n+2-2p}'\right] \tag{2.10}$$

is an interpolation method of trigonometric order p relative to period T. The $\beta_\lambda(v)$ solve the system of linear algebraic equations (2.8) *with $k=2p$, $\beta_0=0$, the $\beta_\lambda^*(v)$ solve the same system with $k=2p-1$ and with no restrictions on the β's. As $v\to0$ the multi-step methods* (2.9) *and* (2.10) *reduce to those of algebraic order $2p$, respectively.*

3. Existence criterion for trigonometric multi-step methods

Theorem 2 establishes the existence of trigonometric multi-step methods only for $v=2\pi h/T$ sufficiently small. A more precise condition on v is furnished by the following

[1] If $r=1$ the coefficient of β_k in the first relation, to be meaningful, must be defined as unity.

Theorem 3. *Multi-step methods* (2.9) *and* (2.10) *of trigonometric order p, relative to period T, exist if*

$$|v| < \min\left(v_p, \frac{2\pi}{2p-1}\right) \qquad (v = 2\pi h/T), \tag{3.1}$$

where v_p is the smallest positive zero of the cosine-polynomial

$$C_p(v) = \begin{cases} \sum_{n=1}^{(p^2+1)/2} \nu_p\left(p^2 - \frac{1}{2}p + \frac{1}{2} - n\right)\cos(2n-1)\frac{v}{2} & (p \text{ odd}) \\ \frac{1}{2}\nu_p\left(p^2 - \frac{1}{2}p\right) + \sum_{n=1}^{p^2/2} \nu_p\left(p^2 - \frac{1}{2}p - n\right)\cos n v & (p \text{ even}). \end{cases} \tag{3.2}$$

Here, $\nu_p(m)$ denotes the number of combinations of p nonnegative[2] *integers not exceeding $2p-1$ which have the sum m.*

Proof. The linear functional associated with the extrapolation method (2.9) is

$$L x = \sum_{\lambda=1}^{2p} \beta_\lambda L_\lambda x + L_{2p+1} x,$$

where $L_\lambda x = -h x'[(2p-\lambda)h]$ $(1 \leq \lambda \leq 2p)$ and L_{2p+1} is given such that $L_{2p+1} 1 = 0$. Similarly,

$$L^* x = \sum_{\lambda=0}^{2p-1} \beta_\lambda^* L_\lambda^* x + L_{2p}^* x$$

with $L_\lambda^* = L_{\lambda+1}$, $L_{2p}^* 1 = 0$, is the functional associated with the interpolation method (2.10). It is apparent, therefore, that the conditions (1.3) of trigonometric order for these particular functionals give rise to a system of $2p$ linear algebraic equations in the unknowns β_λ and β_λ^*, respectively, the matrix of which in either case is given by

$$B(v) = \begin{pmatrix} v\sin(2p-1)v & v\sin(2p-2)v & \dots & v\sin v & 0 \\ -v\cos(2p-1)v & -v\cos(2p-2)v & \dots & -v\cos v & -v \\ 2v\sin 2(2p-1)v & 2v\sin 2(2p-2)v & \dots & 2v\sin 2v & 0 \\ -2v\cos 2(2p-1)v & -2v\cos 2(2p-2)v & \dots & -2v\cos 2v & -2v \\ \dots & \dots & \dots & \dots & \dots \\ p v\sin p(2p-1)v & p v\sin p(2p-2)v & \dots & p v\sin p v & 0 \\ -p v\cos p(2p-1)v & -p v\cos p(2p-2)v & \dots & -p v\cos p v & -p v \end{pmatrix}.$$

The instance $v=0$ (in which B is singular) is sufficiently dealt with by Theorem 2. Theorem 3 will therefore be proved if it is shown that $B(v)$ is nonsingular for all nonvanishing values of v satisfying (3.1).

Replacing the trigonometric functions in $B(v)$ by Euler's expressions and applying a few obvious elementary operations on rows and columns of the

[2] In terms of partitions (more commonly used in combinatorial analysis) which involve *positive* integers with given sum, we have

$$\nu_p(m) = \pi_{p-1}(2p-1, m) + \pi_p(2p-1, m),$$

where $\pi_k(l, m)$ denotes the number of partitions of m into k unequal parts not exceeding l.

resulting matrix, one shows that the determinant of B is equal to

$$\det B(v) = (p!)^2\, 2^{-p}\, i^p\, v^{2p}\, e^{-p^2(2p-1)iv} \begin{vmatrix} w_{2p-1}^{2p} & w_{2p-2}^{2p} & \dots & w_1^{2p} & w_0^{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ w_{2p-1}^{p+1} & w_{2p-2}^{p+1} & \dots & w_1^{p+1} & w_0^{p+1} \\ w_{2p-1}^{p-1} & w_{2p-2}^{p-1} & \dots & w_1^{p-1} & w_0^{p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ w_{2p-1} & w_{2p-2} & \dots & w_1 & w_0 \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix} \quad (w_\lambda = e^{i\lambda v}).$$

The last determinant is a minor of the Vandermonde determinant

$$\begin{vmatrix} u^{2p} & w_{2p-1}^{2p} & \dots & w_1^{2p} & w_0^{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u^p & w_{2p-1}^{p} & \dots & w_1^{p} & w_0^{p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix} = \prod_{\sigma=0}^{2p-1} (u - w_\sigma) \prod_{0 \leqq \sigma < \varrho \leqq 2p-1} (w_\varrho - w_\sigma),$$

namely, up to the sign $(-1)^p$, the coefficient of u^p in the expansion along the first column. From the right-hand side it is seen that this coefficient is equal to

$$(-1)^p\, \sigma_p(w_0, w_1, \dots, w_{2p-1}) \prod_{0 \leqq \sigma < \varrho \leqq 2p-1} (w_\varrho - w_\sigma),$$

where σ_p denotes the p-th elementary symmetric function in $2p$ variables. Therefore,

$$\det B(v) = (p!)^2\, 2^{-p}\, i^p\, v^{2p}\, e^{-p^2(2p-1)iv}\, \sigma_p(w_0, w_1, \dots, w_{2p-1}) \prod_{0 \leqq \sigma < \varrho \leqq 2p-1} (w_\varrho - w_\sigma) \qquad (w_\lambda = e^{i\lambda v}). \tag{3.3}$$

For the product in (3.3) we have

$$\prod_{\sigma<\varrho} (w_\varrho - w_\sigma) = \prod_{\sigma<\varrho} e^{\frac{1}{2}(\varrho+\sigma)iv} \prod_{\sigma<\varrho} \left[e^{\frac{1}{2}(\varrho-\sigma)iv} - e^{-\frac{1}{2}(\varrho-\sigma)iv}\right]$$
$$= (2i)^{p(2p-1)}\, e^{\frac{1}{2}p(2p-1)^2 iv} \prod_{\sigma<\varrho} \sin \tfrac{1}{2}(\varrho - \sigma)\, v.$$

Also,

$$\sigma_p(w_0, w_1, \dots, w_{2p-1}) = \sum e^{(\lambda_1+\lambda_2+\cdots+\lambda_p)iv},$$

where the sum extends over all combinations $(\lambda_1, \lambda_2, \dots, \lambda_p)$ of p nonnegative integers not greater than $2p-1$. Thus,

$$\det B(v) = (-1)^p\, (p!)^2\, 2^{2p(p-1)}\, v^{2p} \times \times \left[e^{-\frac{1}{2}p(2p-1)iv} \sum e^{(\lambda_1+\cdots+\lambda_p)iv}\right] \prod_{0 \leqq \sigma < \varrho \leqq 2p-1} \sin \tfrac{1}{2}(\varrho - \sigma)\, v. \tag{3.4}$$

It is seen from this that $B(v)$ for $v \neq 0$ is singular if and only if either the expression in brackets or the product following this expression vanishes.

As regards the first expression we can write it in the form

$$e^{-\frac{1}{2}p(2p-1)iv} \sum_{n=p(p-1)/2}^{p(3p-1)/2} \nu_p(n)\, e^{inv} = \sum_{n=p(p-1)/2}^{p(3p-1)/2} \nu_p(n)\, e^{[n-\frac{1}{2}p(2p-1)]iv},$$

with $\nu_p(n)$ as defined in Theorem 3. Consider first the case p even. Then, by a shift of the summation index, the last sum is seen to be

$$\sum_{n=-p^2/2}^{p^2/2} \nu_p(p^2 - \tfrac{1}{2}p + n)\, e^{niv}.$$

Since the determinant (3.4) is real, this sum must be real too, which is only possible if

$$\nu_p(p^2 - \tfrac{1}{2}p + n) = \nu_p(p^2 - \tfrac{1}{2}p - n) \quad (p \text{ even}).$$

Our sum then becomes

$$\nu_p(p^2 - \tfrac{1}{2}p) + 2\sum_{n=1}^{p^2/2} \nu_p(p^2 - \tfrac{1}{2}p - n)\cos nv \quad (p \text{ even}). \tag{3.5}$$

Analogously, if p is odd, the sum in question is

$$\left[\sum_{n=p(p-1)/2}^{(p(2p-1)-1)/2} + \sum_{n=(p(2p-1)+1)/2}^{p(3p-1)/2}\right] \nu_p(n)\, e^{[n-\frac{1}{2}p(2p-1)]iv}$$
$$= \sum_{n=1}^{(p^2+1)/2} \left[\nu_p(p^2 - \tfrac{1}{2}p + \tfrac{1}{2} - n)\, e^{-(2n-1)iv/2} + \nu_p(p^2 - \tfrac{1}{2}p - \tfrac{1}{2} + n)\, e^{(2n-1)iv/2}\right].$$

Since this again must be real we also have

$$\nu_p(p^2 - \tfrac{1}{2}p + \tfrac{1}{2} - n) = \nu_p(p^2 - \tfrac{1}{2}p - \tfrac{1}{2} + n) \quad (p \text{ odd}),$$

and our sum becomes

$$2\sum_{n=1}^{(p^2+1)/2} \nu_p\left(p^2 - \frac{1}{2}p + \frac{1}{2} - n\right)\cos(2n-1)\frac{v}{2} \quad (p \text{ odd}). \tag{3.6}$$

Substituting (3.5) and (3.6) for the bracketed expression in (3.4) we finally obtain

$$\det B(v) = (-1)^p (p!)^2\, 2^{2p^2-2p+1} v^{2p} C_p(v) \prod_{0 \leq \sigma < \varrho \leq 2p-1} \sin \tfrac{1}{2}(\varrho - \sigma) v, \tag{3.7}$$

with $C_p(v)$ as defined in (3.2).

Now, $C_p(v) \neq 0$ for $0 < |v| < v_p$ if v_p is the smallest positive zero of C_p. Also, the sine-product in (3.7) is certainly nonvanishing for $0 < |v| < 2\pi/(2p-1)$. Therefore, $\det B(v)$ is nonvanishing for

$$0 < |v| < \min\left(v_p, \frac{2\pi}{2p-1}\right),$$

which proves our theorem.

For reference we list the cosine-polynomials $C_p(v)$ for $p = 1, 2, 3$:

$$C_1(v) = \cos\frac{v}{2},$$

$$C_2(v) = 1 + \cos v + \cos 2v,$$

$$C_3(v) = 3\cos\frac{v}{2} + 3\cos 3\frac{v}{2} + 2\cos 5\frac{v}{2} + \cos 7\frac{v}{2} + \cos 9\frac{v}{2}.$$

One finds easily that

$$v_1 = \pi, \qquad v_2 = v_3 = \frac{\pi}{2}$$

so that the bounds in (3.1) for $p = 1, 2, 3$ are respectively π, $\pi/2$, $2\pi/5$

We also note from (3.2) that

$$0 < |v| < \frac{\pi}{p^2} \tag{3.8}$$

is a sufficient condition for nonvanishing of det $B(v)$.

4. Trigonometric extrapolation and interpolation methods of Adams' type

Multi-step methods with

$$\alpha_0 = -\alpha_1 = 1, \qquad \alpha_\lambda = 0 \qquad (\lambda > 1)$$

and maximal algebraic order for fixed index are called Adams methods. In this section we list methods of trigonometric order that correspond to Adams' extrapolation and interpolation methods in the sense of Theorem 2. The coefficients $\beta_\lambda(v)$ and $\beta_\lambda^*(v)$ are obtained as the power series solution of the appropriate system of equations (2.8) where coefficient matrix and known vector are expanded into their Taylor series.

Adams extrapolation methods of trigonometric order p

$$x_{n+1} = x_n + h \sum_{\lambda=1}^{2p} \beta_{p\lambda}(v)\, x'_{n+1-\lambda} \qquad (v = 2\pi h/T)$$

$$\beta_{11} = \frac{3}{2}\left(1 - \frac{1}{4} v^2 + \frac{1}{120} v^4 + \cdots\right), \qquad \beta_{12} = -\frac{1}{2}\left(1 + \frac{1}{12} v^2 + \frac{1}{120} v^4 + \cdots\right);$$

$$\beta_{21} = \frac{55}{24}\left(1 - \frac{95}{132} v^2 + \frac{79}{792} v^4 + \cdots\right), \qquad \beta_{22} = -\frac{59}{24}\left(1 - \frac{923}{708} v^2 + \frac{15647}{21240} v^4 + \cdots\right),$$

$$\beta_{23} = \frac{37}{24}\left(1 - \frac{421}{444} v^2 + \frac{1921}{13320} v^4 + \cdots\right), \quad \beta_{24} = -\frac{9}{24}\left(1 + \frac{1}{4} v^2 + \frac{11}{120} v^4 + \cdots\right);$$

$$\beta_{31} = \frac{4277}{1440}\left(1 - \frac{5257}{3666} v^2 + \frac{196147}{439920} v^4 + \cdots\right),$$

$$\beta_{32} = -\frac{7923}{1440}\left(1 - \frac{48607}{15846} v^2 + \frac{2341619}{633840} v^4 + \cdots\right),$$

$$\beta_{33} = \frac{9982}{1440}\left(1 - \frac{107647}{29946} v^2 + \frac{2791381}{513360} v^4 + \cdots\right),$$

$$\beta_{34} = -\frac{7298}{1440}\left(1 - \frac{69473}{21894} v^2 + \frac{10276973}{2627280} v^4 + \cdots\right),$$

$$\beta_{35} = \frac{2877}{1440}\left(1 - \frac{10433}{5754} v^2 + \frac{20683}{32880} v^4 + \cdots\right),$$

$$\beta_{36} = -\frac{475}{1440}\left(1 + \frac{55}{114} v^2 + \frac{1015}{2736} v^4 + \cdots\right);$$

. . . .

Adams interpolation methods of trigonometric order p

$$x_{n+1} = x_n + h \sum_{\lambda=0}^{2p-1} \beta^*_{p\lambda}(v)\, x'_{n+1-\lambda} \qquad (v = 2\pi h/T)$$

$$\beta^*_{10} = \beta^*_{11} = \frac{1}{2}\left(1 + \frac{1}{12} v^2 + \frac{1}{120} v^4 + \cdots\right);$$

$$\beta^*_{20} = \frac{9}{24}\left(1 + \frac{1}{4} v^2 + \frac{11}{120} v^4 + \cdots\right), \qquad \beta^*_{21} = \frac{19}{24}\left(1 - \frac{43}{228} v^2 + \frac{13}{360} v^4 + \cdots\right),$$

$$\beta^*_{22} = -\frac{5}{24}\left(1 - \frac{1}{12} v^2 - \frac{7}{72} v^4 + \cdots\right), \qquad \beta^*_{23} = \frac{1}{24}\left(1 + \frac{11}{12} v^2 + \frac{193}{360} v^4 + \cdots\right);$$

$$\beta^*_{30} = \frac{475}{1440}\left(1 + \frac{55}{114} v^2 + \frac{500267}{22800} v^4 + \cdots\right),$$

$$\beta^*_{31} = \frac{1427}{1440}\left(1 - \frac{5149}{8562} v^2 - \frac{15139837}{342480} v^4 + \cdots\right),$$

$$\beta^*_{32} = -\frac{798}{1440}\left(1 - \frac{163}{114} v^2 - \frac{1964441}{10640} v^4 + \cdots\right),$$

$$\beta^*_{33} = \frac{482}{1440}\left(1 - \frac{1697}{1446} v^2 - \frac{20178851}{57840} v^4 + \cdots\right),$$

$$\beta^*_{34} = -\frac{173}{1440}\left(1 + \frac{29}{1038} v^2 - \frac{22688263}{41520} v^4 + \cdots\right),$$

$$\beta^*_{35} = \frac{27}{1440}\left(1 + \frac{13}{6} v^2 - \frac{187111}{240} v^4 + \cdots\right);$$

. . . .

As shown in Section 3 the series for $\beta_{p\lambda}$ and $\beta^*_{p\lambda}$ certainly converge for $|v| < r_p$ where $r_1 = \pi$, $r_2 = \pi/2$, $r_3 = 2\pi/5$.

We also note the explicit formulae

$$\beta_{11} = \frac{\sin\frac{3}{2}v}{v\cos\frac{1}{2}v}, \qquad -\beta_{12} = \beta^*_{10} = \frac{\tan\frac{1}{2}v}{v}.$$

5. Trigonometric extrapolation and interpolation methods of Störmer's type

Linear multi-step methods are also used in connection with differential equations of higher order, in particular with second order differential equations in which the first derivative is absent,

$$(5.1) \qquad x'' = f(t, x), \qquad x(t_0) = x_0, \qquad x'(t_0) = x'_0.$$

They take here the form

$$(5.2) \qquad x_{n+1} + \alpha_1 x_n + \cdots + \alpha_k x_{n+1-k} = h^2(\beta_0 x''_{n+1} + \beta_1 x''_n + \cdots + \beta_k x''_{n+1-k}),$$
$$x''_m = f(t_0 + m h, x_m).$$

The terminology introduced in Section 2 extends in an obvious manner to this new situation. With the multi-step method (5.2) there is now associated the functional

$$L x = \sum_{\lambda=0}^{k} [\alpha_\lambda x(t_0 + (n+1-\lambda) h) - h^2 \beta_\lambda x''(t_0 + (n+1-\lambda) h)] \qquad (\alpha_0 = 1).$$

Theorem 1 (with the modification mentioned in Remark 1 on p. 384) can then be applied to this functional provided that not all the values of α_λ are fixed in advance. Otherwise our assumption (1.5′) would not hold. Except for this provision, however, the construction of multi-step methods (5.2) of trigonometric order follows the same pattern as outlined in Sections 2 and 4 for first order differential equations.

We content ourselves in this section with listing a few methods that result if one takes

$$\alpha_\lambda = 0 \quad \text{for} \quad \lambda > 2. \tag{5.3}$$

In the algebraic case such methods of maximal order (for given index k) are called Störmer methods (cf., e.g., [3, p. 125]).

Störmer extrapolation methods of trigonometric order p

$$x_{n+1} + \alpha_{p1}(v)\, x_n + \alpha_{p2}(v)\, x_{n-1} = h^2 \sum_{\lambda=1}^{2p-1} \beta_{p\lambda}(v)\, x''_{n+1-\lambda} \qquad (v = 2\pi h/T)$$

$$\alpha_{11} = -2, \qquad \alpha_{12} = 1, \qquad \beta_{11} = 1 - \frac{1}{12} v^2 + \frac{1}{360} v^4 + \cdots;$$

$$\alpha_{21} = -2\left(1 - \frac{1}{6} v^4 + \frac{1}{36} v^6 + \cdots\right), \qquad \alpha_{22} = -\alpha_{21} - 1,$$

$$\beta_{21} = \frac{13}{12}\left(1 - \frac{19}{52} v^2 + \frac{7}{120} v^4 + \cdots\right), \qquad \beta_{22} = -\frac{2}{12}\left(1 - \frac{9}{4} v^2 + \frac{37}{120} v^4 + \cdots\right),$$

$$\beta_{23} = \frac{1}{12}\left(1 + \frac{1}{4} v^2 + \frac{7}{120} v^4 + \cdots\right);$$

$$\alpha_{31} = -2\left(1 - \frac{27}{20} v^6 + \cdots\right), \qquad \alpha_{32} = -\alpha_{31} - 1,$$

$$\beta_{31} = \frac{299}{240}\left(1 - \frac{4315}{5382} v^2 + \frac{7357}{49680} v^4 + \cdots\right),$$

$$\beta_{32} = -\frac{176}{240}\left(1 - \frac{3181}{792} v^2 + \frac{264593}{47520} v^4 + \cdots\right),$$

$$\beta_{33} = \frac{194}{240}\left(1 - \frac{2047}{582} v^2 + \frac{38129}{7760} v^4 + \cdots\right),$$

$$\beta_{34} = -\frac{96}{240}\left(1 - \frac{913}{432} v^2 + \frac{6923}{25920} v^4 + \cdots\right),$$

$$\beta_{35} = \frac{19}{240}\left(1 + \frac{221}{342} v^2 + \frac{17521}{41040} v^4 + \cdots\right);$$

. . . .

Störmer interpolation methods of trigonometric order p

$$x_{n+1} + \alpha^*_{p1}(v)\, x_n + \alpha^*_{p2}(v)\, x_{n-1} = h^2 \sum_{\lambda=0}^{2p-2} \beta^*_{p\lambda}\, x''_{n+1-\lambda} \qquad (v = 2\pi h/T)$$

$$\alpha^*_{11} = -2\left(1 + \frac{1}{2} v^2 + \frac{11}{24} v^4 + \frac{301}{720} v^6 + \cdots\right),$$

$$\alpha^*_{12} = -\alpha^*_{11} - 1, \quad \beta^*_{10} = 1 + \frac{11}{12} v^2 + \frac{301}{360} v^4 + \cdots;$$

$$\alpha^*_{21} = -2, \quad \alpha^*_{22} = 1, \quad \beta^*_{20} = \frac{1}{12}\left(1 + \frac{1}{4}v^2 + \frac{7}{120}v^4 + \cdots\right),$$

$$\beta^*_{21} = \frac{10}{12}\left(1 - \frac{1}{20}v^2 + \frac{1}{120}v^4 + \cdots\right), \quad \beta^*_{22} = \frac{1}{12}\left(1 + \frac{1}{4}v^2 + \frac{7}{120}v^4 + \cdots\right);$$

$$\alpha^*_{31} = -2\left(1 + \frac{3}{40}v^6 + \cdots\right), \qquad \alpha^*_{32} = -\alpha^*_{31} - 1,$$

$$\beta^*_{30} = \frac{19}{240}\left(1 + \frac{221}{342}v^2 + \frac{17\,521}{41\,040}v^4 + \cdots\right),$$

$$\beta^*_{31} = \frac{204}{240}\left(1 - \frac{79}{459}v^2 + \frac{11\,039}{110\,160}v^4 + \cdots\right),$$

$$\beta^*_{32} = \frac{14}{240}\left(1 + \frac{95}{42}v^2 - \frac{103}{80}v^4 + \cdots\right), \quad \beta^*_{33} = \frac{4}{240}\left(1 - \frac{16}{9}v^2 - \frac{4711}{2160}v^4 + \cdots\right),$$

$$\beta^*_{34} = -\frac{1}{240}\left(1 + \frac{31}{18}v^2 + \frac{3899}{2160}v^4 + \cdots\right);$$

. . . .

The series for $\alpha_{p\lambda}, \beta_{p\lambda}$ converge if $|v| < r_p$ where $r_1 = \infty$, $r_2 = \pi/2$, those for $\alpha^*_{p\lambda}, \beta^*_{p\lambda}$ converge if $|v| < r^*_p$ where $r^*_1 = \pi/3$, $r^*_2 = \pi/2$. This can be shown by reasonings similar to, but more complicated than, those in Section 3. The values of r_3, r^*_3 were not obtained because of the complexity of the calculations required.

We also note the explicit formulae

$$\beta_{11} = \left(\frac{2\sin\frac{1}{2}v}{v}\right)^2, \quad \alpha^*_{11} = -\frac{2\cos v}{2\cos v - 1}, \quad \beta^*_{10} = \frac{2(1-\cos v)}{v^2(2\cos v - 1)}.$$

6. Effect of uncertainty in the choice of T

Multi-step methods of trigonometric order presuppose the knowledge of the period T of the solution, if it is periodic, or of a suitable substitute, if the solution is only oscillatory. Precise knowledge of this kind is usually not available in advance, so that one has to rely on suitable estimates of T. Since T enters only through the parameter $v = 2\pi h/T$ and $T = \infty$ gives the classical multi-step methods, one expects that uncertainties in the value of T should not seriously impair the effectiveness of trigonometric multi-step methods (when applicable) as long as T is not significantly underestimated.

It is instructive to study from this point of view the simple initial value problem

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6.1}$$

which has the solution

$$x(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Every multi-step method of trigonometric order $\geqq 1$ relative to period 2π is exact in this case, so that the example allows us to isolate the effect of inaccurately estimating the period.

Let us select Adams' interpolation method of trigonometric order 1, which can be written in the form

$$x_{n+1} = x_n + h\,\frac{\tan\frac{1}{2}v}{v}\,(x'_{n+1} + x'_n) \qquad (v = 2\pi h/T). \tag{6.2}$$

The correct choice of T is 2π, giving $v=h$. We consider now T to be some "estimate" of 2π and use

$$\lambda = \frac{2\pi}{T}$$

to measure the quality of the estimate (underestimation, if $\lambda > 1$, overestimation, if $\lambda < 1$, precise estimate, if $\lambda = 1$).

Letting

$$\tau = \frac{1}{\lambda}\tan\frac{\lambda h}{2},$$

application of (6.2) to (6.1) then gives

$$x_{n+1} = x_n + \tau\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(x_{n+1} + x_n),$$

or else, collecting terms,

$$\begin{pmatrix}1 & \tau\\ -\tau & 1\end{pmatrix} x_{n+1} = \begin{pmatrix}1 & -\tau\\ \tau & 1\end{pmatrix} x_n, \qquad x_{n+1} = \frac{1}{1+\tau^2}\begin{pmatrix}1-\tau^2 & -2\tau\\ 2\tau & 1-\tau^2\end{pmatrix} x_n.$$

If we set

$$\tau = \tan\tfrac{1}{2}\vartheta,$$

we get

$$x_{n+1} = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix} x_n.$$

Obviously,

$$\vartheta = 2\arctan\left(\frac{1}{\lambda}\tan\frac{\lambda h}{2}\right). \tag{6.3}$$

The n-th approximation x_n to the solution of (6.1) is thus obtained by rotating the initial vector $x_0 = \begin{pmatrix}1\\0\end{pmatrix}$ n-times through the angle ϑ, where ϑ is given by (6.3). Therefore

$$x_n = \begin{pmatrix}\cos n\vartheta\\ \sin n\vartheta\end{pmatrix},$$

which shows that the approximations have the correct amplitude, but phase errors

$$\varepsilon_n = n(\vartheta - h) = n h\left\{\frac{2}{h}\arctan\left(\frac{1}{\lambda}\tan\frac{\lambda h}{2}\right) - 1\right\}. \tag{6.4}$$

If $\lambda = 1$ then $\varepsilon_n = 0$, as we expect. In the limit as $\lambda \to 0$ we obtain the phase error of the method of algebraic order 1, which in our example is the trapezoidal rule. The expression in curled brackets, as function of λ, has a behavior as shown in Figure 1. It is seen from this, in particular, that the error in absolute value is less than the error at $\lambda=0$ for all λ with $0<\lambda<\lambda_0$ where $\lambda_0>1$. This means that in using the modified trapezoidal rule (6.2) we may overestimate the period as much as we wish, and even underestimate it somewhat, and still get better results than with the ordinary trapezoidal rule. On the other hand, the curve in Figure 1 also shows that the error reduction is not very substantial unless λ is close to 1. If $h=.1$, for example, there is a gain of at least one decimal digit only if the estimated period differs from the true period by 5% or less.

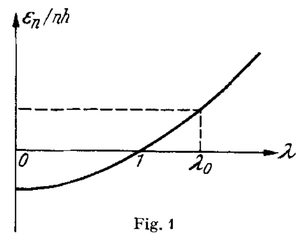

Fig. 1

7. Numerical examples

An important class of differential equations to which trigonometric multi-step methods may advantageously be applied is given by equations of the form

$$x''+P(t)\,x=0, \tag{7.1}$$

where $P(t)$ is a nearly constant nonnegative function,

$$P(t)=P_0[1+p(t)]\geqq 0 \qquad (t\geqq t_0). \tag{7.2}$$

Here, P_0 is a positive constant and $p(t)$ a function which is "small" in some sense for $t\geqq t_0$.

Equation (7.1) may be considered a perturbation of $x''+P_0x=0$, the differential equation of a harmonic oscillator with angular frequency $\sqrt{P_0}$. This suggests the following values of T (and thus of v) as natural choices in methods of trigonometric order,

$$T=2\pi/\sqrt{P_0}, \qquad v=h\sqrt{P_0}. \tag{7.3}$$

If one is willing to select these values anew at each step of integration, one can improve upon (7.3) by using

$$T=T_n=2\pi/\sqrt{P(t_n)}, \qquad v=v_n=h\sqrt{P(t_n)} \tag{7.4}$$

in the computation of x_{n+1}.

Particularly favorable results are expected if t_0 is relatively large and $p(t)$ such that

$$\int^{\infty}|p(t)|\,dt<\infty, \tag{7.5}$$

in which case it is known that $x=c_1\cos\sqrt{P_0}t+c_2\sin\sqrt{P_0}t+o(1)$ (c_1, c_2 constants, $t\to\infty$) for every solution of (7.1). Our first example belongs to this type.

Example 1. $x''+\left(100+\frac{1}{4t^2}\right)x=0, \qquad 0<t_0\leqq t\leqq 10.$

The general solution can be expressed in terms of Bessel functions, $x=c_1\sqrt{t}J_0(10t)+c_2\sqrt{t}Y_0(10t)$. We single out the particular solution $\sqrt{t}\,J_0(10t)$ by choosing the initial values accordingly. Table 1 below shows selected results (every 50th value, using $t_0=1$, $h=.02$) obtained by the Störmer extrapolation methods of algebraic order 2 and 4, and of trigonometric order 1 and 2, in this order[3]. In the latter two methods the constant value (7.3) of T was used, that is, $T=\pi/5$, $v=.2$.

Table 1 reveals an average increase in accuracy of about three decimal digits in favor of the trigonometric extrapolation methods. This — it should be noted — is at practically no extra cost in computation, since the modified coefficients of the trigonometric methods, if (7.3) is used, need only be computed once, at

Table 1. *Störmer extrapolation method of various algebraic and trigonometric orders. Example 1 with* $t_0=1$

t	alg. ord. $p=2$	alg. ord. $p=4$	trig. ord. $p=1$	trig. ord. $p=2$	exact 7 D values
1	−.245 935 8	−.245 935 8	−.245 935 8	−.245 935 8	−.245 935 8
2	.234 590 1	.235 433 7	.236 205 5	.236 211 5	.236 208 5
3	−.142 536 8	−.148 524 7	−.149 587 1	−.149 596 6	−.149 593 7
4	.001 887 5	.014 388 0	.014 725 7	.014 734 9	.014 733 8
5	.139 324 7	.123 416 7	.124 806 8	.124 801 5	.124 800 2
6	−.233 007 6	−.220 565 0	−.224 061 9	−.224 063 0	−.224 059 2
7	.247 293 5	.246 130 4	.251 102 4	.251 110 1	.251 104 9
8	−.177 353 9	−.192 402 2	−.197 253 6	−.197 265 9	−.197 260 6
9	.047 026 8	.077 194 0	.079 880 6	.079 893 8	.079 890 0
10	.099 305 5	.062 054 8	.063 209 7	.063 199 7	.063 200 7

the beginning of the computations. If the choice (7.4) is made an additional $\frac{3}{4}$ decimal digit is gained on the average, the amount of computing being somewhat larger than before.

Störmer interpolation methods of algebraic order 2 and of trigonometric order 1, applied to Example 1, gave results which are 10—20 times worse than the corresponding results in Table 1, the trigonometric method being, on the average, more accurate by $2\frac{1}{2}$ decimal digits. The interpolation method of algebraic order 4, however, is almost 100 times better than the corresponding extrapolation method. Nevertheless there is also here an improvement of about $1\frac{1}{2}$ decimal digits in favor of the trigonometric modification.

Larger values of t_0 would put trigonometric methods into an even more favorable light. As t_0 decreases from 1 to 0, trigonometric methods gradually lose their superiority.

In our next example — a Mathieu differential equation — the relation (7.5) is not satisfied any more.

Example 2. $x''+100(1-\alpha\cos 2t)\,x=0,\ t_0=0,\ x_0=1,\ x_0'=0\ (0<\alpha\leqq 1)$. We integrated this equation for various values of α using the same methods and the same step length $h=.02$ as in Example 1. An independent calculation was done with the help of Nyström's method, which was also used to obtain starting values. Selected results (every 25th value) of the Störmer extrapolation methods, in the case $\alpha=.1$, are displayed in Table 2[3]. Trigonometric order, also in this example, is to be understood relative to period $T=\pi/5$.

[3] Calculations were done on ORACLE in 32 binary bit floating point arithmetic (the equivalent of about 9 significant decimal digits). The final results were rounded to 7 decimal places. — The author takes the opportunity to acknowledge the able assistance of Miss RUTH BENSON in performing these calculations.

The results in Table 2 follow a similar pattern as those above in Table 1, the main difference being a reduction, to roughly half the size, of the improvement of trigonometric methods over the algebraic ones. The average gain in accuracy is now about $1\frac{1}{2}$ decimal digits. The remarks made above on interpolation methods hold true also in Example 2, except for the reduction just mentioned. Obviously, as α decreases to 0, trigonometric methods become increasingly superior to algebraic methods. We have experienced only a slight decrease in this superiority when we let α increase from .1 to 1.

Table 2. *Störmer extrapolation method of various algebraic and trigonometric orders. Example 2 with* $\alpha = .1$

t	alg. ord. $p=2$	alg. ord. $p=4$	trig. ord. $p=1$	trig. ord. $p=2$	exact $7D$ values
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.5	.0767165	.0690295	.0685134	.0691273	.0692085
1.0	−.9035098	−.9056448	−.9089870	−.9080120	−.9084179
1.5	−.7105151	−.6908656	−.6942472	−.6938453	−.6939608
2.0	.1985482	.2287643	.2304036	.2311394	.2309590
2.5	.9715966	.9679083	.9764633	.9767822	.9763699
3.0	.2552862	.2045198	.2060842	.2056667	.2057667
3.5	−.9456869	−.9505080	−.9618456	−.9613337	−.9616794
4.0	−.4833155	−.4221211	−.4260400	−.4262622	−.4265317
4.5	.5453242	.5922666	.6026736	.6021053	.6022367
5.0	.9517667	.9263164	.9422702	.9418659	.9417373

It is anticipated that trigonometric methods can be applied, with similar success, also to nonlinear differential equations describing oscillation phenomena.

References

[1] Antosiewicz, H. A., and W. Gautschi: Numerical methods in ordinary differential equations, Chap. 9 of "Survey of numerical analysis" (ed. J. Todd). New York-Toronto-London: McGraw-Hill Book Co. (in press).

[2] Brock, P., and F. J. Murray: The use of exponential sums in step by step integration. Math. Tables Aids Comput. **6**, 63—78 (1952).

[3] Collatz, L.: The numerical treatment of differential equations, 3rd ed. Berlin-Göttingen-Heidelberg: Springer 1960.

[4] Dennis, S. C. R.: The numerical integration of ordinary differential equations possessing exponential type solutions. Proc. Cambridge Philos. Soc. **56**, 240—246 (1960).

[5] Urabe, M., and S. Mise: A method of numerical integration of analytic differential equations. J. Sci. Hiroshima Univ., Ser. A **19**, 307—320 (1955).

Oak Ridge National Laboratory
Mathematics Panel
P. O. Box X
Oak Ridge, Tennessee

(Received September 18, 1961)