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## NUMERICAL INTEGRATION WITH HIGHLY OSCILLATING WEIGHT FUNCTIONS

JOZEF MIKLOŠKO

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### I. INTRODUCTION

For the numerical computation of integrals with an oscillating weight function (further only WF)  $w(kx) = \begin{cases} \cos kx \\ \sin kx \end{cases}$  ( $k$  - integer), i.e. for

$$(1) \quad J = \int_0^T f(x) w(kx) dx, \quad T = \frac{2\pi}{t}, \quad t = 1, 2, \dots,$$

consider the quadrature formula

$$(2) \quad \int_0^1 f(x) [r + w(2\pi kx)] dx = \sum_{i=1}^n A_i f(x_i^{(n)}) + R_n(f).$$

In [2], formula (2) of Newton-Cotes type is described for  $r = 0$ . For some  $x_i^{(n)}$ ,  $k$ ,  $n$  the coefficients  $A_i$  are tabulated, the convergence of (2) and the method of computation of (1) being investigated on the basis of their specific properties.

The aim of this work is:

a) to suggest for  $r = 1$  the Gauss type of quadrature (2) by using information about  $w(kx)$ ,

b) to investigate the properties of its compound rule (11) and to demonstrate the results by numerical experiments.

For the calculation of  $x_i^{(n)}$  and  $A_i$ , we must have a system of polynomials  $\{\omega_n(x)\}$ ,  $n = 1, 2, \dots$ , orthogonal with nonnegative WF  $1 + w(2\pi kx)$  on  $[0, 1]$ . Since such polynomials are not known, it was necessary to compute them.

### II. COMPUTATION OF ORTHOGONAL POLYNOMIALS

The numerical computation of the system of orthogonal polynomials (further only OP) with WF  $W(x)$  will be described for an arbitrary interval  $[a, b]$ .

Let  $W(x)$  be nonnegative, on  $[a, b]$  measurable and not identically equal to zero whereby the moments

$$(3) \quad W_n = \int_a^b x^n W(x) dx$$

exist for each nonnegative integer  $m$ .

Define the inner product  $(f, g) = \int_a^b f(x) g(x) W(x) dx$ .

An algorithm for the computation of  $\{\omega_n(x)\}$  – a system of polynomials with main coefficients equal 1, orthogonal with  $W(x)$  on  $[a, b]$  – is given in the recurrent form in [1]:

$$(4) \quad \omega_n(x) = (x + B_n) \omega_{n-1}(x) + C_n \omega_{n-2}(x), \quad n = 1, 2, \dots$$

where

$$B_n = - \left( a_1^{(n-1)} + \frac{(x^n, \omega_{n-1})}{(x^{n-1}, \omega_{n-1})} \right), \quad C_n = - \frac{(x^{n-1}, \omega_{n-1})}{(x^{n-2}, \omega_{n-2})},$$

$a_1^{(n-1)}$  is the coefficient of  $x^{n-2}$  in  $\omega_{n-1}(x)$ .

If we put  $\omega_{-1}(x) = 0$ ,  $\omega_0(x) = 1$  then from (4) we may compute the other polynomials  $\omega_n(x)$  for which  $(\omega_i, \omega_j) = 0$ ,  $i \neq j$ .

Let the OP  $\omega_n(x)$  computed from (4) be

$$(5) \quad \omega_n(x) = x^n + a_1^{(n)} x^{n-1} + a_2^{(n)} x^{n-2} + \dots + a_n^{(n)}.$$

We now show that if  $W(x)$  is an even function on  $[a, b]$  then some simplifications occur in (4) and (5).

**Theorem 1.** *If  $W(x)$  is an even function on  $[a, b]$  ( $W(x) = W(a + b - x)$ ) then for  $n = 1, 2, \dots$*

- a)  $x_i^{(n)} = a + b - x_{n-i+1}^{(n)}$ ,  $i = 1, 2, \dots, n$
- where  $x_i^{(n)}$  are the knots of OP  $\omega_n(x)$ ,
- b)  $a_1^{(n)} = -\frac{1}{2}n(a + b)$  in (5),
- c)  $B_n = -\frac{1}{2}(a + b)$  in (4).

Proof. The assertion a) is evident since in this case for the OP  $\omega_n(x)$ ,  $n = 1, 2, \dots$ , it holds [3]

$$(6) \quad \omega_n(x) = (-1)^n \omega_n(a + b - x).$$

The other conclusions can also be easily proved. The knots  $x_i^{(n)}$  of the polynomial  $\omega_n(x)$  satisfy  $\sum_{i=1}^n x_i^{(n)} = -a_1^{(n)}$ . Since  $x_i^{(n)} + x_{n-i+1}^{(n)} = a + b$ ,  $n = 1, 2, \dots$  we have  $\sum_{i=1}^n x_i^{(n)} = \frac{1}{2}n(a + b)$  and thus  $a_1^{(n)} = -\frac{1}{2}n(a + b)$  for an arbitrary  $n$ . By putting into

(4) the OP  $\omega_n(x)$ ,  $\omega_{n-1}(x)$  from (5) and by comparing the coefficients of  $x^{n-1}$  we get for each  $n$

$$B_n = a_1^{(n)} - a_1^{(n-1)} = -\frac{a+b}{2}.$$

Remark: For  $n$  odd,  $x_{(n+1)/2}^{(n)} = \frac{1}{2}(a+b)$  is always a knot of  $\omega_n(x)$ , i.e. it is possible to prove the conclusion c) by putting  $x = \frac{1}{2}(a+b)$  (knot of  $\omega_n(x)$ ,  $\omega_{n-2}(x)$ ) into (4).

From the system  $\{\omega_n(x)\}$  we get an orthonormal system  $\{\bar{\omega}_n(x)\}$  in this manner:  $\bar{\omega}_n(x) = N_n \omega_n(x)$ ,  $n = 1, 2, \dots$ , where  $N_n = (x^n, \omega_n)^{-1/2}$ . The roots of  $\omega_n(x)$  are knots of a Gauss type quadrature. Its coefficients  $A_i$  can be computed from the well known relation modified to

$$(7) \quad A_i = \frac{(x^{n-1}, \omega_{n-1})}{\omega_n'(x_i^{(n)}) \omega_{n-1}(x_i^{(n)})}.$$

For the coefficients (7), the following assertion holds [2]:

**Theorem 2.** *If WF  $W(x)$  is even on  $[a, b]$  then*

$$A_i = A_{n-i+1}, \quad i = 1, 2, \dots, n.$$

The main problem of computing the parameters of the Gauss type method is the precise computation of  $a_i^{(n)}$ ,  $i = 1, 2, \dots, n$  in  $\omega_n(x)$ . If  $\omega_n(x)$ ,  $n = 1, 2, \dots$  are not known in an explicit form then the accuracy of their coefficients depends on

1. the suitable choice of  $[a, b]$  in (4),
2. the accuracy of the computation of the moments (3).

It follows from Theorem 1 that if  $W(x)$  is an even function on  $[a, b]$  then the most suitable interval of orthogonalization is obtained if  $-a = b$  because then  $B_n = 0$  in (4), as  $a_i^{(n)} = 0$ ,  $i = 1, 3, 5, \dots$ , for each  $n$  and  $W_m = 0$  for  $m$  odd.

We investigate now on a concrete example the actual influence of 1. upon the accuracy of  $\omega_n(x)$ ,  $n = 1, 2, \dots$ . We compute on  $[-1, 1]$  and  $[0, 1]$  the system of OP with  $W(x) = 1$ , their roots and the coefficients (7) (in this case we have accurate

Table 1.

		$a_i^{(n)}$		$x_i^{(n)}$		$A_i$	
		$[0, 1]$	$[-1, 1]$	$[0, 1]$	$[-1, 1]$	$[0, 1]$	$[-1, 1]$
$n$	$[a, b]$						
	12	$3.5 \cdot 10^{-3}$	$2.4 \cdot 10^{-16}$	$1.5 \cdot 10^{-4}$	$1.6 \cdot 10^{-16}$	$1.1 \cdot 10^{-4}$	$8.8 \cdot 10^{-17}$
	13	1.1	$5.6 \cdot 10^{-15}$	$5.3 \cdot 10^{-1}$	$1.5 \cdot 10^{-16}$	$1.5 \cdot 10^{-1}$	$2.2 \cdot 10^{-16}$
	20	—	—	—	$7.1 \cdot 10^{-11}$	—	$1.4 \cdot 10^{-10}$

moments), i.e. the Legendre polynomials and the parameters of the Gauss method of numerical integration. Maximal absolute errors  $a_i^{(n)}$ ,  $x_i^{(n)}$  and  $A_i$  for  $n = 12, 13, 20$  are in Table 1.

It is interesting, that at  $n = 12$  for  $[0, 1]$  the symmetry of  $x_i^{(n)}$  and  $A_i$  ( $x_i^{(n)} = 1 - x_{n-i+1}^{(n)}$ ,  $A_i = A_{n-i+1}$ ) when calculated independently is preserved up to 13 digits, whilst the error is already at the 4-th digit. An accuracy checking of these parameters is therefore not possible in this way.

### III. COMPUTATION OF $x_i^{(n)}$ AND $A_i$ FOR (2)

The moments (3) will be in our case

$$(8) \quad W_m = \int_{-1}^1 x^m \left[ 1 + \frac{\cos \pi k(x+1)}{\sin \pi k(x+1)} \right] dx = \frac{1 - (-1)^{m+1}}{m+1} + \left\langle \frac{W_m(c)}{W_m(s)} \right\rangle.$$

The computation of  $W_m(c)$  and  $W_m(s)$  was carried out with the recurrent algorithms

$$(9) \quad W_0(c) = W_1(c) = 0, \quad W_m(c) = \frac{m}{(\pi k)^2} [2 - (m-1)W_{m-2}(c)], \quad m = 2, 3, \dots$$

$$W_m(s) = \frac{-\pi k}{m+1} W_{m+1}(c), \quad m = 0, 1, 2, \dots$$

and thus  $W_m = 0$  for  $m$  odd,  $W(x) = 1 + \cos \pi k(x+1)$  ( $m$  even,  $W(x) = 1 + \sin \pi k(x+1)$ ). The algorithm (9) is very unstable for higher  $m$ . This fact limited our calculations.

The OP  $\omega_n(x)$  were computed for various  $k$ ,  $n$  on  $[-1, 1]$ , i.e. with WF  $1 + \cos \pi k(x+1)$  and  $1 + \sin \pi k(x+1)$ . If  $\omega_n(x)$  is in the form (5) then in (4)  $(x^n, \omega_n) = \sum_{j=0}^n a_j^{(n)} W_{2n-j}$ , where  $a_0^{(n)} = 1$ ,  $n = 1, 2, \dots$

The knots  $x_i^{(n)}$  of the method (2) were computed by the Newton method, the first approximation for the  $i$ -th root being taken on the basis of the separation theorem for roots of OP as the central point of the interval  $[x_{i-1}^{(n-1)}, x_i^{(n-1)}]$  where  $x_0^{(n-1)} = -1$ ,  $x_n^{(n-1)} = 1$ .

The coefficients  $A_i$ ,  $i = 1, 2, \dots, n$  were calculated for various  $k$  and  $n$  from (7). For  $k = 1, 2, 3, 5$  and for given  $n$  the  $x_i^{(n)}$ ,  $A_i$  and some  $N_n^{-2}$  are in Table 2 (for direct use in (2), they are tabulated after the transformation into  $[0, 1]$ ). For  $W(x) = 1 + \cos 2\pi kx$  Theorem 2 holds, i.e.  $x_i^{(n)} = 1 - x_{n-i+1}^{(n)}$  and  $A_i = A_{n-i+1}$ . The parameters  $x_i^{(n)}$  and  $A_i$  were checked by the calculation (2) for  $f(x) = x^m$ ,  $m = 0, 1, \dots, 2n - 1$ . The given number of digits satisfied this check.

Table 2.

$k = 1$	$n = 6$	$n = 8$	$n = 11$	$n = 13$	
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0-02863 48830 20766	0-01753 03231 216	0-01006 52870 0	0-00805 28579
	$x_2^{(n)}$	0-13949 33627 14495	0-08827 98904 606	0-05182 16816 2	0-04157 97384
	$x_3^{(n)}$	0-30660 33696 75395	0-20222 98826 934	0-12259 77507 2	0-09871 28242
	$x_4^{(n)}$		0-34428 61284 048	0-21675 25909 3	0-17486 30914
	$x_5^{(n)}$			0-32913 26344 2	0-26459 67249
	$x_6^{(n)}$			0-50000 00000 0	0-36265 92011
	$x_7^{(n)}$				0-50000 00000
	$A_1$	0-14277 91667 13474	0-08868 89962 796	0-05132 05083 1	0-04110 59282
	$A_2$	0-23673 33598 68050	0-17544 74813 997	0-11126 79178 8	0-09041 70028
	$A_3$	0-12048 74734 18475	0-16908 59993 932	0-14333 58219 4	0-12242 57821
	$A_4$		0-06677 75229 274	0-12549 19656 2	0-12203 31949
	$A_5$			0-06318 45454 6	0-08597 83271
	$A_6$			0-01079 84815 4	0-03543 02838
	$A_7$				0-00521 89617
	$N_n^{-2}$	0-71747 <sub>10</sub> - 7	0-28249 <sub>10</sub> - 9	0-82155 <sub>10</sub> - 13	
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0-03530 06884 48574	0-02097 58516 990	0-01136 47917 2	
	$x_2^{(n)}$	0-16148 16415 72301	0-10192 01213 173	0-05751 69940 0	
	$x_3^{(n)}$	0-33830 76867 80632	0-22559 97328 166	0-13319 61323 5	
	$x_4^{(n)}$	0-53487 55947 36473	0-37453 07782 626	0-23065 49230 8	
	$x_5^{(n)}$	0-86940 35455 13604	0-53567 43606 864	0-34282 53698 9	
	$x_6^{(n)}$	0-97428 69518 95854	0-74704 91554 413	0-46339 55776 1	
	$x_7^{(n)}$		0-91486 92699 091	0-58725 89369 7	
	$x_8^{(n)}$		0-98350 36953 655	0-74712 57367 4	
	$x_9^{(n)}$			0-88243 65014 7	
	$x_{10}^{(n)}$			0-95117 94915 7	
	$x_{11}^{(n)}$			0-99059 68501 1	
	$A_1$	0-10570 77677 32134	0-05953 26585 872	0-03092 39865 0	
	$A_2$	0-29173 73990 48907	0-16875 57629 355	0-08423 55904 9	
	$A_3$	0-35320 74394 86786	0-27603 16214 148	0-15302 31203 8	
	$A_4$	0-15547 90252 49368	0-26804 58925 718	0-21107 30891 3	
$A_5$	0-03915 00317 31642	0-12791 94150 496	0-21531 70781 2		
$A_6$	0-05471 83367 51160	0-01552 74054 150	0-15089 50920 3		
$A_7$		0-04645 98850 318	0-05972 81467 1		
$A_8$		0-03772 73589 940	0-00702 82183 0		
$A_9$			0-02717 10504 2		
$A_{10}$			0-03798 09019 6		
$A_{11}$			0-02262 37259 2		
$N_n^{-2}$	0-57388 <sub>10</sub> - 7	0-25705 <sub>10</sub> - 9	0-60242 <sub>10</sub> - 13		

Table 2 — continued.

$k = 2$	$n = 6$	$n = 8$	$n = 11$	$n = 13$	
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0-02621 42279 62297	0-01724 90454 829	0-00975 29784 1	0-00711 42590
	$x_2^{(n)}$	0-12248 70994 98892	0-08534 67596 210	0-04961 49900 4	0-03653 96257
	$x_3^{(n)}$	0-41749 31418 77675	0-21770 00764 796	0-11626 09693 6	0-08632 25874
	$x_4^{(n)}$		0-42795 96275 387	0-22378 10169 7	0-15311 56757
	$x_5^{(n)}$			0-38811 25216 4	0-30409 54385
	$x_6^{(n)}$			0-50000 00000 0	0-40332 02724
	$x_7^{(n)}$				0-50000 00000
	$A_1$	0-12628 03067 04920	0-08590 48166 871	0-04941 79793 0	0-03621 05141
	$A_2$	0-12606 91883 24699	0-13430 08262 584	0-09792 41878 7	0-07623 08456
	$A_3$	0-24765 05049 70380	0-04611 66617 129	0-08756 03628 6	0-08622 09390
	$A_4$		0-23367 76953 414	0-02268 93352 2	0-04881 48976
	$A_5$			0-13048 68128 3	0-02565 12294
	$A_6$			0-22384 26437 8	0-13000 47088
	$A_7$				0-19373 37303
$N_n^{-2}$	0-64442 <sub>10</sub> — 7	0-34884 <sub>10</sub> — 9	0-78860 <sub>10</sub> — 13		
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0-03592 86695 98318	0-02129 01288 094	0-01155 78217 7	
	$x_2^{(n)}$	0-15129 48931 36651	0-09727 47642 825	0-05651 11899 4	
	$x_3^{(n)}$	0-35185 71212 29824	0-20770 47727 717	0-12691 45107 1	
	$x_4^{(n)}$	0-60198 23664 44602	0-42657 38127 647	0-21600 74190 7	
	$x_5^{(n)}$	0-76285 56835 09764	0-59014 11267 487	0-33435 80686 4	
	$x_6^{(n)}$	0-97701 42180 86842	0-72615 93738 565	0-51574 83346 6	
	$x_7^{(n)}$		0-90644 45086 405	0-62578 98529 0	
	$x_8^{(n)}$		0-98435 83087 316	0-73007 98480 3	
	$x_9^{(n)}$			0-83242 77579 3	
	$x_{10}^{(n)}$			0-95437 67078 4	
	$x_{11}^{(n)}$			0-99119 38665 6	
	$A_1$	0-12097 09902 18188	0-06584 15977 469	0-03322 62096 5	
	$A_2$	0-27392 15366 29941	0-18587 87259 859	0-09770 68294 2	
	$A_4$	0-10283 81348 07334	0-18664 93199 115	0-16101 16644 1	
$A_3$	0-33302 99459 60522	0-07002 81161 997	0-13786 16927 3		
$A_5$	0-12784 22342 70726	0-26949 90052 615	0-03272 70716 3		
$A_6$	0-04139 71581 13287	0-16961 13097 521	0-13511 61643 1		
$A_7$		0-02026 37244 129	0-21472 03669 8		
$A_8$		0-03222 82007 291	0-12607 38467 8		
$A_9$			0-01823 31399 2		
$A_{10}$			0-02331 69822 4		
$A_{11}$			0-02000 60318 7		
$N_n^{-2}$	0-65264 <sub>10</sub> — 7	0-27333 <sub>10</sub> — 9	0-61021 <sub>10</sub> — 13		

Table 2 — continued.

$k = 3$	$n = 6$	$n = 8$	$n = 11$	$n = 13$	
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0-02698 77484 52905	0-01556 82021 385	0-00931 13558 1	0-00703 50999
	$x_2^{(n)}$	0-16073 83677 56222	0-07429 88831 243	0-04660 05425 6	0-03580 06393
	$x_3^{(n)}$	0-35290 13292 58910	0-26601 28305 654	0-10757 58530 7	0-08402 85775
	$x_4^{(n)}$		0-38078 57215 155	0-26480 90698 9	0-19335 99001
	$x_5^{(n)}$			0-36086 29828 8	0-28711 24962
	$x_6^{(n)}$			0-50000 00000 0	0-37485 58904
	$x_7^{(n)}$				0-50000 00000
	$A_1$	0-12411 79475 50255	0-07596 35735 586	0-04676 90983 6	0-03562 33775
	$A_2$	0-09177 76392 27720	0-08855 82910 740	0-08175 99950 7	0-06950 47430
	$A_3$	0-28410 44132 22024	0-14401 23212 529	0-04068 66618 4	0-05648 78852
	$A_4$		0-19146 58141 143	0-11785 09413 2	0-02403 72751
	$A_5$			0-18742 07286 9	0-14087 32868
	$A_6$			0-05102 51493 7	0-15505 09246
	$A_7$				0-03684 50151
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0-03383 30769 41764	0-02149 01334 279	0-01161 02841 6	
	$x_2^{(n)}$	0-13213 20319 77895	0-09335 02047 246	0-05484 75186 7	
	$x_3^{(n)}$	0-39653 62364 54584	0-21200 46011 906	0-12052 06667 4	
	$x_4^{(n)}$	0-62884 20427 70207	0-40627 09154 397	0-21370 21421 5	
	$x_5^{(n)}$	0-79068 57134 66831	0-57747 24883 805	0-37018 33602 7	
	$x_6^{(n)}$	0-97853 02620 85455	0-74766 60161 224	0-46998 23518 6	
	$x_7^{(n)}$		0-85749 03148 280	0-65772 48041 0	
	$x_8^{(n)}$		0-98578 99497 239	0-75138 56232 1	
	$x_9^{(n)}$			0-83536 26197 9	
	$x_{10}^{(n)}$			0-95667 03548 4	
	$x_{11}^{(n)}$			0-99169 28362 1	
	$A_1$	0-12105 59336 49543	0-07128 76786 248	0-03501 67763 9	
	$A_2$	0-18501 64451 69627	0-17701 85726 353	0-10365 33681 1	
	$A_3$	0-28222 11945 31787	0-07843 81091 935	0-13240 61288 3	
$A_4$	0-15587 03945 69039	0-25804 84317 211	0-04216 20181 9		
$A_5$	0-22287 59298 57312	0 09462 01442 127	0 16355 79703 9		
$A_6$	0-03296 01022 22689	0-23235 39054 300	0-15421 60321 0		
$A_7$		0 06171 62439 080	0 08411 59317 5		
$A_8$		0-02651 69142 742	0-17728 81589 2		
$A_9$			0-07601 88971 8		
$A_{10}$			0 01368 06979 9		
$A_{11}$			0-01788 40201 2		



Table 2 — continued.

$k = 5$	$n = 6$	$n = 8$	$n = 11$	$n = 13$	
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0-02203 81581 15244	0-01569 73522 393	0-00872 28145 6	0-00687 95364
	$x_2^{(n)}$	0-18229 78581 99824	0-09306 95772 270	0-04230 69976 4	0-03449 65616
	$x_3^{(n)}$	0-38713 26801 33243	0-21349 05611 430	0-15398 74863 2	0-09609 33067
	$x_4^{(n)}$		0-40203 89341 177	0-22547 59524 1	0-18308 88482
	$x_5^{(n)}$			0-38364 49027 2	0-25687 81025
	$x_6^{(n)}$			0-50000 00000 0	0-39168 62994
	$x_7^{(n)}$				0-50000 00000
	$A_1$	0-09218 52934 66460	0-07283 37607 982	0-04288 52662 9	0-03438 23742
	$A_2$	0-18623 61447 12392	0-05582 08631 587	0-05447 31076 4	0-05482 65714
	$A_3$	0-22157 85618 21146	0-17643 10214 265	0-07748 96500 3	0-02194 17709
	$A_4$		0-19491 43546 165	0-12606 44429 9	0-12656 49029
	$A_5$			0-15807 66243 3	0-07287 36086
	$A_6$			0-08202 18173 7	0-15686 81930
	$A_7$				0-06508 51575
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0-03217 19447 46941	0-01862 99280 219	0-01068 13147 6	
	$x_2^{(n)}$	0-17762 67720 67531	0-08109 64571 769	0-05414 96059 0	
	$x_3^{(n)}$	0-37661 42529 33109	0-25777 71277 171	0-11591 73777 6	
	$x_4^{(n)}$	0-61458 13508 58542	0-43925 92172 925	0-24660 93718 2	
	$x_5^{(n)}$	0-83653 91197 37377	0-61518 70771 838	0-36715 24021 6	
	$x_6^{(n)}$	0-96909 10851 36180	0-75699 29832 154	0-46676 68785 7	
	$x_7^{(n)}$		0-87420 48924 442	0-64149 03472 2	
	$x_8^{(n)}$		0-98755 87300 457	0-76092 53180 5	
	$x_9^{(n)}$			0-85453 98152 3	
	$x_{10}^{(n)}$			0-92597 76792 2	
	$x_{11}^{(n)}$			0-99242 10785 7	
	$A_1$	0-09411 38680 27047	0-05268 65089 198	0-02232 05470 0	
	$A_2$	0-16367 14143 98607	0-11967 47156 145	0-13806 18245 9	
	$A_3$	0-22974 98031 06718	0-41647 74553 019	0-05263 24692 0	
$A_4$	0-23814 38422 18058	0-25208 20763 164	0-20865 99735 7		
$A_5$	0-20017 87556 30190	0-18416 47851 608	0-08282 32109 5		
$A_6$	0-03608 19219 43819	0-11071 48008 208	0-19167 02419 4		
$A_7$		0-12495 45746 834	0-16683 18994 4		
$A_8$		0-01829 18387 452	0-06455 51671 8		
$A_9$			0-14869 51116 2		
$A_{10}$			0-02112 20527 3		
$A_{11}$			0-00953 43349 6		

IV. DESCRIPTION OF THE COMPUTATION METHOD AND ESTIMATE  
OF ERRORS

There holds for (1)

$$(10) \quad J = \int_0^T f(x) [1 + w(kx)] dx - \int_0^T f(x) dx.$$

The second integral on the right hand side of (10) can be computed with some current numerical method. For computing the first integral we have

**Theorem 3.** Let  $A_i^{[p]}$  be the coefficients (7) calculated for WF  $1 + w(2\pi py)$ , let  $x_i^{[l]} = (2\pi/td)(l - 1 + x_i^{(n)})$  where  $x_i^{(n)}$  are the knots from (2),  $d$  is the number of equal subintervals  $[0, T]$ ,  $T = 2\pi/t$ ,  $t = 1, 2, \dots, k = t \cdot p \cdot d$ . Then it holds

$$(11) \quad \int_0^T f(x) \left(1 + \frac{\cos kx}{\sin kx}\right) dx = \frac{2\pi}{td} \sum_{l=1}^d \sum_{i=1}^n A_i^{[p]} f(x_i^{[l]}) + R_n^{[k]}(f).$$

If  $f(x) \in C^{2n}[0, T]$  and  $|f^{(2n)}(x)| \leq M$ ,  $x \in [0, T]$  then

$$(12) \quad |R_n^{[k]}(f)| \leq \frac{M \cdot T^{2n+1}}{(2n)! d^{2n} N_n^2}$$

where  $N_n^2 = \left(\int_0^1 y^n \omega_{n,p}(y) [1 + w(2\pi py)] dy\right)^{-1}$ ,  $\omega_{n,p}(y) = \omega_n(y)$  is the polynomial orthogonal with the weight function  $1 + w(2\pi py)$  on  $[0, 1]$ .

Proof. Let  $w(kx) = \cos kx$  (the proof for  $1 + \sin kx$  is analogous). Construct a Gauss type quadrature formula with the weight  $1 + \cos kz$  for the interval  $[a_{l-1}, a_l] \equiv [(2\pi(l-1)/td), 2\pi l/td]$ , i.e.

$$(13) \quad \int_{a_{l-1}}^{a_l} f(z) (1 + \cos kz) dz = \sum_{i=1}^n B_i f(z_i^{[l]}) + \frac{f^{(2n)}(\xi_l)}{(2n)!} \int_{a_{l-1}}^{a_l} \omega_{n,k}^2(z) (1 + \cos kz) dz$$

where

$$\omega_{n,k}(z) = (z - z_1^{[l]})(z - z_2^{[l]}) \dots (z - z_n^{[l]}), \quad z_i^{[l]}, \xi_l \in [a_{l-1}, a_l]$$

and the rest is given in the familiar form.

Since

$$(14) \quad \omega_{n,k}(z) = \left(\frac{2\pi}{td}\right)^n \omega_{n,p}(y)$$

where

$$(15) \quad z = \frac{2\pi}{td}(l - 1 + y),$$

it is  $B_i = 2\pi/td \cdot A_i^{[p]}$  for each  $l$ .

Considering formula (13) for  $l = 1, 2, \dots, d$  to which all these equations are added, (11) is obtained in which

$$R_n^{[kl]}(f) = \frac{1}{(2n)!} \sum_{l=1}^d f(\xi_l) R_l$$

where

$$(16) \quad R_l = \int_{a_{l-1}}^{a_l} \omega_{n,k}^2(z) (1 + \cos kz) dz .$$

If we substitute (15) into (16) then, since  $\cos [2\pi k/td(l-1+y)] = \cos 2\pi py$  with regard to (14), we get

$$R_l = \left(\frac{2\pi}{td}\right)^{2n+1} \int_0^1 \omega_{n,p}^2(y) (1 + \cos 2\pi py) dy$$

and thus for  $R_n^{[kl]}(f)$  estimation (12) holds.

Remark: The idea of formula (11) can also be applied to the numerical computation of Fourier transformation, i.e. for evaluating

$$(17) \quad J(k) = \int_0^\infty f(x) w(kx) dx$$

and for computing  $m$ -dimensional integrals which occur e.g. when computing Fourier coefficients of more variables, which e.g. for  $m = 2$  are

$$(18) \quad \int_0^{T_2} \int_0^{T_1} f(x, y) w(k_1x) w(k_2y) dx dy$$

where  $T_i = 2\pi/t_i$ ,  $k_i$ ,  $t_i$  are integer.

For the calculation of (17) the formula

$$(19) \quad \int_0^\infty f(x) [1 + w(kx)] dx = \frac{2\pi}{d} \sum_{l=1}^{ds} \sum_{i=1}^n A_i^{lpl} f(x_i^{[l]}) + R_n^{(s)}(f)$$

holds where  $s$  is integer,  $k = p \cdot d$ ,

$$\int_{2\pi s}^\infty |f(x) [1 + w(kx)]| dx \leq \varepsilon .$$

If  $f(x) \in C^{2n}[0, 2\pi s]$  and  $|f^{(2n)}(x)| \leq M$  for  $x \in [0, 2\pi s]$  then

$$|R_n^{(s)}(f)| \leq \frac{(2\pi)^{2n+1} sM}{(2n)! d^{2n} N_n^2} + \varepsilon .$$

For (18) we have again

$$(20) \quad \int_0^{T_2} \int_0^{T_1} f(x, y) [1 + w(k_1 x)] [1 + w(k_2 y)] dx dy = \\ = \frac{T_1 T_2}{d_1 d_2} \sum_{i=1}^{d_1} \sum_{k=1}^{d_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} A_i^{[p]} A_j^{[q]} f(x_i^{[1]}, y_j^{[k]}) + R_{n_1, n_2}(f)$$

where  $k_1 = p d_1 t_1$ ,  $k_2 = q d_2 t_2$ .

If necessary partial derivatives exist and are again bounded by  $M_1$  and  $M_2$  on the given intervals then

$$(21) \quad |R_{n_1, n_2}(f)| \leq T_1 \left[ \frac{M_2 T_2^{2n_2+1}}{(2n_2)! d_2^{2n_2} N_{n_2}^2} + \frac{2M_1 T_1^{2n_1+1} (\pi + 2)}{t_2 (2n_1)! d_1^{2n_1+1} N_{n_2}^2} \right].$$

## V. NUMERICAL EXAMPLES

Introduce the concept of the so called characteristic of the formula (11). It will be the symbol  $(n . t . p . d)$  consisting of the parameters of the formula (11) ( $k = = t . p . d$ ) which was used in all examples (except 5).

The formula (11) gives very good results for high  $k$ .

1. Compute  $\frac{1}{\pi} \int_0^{2\pi} e^x \cos x \sin kx dx$  for  $k = 1, 10, 50, 100 (100) 500$ . Absolute errors (further only a.e.) together with the characteristic are in Table 3.

If in computing  $\int_0^T f(x) \sin kx dx$  we want to make use of the knots and coefficients for  $1 + \cos 2\pi kx$  then — if  $f'(x)$  on  $[0, T]$  is known — we have

$$(22) \quad \int_0^T f(x) \sin kx dx = \frac{1}{k} [f(T)(1 + \cos kT) - 2f(0)] + \frac{1}{k} \int_0^T f'(x)(1 + \cos kx) dx.$$

2. Compute  $\frac{1}{\pi} \int_0^{2\pi} e^x \sin kx dx$  by means of (22).  
A.e.  $(R_n^{[k]}(f')/k)$  for  $k = 1, 10, 50, 100 (100) 500$  are in Table 3.

Table 3.

$k$		1	10	50	100
char.		(10 . 1 . 1 . 1)	(10 . 1 . 10 . 1)	(10 . 1 . 10 . 5)	(8 . 1 . 50 . 2)
$f(x)$	$e^x \cos x$ $e^x$	$1.55 \cdot 10^{-10}$ $2.63 \cdot 10^{-11}$	$2.07 \cdot 10^{-10}$ $5.55 \cdot 10^{-13}$	$2.11 \cdot 10^{-12}$ $1.84 \cdot 10^{-14}$	$2.50 \cdot 10^{-12}$ $3.45 \cdot 10^{-15}$

Table 3.

	$k$	200	300	400	500
	char.	(8 . 1 . 50 . 4)	(8 . 1 . 50 . 6)	(8 . 1 . 50 . 8)	(8 . 1 . 50 . 10)
$f(x)$	$e^x \cos x$ $e^x$	$5.07 \cdot 10^{-14}$ $4.47 \cdot 10^{-16}$	$3.74 \cdot 10^{-14}$ $5.34 \cdot 10^{-16}$	$3.17 \cdot 10^{-14}$ $3.54 \cdot 10^{-16}$	$2.75 \cdot 10^{-14}$ $2.64 \cdot 10^{-16}$

If  $f(x)$  is an odd function on  $[0, T]$  then in (10)  $\int_0^T f(x) dx = 0$ .

3. Compute the first 50 Fourier coefficients of the function  $f(x) = x \cdot \cos x$  i.e.  $1/\pi \int_0^{2\pi} x \cos x \sin kx dx$ . The results are in Table 4.

Table 4.

$k (= d)$	1	2	3 (1) 9	10 (1) 20	21 (1) 50
char.	(10 . 1 . 1 . 1)	(7 . 1 . 1 . 2)	(7 . 1 . 1 . d)	(5 . 1 . 1 . d)	(5 . 1 . 1 . d)
max. error	$1.42 \cdot 10^{-13}$	$5.43 \cdot 10^{-14}$	$2.53 \cdot 10^{-16}$	$6.06 \cdot 10^{-17}$	$1.06 \cdot 10^{-18}$

4. Let  $f(x) = \sin x$ ,  $t = 2$ . The formula (11) with the characteristic (8 . 2 . 1 . d),  $d = 1(1) \dots$  gives all Fourier coefficients of this function. The first 40 were calculated with the maximal a.e.  $1.08 \cdot 10^{-15}$ .

5. Compute  $J(k) = \int_0^\infty e^{-x} \cos kx dx$  for  $k = p \cdot d$  by means of (18). For  $s = 4$  and given  $p, n, d$  the a.e. are in Table 5. It is not necessary that  $d$  in (18) were integer. If  $d = j/p$ ,  $j = 1, 2, \dots$  then the formula (18) yields  $J(k)$  for  $k = 1, 2, \dots$ . E.g. for  $s = 2$ ,  $p = 2$ ,  $d = \frac{1}{2}j$  the a.e. are also in Table 5.

Table 5.

$n$	$d \setminus j$	1	2	3	4	5	6 ... 20
$p=1$	5 $j$	$1.36 \cdot 10^{-5}$	$2.83 \cdot 10^{-8}$	$5.98 \cdot 10^{-10}$	$4.81 \cdot 10^{-11}$	$1.65 \cdot 10^{-11}$	$< 1.3 \cdot 10^{-11}$
	8 $j$	$4.33 \cdot 10^{-11}$	$1.45 \cdot 10^{-11}$	$1.33 \cdot 10^{-11}$	$1.29 \cdot 10^{-11}$	$1.26 \cdot 10^{-11}$	$< 1.2 \cdot 10^{-11}$
$p=2$	5 $j$	$1.43 \cdot 10^{-5}$	$2.99 \cdot 10^{-8}$	$6.32 \cdot 10^{-10}$	$4.96 \cdot 10^{-11}$	$1.74 \cdot 10^{-11}$	$< 1.2 \cdot 10^{-11}$
	5 $0.5 \cdot j$	$2.29 \cdot 10^{-3}$	$1.39 \cdot 10^{-5}$	$4.83 \cdot 10^{-7}$	$3.69 \cdot 10^{-8}$	$2.73 \cdot 10^{-8}$	$< 7.3 \cdot 10^{-9}$

6. At the experiments carried out no notice was made with growing  $p$  and  $d$  in (11) of any instability of the computation process. Table 6 gives the calculated values and their a.e. of the integrals  $(1/\pi) \int_0^{2\pi} x \cos x \sin kx \, dx$  for  $k = 10, 100, 400$ .

The accuracy of the results is remarkable.

Table 6.

$k$	char.	computed value of integral	abs. error
10	(5 . 1 . 1 . 10)	---0.20202020202020208081	$6.06 \cdot 10^{-17}$
100	(5 . 1 . 5 . 20)	---0.02000200020002000308	$1.08 \cdot 10^{-18}$
400	(5 . 1 . 10 . 40)	---0.00500003125019531465	$9.30 \cdot 10^{-19}$

All calculations were carried out on the Danish computer GIER in GIER-ALGOL III in double precision arithmetics.

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#### Súhrn

### NUMERICKÁ INTEGRÁCIA S RÝCHLOOSCILUJÚCOU VÁHOVOU FUNKCIOU

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Článok opisuje novú numerickú metódu pre výpočet integrálov s váhovou funkciou  $\exp(ikx)$ ,  $k$  celé, ktorú možno použiť aj pre nevlastné a viacnásobné integrály. Metóda používa parametre kvadratúry Gaussovho typu, ktoré sú tabelované pre rôzne  $k$ . Jej aplikácia najmä pri veľkom  $k$  je demonštrovaná numerickými experimentami.

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