

## Numerical Inversion of the Laplace Transform

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*Dedicated to our Friend Professor Milić Stojić*

**Abstract:** We give a short account on the methods for numerical inversion of the Laplace transform and also propose a new method. Our method is inspired and motivated from a problem of the evaluation of the Müntz polynomials (see [1]), as well as the construction of the generalized Gaussian quadrature rules for the Müntz systems (see [2]). As an illustration of our method we consider an example with 100 real poles distributed uniformly on  $(-1/2, 100)$ . A numerical investigation shows the efficiency of the proposed method.

**Keywords:** Laplace transform, Bromwich integral, orthogonal polynomials, Gaussian quadrature formula.

### 1 Introduction

Laplace transforms are powerful tools in many problems of mathematics, physics, and other applied and computational sciences. Primarily, these transforms are very attractive in solving differential equations, and therefore play important role in automatics and control theory.

An integral of the form

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} e^{-st} f(t) dt, \quad \operatorname{Re}(s) > \sigma_0, \quad (1.1)$$

is called the *Laplace transform* of the *original*  $f(t)$ , for which we suppose it grows at most exponentially, i.e., there is  $\sigma > \sigma_0$  such that  $|f(t)| \leq Me^{\sigma t}$  ( $t \rightarrow +\infty$ ). In

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addition, if  $f(t)$  is absolutely integrable on any interval  $[0, a]$  ( $a > 0$ ), then the Laplace integral (1.1) converges for all  $\operatorname{Re}(s) > \sigma$  and defines a single-valued analytic function in this half-plane. Here,  $\sigma_0$  is so-called the *abscissa of convergence* of  $F(s)$ . For an original we put  $f(t) = 0$  for  $t < 0$ . An analysis of properties of  $F(s)$ , as well as several applications, are given in details in [3, pp. 199–261].

The main difficulty in applications is finding the inversion of  $F(s)$ , except cases when it is given in a table (cf. [4]). As an imprecise form of inversion formula is

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st} ds \quad (\sigma > \sigma_0), \quad (1.2)$$

which is correct if  $f(t)$  is differentiable at the point  $t$ . Here, the integration along the Bromwich contour from  $\sigma - i\infty$  to  $\sigma + i\infty$  ( $\sigma > \sigma_0$ ) should be understood as the Cauchy principal value, i.e.,

$$\int_{\sigma-i\infty}^{\sigma+i\infty} = \lim_{\omega \rightarrow +\infty} \int_{\sigma-i\omega}^{\sigma+i\omega}.$$

The Bromwich contour is parallel to the imaginary axis and it is located to the right of all the singularities of  $F(s)$ . If  $F(s)$  is an analytic function in  $\operatorname{Re}(s) > \sigma_0$  and for  $\sigma > \sigma_0$

1.  $\int_{-\infty}^{+\infty} |F(\sigma + i\omega)| d\omega$  converges,
2.  $F(s) \rightarrow 0$  for  $\operatorname{Re}(s) \geq \sigma$ ,  $|s| \rightarrow +\infty$ ,

then (1.2) holds.

As an immediately consequence of Cauchy's theorem, the Cauchy principal value integral on the right side of (1.2) does not depend on  $\sigma$ , so that in its numerical calculation, using some quadrature formula, one can move the Bromwich contour to the left in order to reduce in magnitude the exponential factor  $e^{st}$  in the integrand, but not too close to singularities of  $F(s)$ .

## 2 Methods of Numerical Inversion

There are several algorithms available for the numerical inversion of Laplace transforms. Some surveys were given by Davies and Martin [5], Narayanan and Beskos [6] and Duffy [7]. A bibliography of several hundred such papers is available on the WEB (see [8]). In a recent paper, Abate and Valkó [9] have classified those algorithms into four categories according to the basic approach of the method as

follows: (1) Fourier series expansion; (2) Laguerre function expansion; (3) Combination of Gaver functionals; (4) Deform the Bromwich contour.

The methods from the first category were discussed in the survey paper of Abate and Whitt [10] as well as in the papers of D'Amore, Lacetti and Murli [11].

Another type of algorithms is based on the Laguerre function expansion of the original  $f(t)$  (cf. Weeks [12], Lyness and Giunta [13], Abate, Choudhury, Whitt [14] and Weideman [15]), i.e.,

$$f(t) = e^{\sigma t} \sum_{n=0}^{+\infty} a_n e^{-bt} L_n(2bt), \quad t > 0, \tag{2.1}$$

where  $L_n(t)$  denotes the Laguerre polynomial of degree  $n$ ,  $b$  is a positive number, and the coefficients  $a_n$  are defined by

$$\frac{2b}{1-z} F\left(\sigma + \frac{2b}{1-z} - b\right) = \sum_{n=0}^{+\infty} a_n z^n, \quad |z| < R, \tag{2.2}$$

where  $R$  is the radius of convergence of the series on the right side in (2.2). The method is associated with the name of Weeks, although this expansion was considered several years ago by Tricomi [16]. Two free parameters,  $\sigma$  ( $> \sigma_0$ ) and  $b$  ( $> 0$ ), are included in the expansion. For a given  $\sigma > \sigma_0$ , one can find the optimal  $b$ , which maximizes the rate of convergence of the series, in terms of the location of the singularities of  $F(s)$ , as it is described by Giunta, Lacetti and Rizzadi [17] and Weideman [15]. After a selection of these parameters, the method is based on an  $N$ -term truncation of the series (2.1). Then, it can be evaluated recursively, e.g. by the well-known Clenshaw's algorithm. Such an implementation in FORTRAN was given by Garbow, Giunta, Lyness, and Murli [18].

The third approach to numerical inversion of the Laplace transform is based on the sequence of functionals developed by Gaver [19]. A comparison of sequence accelerators for the Gaver method was given by Valkó and Abate [20].

Finally, one of the best ways for numerical inversion of the Laplace transform is to deform the standard contour in the Bromwich integral (1.2). One of the well-known paper in this direction is given in 1979 by Talbot [21].

The Talbot's contour is illustrated in Figure 2.1. It can be expressed in the form

$$s(z) = \sigma + \lambda s_\nu(z), \quad z \in (-2\pi i, 2\pi i),$$

where

$$s_\nu(z) = \frac{z}{1 - e^{-z}} + \frac{1}{2}(\nu - 1)z.$$

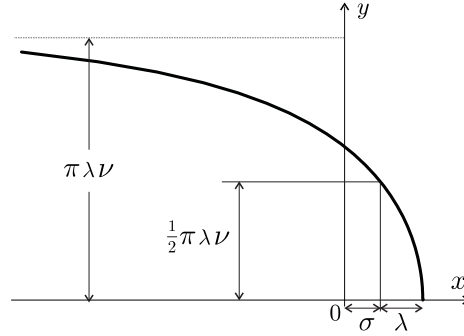


Fig.2.1. Talbot's contour

Putting  $z = 2i\theta$ , the contour can be parameterized using

$$s(\theta) = \sigma + \lambda s_v(\theta), \quad s_v(\theta) = \theta \cot \theta + i\nu\theta, \quad \theta \in (-\pi, \pi). \quad (2.3)$$

The real parameters  $\sigma$ ,  $\lambda (> 0)$  and  $\nu (> 0)$  determine the geometry of the curve. Since  $s'_v(\theta) = \cot \theta - \theta \csc^2 \theta + i\nu$ , the integral (1.2) reduces to

$$f(t) = \frac{\lambda e^{\sigma t}}{2\pi i} \int_{-\pi}^{\pi} F[\sigma + \lambda s_v(\theta)] e^{\lambda s_v(\theta)t} s'_v(\theta) d\theta.$$

In order to calculate this integral, Talbot uses the trapezoidal rule and in that way he gets

$$f(t) \approx -i \frac{\lambda e^{\sigma t}}{n} \sum_{j=0}^n{}'' e^{\lambda t s_v(\theta_j)} F[\sigma + \lambda s_v(\theta_j)] s'_v(\theta_j), \quad (2.4)$$

where  $\sum''$  denotes that the first and last term of the sum (for  $j = 0$  and  $j = n$ ) are taken with factor  $1/2$ .

**Remark.** By replacing the function  $\theta \cot \theta$  with the first two terms in its partial fraction expansion, a similar contour can be obtained (cf. [22]). In this case we have

$$s(\theta) = \sigma + \lambda \left( 1 + \frac{2\theta^2}{\theta^2 - \pi^2} + i\nu\theta \right), \quad -\pi \leq \theta \leq \pi.$$

At the end of this section we mention also the Henrici's real inversion formula. Starting from the (complex) Riemann inversion formula, Henrici [3] derived an explicit representation of the original function in terms of its Laplace transform function evaluated along the real negative axis. Such representation is known as the *real inversion formula*. It requires that the Laplace transform function, which must be considered as the restriction on the half plane of convergence of a complex function, has a branch point at zero and the negative real axis as a branch cut.

Integrating the function  $s \mapsto F(s)e^{st}$  over the contour as in Figure 2.2, with  $R \rightarrow +\infty$  and  $r \rightarrow 0$ , we get the Henrici's formula

$$f(t) = \frac{1}{2\pi i} \int_0^{+\infty} e^{-tx} [F(xe^{-i\pi}) - F(xe^{i\pi})] dx, \quad (2.5)$$

where the values of  $F$  on the upper and lower edges of the cut are denoted by  $F(xe^{i\pi})$  and  $F(xe^{-i\pi})$  ( $x > 0$ ), respectively.

Recently D'Amore, Murli, and Rizzardi (see [23]) introduced some extensions of this formula. Namely, they derived an integral equation of convolution type, whose solution is the inverse Laplace transform function. This formula can be used if the Laplace transform has a finite number of singularities, located everywhere in the complex plane, and provided that their corresponding residues are known. Of course, their formula only requires the knowledge of the Laplace transform function on the real negative axis.

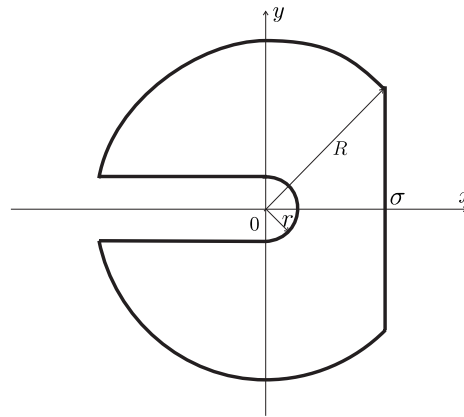


Fig.2.2. Henrici's contour.

### 3 New Method

In this section we present another method for the calculation of the inverse Laplace transform. It can be applied successfully for the specific class of functions  $\mathcal{F}$ . Actually, it is a function class as the one to which Talbot algorithm can be applied (see [21], [24]). Thus, we assume that function  $F$  has the following two properties:

- All singularities of  $F$  are known to be placed in the the region

$$S = \{s \mid |\text{Im}(s)| < K, \text{Re}(s) \leq \sigma_0\},$$

- $|F(s)|$  tends to zero uniformly as  $|s|$  tends to infinity for  $\text{Re}(s) \leq \sigma_0$  and  $|\text{Im}(s)| > K$ .

The main idea is to perform the evaluation of the Bromwich integral on the contour presented in Figure 3.1.

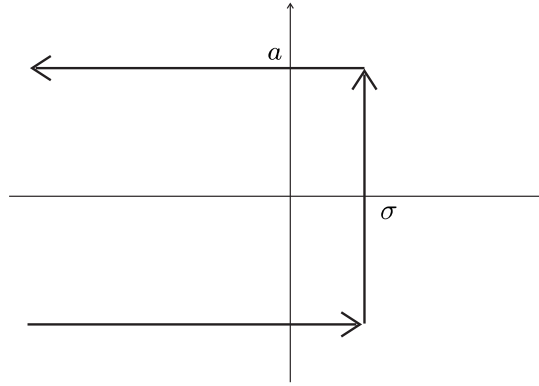


Fig.3.1. Deformed Bromwich contour

For this class of functions  $\mathcal{F}$  we can prove the following result:

**Theorem 3.1** For any  $\sigma > \sigma_0, t > 0$  and  $a > Kt$ , we have

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} F(s) ds = \frac{e^{\sigma t}}{\pi t} \left\{ \text{Re} \int_0^a e^{iu} F\left(\sigma + \frac{iu}{t}\right) du - \text{Im} e^{ia} \int_0^{+\infty} e^{-u} F\left(\sigma + \frac{ia}{t} - \frac{u}{t}\right) du \right\}. \quad (3.1)$$

*Proof.* First we note that, according to fact that every original  $f$  is real, we have that

$$\overline{F(s)} = \int_0^{+\infty} e^{-\bar{s}t} f(t) dt = F(\bar{s}),$$

i.e.,  $F(s) = \overline{F(\bar{s})}$ . Using this fact we can write

$$\begin{aligned} 2\pi i f(t) &= \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} F(s) ds = i e^{\sigma t} \int_{-\infty}^{+\infty} e^{iyt} F(\sigma + iy) dy \\ &= i e^{\sigma t} \left( \int_0^{+\infty} e^{iyt} F(\sigma + iy) dy + \int_0^{+\infty} e^{-iyt} F(\sigma - iy) dy \right) \\ &= 2i e^{\sigma t} \text{Re} \left\{ \int_0^{+\infty} e^{iyt} F(\sigma + iy) dy \right\}. \end{aligned}$$

Now, perform the substitution  $u = yt$  and choose some  $a > Kt$ . Then, we have

$$f(t) = \frac{e^{\sigma t}}{\pi t} \left( \operatorname{Re} \int_0^a e^{iu} F \left( \sigma + \frac{iu}{t} \right) du + \operatorname{Re} \int_a^{+\infty} e^{iu} F \left( \sigma + \frac{iu}{t} \right) du \right).$$

Consider the last integrand in the complex  $u$ -plane over the closed contour

$$C_R = \left\{ u \mid u \in [a, a+R] \right\} \cup \left\{ u \mid u - a = Re^{i\theta}, \theta \in [0, \pi/2] \right\} \\ \cup \left\{ u \mid \operatorname{Re}(u) = a, \operatorname{Im}(u) \in [0, R] \right\},$$

where  $R > 0$ . According to the analytic properties of  $\mathcal{F}$  and Cauchy residue theorem we know that

$$\oint_{C_R} e^{iu} F(\sigma + iu/t) du = 0.$$

For the integral over the arc we get

$$\left| \int_0^{\pi/2} e^{iR\cos\theta} e^{-R\sin\theta} e^{ia} F \left( \sigma + \frac{iu}{t} \right) R e^{i\theta} d\theta \right| \\ \leq R \int_0^{\pi/2} e^{-R\sin\theta} \left| F \left( \sigma + \frac{iu}{t} \right) \right| d\theta \\ \leq \max_{\theta \in [0, \pi/2]} \left| F \left( \sigma + \frac{iu}{t} \right) \right| R \int_0^{\pi/2} e^{-R\sin\theta} d\theta \\ \leq \frac{\pi}{2} (1 - e^{-2R/\pi}) \max_{\theta \in [0, \pi/2]} \left| F \left( \sigma + \frac{iu}{t} \right) \right|,$$

where we used Jordan inequality  $\sin \theta > 2\theta/\pi$ , for  $\theta \in (0, \pi/2)$ . It can be easily seen that, according to assumptions, we have that the integral over the arc tends to zero as  $R$  tends to  $+\infty$ , due to the fact that the function  $F$  tends uniformly to zero as the argument tends to infinity in the region  $\operatorname{Im}(s) > K$  and  $\operatorname{Re}(s) \leq \sigma_0$ . To be completely precise we need to justify the convergence in the region  $\operatorname{Im}(s) > K$  and  $\operatorname{Re}(s) > \sigma_0$ , but this is a consequence of the fact that the Laplace transform tends uniformly to zero as the argument tends to infinity in the region  $\operatorname{Re}(s) > \sigma_0$  (see [25]).

This means that we have

$$\int_a^{+\infty} e^{iu} F \left( \sigma + \frac{iu}{t} \right) du = ie^{ia} \int_0^{+\infty} e^{-u} F \left( \sigma + \frac{ia}{t} - \frac{u}{t} \right) du.$$

Since we need only the real part of the last integral and it is multiplied by  $i$ , actually we need minus imaginary part of the integral which is not multiplied by  $i$ .

This theorem suggests usage of the Gaussian quadrature rules to two integrals which are presented in (3.1). Especially, the second integral can be easily seen to be an integral with respect to the Laguerre measure so that we can apply the Gauss-Laguerre quadrature rule. For the first integral we can apply the Gauss-Legendre quadrature rule. However, in order to have faster convergence it is usually suitable not to perform the Gauss-Legendre quadrature to the first integral as a whole, but to divide interval  $(0, a)$  into the several subintervals and then to apply suitably transformed Gauss-Legendre quadrature rule to each subinterval.

As we know (see [26]), the Gauss-Legendre quadrature rule has the following property

$$\int_{-1}^1 p(x)dx = \sum_{k=1}^n w_k^n p(x_k^n),$$

where  $p$  is a polynomial of degree less than  $2n$ . In order to apply the Gauss-Legendre quadrature rule for the approximation of an integral over the interval  $(a, b)$ , we need to transform it in the following fashion

$$\tilde{w}_k^n = T_a^b(w_k^n) = \frac{b-a}{2} w_k^n, \quad \tilde{x}_k^n = T_a^b(x_k^n) = a + \frac{b-a}{2}(x_k^n + 1), \quad (3.2)$$

where  $k = 1, \dots, n$ .

Then we have the following lemma.

**Lemma 3.1** *Suppose  $w_k^n$  and  $x_k^n$ ,  $k = 1, \dots, n$ , are the nodes and weights of the Gauss-Legendre quadrature rule, and  $\tilde{w}_k^n$  and  $\tilde{x}_k^n$ ,  $k = 1, \dots, n$ , are given by (3.2), then we have*

$$\int_a^b p(x)dx = \sum_{k=1}^n \tilde{w}_k^n p(\tilde{x}_k^n),$$

for all polynomials of degree smaller than  $2n$ .

Now, suppose that we have divided the interval  $[0, a)$  into the following non-overlapping intervals  $[a_\nu, b_\nu)$ ,  $\nu = 1, \dots, m$ , with the property that  $\cup_{\nu=1}^m [a_\nu, b_\nu) = [0, a)$ . Then, we have

$$\begin{aligned} \int_0^a e^{iu} F\left(\sigma + \frac{iu}{t}\right) du &= \sum_{\nu=1}^m \int_{a_\nu}^{b_\nu} e^{iu} F\left(\sigma + \frac{iu}{t}\right) du \\ &\approx \sum_{\nu=1}^m \sum_{k=1}^{n_\nu} T_{a_\nu}^{b_\nu}(w_k^{n_\nu}) e^{iT_{a_\nu}^{b_\nu}(x_k^{n_\nu})} F\left(\sigma + \frac{iT_{a_\nu}^{b_\nu}(x_k^{n_\nu})}{t}\right). \end{aligned}$$

In the previous formula it is assumed that on every subinterval we can take different Gauss-Legendre quadrature rules, this fact produces index  $\nu$ . In the case we



are using the same Gauss-Legendre quadrature rules, on every subinterval, we can safely drop every appearance of  $v$  connected with the nodes and weights.

The Gauss-Laguerre quadrature rule on the other hand has the following property

$$\int_0^{+\infty} e^{-x} p(x) dx = \sum_{k=1}^n W_k p(X_k),$$

for all polynomials  $p$  of degree less than  $2n$  (see [26]).

According to (3.1) and the previous discussion, in total, our quadrature rule can be given by

$$f(t) \approx \frac{e^{\sigma t}}{\pi t} \left\{ \operatorname{Re} \sum_{v=1}^m \sum_{k=1}^{n_v} T_{a_v}^{b_v}(w_k^{n_v}) e^{iT_{a_v}^{b_v}(x_k^{n_v})} F \left( \sigma + \frac{iT_{a_v}^{b_v}(x_k^{n_v})}{t} \right) - \operatorname{Im} e^{ia} \sum_{k=1}^N W_k F \left( \sigma + \frac{ia - X_k}{t} \right) \right\}. \quad (3.3)$$

The real art is how to choose free parameters in the equation (3.3), in order to have as small as possible number of function evaluations and to have as high as possible precision in the result, for the given function  $F$ . Free parameters are  $m$ , the number of subintervals, then  $n_v$ ,  $v = 1, \dots, m$ , the number of nodes of the Gauss-Legendre quadrature in each interval, the point  $a$ , the number of nodes in the Gauss-Laguerre quadrature rule  $N$ , and of course  $\sigma$  which only constraint is to be bigger than  $\sigma_0$ . In general, this problem is still open.

However, there are some general guidelines for choosing parameters of the quadrature rule (3.3). First one is that the parameter  $\sigma$  should be chosen such that the contour is not passing too close to the singularities of the function  $F$ , for if it does, it would produce slow convergence of the quadrature rules. Numerical results suggest it should be at least at the distance 1 from the closest singularity. The parameter  $a$  should be chosen such that singularities of the function are far away from the arc  $\{s \mid \operatorname{Re}(s) \in (-\infty, \sigma), \operatorname{Im}(s) = a\}$ , in order for the Gauss-Laguerre quadrature rule to be efficient. Also, the intervals  $(a_k, b_k)$  should not be too large, for if they are oscillatory nature of the function  $e^{iu}$ , becomes the dominant factor which affects the convergence.

In order to present results easier we adopt the following short notation

$$F_1(u; \sigma, t) = F \left( \sigma + \frac{iu}{t} \right) \quad \text{and} \quad F_2(u; a, \sigma, t) = F \left( \sigma + \frac{ia - u}{t} \right) \quad (3.4)$$

for functions which appear in the quadrature rule (3.3). It can be easily seen that the function  $F_1$  has singularities which depend of  $t$  and  $\sigma$ . Also, the function  $F_2$

has singularities depending on  $t$ ,  $\sigma$  and  $a$ . This means that our contour should also change with  $t$  in order to have the smallest number of function evaluations.

Some surprising results that can be achieved we discuss in the next section.

#### 4 Numerical Example

In this section we give a numerical example of the evaluation of the inverse Laplace transform using Talbot algorithm presented in [21], [24] and, also, using algorithm which uses Gaussian quadrature rules given by equation (3.3).

We present the evaluation of the inverse Laplace transform for the function

$$F(s) = \frac{1}{s + \lambda_n} \prod_{k=1}^{n-1} \frac{s - \lambda_k - 1}{s + \lambda_k}, \quad (4.1)$$

where the nondecreasing sequence  $\lambda_k$ ,  $k = 1, \dots, n$ , is the sequence of real numbers with the property  $\lambda_k > -1/2$ . A calculation of the inverse Laplace transform for the presented function  $F$  is motivated from the problem of the evaluation of the Müntz polynomials (see [1] and [27]) and is quite essential for the construction of the generalized Gaussian quadrature rules for the Müntz function systems (see [2]). The value of  $n$  is typically required to be around 100, while the numbers  $\lambda_k$ ,  $k = 1, \dots, n$ , are usually distributed uniformly on  $(-1/2, 100)$ .

As it can be checked directly, the function  $F$  has all of its singularities on the real line, and all singularities are poles. It cannot be claimed these poles are simple, since it might happen that some elements of the sequence  $\lambda_k$ ,  $k = 1, \dots, n$ , be the same. It can be checked easily that the function  $F$  satisfies the following property

$$\lim_{|s| \rightarrow +\infty} F(s) = 0.$$

This property allows us to use the method presented in Section 3, as well as the Talbot method.

At first, we present results obtained using Talbot algorithm as implemented in [24]. The interesting part is that the contour which is used for the calculation, for the case of the function (4.1), is always driven by  $\lambda_1$  and the actual value of  $t$ . Due to this simple fact, the algorithm is perfectly well-suited, provided we have only one pole, i.e.,  $\lambda_k = \lambda_1$ ,  $k = 1, \dots, n$ . However, if we choose original settings of our problem, the results are quite poor, since the Talbot contour is not optimized in any way to take into account other singularities in our function (4.1). The Talbot contour, chosen for our function, is presented in Figure 4.1.

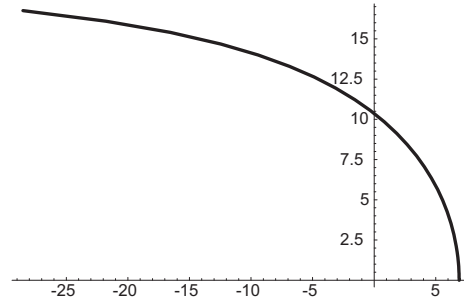


Fig.4.1. Talbot contour for the inversion of function (4.1) for  $t = 1$ , with parameters  $\lambda = 6.4, \sigma = 0.5, \nu = 1$ .

The parameters of the Talbot curve (2.3), according to [24] provided we are working with double precision format, can be calculated in the following way

$$\sigma = \max\{0, -\lambda_1\}, \quad \lambda = 6.4/t, \quad \nu = 1.$$

Choosing some random  $\lambda_k, k = 1, \dots, n$ , uniformly distributed in the interval  $(-1/2, 100)$ , the relative error of the Talbot algorithm, for  $t = 1$ , is of order  $10^6$ , with  $10^4$  nodes in the quadrature rule (2.4).

First we discuss the change of the singularities with respect to  $t$  in the method given by (3.3). Using (3.4), for our example we have

$$F_1(u; \sigma, t) = F\left(\sigma + \frac{iu}{t}\right) = \frac{-it}{u - it(\sigma + \lambda_n)} \prod_{k=1}^{n-1} \frac{u - it(\sigma - \lambda_k - 1)}{u - it(\sigma + \lambda_k)}$$

and

$$\begin{aligned} F_2(u; \sigma, a, t) &= F\left(\sigma + \frac{ia}{t} - \frac{u}{t}\right) \\ &= \frac{-t}{u - t(\sigma + \lambda_n) - ia} \prod_{k=1}^{n-1} \frac{u - t(\sigma - \lambda_k - 1) - ia}{u - t(\sigma + \lambda_k) - ia}. \end{aligned}$$

We note that for  $t \rightarrow 0^+$ , almost all singularities of  $F_1$  are grouped and very close to the number zero. Such behavior is not what we would like, since it would harm the computation of the integral on the interval which has zero as its left boundary. To avoid this, we can choose  $\sigma = -\lambda_1 + 1/t$ , and for any  $t > 0$  we have that the distance between closest singularity and the interval  $(0, a)$  is at least one. There is also one more effect which appears. That is an accumulation of the zeros of the function  $F$  around the point zero or, if we are using  $\sigma = -\lambda_1 + 1/t$ , around the

point one on the imaginary axes. This fact gives a possibility for an easier calculation since zeros and poles of the function  $F$ , loosely speaking, are cancelling each other.

When  $t$  tends to  $+\infty$ , the situation seems to be better since our singularities become more and more sparsely distributed on the imaginary axes, which allows stable calculation with the smaller effort.

The same thing is happening with the function  $F_2$ , however, for  $t \rightarrow 0^+$ . The zeros and poles are accumulated around the point  $1 + ia$ , and are “cancelling” each other. When  $t$  tends to  $+\infty$ , the singularities are sparsely distributed on the line  $\text{Im}(s) = a$ ,  $\text{Re}(s) \geq 1$ . This situation can be harmful for the computation since, until  $t$  is not big enough, density of poles can result in huge values of the function  $F_2$  on the path of integration. This suggests that  $a$  has to be chosen large in order to avoid this phenomenon.

In order to be able to compare results with exact value of the inverse Laplace transform we focus our attention to the example  $\lambda_k = k - 1$ ,  $k = 1, \dots, n$ , in which case we can prove the following simple auxiliary result:

**Lemma 4.1** *If  $\lambda_k = k - 1$ ,  $k = 1, \dots, n$ , and  $F$  is given by (4.1), we have*

$$f(t) = (-1)^n \sum_{k=0}^n (-1)^k e^{-kt} \binom{2k}{k} \binom{n+k}{n-k}, \quad t > 0. \quad (4.2)$$

*Proof.* Direct computation.

This concrete sequence  $\lambda_k = k - 1$ ,  $k = 1, \dots, n - 1$ , is an illustrative example, since  $\lambda_k$ ,  $k = 1, \dots, n$ , are uniformly distributed on  $[0, n - 1]$ . To get the feeling what are we dealing we give the graphic of the inverse function  $f$  in Figure 4.2.

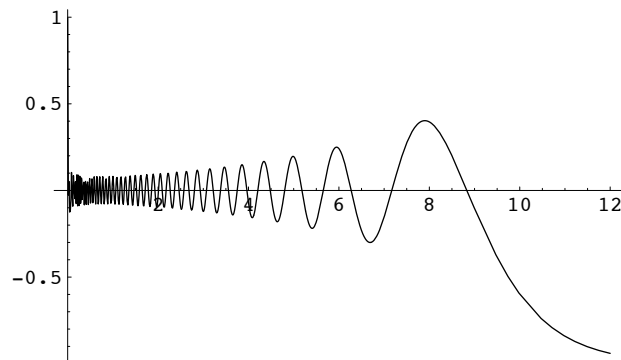


Fig.4.2. Graph of the function  $t \mapsto f(t)$  given by (4.2)

$\log_{10} t$	$f(t)$	Absolute error	contour	$n_v$	total points
-5	0.9034	$10^{-16}$	0, 3, 10	20, 20, 20	60
-4	0.2297	$10^{-15}$	0, 3, 10	20, 20, 20	60
-3	0.2223	$10^{-14}$	0, 3, 6, 10	20, 20, 20, 20	80
-2	0.1743	$10^{-13}$	0, 3, 6, 10, 30	20, 20, 20, 20, 20	100
-1	0.3659(-1)	$10^{-15}$	0, 3, 6, 10, 18, 30, 60	20, 20, 20, 20, 20, 20, 20	140
0	0.8108(-1)	$10^{-15}$	0, 3, 7, 14, 25, 40, 70, 110	20, 20, 20, 20, 20, 20, 20, 20	160
1	-0.5986	$10^{-16}$	0, 3, 9, 40	20, 20, 20, 20	80
2	-1.	$10^{-15}$	0, 3, 9, 40	20, 20, 20, 10	70
4	-1.	$10^{-15}$	0, 3, 18	20, 20, 10	50
5	-1.	$10^{-15}$	0, 3, 20	20, 20, 10	50

Table 4.1. Results of the experiments performed on the function (4.1)

Table 4.1, represents the results of the calculation. The first column represents the actual value of  $t$  at which the function is being evaluated. The second column gives the actual value of the function. The third column represents an absolute error with respect to the exact value of the calculated using (4.2). In the fourth column we give the contour which is used for the integration in the formula (3.3). For example, for  $t = 10^{-5}$ , we have  $a_1 = 0$ ,  $b_1 = a_2 = 3$ ,  $b_2 = a = 10$ . The fifth column shows the number of points in the quadrature rules used on any part of the contour. For example for  $t = 10^{-5}$  we have 20 points in the Gauss-Legendre quadrature rules on  $(0, 3)$  and  $(3, 10)$ , and exactly the same number of points in the Gauss-Laguerre quadrature rule on  $(10, 10 - i\infty)$ . The last column represents the total number of function evaluations we used to achieve the requested accuracy.

As it can be seen the results are in a good correlation with the consideration we had in this and the previous section.

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